

COMPLEXITY AND ALGORITHMS FOR EULER CHARACTERISTIC OF SIMPLICIAL COMPLEXES

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ABSTRACT. We consider the problem of computing the Euler characteristic of an abstract simplicial complex given by its vertices and facets. We show that this problem is $\#P$ -complete and present two new practical algorithms for computing Euler characteristic. The two new algorithms are derived using combinatorial commutative algebra and we also give a second description of them that requires no algebra. We present experiments showing that the two new algorithms can be implemented to be faster than previous Euler characteristic implementations by a large margin.

1. INTRODUCTION

The Euler characteristic of a topological space is an invariant used in a variety of contexts such as category theory, algebraic geometry and differential geometry. In combinatorics, the Euler characteristic of a simplicial complex is related to the Möbius function of a poset and the inclusion-exclusion principle [16] and to valuations on simplicial complexes [12] to name but a few connections.

The *reduced Euler characteristic* of an abstract simplicial complex¹ Δ is

$$\tilde{\chi}(\Delta) \stackrel{\text{def}}{=} - \sum_{\sigma \in \Delta} (-1)^{|\sigma|} = -f_{-1} + f_0 - f_1 + f_2 - f_3 + \cdots$$

where f_i denotes the number of faces (elements) of dimension i in the complex.² The dimension of a face σ is $\dim(\sigma) \stackrel{\text{def}}{=} |\sigma| - 1$.

In Section 2 we prove that computing the Euler characteristic of a simplicial complex specified by its vertices and facets is $\#P$ -complete, which is a formal way of stating that Euler characteristic is a difficult computational problem. We also show that the problem of deciding if $\tilde{\chi}(\Delta) = 0$ is not in NP unless $\#P$ is no harder than NP . This answers two open questions posed by Kaibel and Pfetsch in their survey [11].

In Section 3 we introduce two new practical algorithms for computing Euler characteristic. These two algorithms were conceived of in terms of combinatorial commutative algebra, and Section 3 is written solely in terms of algebra. Section 4 independently describes the same two algorithms in terms of simplicial complexes and without any reference to algebra. Section 5 describes how the algebra was translated to simplicial complexes and how doing so brought up interesting mathematics.

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¹An *abstract simplicial complex* Δ is a family of sets closed under taking subset, so if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. This is closely related to the notion of a *simplicial complex* which is a set of polyhedra with certain properties. All complexes in this paper are abstract. See Section 4.1 for further background.

²The usual definition of Euler characteristic is $\chi(\Delta) \stackrel{\text{def}}{=} f_0 - f_1 + f_2 - f_3 + \cdots$. The difference is that $\chi(\Delta)$ does not count the empty set, while $\tilde{\chi}(\Delta)$ does, so $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$. All Euler characteristics in this paper are $\tilde{\chi}(\Delta)$ rather than $\chi(\Delta)$ because that simplifies the formulas.

Finally, Section 6 presents experiments that show that the two new algorithms can be implemented to be faster than previous Euler characteristic implementations by a large margin.

2. THE COMPLEXITY OF EULER CHARACTERISTIC

We describe the complexity class $\#P$ and then prove that Euler characteristic is $\#P$ -complete. This is a precise way of saying that Euler characteristic is a difficult computational problem. We also consider the complexity of decision problems associated to Euler characteristic. See Section 4.1 for basic definitions relating to simplicial complexes.

The complexity of Euler characteristic has been studied before, but not of a simplicial complex specified by its vertices and facets. It has been studied for the case of the input being a CW-complex specified as a circuit [6] in the context of real valued computation and for the input being a sheaf [1] in the context of algebraic geometry.

2.1. The complexity class $\#P$. The complexity class $\#P$ is the set of counting problems associated to decision problems in NP . For example the decision problem “does a logical formula have *some* satisfying assignment of truth-values?” is in NP , while “*how many* satisfying assignments of truth-values does a logical formula have?” is in $\#P$. The former is called **SAT** while the latter is called **$\#SAT$** . A problem is $\#P$ -complete if it is in $\#P$ and any other problem in $\#P$ can be reduced to it in polynomial time.

There is already a list of problems that are known to be $\#P$ -complete, which is very helpful when proving that a new problem is $\#P$ -complete, as then a problem in $\#P$ is $\#P$ -complete if some other $\#P$ -complete problem reduces to it. For example it is known that **$\#SAT$** is $\#P$ -complete even when restricted to formulas with two literals per clause and no negations [18]. A **SAT** formula is a conjunction of clauses, where each clause is a disjunction of some number of literals. For example

$$(a \vee \neg b) \wedge (a \vee c) \wedge (\neg b \vee c),$$

where a , b and c are boolean variables. Here the satisfying truth assignments (a, b, c) are

$$\{(0, 0, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

The output for **SAT** with this formula as input is “yes” since there is a satisfying truth assignment. The output for **$\#SAT$** is “4”, since there are four satisfying truth assignments. The output for **$\#SAT$** does not include the satisfying truth assignments themselves, only the number of them.

The program for the rest of this section is to formally define a problem **EulerChar** in $\#P$ that represents the Euler characteristic problem, and then to prove that **EulerChar** is $\#P$ -complete.

2.2. Euler Characteristic is in $\#P$. The most straightforward way to define **EulerChar** would be to have the input be the facets and vertices of a simplicial complex Δ and have the output be simply $\tilde{\chi}(\Delta)$. It is immediate that this could never be in $\#P$ because $\tilde{\chi}(\Delta)$ can be negative while $\#P$ is a class of *counting* problems so that their output must be a natural number.

To arrive at a satisfactory definition of **EulerChar**, the first step is to observe that

$$(1) \quad \tilde{\chi}(\Delta) = \sum_{\sigma \in \Delta} (-1)^{\dim(\sigma)} = \#(\text{odd faces}) - \#(\text{even faces}).$$

Read $\#(\text{odd faces})$ as “the number of odd faces of Δ ”, where a set is odd if it has an odd number of elements which is to say that its dimension is even. It is not hard to argue that counting the number of even faces is a problem in $\#P$, and that counting the number of odd faces is a problem in $\#P$ as well. Unfortunately, we know of no theorem stating that a difference of two functions in $\#P$ is again in $\#P$. So we must find an alternative way to express the Euler characteristic.

Let Δ have n vertices. Then

$$\#(\text{even faces}) + \#(\text{even non-faces}) = \#(\text{even sets}) = 2^{n-1},$$

which together with Equation (1) implies that

$$\tilde{\chi}(\Delta) + 2^{n-1} = \#(\text{odd faces}) + \#(\text{even non-faces}).$$

Consider the decision problem “does Δ have an odd face or an even non-face?”. This problem is in NP where a certificate of a “yes”-answer is any concrete odd face or even non-face. Define **EulerChar** to be the counting version of this. The input is then the vertices and facets of Δ and the output is the number of odd faces and even non-faces, that is the output is $\tilde{\chi}(\Delta) + 2^{n-1}$. We can subtract 2^{n-1} in polynomial time, which justifies that **EulerChar** represents the problem of computing the Euler characteristic of a simplicial complex.

We conclude that computing Euler characteristic is a problem in #P when expressed formally in the form of the **EulerChar** problem.

2.3. Euler Characteristic is #P-complete. The main result in this section is Theorem 1 which states that **EulerChar** is #P-complete. This is an example of the fact that even trivial problems can have a counting version that is #P-complete. To see that the problem “does Δ have an odd face or an even non-face” is especially trivial, observe that only $\Delta = \emptyset$ fails to have the even face \emptyset .

Theorem 1. **EulerChar** is #P-complete. That is, the problem of computing the Euler characteristic of a simplicial complex given by its vertices and facets is #P-complete.

Proof. We proved in Section 2.2 that **EulerChar** is in #P. We prove the statement of the theorem by showing that the #P-complete problem **#SAT** reduces to **EulerChar**. We introduce an intermediate problem **IndepSum** and prove that **#SAT** reduces to **IndepSum** and then that **IndepSum** reduces to **EulerChar**. Given a SAT formula S , the combination of these two reductions yields a simplicial complex Δ such that $\tilde{\chi}(\Delta)$ is the number of truth assignments that satisfy S .

We need to introduce some terminology. Let the *parity sum* of a set of sets S be $P(S) \stackrel{\text{def}}{=} \sum_{s \in S} (-1)^{|s|}$. For example the parity sum of a simplicial complex Δ is $P(\Delta) = -\tilde{\chi}(\Delta)$. Let G be a simple graph with vertex set V . Then a set of vertices $S \subseteq V$ is *dependent* if it contains both endpoints of some edge of G . Otherwise S is *independent*.

We can now define the problem **IndepSum**. The input of **IndepSum** is the vertices and edges of a graph G , and the output is the parity sum of the set of independent sets of G .

IndepSum reduces to EulerChar: Let G be a simple graph with vertex set V and define a simplicial complex Δ such that the facets of Δ are the complements of the edges of G . Then a set of vertices is a face of Δ if and only if the complement contains an edge, that is if and only if the complement is a dependent set of G .

If $\sigma \subseteq V$ then let $\bar{\sigma} \stackrel{\text{def}}{=} V \setminus \sigma$ be its complement. Let D be the set of dependent sets of G or equivalently $D = \{\bar{\sigma} \mid \sigma \in \Delta\}$. Since $(-1)^{|\bar{\sigma}|} = (-1)^n (-1)^{|\sigma|}$ and $-\tilde{\chi}(\Delta)$ is the parity sum of the faces of Δ , we get that

$$P(D) = (-1)^n P(\Delta) = -(-1)^n \tilde{\chi}(\Delta).$$

Let I be the set of independent sets. Every set is either dependent or independent, so assuming that $V \neq \emptyset$ we get that $(\mathcal{P}(V))$ is the set of all subsets of V

$$P(I) + P(D) = P(I \cup D) = P(\mathcal{P}(V)) = 0.$$

We conclude that $P(I) = -P(D) = (-1)^n \tilde{\chi}(\Delta)$ so that **IndepSum** reduces to **EulerChar**.

#SAT reduces to IndepSum: Let S be a SAT formula. We construct a graph G such that the number of truth assignments that satisfy S equals $(-1)^n$ times the parity sum of the independent sets of G .

Let v_1, \dots, v_n be the variables that appear in the formula S and let c_1, \dots, c_k be the clauses that appear in S . For each variable v_i we introduce a 3-clique with vertices T_i , F_i and D_i . Here T_i represents v_i having the value true and F_i represents false. For each clause c_j we introduce a vertex C_j . If the literal v_i appears in clause c_j with no negation, then we add an edge between T_i and C_j . If the literal $\neg v_i$ appears in clause c_j then we add an edge between F_i and C_j . We claim that the number of truth assignments that satisfy S equals $(-1)^n$ times the parity sum of the independent sets of this graph G .

For concreteness, consider the SAT formula

$$(v_1 \vee \neg v_2) \wedge (v_1 \vee v_3) \wedge (\neg v_2 \vee v_3).$$

The graph that we construct based on this formula is shown in Figure 1.

Let A be the set of vertices named D_i or C_j and let B be the set of vertices named T_i or F_j . Let I be the set of independent sets of G and let I_B be the set of independent sets that are subsets of B . Define the function $p: I \rightarrow I_B$ by $p(d) \stackrel{\text{def}}{=} d \setminus A$.

We are going to prove that *i*) if $p^{-1}(d) = \{d\}$ then $|d| = n$ and *ii*) that the set of such d is in bijection with the set of truth assignments that satisfy S . We are also going to prove *iii*) that if $p^{-1}(d) \neq \{d\}$ then the parity sum $P(p^{-1}(d))$ is zero. These three statements imply that

$$\#(\text{satisfying truth assignments}) = (-1)^n \sum_{d \in I_B} P(p^{-1}(d)) = (-1)^n P(I),$$

where we use that $\{p^{-1}(d)\}_{d \in I_B}$ is a partition of I . It only remains to prove *i*), *ii*) and *iii*).

i) If $p^{-1}(d) = \{d\}$ then $|d| = n$: Suppose that $d \in I_B$ such that $p^{-1}(d) = \{d\}$. Pick some variable v_i . Then $d \cup \{D_i\}$ is dependent since otherwise it would be an element of $p^{-1}(d)$. As D_i is only adjacent to T_i and F_i , it must be the case that d contains one of T_i and F_i . It cannot contain both as there is an edge between them. If d contains T_i then we assign the value true to v_i and otherwise d contains F_i and we assign the value false to v_i . In this way d encodes a truth assignment to the variables of the formula S .

ii) $\{d \in I_b \mid p^{-1}(d) = \{d\}\}$ is in bijection with the satisfying truth assignments of S : Pick some clause c_j . Then $d \cup \{C_j\}$ is dependent so d must contain some T_i or F_i that is adjacent to C_j and this implies that the truth assignment that d represents satisfies the clause c_j . This

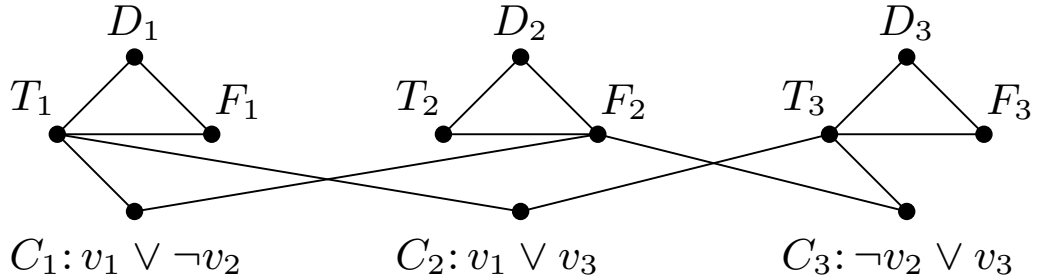


FIGURE 1. Illustration for the proof of Theorem 1.

establishes a bijection between the set of $d \in I_B$ such that $p^{-1}(d) = \{d\}$ and the set of truth assignments that satisfy S .

iii) if $p^{-1}(d) \neq \{d\}$ then $P(p^{-1}(d)) = 0$: Let $d \in I_b$ such that $p^{-1}(d) \neq \{d\}$. Then we can pick some vertex $a \in A$ such that $d \cup \{a\} \in p^{-1}(d)$. Then d does not contain any vertex that is adjacent to a , and there are no edges between the elements of A , so if we let

$$E \stackrel{\text{def}}{=} \{h \in p^{-1}(d) \mid a \notin h\}, \quad F \stackrel{\text{def}}{=} \{h \in p^{-1}(d) \mid a \in h\}$$

then $h \mapsto h \cup \{a\}$ is a bijection from E to F so that $P(F) = -P(E)$. As $\{E, F\}$ is a partition of $p^{-1}(d)$ we then get that $P(p^{-1}(d)) = P(E) + P(F) = 0$. \square

The Euler characteristic is the alternating sum of the entries of the f -vector, so Euler characteristic reduces to f -vector. So we get the following result of Kaibel and Pfetsch [11] as a corollary.

Corollary 2. The problem of computing the f -vector of a simplicial complex given by its vertices and facets is $\#P$ -hard.

2.4. Decision Problems. In this section we investigate the complexity of decision problems associated to Euler characteristic. Kaibel and Pfetsch pose the open problem of whether deciding $\tilde{\chi}(\Delta) = 0$ lies in NP [11]. Theorem 3 answers this question in the negative unless $\#P$ is no harder than NP . It is a central conjecture of computational complexity theory that $\#P$ is harder than NP .

Theorem 3. Let \mathbf{E}_0 be the problem of deciding if $\tilde{\chi}(\Delta) = 0$ where Δ is a simplicial complex given by its facets and vertices. Then \mathbf{E}_0 is co-NP -hard. Also, \mathbf{E}_0 does not lie in NP unless $\#P$ is no harder than NP .

Proof. Let $\mathbf{E}_{<}$ be as in Lemma 5 and assume that \mathbf{E}_0 is in NP . Then $\mathbf{E}_{<}$ is in $\text{NP} \cap \text{co-NP}$ by Lemma 5. This allows us to compute Euler characteristic in $\text{NP} \cap \text{co-NP}$ using binary search. Euler characteristic is $\#P$ -complete by Theorem 1, so then $\#P$ is no harder than $\text{NP} \cap \text{co-NP}$.

\mathbf{E}_0 is co-NP -hard: Let S be a SAT formula. The proof of Theorem 1 constructs a simplicial complex Δ such that $\tilde{\chi}(\Delta)$ is the number of satisfying truth assignments to S . So the NP -complete problem SAT reduces to the decision problem $\tilde{\chi}(\Delta) \neq 0$. So the complement of \mathbf{E}_0 is NP -hard, which implies that \mathbf{E}_0 is co-NP -hard. \square

This leaves an open problem of whether \mathbf{E}_0 lies in co-NP , since Theorem 3 does not rule that out. If \mathbf{E}_0 does lie in co-NP , it would then be proven that $\text{NP} \neq \text{co-NP}$ unless $\#P$ is no harder than NP since \mathbf{E}_0 would then lie in co-NP and not in NP . It is an open problem whether $\text{NP} \neq \text{co-NP}$.

Theorem 4. Let $\mathbf{E}_{>0}$ be the problem of deciding if $\tilde{\chi}(\Delta) > 0$ where Δ is a simplicial complex given by its facets and vertices. Then $\mathbf{E}_{>0}$ is $\#P$ -hard.

Proof. Let $\mathbf{E}_{>}$ and $\mathbf{E}_{<}$ be as in Lemma 5. The argument used to prove the equivalence of \mathbf{E}_0 and $\mathbf{E}_{=}$ in Lemma 5 also works to show that $\mathbf{E}_{>0}$ and $\mathbf{E}_{>}$ are equivalent. Then in particular both $\mathbf{E}_{>}$ and $\mathbf{E}_{<}$ reduce to $\mathbf{E}_{>0}$, so Euler characteristic reduces to $\mathbf{E}_{>0}$ using binary search. Euler characteristic is $\#P$ -complete by Theorem 1 so then $\mathbf{E}_{>0}$ is $\#P$ -hard. \square

Lemma 5. Consider the following decision problems, where Δ is a simplicial complex given by its facets and vertices and k is an integer,

$$\begin{aligned} \mathbf{E}_0: \quad & \tilde{\chi}(\Delta) = 0, & \mathbf{E}_{<}: \quad & \tilde{\chi}(\Delta) < k, \\ \mathbf{E}_{=}: \quad & \tilde{\chi}(\Delta) = k, & \mathbf{E}_{>}: \quad & \tilde{\chi}(\Delta) > k. \end{aligned}$$

If any one of these problems are in NP then they are all in $\text{NP} \cap \text{co-NP}$. There are polynomial time reductions in both directions between \mathbf{E}_0 and $\mathbf{E}_{=}$ and between $\mathbf{E}_{<}$ and $\mathbf{E}_{>}$.

Proof. E_0 and $E_=$ are equivalent: Assume without loss of generality that $\Delta \neq \emptyset$. Use Lemma 6 to construct a polynomial size simplicial complex Γ such that $\tilde{\chi}(\Gamma) = k - 1$ and $\Delta \cap \Gamma = \{\emptyset\}$. Then $\Psi \stackrel{\text{def}}{=} \Delta \cup \Gamma$ has $\tilde{\chi}(\Psi) = \tilde{\chi}(\Delta) + \tilde{\chi}(\Gamma) + 1 = \tilde{\chi}(\Delta) + k$ so $\tilde{\chi}(\Delta) = 0$ if and only if $\tilde{\chi}(\Psi) = k$. This gives a polynomial time reduction in both directions between E_0 and $E_=$ and also shows that E_0 is in NP or co-NP if and only if $E_=$ is in NP or co-NP respectively.

$E_<$ and $E_>$ are equivalent: Let Γ be a simplicial complex such that $\tilde{\chi}(\Gamma) = -1$ and such that the set of vertices of Γ is disjoint from the set of vertices of Δ . Let $\Psi \stackrel{\text{def}}{=} \Delta \oplus \Gamma$ as in Theorem 15 whereby $\tilde{\chi}(\Psi) = -\tilde{\chi}(\Delta)$. Then $\tilde{\chi}(\Delta) < k$ if and only if $\tilde{\chi}(\Psi) > -k$. This gives a polynomial time reduction in both directions between $E_<$ and $E_>$ and also shows that $E_<$ is in NP or co-NP if and only if $E_>$ is in NP or co-NP respectively.

$E_=$ in NP $\Leftrightarrow E_<$ in NP: If $E_=$ is in NP then we can certify the exact value of $\tilde{\chi}(\Delta)$, which will also serve as a certificate for $\tilde{\chi}(\Delta) > k$. If $E_<$ is in NP, then so is $E_>$ in which case we can certify that $k - 1 < \tilde{\chi}(\Delta) < k + 1$ which serves as a certificate of $\tilde{\chi}(\Delta) = k$.

If one problem is in NP, then they are all in co-NP: Assume that one of E_0 , $E_=$, $E_<$ and $E_>$ is in NP. Then we have shown that they are all in NP. So we know that $E_=$ is in NP, which allows us to certify the exact value of $\tilde{\chi}(\Delta)$. This also serves as a certificate for when $\tilde{\chi}(\Delta) < k$ is not true and when $\tilde{\chi}(\Delta) = k$ is not true, so $E_<$ and $E_=$ are in co-NP. We have already proven that this implies that $E_>$ and E_0 are also in co-NP. \square

Lemma 6 constructs a simplicial complex with a given Euler characteristic k such that the bit size of the complex is bounded by a fixed polynomial in the bit size of the Euler characteristic which is $\lceil \log k \rceil$. It is necessary to bound the bit size of the simplicial complex in this way since otherwise the proof of Lemma 5 would not go through.

Lemma 6. Let k be an integer. Then there is a simplicial complex Δ such that $\tilde{\chi}(\Delta) = k$ and Δ has no more facets and no more vertices than $2l^2 + 3l + 7$ where $l = \lceil \log_2(|k|) \rceil$ or $l = 0$ if $k = 0$.

Proof. The proof is based on inclusion-exclusion along with Theorem 15.

The case $k = 0, 1$: Let $\Delta_0 \stackrel{\text{def}}{=} \emptyset$ and $\Delta_1 \stackrel{\text{def}}{=} \langle \{t_1\}, \{t_2\} \rangle$.

The case $k = 2^n$: Let n be a positive integer and define $\Gamma_i \stackrel{\text{def}}{=} \langle \{x_{ni1}\}, \{x_{ni2}\}, \{x_{ni3}\} \rangle$. Let $\Psi_n \stackrel{\text{def}}{=} \Gamma_1 \oplus \dots \oplus \Gamma_n$. We have $\tilde{\chi}(\Gamma_i) = 2$ so $\tilde{\chi}(\Psi_n) = 2^n$. Also Ψ_n has $3n$ vertices and $3n$ facets.

The case $k > 0$: Write $k = (b_l \dots b_0)_2$ in binary such that $k = \sum_{n=0}^l b_n 2^n$, $b_n \in \{0, 1\}$ and $b_l \neq 0$. This implies that $l = \lceil \log_2(k) \rceil$.

Let W be a finite set of non-empty simplicial complexes with disjoint vertex sets. Then the only face in more than one element of W is \emptyset , so $\tilde{\chi}(\cup W) = \sum_{A \in W} \tilde{\chi}(A) + |W| - 1$. So for $W'_n \stackrel{\text{def}}{=} \{\Psi_n | b_n = 1\}$ we have $\tilde{\chi}(\cup W'_n) = k + |W'_n| - 1$. Let $p \stackrel{\text{def}}{=} |W'_n| + 1$ and

$$\Phi \stackrel{\text{def}}{=} \langle \{y_1\} \dots, \{y_p\} \rangle \oplus \langle \{a, b\}, \{a, c\}, \{b, c\} \rangle.$$

Then $\tilde{\chi}(\Phi) = (p - 1) * (-1) = -|W'_n|$ by Theorem 15. Observe that Φ has no more facets and no more vertices than $|W'_n| + 4 \leq l + 4$. Let $W_n \stackrel{\text{def}}{=} W'_n \cup \{\Phi\}$ and $\Delta_k \stackrel{\text{def}}{=} \cup W_n$. Then

$$\tilde{\chi}(\Delta_k) = \tilde{\chi}(\cup W_n) = (k + \tilde{\chi}(\Phi)) + (|W'_n| + 1) - 1 = k.$$

Observe that Δ_k has no more vertices and no more facets than

$$(l + 4) + \sum_{n=0}^l 3n = l + 4 + \frac{3}{2}(l(l + 1)).$$

The case $k < 0$: Let $\Omega \stackrel{\text{def}}{=} \langle \{z_1, z_2\}, \{z_1, z_3\}, \{z_2, z_3\} \rangle$ and observe that $\tilde{\chi}(\Omega) = -1$. Let $\Delta_k \stackrel{\text{def}}{=} \Omega \oplus \Delta_{-k}$ so that $\tilde{\chi}(\Delta_k) = -\tilde{\chi}(\Delta_{-k}) = k$. Observe that Δ_k has no more vertices and no more facets than

$$l + 4 + \frac{3}{2}(l(l+1)) + 3 = \frac{3}{2}l^2 + \frac{5}{2}l + 7 \leq 2l^2 + 3l + 7. \quad \square$$

3. ALGEBRAIC ALGORITHMS FOR EULER CHARACTERISTIC

In this section we describe two new algorithms for computing Euler characteristic of a simplicial complex using algebraic techniques. In Section 4 we present these same two algorithms in the language of simplicial complexes. This section is independent from Section 4 and it only uses algebra.

More precisely the two algorithms we present in this section compute the coefficient of $\mathfrak{x} \stackrel{\text{def}}{=} x_1 \cdots x_n$ in the multigraded Hilbert-Poincaré series numerator $H(I)$ of a square free monomial ideal I . In Section 5 we show that this is equivalent to computing the Euler characteristic of a simplicial complex. For that reason we define

$$\tilde{\chi}(I) \stackrel{\text{def}}{=} \text{coefficient of } \mathfrak{x} \text{ in } H(I).$$

The summary of what Section 5 shows in detail is that given a simplicial complex Δ we can define a monomial ideal I such that $\tilde{\chi}(I) = \tilde{\chi}(\Delta)$. Computing I from Δ takes little time. See Section 5 for details on the relationship between the algebraic algorithms in this section and the simplicial algorithms in Section 4.

3.1. Background and Notation. We work in a polynomial ring $\kappa[x_1, \dots, x_n]$ over a field κ and with variables x_1, \dots, x_n . A *monomial ideal* is a polynomial ideal generated by monomials. Let I be a monomial ideal. Each monomial ideal has a unique minimal set of monomial generators $\min(I)$. The *exponent vector* of a monomial m is a vector v such that $m = \prod_{i=1}^n x_i^{v_i}$. A monomial has full support if it is divisible by \mathfrak{x} . A monomial ideal has full support if $\text{lcm}(\min(I))$ has full support. The colon of two monomials a and b is $a : b \stackrel{\text{def}}{=} \frac{\text{lcm}(a,b)}{b}$. The colon of a monomial ideal I by a monomial a is

$$I : a = \{m \mid ma \in I\} = \langle m : a \mid m \in I \text{ and } m \text{ is a monomial} \rangle.$$

The *multigraded Hilbert-Poincaré series* $\text{hilb}(I)$ is the possibly infinite sum of standard monomials of I , that is $\text{hilb}(I) \stackrel{\text{def}}{=} \sum_{m \notin I} m$ where the sum is taken over monic monomials m . The multigraded Hilbert-Poincaré series can be written as a rational function

$$\text{hilb}(I) = \frac{H(I)}{(1-x_1) \cdots (1-x_n)}$$

where $H(I)$ is a polynomial called the *multigraded Hilbert-Poincaré series numerator*.

3.2. Divide... Both algorithms we present are divide-and-conquer algorithms. They take a monomial ideal I and split it into two simpler monomial ideals J and K such that $\tilde{\chi}(I) = \tilde{\chi}(J) + \tilde{\chi}(K)$. This process proceeds recursively until all the remaining ideals are simple enough that they can be processed directly.

Let p be a square free monomial and let I be a square free monomial ideal. The divide steps for the two algorithms are derived from the equation

$$(2) \quad \text{hilb}(I) = (\text{hilb}(I : p))p + \text{hilb}(I + \langle p \rangle).$$

By giving these three Hilbert-Poincaré series the same denominator $(1 - x_1) \cdots (1 - x_n)$, we get a similar equation for the Hilbert-Poincaré series numerators

$$H(I) = (H(I : p))p + H(I + \langle p \rangle).$$

By considering the coefficient of \mathfrak{x} on both sides, we then get that

$$\tilde{\chi}(I) = \tilde{\chi}((I : p)p) + \tilde{\chi}(I + \langle p \rangle).$$

It would simplify this expression if we could write $\tilde{\chi}(I : p)$ instead of $\tilde{\chi}((I : p)p)$. This does not work directly since in general $\tilde{\chi}(I : p)$ will be zero since $I : p$ will not have full support. We are working within a polynomial ring $R \stackrel{\text{def}}{=} \kappa[x_1, \dots, x_n]$. We will argue that if we change the ring that $I : p$ is embedded in, then it becomes true that $\tilde{\chi}(I : p) = \tilde{\chi}((I : p)p)$.

Since the variables that divide p do not appear in $\min(I : p)$ we can embed $I' \stackrel{\text{def}}{=} I : p$ into a ring $R' \stackrel{\text{def}}{=} \kappa[P]$ where $P \stackrel{\text{def}}{=} \{x_i \mid x_i \text{ does not divide } p\}$. Let $\mathfrak{x}' \stackrel{\text{def}}{=} \prod P = \frac{\mathfrak{x}}{p}$ be the product of the variables in R' . Then $\tilde{\chi}(I') = \tilde{\chi}((I : p)p)$ so we consider $I : p$ to be embedded in R' , which gives us the final equation behind splitting

$$(3) \quad \tilde{\chi}(I) = \tilde{\chi}(I : p) + \tilde{\chi}(I + \langle p \rangle).$$

There are two different algorithms for Hilbert-Poincaré series that are based on Equation (2). We present two analogous algorithms for Euler characteristic that are based on Equation (3).

The Hilbert-Poincaré series algorithm due to Bigatti, Conti, Robbiano and Traverso [5, 4] uses Equation (2) directly as we have written it. It is a divide-and-conquer algorithm that splits a monomial ideal I into the two simpler monomial ideals $I : p$ and $I + \langle p \rangle$. We call this algorithm the *BCRT algorithm for Hilbert-Poincaré series*. In analogy with that algorithm, we propose a BCRT algorithm for Euler characteristic that uses Equation (3) directly as written – it splits I into the two simpler ideals $I : p$ and $I + \langle p \rangle$. We call it the *algebraic BCRT algorithm for Euler characteristic*.

We call the p in Equation (3) the *pivot*. Section 3.5 explores strategies for selecting pivots.

There is also a Hilbert-Poincaré series algorithm due to Dave Bayer and Michael Stillman [3] that we will call the *DBMS algorithm for Hilbert-Poincaré series*. It is based on writing Equation (2) as

$$\text{hilb}(I + \langle p \rangle) = \text{hilb}(I) - \text{hilb}(I : p).$$

Given an ideal J , the idea is to choose $p \in \min(J)$ and let $I \stackrel{\text{def}}{=} \langle \min(J) \setminus \{p\} \rangle$ such that

$$\text{hilb}(J) = \text{hilb}(I + \langle p \rangle) = \text{hilb}(I) - \text{hilb}(I : p).$$

In this way J splits into the two simpler ideals I and $I : p$. In the same way, we can rewrite Equation (3) as

$$(4) \quad \tilde{\chi}(J) = \tilde{\chi}(I + \langle p \rangle) = \tilde{\chi}(I) - \tilde{\chi}(I : p).$$

We propose a DBMS algorithm for Euler characteristic that uses this equation to split J into I and $I : p$. We call it the *algebraic DBMS algorithm for Euler characteristic*. Note that the pivots in the DBMS algorithm are minimal generators of the ideal, which would not make sense for the BCRT algorithm.

3.3. ... and Conquer. A square free monomial ideal I is a base case for both algorithms when Theorem 7 applies. Note that the improvements in Section 3.6 enables further base cases.

Theorem 7. Let I be a square free monomial ideal. Then

- (1) if I does not have full support then $\tilde{\chi}(I) = 0$,

- (2) if I has full support and the minimal generators $\min(I)$ of I are pairwise prime monomials then $\tilde{\chi}(I) = (-1)^{|\min(I)|}$,
- (3) if I has full support and $|\min(I)| = 2$ then $\tilde{\chi}(I) = 1$.

Proof. (1): All the monomials with non-zero coefficient in $H(I)$ can be written as $\text{lcm}(M)$ for some $M \subseteq \min(I)$. If I does not have full support then neither does any monomial of the form $\text{lcm}(M)$. Then \mathbf{x} must have a zero coefficient since it has full support.

(2): If the elements of $\min(I)$ are relatively prime then $H(I) = \prod_{m \in \min(I)} (1 - m)$. As the ideal has full support we then get that $\mathbf{x} = \prod_{m \in \min(I)} m$ so the coefficient of \mathbf{x} is $(-1)^{|\min(I)|}$.

(3): If $\{a, b\} \stackrel{\text{def}}{=} \min(I)$ and $g \stackrel{\text{def}}{=} \gcd(a, b)$ then $I = g \langle c, d \rangle$ where $c \stackrel{\text{def}}{=} a : g$ and $d \stackrel{\text{def}}{=} b : g$. As c and d are relatively prime by construction, we get that

$$H(I) = g H(\langle c, d \rangle) = g(1 - c)(1 - d) = cdg - cg - dg + g.$$

So the coefficient of $\mathbf{x} = cdg$ in $H(I)$ is 1. □

These base cases improve on the ones for the Hilbert-Poincaré series algorithms in that they apply more often and can be processed more quickly. For example the Hilbert-Poincaré series algorithms as well have a base case when the elements of $\min(I)$ are relatively prime, but it takes exponential time to process that base case since $\prod_{m \in \min(I)} (1 - m)$ can have $2^{|\min(I)|}$ terms. Here all that is required is to determine if $|\min(I)|$ is even or odd. The base case when I does not have full support does not exist for the Hilbert-Poincaré series algorithms.

3.4. Termination and Complexity. It is clear that the DBMS algorithm terminates since $|\min(I)|$ decreases strictly at each step. For the BCRT algorithm, termination requires that we choose the pivots p such that $1 \neq p \notin I$ since otherwise we get an infinite number of steps from $I = I : p$ or $I = I + \langle p \rangle$.

If we cannot choose a p such that $1 \neq p \notin I$ then $I = \langle 1 \rangle$ which is a base case. If $1 \neq p \notin I$ and I has full support, then $I \subsetneq I + \langle p \rangle$ and $I \subsetneq I : p$. So if the BCRT algorithm does not terminate then there would be an infinite sequence of strictly increasing ideals in contradiction to the fact that the ambient polynomial ring is Noetherian. So both algorithms terminate.

We have seen that the DBMS algorithm gets rid of at least one minimal generator at each step. The BCRT algorithm gets rid of at least one variable at each step if the pivot is chosen to be a single variable x_i . It is immediate that x_i is not a variable of $I : x_i$. To see that x_i can also be removed from $I + \langle x_i \rangle$, observe that the only minimal generator that is divisible by x_i is x_i itself, so x_i is $(I + \langle x_i \rangle)$ -independent from the other variables and so can be removed using the independent variables technique from Section 3.6.

Since the Euler characteristic problem is #P-complete we expect both algorithms to run in at least single exponential time. Using the transpose technique from Section 3.6, we can interchange the number of variables n with the number of minimal generators $|\min(I)|$. So if $l \stackrel{\text{def}}{=} \min(n, |\min(I)|)$ then both the BCRT and DBMS algorithms have $O(q2^l)$ asymptotic worst case time complexity where q is a polynomial. So both algorithms run in single exponential time. We expect that a more careful analysis could reduce the base of the exponential.

3.5. Pivot Selection. We have proven that the BCRT and DBMS algorithms terminate in a finite amount of time. To be useful in practice, the amount of time until termination should be small rather than just finite. The strategy used for selecting pivots when splitting an ideal has a significant impact on performance. We describe several different pivot selection strategies here and compare them empirically in Section 6.

A *popular variable* is a variable that divides a maximum number of minimal generators of the ideal. In other words, a popular variable x_i maximizes $|\min(I) \cap \langle x_i \rangle|$. A *rare variable* is a variable x_i that minimizes $|\min(I) \cap \langle x_i \rangle|$ with the constraint that $x_i \notin \min(I)$.

If there are several candidate pivots that fit a given pivot selection strategy, then the pivot used is chosen in an arbitrary deterministic way among the tied candidates.

BCRT Pivot Selection Strategies. Recall that BCRT pivots p satisfy $1 \neq p \notin I$.

popvar: The pivot is a popular variable.

rarevar: The pivot is a rare variable.

random: The pivot is a random variable e such that $e \notin \min(I)$.

popgcd: Let x_i be a popular variable. The pivot is the gcd of three minimal generators chosen uniformly at random among those minimal generators that x_i divides.

The strategies **popvar** and **popgcd** have been found to work well for the BCRT algorithm for Hilbert-Poincaré series, so we also try them here. **rarevar** and **random** have been included to have something to compare to.

DBMS Pivot Selection Strategies. Recall that DBMS pivots are elements of $\min(I)$.

rarevar: The pivot is a minimal generator divisible by a rare variable.

popvar: The pivot is a minimal generator divisible by a popular variable.

maxsupp: The pivot is a minimal generator with maximum support.

minsupp: The pivot is a minimal generator with minimum support.

random: The pivot is a minimal generator chosen uniformly at random.

rarest: The pivot is a generator that is divisible by a maximum number of rare variables. Break ties by picking the generator that is divisible by the maximum number of second-most-rare variables and so on.

raremax: The pivot is chosen according to **rarevar** where ties are broken according to **maxsupp**.

3.6. Improvements. We present several improvements to the DBMS and BCRT algorithms.

Independent Variables. We say that two subsets $A, B \subseteq \{x_1, \dots, x_n\}$ are *I-independent* if $\min(I)$ is the disjoint union of $\min(I \cap \kappa[A])$ and $\min(I \cap \kappa[B])$. This is another way of saying that the minimal generators of I can be partitioned into two subsets such that every generator in one set is relatively prime to every generator from the other set. If A and B are *I-independent* then

$$\text{hilb}(I) = \text{hilb}(I \cap \kappa[A]) \cdot \text{hilb}(I \cap \kappa[B]).$$

This is a standard technique for computing Hilbert-Poincaré series [3, 5, 15], and it applies to computing Euler characteristic as well since

$$\tilde{\chi}(I) = \tilde{\chi}(I \cap \kappa[A]) \cdot \tilde{\chi}(I \cap \kappa[B]).$$

The existence of an *I-independent* pair (A, B) can be determined in nearly linear time [15], and such a pair can cut down on computation time dramatically. However, independence rarely occurs for large random ideals and detecting it does take some time, so this technique is not worth it unless there is some reason to suspect that it will apply to a given ideal.

Eliminate Unique Variables. Suppose x_i divides only one minimal generator $p \in \min(J)$. Use Equation (4) with p as the pivot to get that

$$\tilde{\chi}(J) = \tilde{\chi}(I + \langle p \rangle) = \tilde{\chi}(I) - \tilde{\chi}(I : p) = -\tilde{\chi}(I : p),$$

since $\tilde{\chi}(I) = 0$ as no generator of I is divisible by x_i so I does not have full support.

Transpose Ideals. Let M be a matrix whose entries are 0 or 1. Each row of M is then a 0-1 vector that we can interpret as the exponent vector of a square free monomial. Define $\langle M \rangle$ to be the monomial ideal generated by the monomials whose exponent vectors are the rows of M . On the other hand, given a square free monomial ideal I , we can take the exponent vectors of the minimal generators of I and put them in a matrix. Define M_I to be that matrix. We label the rows of M_I with the elements of $\min(I)$ and we label the columns of M_I with the variables in the ambient polynomial ring of I . Let the *transpose ideal* I^T of I be the ideal generated by the transpose of the matrix of I , so $\text{trans}(I) \stackrel{\text{def}}{=} \langle M_I^T \rangle$.

Theorem 8 states that $\tilde{\chi}(I) = \tilde{\chi}(\text{trans}(I))$, so we can transpose the ideal without changing the Euler characteristic. The BCRT algorithm is more sensitive to the number of variables than it is to the number of generators, so it can be beneficial to transpose the ideal if it has fewer generators than variables. The DBMS algorithm is opposite of this in that it is more sensitive to the number of generators than the number of variables.

Another situation where transposing can be beneficial is in the case where a column of M_I dominates another column. If we take the transpose, those two columns will become generators and the dominating generator will not be minimal. When we then take the transpose again, we will have fewer variables than we started with. This process can repeat itself several times as the removal of dominating columns from the matrix can cause rows to be dominating, and removing those dominating rows can then cause yet more columns to become dominating.

Theorem 8. If I is a square free monomial ideal then $\tilde{\chi}(I) = \tilde{\chi}(\text{trans}(I))$.

Proof. The proof is by induction on the number of variables n . Let I be a square free monomial ideal. Choose a variable x_i and let

$$J \stackrel{\text{def}}{=} \langle \min(I) \setminus \langle x_i \rangle \rangle = \langle m \in \min(I) \mid x_i \text{ does not divide } m \rangle.$$

The plan of the proof is to show that

$$(5) \quad \tilde{\chi}(I) = \tilde{\chi}(I : x_i) - \tilde{\chi}(J : x_i)$$

and that

$$(6) \quad \tilde{\chi}(\text{trans}(I)) = \tilde{\chi}(\text{trans}(I : x_i)) - \tilde{\chi}(\text{trans}(J : x_i)).$$

Recall that we embed $I : x_i$ and $J : x_i$ in a subring that does not have the variable x_i , so the result follows from these two equations by applying the induction assumption to $I : x_i$ and $J : x_i$. It only remains to prove Equation (5) and Equation (6).

Equation (5): Equation 3 with x_i as the pivot implies that

$$\tilde{\chi}(I) = \tilde{\chi}(I : x_i) + \tilde{\chi}(I + \langle x_i \rangle).$$

Now J does not have full support and $I + \langle x_i \rangle = J + \langle x_i \rangle$, so we get by Equation (4) that

$$\tilde{\chi}(I + \langle x_i \rangle) = \tilde{\chi}(J + \langle x_i \rangle) = \tilde{\chi}(J) - \tilde{\chi}(J : x_i) = -\tilde{\chi}(J : x_i).$$

Equation (6): This part of the proof is easier to follow by the reader drawing pictures of the matrices involved. Let v be column x_i of M_I . Then the entries of v are indexed by $\min(I)$ and if

$m \in \min(I)$ then $v_m = 1$ if and only if $x_i | m$. Let A be the result of removing column x_i from M_I . Then $\langle M_I^T \rangle = \langle A^T \rangle + \langle x^v \rangle$ so Equation (4) implies that

$$\tilde{\chi}(\text{trans}(I)) = \tilde{\chi}(\langle M_I^T \rangle) = \tilde{\chi}(\langle A^T \rangle + \langle x^v \rangle) = \tilde{\chi}(\langle A^T \rangle) - \tilde{\chi}(\langle A^T \rangle : x^v).$$

The colon $I : x_i$ corresponds to removing column x_i of M_I so $\langle A \rangle = I : x_i$. The colon can also reduce the number of minimal generators, so $M_{I:x_i}$ can have fewer rows than A does. However, those extra rows are exponent vectors of non-minimal generators so they do not impact $\langle A \rangle$. Then Lemma 9 implies that

$$\tilde{\chi}(\text{trans}(I : x_i)) = \tilde{\chi}(\text{trans}(\langle A \rangle)) = \tilde{\chi}(\langle A^T \rangle).$$

It now suffices to prove that $\text{trans}(J : x_i) = \langle A^T \rangle : x^v$. Let B be the result of removing column x_i of M_I and also those rows $m \in \min(I)$ such that $v_m = 1$. We see that M_J is M_I with the same rows removed as for B . Doing a colon by x_i removes column x_i so $J : x_i = \langle B \rangle$.

It remains to prove that $\langle B^T \rangle = \langle A^T \rangle : x^v$. Observe that $\langle A^T \rangle : x^v$ removes those columns $m \in \min(I)$ of A^T where $v_m = 1$ which are the same columns that are removed from B^T . Both A^T and B^T have had row x_i removed so $\langle A^T \rangle : x^v = \langle B^T \rangle$. We have proven that

$$\text{trans}(J : x_i) = \text{trans}(\langle B \rangle) = \langle B^T \rangle = \langle A^T \rangle : x^v$$

where $\text{trans}(\langle B \rangle) = \langle B^T \rangle$ depends on the observation that no row of B dominates any other. \square

Lemma 9. If $\langle A \rangle = \langle B \rangle$ then $\tilde{\chi}(\langle A^T \rangle) = \tilde{\chi}(\langle B^T \rangle)$ for matrices A and B .

Proof. Assume without loss of generality that no row of B dominates any other. Then A has all the rows that B does and also some additional non-minimal rows. Assume by induction that there is only one additional row r . Let d be some other row of A that is dominated by r . Then d and r contribute variables v_d and v_r to the ambient ring of $\langle A^T \rangle$. We get by Equation 3 that

$$\tilde{\chi}(\langle A^T \rangle) = \tilde{\chi}(\langle A^T \rangle : v_r) + \tilde{\chi}(\langle A^T \rangle + \langle v_r \rangle).$$

All generators of $\langle A^T \rangle$ that are divisible by v_d are also divisible by v_r , so $\langle A^T \rangle + \langle v_r \rangle$ does not have full support at v_r so $\tilde{\chi}(\langle A^T \rangle + \langle v_r \rangle) = 0$. Therefore we have that

$$\tilde{\chi}(\langle A^T \rangle) = \tilde{\chi}(\langle A^T \rangle : v_r) = \tilde{\chi}(\langle B^T \rangle),$$

where we are using that the colon $\langle A^T \rangle : v_r$ corresponds to removing row r from A . \square

Base Case for $|\min(I)| = 3$. Assume that all unique variables have been eliminated, that I has full support and that $|\min(I)| = 3$. Then every variable x_i divides 2 or 3 elements of $\min(I)$. We can ignore the variables that divide all 3 minimal generators as they make no difference to the Euler characteristic. For every minimal generator there must be at least one variable that does not divide it. So after removing repeated variables by taking the transpose twice, we see that I must have the same Euler characteristic as $\langle xy, xz, yz \rangle$. So $\tilde{\chi}(I) = 2$.

Partial Base Case for $|\min(I)| = 4$. Suppose that $|\min(I)| = 4$, that every variable divides exactly two elements of $\min(I)$ and that the number of variables is 4. Then $\tilde{\chi}(I) = -1$. There should be more rules like this, though identifying them by hand is laborious and error prone.

Make a Table. It would be beneficial for each small k to perform a computer search to make a table of all ideals I with $|\min(I)| = k$ up to reordering of the variables and the various techniques for simplifying an ideal that we have presented. Then the Euler characteristic of ideals with few generators could be computed through a table look-up.

Data Structures. The exponents of I are all 0 or 1, so we can pack 32 or 64 exponents into a single 32 or 64 bit machine word, and in that representation many operations become much faster. We used this standard technique in our implementation and while the general concept is simple we warn that the implementation details are tricky.

For sparse or complement-of-sparse exponent vectors, it might pay off to only record the zero entries and one entries respectively, though this is not something that we have pursued.

4. SIMPLICIAL ALGORITHMS FOR EULER CHARACTERISTIC

In this section we present two algorithms for Euler characteristic that work directly with simplicial complexes. These two algorithms are equivalent to the monomial ideal based algorithms from Section 3. We introduce the simplicial versions from the ground up, so this section is independent from Section 3 and does not use any algebra. See Section 5 for more on the connection between the algebraic and simplicial versions of the two algorithms.

4.1. Background and Notation. In the introduction we wrote that a simplicial complex is a finite set of sets that is closed under subset. We are going to set up an algebra-simplicial dictionary, and for that to work we associate a set of vertices to a simplicial complex. This way a simplicial complex can have vertices that are not an element of any of its faces. Section 5.1 shows why this is necessary. Definition 10 adds this vertex set structure to simplicial complexes.

Let $\mathcal{P}(V)$ be the set of all subsets of V for any set V .

Definition 10. Given a finite set V , a simplicial complex Δ is a subset of $\mathcal{P}(V)$ that is closed under taking subsets. In other words, Δ is a set of subsets of V such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. The elements of Δ are called *faces* and the elements of $V_\Delta \stackrel{\text{def}}{=} V$ are called *vertices* of Δ .

The *facets* of Δ are the maximal faces of Δ with respect to inclusion. The set of facets is denoted by $\text{fac}(\Delta)$. The *closure* $\langle D \rangle$ of a set of sets $D \subseteq \mathcal{P}(V)$ is the smallest simplicial complex that contains D , namely $\langle D \rangle \stackrel{\text{def}}{=} \cup_{d \in D} \mathcal{P}(d)$. A simplicial complex is uniquely given by its facets since $\Delta = \langle \text{fac}(\Delta) \rangle$. The *complement* $\bar{\sigma}$ of a set $\sigma \subseteq V_\Delta$ is $\bar{\sigma} \stackrel{\text{def}}{=} V_\Delta \setminus \sigma$. Two sets $\sigma, \tau \subseteq V_\Delta$ are *co-disjoint* if their complements are disjoint, or equivalently if $\sigma \cup \tau = V_\Delta$. If $\tau \subseteq V_\Delta$ then

$$\Delta \ominus \tau \stackrel{\text{def}}{=} \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset\} = \{\sigma \setminus \tau \mid \sigma \in \Delta\}.$$

The vertex set of $\Delta \ominus \tau$ is $V_{\Delta \ominus \tau} \stackrel{\text{def}}{=} V_\Delta \setminus \tau$.

We will use that Euler characteristic respects inclusion-exclusion in the sense that

$$\tilde{\chi}(\Delta \cup \Delta') = \tilde{\chi}(\Delta) + \tilde{\chi}(\Delta') - \tilde{\chi}(\Delta \cap \Delta').$$

4.2. Divide... Both algorithms we present are divide-and-conquer algorithms. They take a simplicial complex Δ and split it into two simpler complexes Δ' and Δ'' such that $\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta') + \tilde{\chi}(\Delta'')$. This process proceeds recursively until all the remaining complexes are simple enough that they can be processed directly. Splitting (the divide step) is based on Theorem 11.

Theorem 11. If σ is a non-empty set of vertices then

$$\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta \ominus \bar{\sigma}) + \tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)).$$

Proof. First observe that

$$\Delta \cap \mathcal{P}(\sigma) = \{\tau \in \Delta \mid \tau \subseteq \sigma\} = \{\tau \in \Delta \mid \tau \cap \bar{\sigma} = \emptyset\} = \Delta \ominus \bar{\sigma}.$$

Euler characteristic respects inclusion-exclusion so then

$$\begin{aligned}\tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)) &= \tilde{\chi}(\Delta) + \tilde{\chi}(\mathcal{P}(\sigma)) - \tilde{\chi}(\Delta \cap \mathcal{P}(\sigma)) \\ &= \tilde{\chi}(\Delta) - \tilde{\chi}(\Delta \ominus \bar{\sigma}),\end{aligned}$$

where $\tilde{\chi}(\mathcal{P}(\sigma)) = 0$ as σ is non-empty. \square

In analogy with the algebraic BCRT algorithm from Section 3, we can use the equation in Theorem 11 as written to split a simplicial complex Δ into the simpler complexes $\Delta \ominus \bar{\sigma}$ and $\Delta \cup \mathcal{P}(\sigma)$. We will refer to this algorithm as the *simplicial BCRT algorithm*. It is not immediately clear that $\Delta \ominus \bar{\sigma}$ and $\Delta \cup \mathcal{P}(\sigma)$ are simpler than Δ is. For now, consider that $\Delta \ominus \bar{\sigma}$ has fewer vertices if $\sigma \subsetneq V_\Delta$ and that $\Delta \cup \mathcal{P}(\sigma)$ has fewer non-faces if $\sigma \notin \Delta$.

We call the σ in Theorem 11 the *pivot*. Section 4.5 explores strategies for selecting pivots.

We can also write the equation in Theorem 11 as

$$\tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)) = \tilde{\chi}(\Delta) - \tilde{\chi}(\Delta \ominus \bar{\sigma}).$$

Let D be a simplicial complex and let $\sigma \in \text{fac}(D)$. If $\Delta \stackrel{\text{def}}{=} \langle \text{fac}(D) \setminus \{\sigma\} \rangle$ then

$$(7) \quad \tilde{\chi}(D) = \tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)) = \tilde{\chi}(\Delta) - \tilde{\chi}(\Delta \ominus \bar{\sigma}).$$

In analogy with the algebraic DBMS algorithm from Section 3, we can use this equation to split a simplicial complex D into the simpler complexes Δ and $\Delta \ominus \bar{\sigma}$. We will refer to this algorithm as the *simplicial DBMS algorithm*. Note that the pivots in the simplicial DBMS algorithm are facets of the complex, which would not make sense for the simplicial BCRT algorithm.

4.3. ... and Conquer. A simplicial complex Δ is a base case for both algorithms when Theorem 12 applies. The improvements in Section 4.6 enable further base cases.

Theorem 12. Let $\Delta \neq \emptyset$ be a simplicial complex. Then

- (1) if Δ is a cone then $\tilde{\chi}(\Delta) = 0$,
- (2) if Δ is not a cone and the facets of Δ are pairwise co-disjoint then $\tilde{\chi}(\Delta) = (-1)^{|\text{fac}(\Delta)|}$,
- (3) if Δ is not a cone and $|\text{fac}(\Delta)| = 2$ then $\tilde{\chi}(\Delta) = 1$.

Proof. (1): This is well known and not hard to prove.

(2): By induction on $|\text{fac}(\Delta)|$ using Lemma 13.

(3): The two facets are co-disjoint when ignoring unused vertices. \square

Lemma 13. Let D be a simplicial complex and let $\sigma \in \text{fac}(D)$ such that $\sigma \neq V_D$ is co-disjoint to every other facet of D . Then $\tilde{\chi}(D) = -\tilde{\chi}(\Delta \ominus \bar{\sigma})$ where $\Delta \stackrel{\text{def}}{=} \langle \text{fac}(D) \setminus \{\sigma\} \rangle$.

Proof. Let $v \notin \sigma$ and $\tau \in \text{fac}(\Delta)$. Then $v \in \tau$ so Δ is a cone and $\tilde{\chi}(\Delta) = 0$. By Theorem 11

$$\tilde{\chi}(D) = \tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)) = \tilde{\chi}(\Delta) - \tilde{\chi}(\Delta \ominus \bar{\sigma}) = -\tilde{\chi}(\Delta \ominus \bar{\sigma}). \quad \square$$

4.4. Termination and Complexity. It is clear that the simplicial DBMS algorithm terminates since $|\text{fac}(\Delta)|$ decreases strictly at each step. For the simplicial BCRT algorithm, termination requires that we choose the pivots σ such that $\sigma \subsetneq V_\Delta$ and $\sigma \notin \Delta$ since otherwise we get an infinite number of steps from $\Delta = \Delta \ominus \bar{\sigma}$ or $\Delta = \Delta \cup \mathcal{P}(\sigma)$.

If we cannot choose a σ such that $\sigma \subsetneq V_\Delta$ and $\sigma \notin \Delta$ then $\Delta = \mathcal{P}(V_\Delta)$ which is a base case. If we always choose pivots σ such that $\sigma \subsetneq V_\Delta$ and $\sigma \notin \Delta$ then termination of the simplicial BCRT algorithm follows from the fact that either the number of vertices $|V_\Delta|$ or the number of non-faces

$|\overline{\Delta}| = 2^{|V_\Delta|} - |\Delta|$ decreases between any two steps. In fact some consideration shows that $|\overline{\Delta}|$ decreases strictly at each step.

We have seen that the DBMS algorithm gets rid of at least one facet at each step. The BCRT algorithm gets rid of at least one vertex at each step if the pivot is chosen to be of the form $\sigma \stackrel{\text{def}}{=} \overline{\{e\}}$. It is immediate that e is not a vertex of $\Delta \ominus \overline{\sigma}$. To see that e can also be removed from $\Delta \cup \mathcal{P}(\sigma)$, observe that σ is the only facet that does not contain e , so e is $(\Delta \cup \mathcal{P}(\sigma))$ -independent from the other vertices and so can be removed using the independent vertices technique from Section 4.6.

Since the Euler characteristic problem is $\#P$ -complete we expect both algorithms to run in at least single exponential time. Taking nerves as described in Section 4.6, we can interchange the number of vertices $|V_\Delta|$ with the number of facets $|\text{fac}(\Delta)|$. So if $l \stackrel{\text{def}}{=} \min(|V_\Delta|, |\text{fac}(\Delta)|)$ then both the BCRT and DBMS algorithms have $O(q^{2^l})$ asymptotic worst case time complexity where q is a polynomial. So both algorithms run in single exponential time. We expect that a more careful analysis could reduce the base of the exponential.

4.5. Pivot Selection. We have proven that the BCRT and DBMS algorithms terminate in a finite amount of time. To be useful in practice, the amount of time until termination should be small rather than just finite. The strategy used for selecting pivots when splitting an ideal has a significant impact on performance. We describe several different pivot selection strategies here and compare them empirically in Section 6.

A *rare vertex* is a vertex that belongs to a minimum number of facets. A *popular vertex* is a vertex e that is an element of a maximum number of facets with the constraint that $\overline{\{e\}} \notin \text{fac}(\Delta)$.

If there are several candidate pivots that fit a given pivot selection strategy, then the pivot used is chosen in an arbitrary deterministic way among the tied candidates.

BCRT Pivot Selection Strategies. Recall that BCRT pivots σ satisfy $\sigma \subsetneq V_\Delta$ and $\sigma \notin \Delta$.

The reason that some of these names seem opposite of their definition is that the names come from the algebraic setting and the translation to simplicial complexes involves taking a complement. For example a rare variable of an ideal translates to a popular vertex of the corresponding simplicial complex.

popvar: The pivot is $\overline{\{e\}}$ for e a rare vertex.

rarevar: The pivot is $\overline{\{e\}}$ for e a popular vertex.

random: The pivot is $\overline{\{e\}}$ for e a random vertex such that $\overline{\{e\}} \notin \text{fac}(\Delta)$.

popgcd: Let e be a rare vertex. The pivot is the union of three facets chosen uniformly at random among those facets that do not contain e .

The simplicial BCRT algorithm presented here is based on the algebraic BCRT algorithm for Hilbert-Poincaré series, and the strategies **popvar** and **popgcd** have been found to work well for Hilbert-Poincaré series computation. So we also try them here. **rarevar** and **random** have been included to have something to compare to.

DBMS Pivot Selection Strategies. Recall that DBMS pivots are elements of $\text{fac}(\Delta)$.

rarevar: The pivot is a facet that does not contain some popular vertex.

popvar: The pivot is a facet that does not contain some rare vertex.

maxsupp: The pivot is a facet of minimum size.

minsupp: The pivot is a facet of maximum size.

random: The pivot is a facet chosen uniformly at random.

rarest: The pivot is a facet that lacks (does not contain) a maximum number of popular vertices. Ties are broken by picking the facet that lacks the maximum number of second-most-popular vertices and so on.

raremax: The pivot is chosen according to **rarevar** where ties are broken according to **maxsupp**.

4.6. Improvements. We present several improvements to the DBMS and BCRT algorithms.

Independent Vertices. If Δ and Γ are simplicial complexes with disjoint vertex sets V_Δ and V_Γ , then

$$\Delta \oplus \Gamma \stackrel{\text{def}}{=} (\Delta \times \mathcal{P}(V_\Gamma)) \cup (\mathcal{P}(V_\Delta) \times \Gamma).$$

So if $\Delta \stackrel{\text{def}}{=} \langle \{x\}, \{y\} \rangle$ and $\Gamma \stackrel{\text{def}}{=} \langle \{a\}, \{b\}, \{c\} \rangle$ then

$$\Delta \oplus \Gamma = \langle \{x, a, b, c\}, \{y, a, b, c\}, \{x, y, a\}, \{x, y, b\}, \{x, y, c\} \rangle.$$

Using Theorem 15 we can compute $\tilde{\chi}(\Delta \oplus \Gamma)$ in terms of $\tilde{\chi}(\Delta)$ and $\tilde{\chi}(\Gamma)$. So if we are computing $\tilde{\chi}(\Psi)$ for a simplicial complex Ψ , then we could significantly simplify the task by finding simplicial complexes Δ and Γ such that $\Psi = \Delta \oplus \Gamma$.

We say that two subsets $A, B \subseteq V_\Psi$ are Ψ -independent if $\text{fac}(\Psi)$ is the disjoint union of

$$F_A \stackrel{\text{def}}{=} \{\sigma \in \text{fac}(\Psi) \mid \sigma \setminus A = \overline{A}\} \quad \text{and} \quad F_B \stackrel{\text{def}}{=} \{\sigma \in \text{fac}(\Psi) \mid \sigma \setminus B = \overline{B}\}.$$

If $\Psi = \Delta \oplus \Gamma$ then it is immediate that V_Δ and V_Γ are Ψ -independent. On the other hand, if A and B are Ψ -independent, then $\Psi = \Delta \oplus \Gamma$ where

$$\Delta \stackrel{\text{def}}{=} \langle F_A \rangle \ominus \overline{A} \quad \text{and} \quad \Gamma \stackrel{\text{def}}{=} \langle F_B \rangle \ominus \overline{B}.$$

We have proven Theorem 14, which together with Theorem 15 generalizes Lemma 13.

Theorem 14. A simplicial complex Ψ can be written as $\Psi = \Delta \oplus \Gamma$ if and only if there is a Ψ -independent pair (A, B) .

Theorem 15. Let Δ and Γ be simplicial complexes with disjoint non-empty vertex sets V_Δ and V_Γ . Then

$$\tilde{\chi}(\Delta \oplus \Gamma) = \tilde{\chi}(\Delta) \tilde{\chi}(\Gamma).$$

Proof. Choose any set D such that $\Delta = \langle D \rangle$. Then we get by inclusion-exclusion that

$$\begin{aligned} \tilde{\chi}(\Delta) &= \tilde{\chi}\left(\bigcup_{\sigma \in D} \mathcal{P}(\sigma)\right) = \sum_{v \subseteq D} (-1)^{|v|+1} \tilde{\chi}\left(\bigcap_{\sigma \in v} \mathcal{P}(\sigma)\right) \\ &= \sum_{v \subseteq D} (-1)^{|v|+1} \tilde{\chi}(\mathcal{P}(\cap v)). \end{aligned}$$

Let $F_\Delta \stackrel{\text{def}}{=} \text{fac}(\Delta) \times \{V_\Gamma\}$ and $F_\Gamma \stackrel{\text{def}}{=} \{V_\Delta\} \times \text{fac}(\Gamma)$. Then $\Delta \oplus \Gamma = \langle F_\Delta \cup F_\Gamma \rangle$ so in the same way we get by inclusion-exclusion that

$$\begin{aligned} \tilde{\chi}(\Delta \oplus \Gamma) &= \sum_{v \subseteq F_\Delta \cup F_\Gamma} (-1)^{|v|+1} \tilde{\chi}(\mathcal{P}(\cap v)) \\ &= \sum_{d \subseteq F_\Delta} \sum_{g \subseteq F_\Gamma} (-1)^{|d|+|g|+1} \tilde{\chi}(\mathcal{P}(\cap(d \cup g))). \end{aligned}$$

Let $d' \subseteq \text{fac}(\Delta)$ and $g' \subseteq \text{fac}(\Gamma)$ such that $d = d' \cup V_\Gamma$ and $g = g' \cup V_\Delta$. We will prove that

$$(8) \quad \tilde{\chi}(\mathcal{P}(\cap(d \cup g))) = -\tilde{\chi}(\mathcal{P}(\cap d')) \tilde{\chi}(\mathcal{P}(\cap g')).$$

Since $|d| = |d'|$ and $|g| = |g'|$ we then get that

$$\begin{aligned}\tilde{\chi}(\Delta \oplus \Gamma) &= \sum_{d' \subseteq \text{fac}(\Delta)} \sum_{g' \subseteq \text{fac}(\Gamma)} (-1)^{|d'|+|g'|} \tilde{\chi}(\mathcal{P}(\cap d')) \tilde{\chi}(\mathcal{P}(\cap g')) \\ &= \sum_{d' \subseteq \text{fac}(\Delta)} (-1)^{|d'|+1} \tilde{\chi}(\mathcal{P}(\cap d')) \sum_{g' \subseteq \text{fac}(\Gamma)} (-1)^{|g'|+1} \tilde{\chi}(\mathcal{P}(\cap g')) \\ &= \tilde{\chi}(\Delta) \tilde{\chi}(\Gamma).\end{aligned}$$

It only remains to prove Equation (8). Observe that $\tilde{\chi}(\mathcal{P}(\tau)) = 0$ if τ is non-empty and otherwise $\tilde{\chi}(\mathcal{P}(\tau)) = -1$. So Equation (8) is equivalent to the statement that

$$\cap(d \cup g) = \emptyset \quad \Leftrightarrow \quad \cap d' = \emptyset \text{ and } \cap g' = \emptyset.$$

As every face in d contains V_Γ and every face in g contains V_Δ , the only way for $\cap(d \cup g)$ to be empty is if $\cap d = V_\Gamma$ and $\cap g = V_\Delta$. The statement then follows from the observation that $\cap d = V_\Gamma$ if and only if $\cap d' = \emptyset$ and likewise $\cap g = V_\Delta$ if and only if $\cap g' = \emptyset$. \square

Eliminate Abundant Vertices. Suppose e is an element of every facet of D except $\sigma \in \text{fac}(D)$. Use Equation (7) with σ as the pivot to get that

$$\tilde{\chi}(D) = \tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)) = \tilde{\chi}(\Delta) - \tilde{\chi}(\Delta \ominus \bar{\sigma}) = \tilde{\chi}(\Delta \ominus \bar{\sigma}),$$

since $\tilde{\chi}(\Delta) = 0$ as every facet of Δ contains e so Δ is a cone.

Take Nerves. The *nerve* of a simplicial complex Δ is

$$\text{nerve}(\Delta) \stackrel{\text{def}}{=} \{v \subseteq \text{fac}(\Delta) \mid \cap v \neq \emptyset\}.$$

A complex and its nerve have the same homotopy type [10, Thm. 10], so $\tilde{\chi}(\Delta) = \tilde{\chi}(\text{nerve}(\Delta))$. Taking the nerve corresponds to transposing the facet-vertex incidence matrix M . Taking the nerve twice will remove dominated columns from M and then remove any dominated rows that might have appeared. This process can continue as removing rows of M can cause yet more columns to be dominated.

If no columns are dominated then taking the nerve once will exchange the number of vertices and facets. The BCRT algorithm is more sensitive to the number of vertices than it is to the number of facets, so it can be beneficial to do computations on the nerve if the complex has fewer facets than vertices. The DBMS algorithm is opposite of this in that it is more sensitive to the number of facets than the number of vertices.

Base Case for $|\text{fac}(\Delta)| = 3$. Assume that all abundant vertices have been removed, that Δ is not a cone and that $|\text{fac}(\Delta)| = 3$. Then every vertex that actually occurs in the complex is an element of one facet. The nerve of the nerve of Δ will then have the form $\langle \{a\}, \{b\}, \{c\} \rangle$. So $\tilde{\chi}(\Delta) = 2$.

Partial Base Case for $|\text{fac}(\Delta)| = 4$. Suppose that $|\text{fac}(\Delta)| = 4$, that every vertex is an element of exactly two facets and that the number of vertices is 4. Then $\tilde{\chi}(\Delta) = -1$. There should be more rules like this, though identifying them by hand is laborious and error prone.

Make a Table. It would be beneficial for each small k to perform a computer search to make a table of all simplicial complexes Δ with $|\text{fac}(\Delta)| = k$ up to reordering of the vertices and the various techniques for simplifying a simplicial complex that we have presented. Then the Euler characteristic of simplicial complexes with few facets could be computed through a table look-up.

Data Structures. In our implementation we have represented a facet as a 0-1 vector, where we pack 32 or 64 entries into a single 32 or 64 bit word. Many operations are very fast in this representation. While the general concept is simple we warn that the implementation details are tricky.

For sparse or complement-of-sparse vectors, it might pay off to only record the zero entries or one entries respectively, though this is not something that we have pursued.

5. IDEALS TO SIMPLICIAL COMPLEXES

In this section we describe in more detail how the simplicial algorithms in Section 4 are equivalent to the algebraic algorithms in Section 3. We treat this topic in detail in part to support our claim that the algebraic and simplicial algorithms are equivalent and in part because the algebra-simplicial translation brings up interesting mathematics.

To set up the algebra-simplicial translation, let Δ be a simplicial complex with vertices x_1, \dots, x_n that are simultaneously the variables in a polynomial ring. The algebra-simplicial connection is via the function ϕ from sets to monomials defined by $\phi(\sigma) \stackrel{\text{def}}{=} \prod \sigma$. In other words, $\phi(\sigma)$ is the product of variables (vertices) that are not in σ . So if $n = 3$ then $\phi(\emptyset) = x_1 x_2 x_3$ and $\phi(\{x_1, x_3\}) = x_2$.

Extend ϕ from sets to simplicial complexes by $\phi(\Delta) \stackrel{\text{def}}{=} \langle \phi(\sigma) \mid \sigma \in \Delta \rangle$. Then ϕ is a bijection from the facets of Δ to the minimal generators of $\phi(\Delta)$, so $\phi(\Delta)$ can be quickly computed given Δ . We can also describe $\phi(\Delta)$ as the *Stanley-Reisner ideal* of the *Alexander dual* of Δ , though we will not use this alternative description of $\phi(\Delta)$ here.

The fundamental observation that we have been using is that $\tilde{\chi}(\phi(\Delta)) = \tilde{\chi}(\Delta)$ so that we can compute $\tilde{\chi}(\Delta)$ with monomial ideal methods on $\phi(\Delta)$.

Theorem 16. If Δ is a simplicial complex then $\tilde{\chi}(\Delta) = \tilde{\chi}(\phi(\Delta))$, where $\tilde{\chi}(\phi(\Delta))$ is the coefficient of \mathbf{x} in the multivariate Hilbert-Poincaré series numerator $H(I)$ of I .

Proof. The proof is based on a formula of Bayer concerning upper Koszul simplicial complexes.

Let m be a monomial and let I be a monomial ideal. Then the *upper Koszul simplicial complex*³ [13, 2] of I at m is

$$\Delta_m^I \stackrel{\text{def}}{=} \left\{ \sigma \subseteq \{x_1, \dots, x_n\} \mid \frac{x^m}{\prod \sigma} \in I \right\}.$$

The numerator of the multivariate Hilbert-Poincaré series of I is related to the Euler characteristics of the Koszul complexes of I by the following formula due to Bayer [2, Proposition 3.2]

$$H(I) - 1 = \sum_{v \in \mathbb{N}^n} \tilde{\chi}(\Delta_{x^v}^I) x^v.$$

Since $\Delta_{\mathbf{x}}^{\phi(\Delta)} = \Delta$, we get that

$$\text{Coefficient of } \mathbf{x} \text{ in } H(\phi(\Delta)) = \tilde{\chi}(\Delta_{\mathbf{x}}^{\phi(\Delta)}) = \tilde{\chi}(\Delta). \quad \square$$

5.1. The Inverse of ϕ . In Section 4.1 we modified the definition of a simplicial complex to have an associated set of vertices so that a vertex need not actually appear in any of the faces of the complex. Here we give an example that shows that this change is necessary for ϕ to be a bijection between simplicial complexes and square free monomial ideals. In short, the vertex set definition is necessary to preserve information about the variables in the ambient polynomial ring.

³This notion of Koszul complex is not to be confused with the chain complex notion of a Koszul complex.

We have two functions ϕ , one that maps sets to square free monomials and one that maps ideals to square free monomial ideals. They are defined by $\phi(v) \stackrel{\text{def}}{=} \Pi v$ and $\phi(\Delta) \stackrel{\text{def}}{=} \langle \phi(v) \mid v \in \Delta \rangle$ respectively. The inverse of $\Delta \mapsto \phi(\Delta)$ is given by

$$\phi^{-1}(I) = \{v \subseteq \{x_1, \dots, x_n\} \mid \phi(v) \in I\}.$$

We define the vertex set of $\phi^{-1}(I)$ to be the set of variables in the ambient ring of I , even if some of those variables do not appear in any face of $\phi^{-1}(I)$. We give an example that shows that $\Delta \mapsto \phi(\Delta)$ would not have an inverse if the vertex set of $\phi^{-1}(I)$ were the union of its faces.

Consider the ideal $I \stackrel{\text{def}}{=} \langle x_1x_2, x_1x_3 \rangle \subseteq \kappa[x_1, x_2, x_3]$ and let $\Delta \stackrel{\text{def}}{=} \phi^{-1}(I)$ so that $\text{fac}(\Delta) = \{\{x_3\}, \{x_2\}\}$. The question now is what the vertex set of Δ is. The union of faces is $\{x_2, x_3\}$ so if that were the set of vertices then $\phi(\{x_2\}) = x_3$ and $\phi(\{x_3\}) = x_2$ so then $\phi(\phi^{-1}(I)) = \phi(\Delta) = \langle x_3, x_2 \rangle \neq I$. We observe that if the vertex set were the union of faces, then there would be no simplicial complex Δ such that $\phi(\Delta) = I$. So ϕ would be a bijection not from simplicial complexes to square free monomial ideals, but from simplicial complexes to square free monomial ideals with the property that $\gcd(\min(I)) = 1$.

Using the definition that we have given, the vertex set of Δ is still $\{x_1, x_2, x_3\}$ even though x_1 does not appear in any face of Δ . With our definition ϕ does have ϕ^{-1} as an inverse, and indeed we see that $\phi(\phi^{-1}(I)) = \phi(\Delta) = I$.

Even if the input ideal I to the algebraic BCRT and DBMS algorithms has the property that $\gcd(\min(I)) = 1$, the intermediate ideals that these algorithms generate do not necessarily have that property. So if we had defined the vertex set to be the union of faces, then we would have had to deal with this issue in some other way.

5.2. Divide... The algebra algorithms are based on Equation 3 which states that for monomials p

$$\tilde{\chi}(I) = \tilde{\chi}(I : p) + \tilde{\chi}(I + \langle p \rangle).$$

The simplicial algorithms are based on Theorem 11 which states that for sets σ

$$\tilde{\chi}(\Delta) = \tilde{\chi}(\Delta \ominus \overline{\sigma}) + \tilde{\chi}(\Delta \cup \mathcal{P}(\sigma)).$$

These two equations are equivalent since if $\Delta = \phi^{-1}(I)$ and $\sigma = \phi^{-1}(p)$ then

$$\phi^{-1}(I : p) = \Delta \ominus \overline{\sigma}, \quad \phi^{-1}(I + \langle p \rangle) = \Delta \cup \mathcal{P}(\sigma).$$

5.3. ... and Conquer. Theorem 7 specifies base cases for the algebra algorithms when

- (1) I does not have full support,
- (2) I has full support and the minimal generators $\min(I)$ are pairwise prime,
- (3) I has full support and $|\min(I)| = 2$.

Theorem 12 specifies base cases for the simplicial algorithms when

- (1) Δ is a cone,
- (2) Δ is not a cone and the facets of Δ are pairwise co-disjoint,
- (3) Δ is not a cone and $|\text{fac}(\Delta)| = 2$.

If $\Delta = \phi^{-1}(I)$ then the algebra and simplicial conditions are equivalent. To be more precise

- (1) I has full support if and only if Δ is not a cone,
- (2) monomials a and b are relatively prime if and only if $\phi^{-1}(a)$ and $\phi^{-1}(b)$ are co-disjoint,
- (3) $|\min(I)| = |\text{fac}(\Delta)|$.

5.4. Termination and Complexity. The algebraic DBMS algorithm terminates as $|\min(I)|$ decreases strictly at each step. The simplicial DBMS algorithm terminates as $|\text{fac}(\Delta)|$ decreases strictly at each step. These reasons are equivalent when $\Delta = \phi^{-1}(I)$ as then $|\min(I)| = |\text{fac}(\Delta)|$.

The algebraic BCRT algorithm terminates if the pivots p are chosen such that $p \neq 1$ and $p \notin I$. The simplicial BCRT algorithm terminates if the pivots σ are chosen such that $\sigma \subsetneq V_\Delta$ and $\sigma \notin \Delta$. If $\Delta = \phi^{-1}(I)$ and $\sigma = \phi^{-1}(p)$ then these conditions are equivalent.

5.5. Pivot Selection. The pivot selection strategies that have the same name in the two sections on pivot selection are equivalent. The main points in proving this are that for a monomial ideal I and a simplicial complex $\Delta \stackrel{\text{def}}{=} \phi^{-1}(I)$

- a variable x_i is rare if and only if the vertex x_i is popular,
- a variable x_i is popular if and only if the vertex x_i is rare,
- a minimal generator m has maximum support if and only if $\phi^{-1}(m)$ has minimum size,
- a minimal generator m has minimum support if and only if $\phi^{-1}(m)$ has maximum size.

5.6. Improvements. Let I be a square free monomial ideal and let $\Delta \stackrel{\text{def}}{=} \phi^{-1}(I)$. The two sections on improvements present equivalent techniques in the same order.

Independent Variables — Independent Vertices. Let A and B be disjoint sets of variables. Let $I \subseteq \kappa[A]$ and $J \subseteq \kappa[B]$. Then $I + J \subseteq \kappa[A \cup B]$ and the algebraic section on independent variables proves that $\tilde{\chi}(I + J) = \tilde{\chi}(I)\tilde{\chi}(J)$. Furthermore, an ideal K can be written as a sum $I + J$ if and only if there are disjoint sets A and B such that $\min(K)$ is the union of $\min(K) \cap \kappa[A]$ and $\min(K) \cap \kappa[B]$. In that case we say that A and B are K -independent and then $K = I + J$ for $I \stackrel{\text{def}}{=} \langle \min(K) \cap \kappa[A] \rangle$ and $J \stackrel{\text{def}}{=} \langle \min(K) \cap \kappa[B] \rangle$.

For the simplicial side of things, let Δ and Γ be simplicial complexes such that the vertex sets V_Δ and V_Γ are disjoint. Then

$$\Delta \oplus \Gamma \stackrel{\text{def}}{=} (\Delta \times \mathcal{P}(V_\Gamma)) \cup (\mathcal{P}(V_\Delta) \times \Gamma) \quad \text{and} \quad V_{\Delta \oplus \Gamma} \stackrel{\text{def}}{=} V_\Delta \cup V_\Gamma$$

The simplicial section on independent vertices proves that $\tilde{\chi}(\Delta \oplus \Gamma) = \tilde{\chi}(\Delta)\tilde{\chi}(\Gamma)$. Furthermore, an ideal Ψ can be written as $\Delta \oplus \Gamma$ if and only if there are disjoint sets A and B such that $\text{fac}(\Psi)$ is the disjoint union of

$$F_A \stackrel{\text{def}}{=} \{\sigma \in \text{fac}(\Psi) \mid \sigma \setminus A = \overline{A}\} \quad \text{and} \quad F_B \stackrel{\text{def}}{=} \{\sigma \in \text{fac}(\Psi) \mid \sigma \setminus B = \overline{B}\}.$$

In that case $\Psi = \Delta \oplus \Gamma$ for $\Delta \stackrel{\text{def}}{=} \langle F_A \rangle$ and $\Gamma \stackrel{\text{def}}{=} \langle F_B \rangle$.

Addition of ideals and \oplus of simplicial complexes are equivalent. If $\Delta \stackrel{\text{def}}{=} \phi^{-1}(I)$ and $\Gamma \stackrel{\text{def}}{=} \phi^{-1}(J)$ then $\Delta \oplus \Gamma = \phi^{-1}(I + J)$. Observe that we are using three different functions ϕ^{-1} here as the ambient polynomial ring is different for each of them. Recall that the definition of ϕ^{-1} involves taking a complement, and if v is a set of variables then the meaning of the complement \overline{v} depends on what the variables in the ambient polynomial ring are.

Independence of variables and vertices are also equivalent. If $\Psi \stackrel{\text{def}}{=} \phi^{-1}(K)$ and A and B are sets of variables/vertices, then A and B are K -independent if and only if A and B are Ψ -independent.

Let A and B be K -independent and let $\Psi = \phi^{-1}(K)$ so that A and B are also Ψ -independent. Let $\Delta \stackrel{\text{def}}{=} \langle F_A \rangle$ and $\Gamma \stackrel{\text{def}}{=} \langle F_B \rangle$. Let $I \stackrel{\text{def}}{=} \langle \min(K) \cap \kappa[A] \rangle$ and $J \stackrel{\text{def}}{=} \langle \min(K) \cap \kappa[B] \rangle$. Then $K = I + J$ and $\Psi = \Delta \oplus \Gamma$. Furthermore, $\Delta = \phi^{-1}(I)$ and $\Gamma = \phi^{-1}(J)$.

The notion of I -independence is standard, though we have not been able to find any reference in the literature to Ψ -independence or the operation \oplus on simplicial complexes.

Eliminate Unique Variables — Eliminate Abundant Vertices. A variable x_i is unique in I if and only if x_i is an abundant vertex of Δ .

Transpose Ideals — Take Nerves. Both the nerve and the transpose of an ideal are transposing a matrix where the columns are variables/vertices and the rows are generators/facets. One of these matrices can be derived from the other by replacing all 0's by 1 and simultaneously replacing all 1's by 0. This shows that $\phi^{-1}(\text{trans}(I)) = \text{nerve}(\Delta)$.

As far as we know, the transpose operation has not been applied to monomial ideals before – we are investigating if a monomial ideal and its transpose have any interesting relations between them when the ideal is not square free.

Base Case for $|\min(I)| = 3$ — Base Case for $|\text{fac}(\Delta)| = 3$. These are equivalent as $|\min(I)| = |\text{fac}(\Delta)|$ and both base cases give the same Euler characteristic of 2.

Partial Base Case for $|\min(I)| = 4$ — Partial Base Case for $|\text{fac}(\Delta)| = 4$. These are equivalent.

Make a Table. This is the same idea.

Data Structures. These are the same considerations. Sparse exponent vectors correspond to large facets, while small facets correspond to mostly-one exponent vectors. In particular, it is not the case that sparse exponent vectors correspond to small facets.

6. EMPIRICAL EVALUATION OF EULER CHARACTERISTIC ALGORITHMS

We have implemented the BCRT and DBMS algorithms for Euler characteristic in the program **Frobby** [14]. In this section we explore the pivot selection strategies for both algorithms, and we compare the BCRT and DBMS implementations to several other systems with Euler characteristic functionality.

We use simplicial terminology in this section. To recover equivalent statements using algebraic terminology read “monomial ideal” for “complex”, read “variables” for “vertices”, read “minimal generators” for “facets”, read “having not full support” for “being a cone” and read “transpose” for “nerve”.

We compare the following implementations, listed in alphabetical order.

Frobby version 0.9.0 [14]: **Frobby** is a free and open source system for computations on monomial ideals. We have written a C++ implementation of the BCRT and DBMS algorithms for Euler characteristic in **Frobby**. We ran **Frobby** with the option `-minimal` turned on in Table 6 as all faces given in the input are facets.

GAP version 4.4.12 [9]: **GAP** is a free and open source system for computational discrete algebra. It computes the Euler characteristic of a complex by enumerating all faces. The implementation is written in the **GAP** scripting language. We use the `SCEulerCharacteristic` function from version 1.4.0 of the **simpcomp** package [7]. We extend the memory limit for **GAP** to 2 GB.

Macaulay 2 version 1.4 [8]: **Macaulay 2** is a free and open source computer algebra system. It computes the Euler characteristic of a simplicial complex using the BCRT algorithm for Hilbert-Poincaré series to get the f -vector. The time consuming parts of the code are written in C++. Due to some inefficiencies in the **SimplicialComplexes** package it is faster to compute the Euler characteristic using the `poincare` function directly on a monomial ideal instead of going through a simplicial complex. We have used this faster method.

Sage version 4.7 [17]: Sage is a free and open source computer algebra system. It computes the Euler characteristic of a simplicial complex by enumerating all faces of the complex. The implementation is written in Python. We use the `eulerCharacteristic` function from the `SimplicialComplexes` package.

We use the following complexes for the comparison.

random- a - b : A randomly generated complex with a vertices and b facets. The complex is generated one facet at a time. A prospective facet σ is generated at random so that each vertex has a 50% chance to be an element of σ . If σ is contained in or contains any previously generated facet then σ is discarded and another prospective facet is generated in its stead.

We generated these example using the `genideal` action of `Frobby`.

nicgraph- a - b : The simplicial complex of all not b -connected graphs on a given set of a vertices. Here we view each possible edge as a vertex of the simplicial complex. A graph is b -connected if it is connected and it cannot be disconnected by removing $b - 1$ edges.

We generated these examples with the `NotIConnectedGraphs` function in `Sage`.

rook- a - b : The $a \times b$ chessboard complex.

Let V and W be sets of vertices such that $|V| = a$ and $|W| = b$. A *partial matching* between V and W is a set of pairs (v, w) with $v \in V$ and $w \in W$ such that each vertex is in at most one pair. The set of partial matchings between V and W forms a simplicial complex called the *chessboard complex*. It corresponds to ways of placing rooks on an $a \times b$ chessboard so that no rook attacks any other rook.

We initially generated these examples with the `ChessboardComplex` function in `Sage`. We could not generate large enough examples with that function so we added the same functionality to `Frobby` and used that.

match- a : The matching complex on a vertices.

Let V be a set of vertices with $|v| = a$. A *partial matching* on V is a set of pairs $\{v, w\}$ for $v, w \in V$ such that each vertex is in at most one pair. The set of partial matchings on V forms a simplicial complex called the *matching complex*.

We initially generated these examples with the `MatchingComplex` function in `Sage`. We could not generate large enough examples with that function so we added the same functionality to `Frobby` and used that.

In all tables $|V_\Delta|$ refers to the number of vertices and $|\text{fac}(\Delta)|$ refers to the number of facets. OOM stands for “out of memory” and indicates that the computation was terminated due to the program reporting an out of memory error. All experiments were run on an Intel® Core™ 2 Duo CPU T7500 at 2.20GHz with 4GB of RAM running Mandriva Linux 2010.1.

We checked that all the programs gave the correct answer for every input. All times are the median time of three runs. Time is in each case measured by the programs themselves, except for `Frobby` that was timed using the `Unix` time command line utility. All times exclude the time taken to start the program and read the input file, except for the `Frobby` times that do include startup and input time. We had to cut out startup and input time as it was taking more time than the Euler characteristic computation itself in some cases, and these experiments are not intended to be about starting programs or reading input. We did not do this for `Frobby` as `Frobby` starts up and reads input in a tiny fraction of the time taken for the Euler characteristic computation.

6.1. Pivot Selection Strategies for the BCRT and DBMS Algorithms. The BCRT and DBMS algorithms are parameterized on a pivot selection strategy that determines which pivot to

$ V_\Delta $	$ \text{fac}(\Delta) $	random	popvar	rarevar	popgcd
4000	20	0.05	0.05	0.06	0.06
6000	20	0.07	0.07	0.07	0.07
8000	20	0.09	0.09	0.08	0.09
10000	20	0.11	0.11	0.11	0.12
4000	30	3.70	2.31	5.73	9.83
6000	30	4.84	3.25	6.31	13.41
8000	30	5.87	4.18	7.61	15.56
10000	30	7.25	5.34	9.59	20.53
20	4000	0.13	0.12	0.12	0.24
20	6000	0.24	0.24	0.22	0.45
20	8000	0.37	0.37	0.36	0.65
20	10000	0.55	0.55	0.52	0.95
30	4000	4.11	2.63	5.78	14.53
30	6000	6.04	4.06	7.69	22.01
30	8000	7.48	5.37	9.11	28.85
30	10000	9.48	6.99	10.83	33.41
100	100	7.07	1.38	40.15	166.56
120	120	32.01	5.92	170.28	2109.31
140	140	130.44	23.19	740.48	>7200
160	160	491.43	89.87	2729.22	>7200
180	180	1497.50	261.09	>7200	>7200
200	200	4965.33	796.03	>7200	>7200
220	220	>7200	1756.16	>7200	>7200
240	240	>7200	5051.55	>7200	>7200

TABLE 1. BCRT pivot selection strategies. All times are in seconds.

use at each step. We compare the strategies described in Section 3.5/Section 4.5 on a battery of randomly generated complexes.

BCRT Pivots. Table 1 shows that **popvar** is the best BCRT pivot selection strategy for these randomly generated complexes. **popvar** is already faster for the simpler ideals and its lead over the other strategies increases with the number of facets and especially with the number of vertices.

random: This strategy is here to be able to tell if the other strategies are better or worse than a random choice of vertex.

popvar: We believe that **popvar** performs well because a rare vertex e gives $\Delta \cup \langle \overline{\{e\}} \rangle$ as few facets as possible.

rarevar: **rarevar** does the opposite of what **popvar** does. It is never far ahead and is mostly significantly behind **popvar**. For the balanced examples it is much worse than even a random choice of vertex.

popgcd: **popgcd** works well for the Hilbert-Poincaré series algorithms that the algorithms we present here are based on. However, the data shows that it is often significantly worse than choosing a random vertex when it comes to computing Euler characteristics.

DBMS Pivots. Table 2 shows that **raremax** is the best DBMS pivot selection strategy for these randomly generated complexes.

random: This strategy is here to be able to tell if the other strategies are better or worse than a random choice of pivot.

raremax: **raremax** combines the benefits of removing a facet that lacks a popular vertex with the benefit of removing a small facet. Table 2 shows that this combination is better than the pure strategies **rarevar** and **maxsupp**.

rarest: We believe that **rarest** performs well because removing facets that lack popular vertices tends to create complexes that are close to being a cone.

rarevar: **rarevar** is almost as good as **rarest**, which is reasonable since **rarest** is a more sophisticated way of breaking ties in **rarevar**.

popvar: **popvar** does the opposite of what **rarevar** does, so it is reasonable that it does poorly.

maxsupp: We believe that **maxsupp** performs better than **random** because removing small facets tends to create complexes that are close to being a cone.

minsupp: **minsupp** performs worse than **random**, confirming that it is beneficial to select small pivots.

6.2. Variations of the BCRT and DBMS Algorithms. Table 3 and Table 4 show times for the pivot selection strategies when the nerve technique from Section 3.6/Section 4.6 is turned off. First of all we observe that turning the nerve technique off does not change the ranking of the pivot selection strategies. Furthermore, we see that without nerves the BCRT algorithm is much more sensitive to the number of vertices while the DBMS technique is much more sensitive to the number of facets. The nerve technique hides this sensitivity as it allows to interchange the number of facets and the number of vertices, so the DBMS algorithm can adjust the input to make the number of facets less than the number of vertices and vice versa for the BCRT algorithm.

An alternative to the nerve technique is to use a hybrid approach where the BCRT algorithm is used for ideals with more facets and the DBMS algorithm is used for ideals with more vertices. If we compare the tables we see that the DBMS algorithm with the nerve technique turned on is faster than the hybrid approach even for ideals with more facets than vertices, so in this experiment the hybrid approach is inferior to the nerve technique.

6.3. Comparison of Euler Characteristic Implementations. Table 6 compares several implementations of Euler characteristic algorithms. This also serves as a comparison of the algorithms used by these implementations. Evaluating the practicality of an algorithm as opposed to an implementation is difficult because quality of implementation has a significant effect on performance yet quality of implementation cannot easily if at all be measured or corrected for.

An example of an implementation (as opposed to algorithm) difference is that Frobbly and Macaulay 2 are written in C++ that compiles to native code while Gap and Sage do not compile to native code due to the languages that they are written in.⁴ While the choice of implementation language can make a significant difference for performance, the magnitude of the differences in Table 6 is so large that choice of language is unlikely to be the main factor in our estimation.

⁴Sage does have modules written in Cython which is similar to Python but that does compile to native code. Many components of both Sage and Gap are written in native languages such as C and C++. However, that is not the case for the Euler characteristic components of Gap and Sage. Parts of Macaulay 2 are written in the interpreted Macaulay 2 language, but the Hilbert-Poincaré series code is written in C++.

$ V_\Delta $	$ \text{fac}(\Delta) $	random	popvar	maxsupp	rarest	raremax	minsup	rarevar
4000	20	0.07	0.06	0.06	0.04	0.04	0.08	0.04
6000	20	0.08	0.08	0.07	0.06	0.06	0.09	0.05
8000	20	0.09	0.09	0.08	0.07	0.07	0.11	0.07
10000	20	0.11	0.11	0.10	0.09	0.09	0.15	0.10
4000	30	4.53	4.46	2.36	1.14	1.06	13.34	1.24
6000	30	5.96	6.77	3.58	1.47	1.46	15.62	1.63
8000	30	7.83	8.28	4.46	1.97	1.90	21.17	2.26
10000	30	10.50	10.18	5.64	2.59	2.53	27.11	3.24
20	4000	0.19	0.18	0.16	0.12	0.12	0.22	0.12
20	6000	0.32	0.31	0.29	0.23	0.24	0.39	0.24
20	8000	0.48	0.48	0.46	0.37	0.37	0.57	0.38
20	10000	0.66	0.65	0.63	0.52	0.54	0.74	0.54
30	4000	5.08	4.97	3.10	1.88	1.76	10.68	2.06
30	6000	7.70	7.44	4.81	2.97	2.85	16.25	3.29
30	8000	10.16	10.02	6.46	3.99	3.98	19.61	4.48
30	10000	12.63	13.17	8.90	5.00	4.98	25.35	5.58
100	100	5.89	5.84	1.29	1.05	0.79	40.83	1.13
120	120	25.42	26.19	5.79	4.72	3.44	173.82	5.06
140	140	108.40	119.19	22.43	18.43	13.03	749.66	21.34
160	160	427.46	464.28	87.13	69.92	49.59	2767.93	78.52
180	180	1291.26	1241.91	253.15	192.95	143.14	>7200	232.63
200	200	4211.65	4137.68	769.79	614.39	434.43	>7200	755.41
220	220	>7200	>7200	1704.65	1440.19	980.31	>7200	1665.74
240	240	>7200	>7200	4871.15	4022.04	2697.84	>7200	4811.50

TABLE 2. DBMS pivot selection strategies. All times are in seconds.

$ V_\Delta $	$ \text{fac}(\Delta) $	random	popvar	rarevar	popgcd
30	10000	7.50	5.36	9.35	30.24
10000	30	>7200	62.13	>7200	>7200
240	240	>7200	6961.76	>7200	>7200

TABLE 3. BCRT pivot selection strategies without nerves. All times are in seconds.

$ V_\Delta $	$ \text{fac}(\Delta) $	random	raremax	rarest	rarevar	popvar	maxsupp	minsupp
30	10000	318.41	52.39	64.01	77.87	243.41	71.85	2382.40
10000	30	10.64	2.58	2.66	3.26	11.30	5.97	25.95
240	240	>7200	4184.07	6212.34	>7200	>7200	>7200	>7200

TABLE 4. DBMS pivot selection strategies without nerves. All times are in seconds.

We can draw the firm conclusion from Table 6 that the BCRT and DBMS algorithms presented in this paper can be implemented to be faster on this set of complexes than all the other Euler characteristic implementations that we have compared. We have found no faster Euler characteristic implementations than these, so we believe that the comparison we have made is the most fair and informative comparison that can be made using the implementations that exist today.

From Table 1, Table 2 and Table 6, we see that the DBMS algorithm is faster than the BCRT algorithm for all the complexes when both algorithms use their best pivot selection strategy. So we can recommend using the DBMS algorithm over the BCRT algorithm, even though the difference is slight. The nerve technique is vital to make the DBMS algorithm always be faster – without it, good performance could only be reached by implementing both algorithms and choosing which to use based on the ratio of facets to vertices.

The implementations in **Sage** and **GAP** are based on enumerating faces. Table 5 shows the number of faces of each complex, and there is a clear trend that the time taken by **Sage** and **GAP** is related to the number of faces of the complex. In contrast the times for the implementations in **Frobby** and **Macaulay 2** do not have a clear relationship to the number of faces.

For example **nicgraph-8-2** has 2928 times more faces than **match-13** does, and as expected **Sage** and **GAP** take much longer to compute the Euler characteristic of **nicgraph-8-2** than of **match-13**. In contrast **Frobby** computes the Euler characteristic of **nicgraph-8-2** in 0.06s while it takes 7.84s for **match-13**. So **nicgraph-8-2** has 2928 times more faces than **match-13** does yet it takes 260 times less time to compute its Euler characteristic using **Frobby**. We give this as evidence that the time taken by the BCRT and DBMS algorithms for Euler characteristic depends more on the structure of the complex than on the number of faces of the complex.

We find that the performance of **Frobby** and **Macaulay 2** in this comparison lends credence to the idea of using algebraic formulations and implementations for combinatorial problems.

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Example	Vertices	Facets	Faces	$\tilde{\chi}(\Delta)$
rook-6-6	36	720	13,327	185
rook-7-7	49	5,040	130,922	-204
rook-8-8	64	40,320	1,441,729	-6,209
match-9	36	945	2,620	-28
match-10	45	945	9,496	-1,216
match-11	55	10,395	35,696	-936
match-12	66	10,395	140,152	12,440
match-13	78	135,135	568,503	23,672
nicgraph-7-2	21	217	1,014,888	-120
nicgraph-8-2	28	504	166,537,616	-720
nicgraph-9-2	36	1,143	50,680,432,112	-5,040
randomv20g100	20	100	86,116	-25
randomv20g500	20	500	227,792	1,166
randomv20g1000	20	1,000	287,689	-1,007
randomv25g100	25	100	1,223,224	-202
randomv25g500	25	500	3,628,979	-3,815
randomv25g1000	25	1,000	5,368,430	3,666

TABLE 5. Characteristics of the examples used in Table 6.

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Example	Frobby DBMS	Frobby BCRT	Sage	Macaulay 2	GAP
rook-6-6	0.01	0.01	1.04	0.24	0.13
rook-7-7	0.13	0.14	12.59	3.37	3.86
rook-8-8	2.43	6.39	>223.11*	58.16	>7200
match-9	0.00	0.00	0.16	0.13	0.08
match-10	0.02	0.01	0.69	0.29	0.12
match-11	0.21	0.15	2.90	2.41	6.97
match-12	0.33	0.47	12.31	8.32	9.15
match-13	7.84	11.26	>7200	101.49	2401.58
nicgraph-7-2	0.00	0.00	322.94	0.43	22.41
nicgraph-8-2	0.03	0.06	>7200	10.33	>7200
nicgraph-9-2	0.40	0.65	>7200	306.28	>7200
randomv20f100	0.00	0.00	11.37	0.07	0.42
randomv20f500	0.01	0.01	35.99	0.43	3.28
randomv20f1000	0.02	0.02	47.32	0.72	7.60
randomv20f100	0.00	0.00	322.80	0.37	58.17
randomv20f500	0.02	0.04	>7200	3.16	592.70
randomv20f1000	0.02	0.04	>7200	6.43	>7200

* Sage reports taking 223.11s on rook-8-8, but the actual time was in excess of half an hour. The discrepancy persisted across several runs. We do not know how to explain the discrepancy since the times that Sage reports is usually in line with external measurements.

TABLE 6. Comparison of Euler characteristic implementations. All times are in seconds.