

Point Equivalence of Second-Order ODEs: Maximal Invariant Classification Order

Robert Milson¹

Department of Mathematics and Statistics
Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J5
milson@mathstat.dal.ca
<http://www.mathstat.dal.ca/~milson/>

Francis Valiquette²

Department of Mathematics
SUNY at New Paltz
New Paltz, NY 12561
valiquef@newpaltz.edu
<http://www2.newpaltz.edu/~valiquef>

Keywords: Differential invariants, moving frames, Painlevé equations, point transformations, second-order ordinary differential equations.

Mathematics subject classification (MSC2010): 53A55

Abstract

We show that the local equivalence problem of second-order ordinary differential equations under point transformations is completely characterized by differential invariants of order at most 10 and that this upper bound is sharp. We also demonstrate that, modulo Cartan duality and point transformations, the Painlevé–I equation can be characterized as the simplest second-order ordinary differential equation belonging to the class of equations requiring 10th order jets for their classification.

1 Introduction

This paper is concerned with the local equivalence of second-order ordinary differential equations (ODEs) under point transformations. This is a classical problem that has been extensively studied. This is particularly true of the linearization problem which consists of determining when an equation $u_{xx} = f(x, u, u_x)$ is locally equivalent to $u_{xx} = 0$. Sophus Lie was the first to observe that the equation had to be cubic in the first order derivative

$$u_{xx} = K(x, u) u_x^3 + L(x, u) u_x^2 + M(x, u) u_x + N(x, u)$$

to be linearizable, [18]. Precise conditions on the coefficients K , L , M , N were later determined by Liouville, [19]. Subsequently, equivalent linearization conditions were found by many authors, [4, 8, 9, 10, 12, 32, 35, 36, 38, 40]. Tresse was the first to give a complete generating set of differential invariants for generic second-order ODEs not constraint by differential relations, [38]. He also fully characterized the equations admitting a point symmetry group (For a modern geometrical account of Tresse’s paper we refer the reader to [17].). Another facet of the problem that has attracted considerable attention is the local classification of the Painlevé transcendents, [11, 12, 15]. To

¹Supported by NSERC grant RGPIN-228057-2009.

²Supported in part by an AARMS Postdoctoral Fellowship.

the best of our knowledge, none of the aforementioned references studies all the different branches of the equivalence problem. This is understandable as this is a computationally demanding task to do manually. But with computer algebra systems becoming more and more efficient, a wide range of equivalence problems can now be codified. It then becomes important to establish an upper bound on the number of iterations the algorithm has to go through to guarantee a complete solution. In geometry, and particularly in general relativity, [13, 21], it is common to search for the highest order differential invariants encountered in the solution of an equivalence problem. In this paper we do the same for the point equivalence problem of second-order ODEs. To determine the highest order differential invariants occurring in the solution of the equivalence problem we survey the different branches of the equivalence problem and focus our attention on the most singular ones as the highest order invariants will occur in these branches.

There are several ways of finding these highest order invariants. Based on one's preference, it is possible to use Lie's infinitesimal method, [24], the theory of G -structures, [12, 25], or the method of equivariant moving frames, [29, 39]. We decided to employ the theory of equivariant moving frames to take advantage of the symbolic and algorithmic nature of the method. The solution relies on the *universal recurrence relations* which symbolically determine the exterior derivative of differential invariants. Very little information is needed to write down these equations. It only requires the knowledge of the infinitesimal generator of the equivalence pseudo-group and the choice of a cross-section to the pseudo-group orbits. In particular, the coordinate expressions for the invariants are not required. Also, the computations only involve differentiation and linear algebra and these are well handled by symbolic softwares. In our case, we used MATHEMATICA to implement the computations.

The main result of this paper establishes that any regular second order ODE is classified, relative to point transformations, by its 10th order jets. Furthermore, this bound is sharp, meaning that there exist regular ODEs that are not classified by 9th order jets. Furthermore, we show that every equation having maximal invariant classification (IC) order of 10 is equivalent, modulo point transformations and Cartan duality, to an equation of the form

$$u_{xx} = 6u^2 + g(x) \quad \text{with} \quad D_x^2[g(x)^{-1/4}] \neq 0, \quad (1.1)$$

the “simplest” of which is the Painlevé-I equation $u_{xx} = 6u^2 + x$ whose 10th order classifying invariant vanishes. The branch in which equation (1.1) occurs can also be found in the works of Kamran, [12], Morozov, [22], and Sharipov, [36], though none of them have studied the question of maximal invariant classification order.

2 Formalization of the problem and results

Following standard practices, [12, 25], we let

$$p = u_x, \quad q = u_{xx}$$

denote the first and second order derivatives of a single variable function $u = u(x)$. Then, two second-order ordinary differential equations

$$q = f(x, u, p) \quad \text{and} \quad Q = F(X, U, P), \quad p = u_x, q = u_{xx}, P = U_X, Q = U_{XX},$$

are (locally) point equivalent if there exists a local diffeomorphism of the plane

$$(X, U) = \psi(x, u), \quad \psi \in \text{Diff}(\mathbb{R}^2), \quad (2.1)$$

such that

$$F(X, U, P) = \widehat{Q}(p, f(x, u, p), X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, U_{xu}, U_{uu}) \quad (2.2)$$

where

$$\begin{aligned} P &= \widehat{P}(p, X_x, X_u, U_x, U_u) = \frac{\widehat{D}_x U}{\widehat{D}_x X} = \frac{p U_u + U_x}{p X_u + X_x}, \\ Q &= \widehat{Q}(p, q, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, U_{xu}, U_{uu}) \\ &= \frac{\widehat{D}_x P}{\widehat{D}_x X} = \frac{\widehat{D}_x^2 U \cdot \widehat{D}_x X - \widehat{D}_x U \cdot \widehat{D}_x^2 X}{(\widehat{D}_x X)^3} \\ &= \frac{(p X_u + X_x) (p^2 U_{uu} + 2p U_{ux} + U_{xx}) - (p U_u + U_x) (p^2 X_{uu} + 2p X_{ux} + X_{xx})}{(p X_u + X_x)^3} \\ &\quad + \frac{U_u X_x - U_x X_u}{(p X_u + X_x)^3} q, \end{aligned} \quad (2.3)$$

describe the transformation law for $p = u_x$ and $q = u_{xx}$, and

$$\widehat{D}_x = \partial_x + p \partial_u + q \partial_p$$

is the truncation of the usual total derivative operator. The equations (2.3) together with the usual contact conditions, constitute a quasi-linear system of PDEs in the pseudo-group jet variables X, U, X_x, X_u, U_x, U_u . This system is over-determined, owing to higher order integrability conditions that, for sufficiently high order, govern the outcome of the equivalence problem. These integrability conditions take the form of equalities between differential invariants of the two equations leading to Definition 2.1 below.

Taking repeated derivatives of (2.2) with respect to X, U, P yields the following necessary conditions for equivalence:

$$F_{ijk}(X, U, P) = \frac{\partial^{i+j+k} F(X, U, P)}{\partial X^i \partial U^j \partial P^k} = \widehat{Q}_{ijk}(p, q, q_x, q_u, q_p, \dots; X_x, X_u, U_x, U_u, \dots),$$

where

$$\widehat{Q}_{ijk}(p, q^{(i+j+k)}; \psi_0^{(i+j+k+2)}), \quad \psi \in \text{Diff}(\mathbb{R}^2),$$

describes the transformation law for the partial derivatives

$$q_{ijk} = \frac{\partial^{i+j+k} q}{\partial x^i \partial u^j \partial p^k}$$

under the point transformations (2.1), and where

$$\begin{aligned} q^{(n)} &= \{q_{ijk} : 0 \leq i + j + k \leq n\}, & q_{ijk} &:= q_{x^i u^j p^k}; \\ \psi^{(n)} &= \{(X_{ij}, U_{ij}) : 0 \leq i + j \leq n\}, & X_{ij} &:= X_{x^i u^j}, \quad U_{ij} := U_{x^i u^j}; \\ \psi_0^{(n)} &= \{(X_{ij}, U_{ij}) : 1 \leq i + j \leq n\}; \end{aligned} \quad (2.4)$$

denote the indicated jets.

Definition 2.1. We say that a second-order ODE $q = f(x, u, p)$ is classified by n th order jets if the *algebraic* consistency of the system

$$P = \hat{P}(p, \psi_0^{(1)}),$$

$$F_{ijk}(X, U, P) = \hat{Q}_{ijk}(p, f^{(i+j+k)}(x, u, p), \psi_0^{(i+j+k+2)}), \quad 0 \leq i + j + k \leq n;$$

is sufficient for the existence of a point transformation relating $q = f(x, u, p)$ and $Q = F(X, U, P)$. We call the smallest such n the *IC (invariant classification) order* of the differential equation.

Note: In the formulation of the above definition it must be understood that the pseudo-group variables $(X_x, X_u, U_x, U_u, \dots)$ are to be treated as auxiliary independent variables rather than functions of x and u .

The question that motivates us here is the following:

What is the maximal jet order required for the invariant classification of a second-order ordinary differential equation up to local point transformations?

By way of an example, the well-known Linearization Theorem for second-order ordinary differential equations, [8, 9, 25, 34], states that $q = f(x, u, p)$ is point equivalent to the trivial equation $Q = U_{XX} = 0$ if and only if the fourth-order relative¹ invariants

$$q_{pppp} \equiv 0,$$

$$\hat{D}_x^2(q_{pp}) - 4\hat{D}_x(q_{up}) - q_p\hat{D}_x(q_{pp}) + 6q_{uu} - 3q_uq_{pp} + 4q_pq_{up} \equiv 0, \quad (2.5)$$

are identically zero. From this it follows that the linearizable class has IC order equal to 4. On the other hand, the invariant classification of general second-order equations will require higher order jets. The complete answer regarding the maximal order is given below in Theorem 2.4.

A well-posed equivalence problem requires some notion of regularity. Therefore, before proceeding further, we must impose a technical rank assumption. Owing to the covariance of the transformation laws, [29], the functions \hat{Q}_{ijk} are invariant with respect to point transformations. For a smoothly defined ODE $q = f(x, u, p)$, let

$$q^{(n)} = f^{(n)}(x, u, p)$$

denote the n th order jet of the defining function. The composition of \hat{Q}_{ijk} and $f^{(n)}$ produces an invariant of the ODE,

$$Q_{ijk} = \hat{Q}_{ijk}\left(p, f^{(i+j+k)}(x, u, p), \psi_0^{(i+j+k+2)}\right), \quad (2.6)$$

which we call a *lifted invariant*² to signify its dependence on the pseudo-group variables $\psi_0^{(n)} = (X_x, X_u, U_x, U_u, \dots)$.

The above remarks lead us to the following definition, after which we will be ready to state our main result.

¹A relative invariant is a function whose value is multiplied by a certain factor, known as a multiplier, under pseudo-group transformations. An invariant is a relative invariant with multiplier equal to one.

²By contrast, *absolute differential invariants* are functions of x, u, p only. To construct absolute invariants one eliminates, by normalization, the pseudo-group variables from the lifted invariants of sufficiently high order. This is the essence of the (equivariant) moving frame method, [29].

Definition 2.2. We say that a smoothly defined ODE $q = f(x, u, p)$ is *regular* if for every $n \geq 0$ the rank of the lifted invariants $\{Q_{ijk} : 0 \leq i + j + k \leq n\}$ is constant on the differential equation $q = f(x, u, p)$.

Remark 2.3. Irregular differential equations that fail to satisfy Definition 2.2 are more difficult to classify and require more care, [25]. Following customary practice, these equations are omitted in this paper.

We now can state the main result of this paper.

Theorem 2.4. Every regular second-order ODE is classified, relative to point transformations, by its 10th order jets. This bound is sharp — meaning that there exist regular ODEs that are not classified by 9th order jets. Furthermore, every ODE having the maximal IC order of 10 is equivalent, modulo point transformations and Cartan duality (see Appendix A for the definition), to an equation of the form

$$u_{xx} = 6u^2 + g(x) \quad \text{with} \quad D_x^2[g(x)^{-1/4}] \neq 0. \quad (2.7)$$

The proof of Theorem 2.4 boils down to identifying the branch(es) of the equivalence problem where non-constant absolute invariants appear as late and slowly as possible during the course of Cartan's normalization procedure. Taking advantage of the *universal recurrence relations*, we first give a proof of Theorem 2.4 which does not require explicit coordinate expressions for the differential invariants. This is possible since the universal recurrence relations can be written down knowing only the expression for the prolonged infinitesimal generators of the pseudo-group action and the choice of a cross-section. Using the fact that there is a notion of duality among second-order ordinary differential equations (see Appendix A) our conclusion is that there exist two families of differential equations (dual to each other) that achieve the maximal IC order. By a generalization of Cartan's Integration Theorem, [3], we are then able to show that one of the two families of differential equations depends on one arbitrary function of the independent variable. Finally, we integrate the structure equations for the canonical coframe and derive form (2.7).

Second-order ordinary differential equations equivalent to (2.7) by point transformations and Cartan duality admit three fundamental absolute invariants I_7, I_8, I_9 of the indicated order, and a tenth order invariant I_{10} which is functionally dependent on I_9 . Relative to the normal form (2.7), these invariants can be expressed as

$$I_7 = \frac{p}{u^{3/2}}, \quad I_8 = \frac{g(x)}{u^2}, \quad I_9 = \frac{(g'(x))^4}{(g(x))^5}, \quad I_{10} = \frac{g(x)g''(x)}{(g'(x))^2}. \quad (2.8)$$

The functional relation between I_9 and I_{10} is the essential classifying relation for equations of maximal IC order.

The class of equations (2.7) requiring 10th order jets for their classification includes the Painlevé-I equation as the subclass when $g(x)$ is linear in x . As a Corollary to Theorem 2.4 we are able to give the following characterization of Painlevé-I.

Theorem 2.5. The equivalence class of the Painlevé-I equation can be characterized as the subclass of second-order ODEs requiring 10th order jets for their classification and satisfying

$$q_{pppp} = 0, \quad I_{10} = 0.$$

The condition $q_{pppp} = 0$ distinguishes the Painlevé-I equation from its Cartan dual. The vanishing of $I_{10} = 0$ means that the Painlevé-I equation can be characterized as the “simplest” second-order differential equation belonging to the class of maximal IC order equations. We will derive (2.8) at the end of Section 5.2. Thereafter Theorem 2.5 follows as a straight-forward Corollary of Theorem 2.4.

Remark 2.6. An invariant characterization of the Painlevé-I and II equations up to fibre preserving transformations and point transformations were given in [11, 12]. In the latter case, the characterization obtained is a particular case of our more general result when $g(x) = x$. Other works devoted to the Painlevé-I and II equations can be found in [1, 6, 14, 15].

3 The geometric setting

In this section we introduce the geometric setting for the equivalence problem of second-order ordinary differential equations under point transformations. The key formalism is a certain groupoid and two sets of fundamental equations: the *universal recurrence relations* for the prolonged jet coordinates, and the *Maurer–Cartan structure equations* of the infinite-dimensional Lie pseudo-group $\mathcal{D} = \text{Diff}(\mathbb{R}^2)$ [28, 29].

Let $N^3 = J^1(\mathbb{R}, \mathbb{R})$ and $M^4 = J^2(\mathbb{R}, \mathbb{R})$ denote, respectively, the first- and second-order jet space of curves $u = u(x)$. Setting $p = u_x$ and $q = u_{xx}$, local coordinates are given by

$$N^3: x, u, p; \quad M^4: x, u, p, q.$$

A smoothly defined second-order ordinary differential equation $q = f(x, u, p)$ is then identified as a smooth section $f: N^3 \rightarrow M^4$. Let $\Gamma(N^3, M^4)$ denote the space of sections $f: N^3 \rightarrow M^4$, then $\mathcal{J}^{(n)} = J^n \Gamma(N^3, M^4)$ can be identified as the bundle of n th-order jets of second-order ODEs, [24]. Local coordinates are given by

$$\mathcal{J}^{(n)}: x, u, p, q^{(n)};$$

where $q^{(n)} = (\dots q_{x^i u^j p^k} \dots)$ collects the derivative coordinates of order $\leq n$. The prolongation of a differential equation $q = f(x, u, p)$ yields a section of $\mathcal{J}^{(n)}$ which we denoted $f^{(n)}: N^3 \rightarrow \mathcal{J}^{(n)}$. Next, let $\mathcal{G} \subset \text{Diff}(\mathbb{R}^4)$ denote the prolongation of $\text{Diff}(\mathbb{R}^2)$ to M^4 . In local coordinates, the pseudo-group \mathcal{G} is given by (2.1) and (2.3). Geometrically, \mathcal{G} specifies how a second-order ordinary differential equation transforms under a point transformation. For $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)} \rightarrow \mathbb{R}^2$ denote the bundle of n th order diffeomorphism jets of $\mathcal{D} = \text{Diff}(\mathbb{R}^2)$ and similarly let $\mathcal{G}^{(n)}$ denote the n th order pseudo-group jet bundle of \mathcal{G} . Local coordinates are given by

$$\mathcal{D}^{(n)}: x, u, X, U, \psi_0^{(n)},$$

where $\psi_0^{(n)} = (\dots X_{x^i u^j}, U_{x^i u^j} \dots)$ denotes the derivative coordinates of order 1 up to n . Now, let $\mathcal{E}^{(n)} \rightarrow \mathcal{J}^{(n)}$ denote the n th order *lifted bundle* obtained by pulling back $\mathcal{D}^{(n+2)} \cong \mathcal{G}^{(n)} \rightarrow \mathbb{R}^2$ via the projection $\mathcal{J}^{(n)} \rightarrow M^4 \rightarrow N^3 \rightarrow \mathbb{R}^2$, [29]. Canonical bundle coordinates on $\mathcal{E}^{(n)}$ are

$$\mathcal{E}^{(n)}: x, u, p, q^{(n)}, X, U, \psi_0^{(n+2)}.$$

We note that the n th-order diffeomorphism jet $\psi^{(n)} = (X, U, \psi_0^{(n)}) = (X, U, X_x, X_u, U_x, U_u, \dots)$ has

$$2 \times (1 + 2 + 3 + \dots + n + 1) = (n + 1)(n + 2)$$

components and that the n th-order jet $q^{(n)} = (q, q_x, q_u, q_p, \dots) \in \mathcal{J}^{(n)}|_{(x,u,p)}$ has $\binom{n+3}{3}$ components. Hence

$$\begin{aligned} \dim \mathcal{J}^{(n)} &= 3 + \binom{n+3}{3}, & \dim \mathcal{D}^{(n)} &= 2 + (n+1)(n+2), \\ \dim \mathcal{E}^{(n)} &= 3 + \binom{n+3}{3} + (n+3)(n+4). \end{aligned}$$

Next, we define the groupoid structure of the lifted bundle $\mathcal{E}^{(n)}$. The source map $\sigma^{(n)}: \mathcal{E}^{(n)} \rightarrow \mathcal{J}^{(n)}$ is the standard projection given by $(x, u, p, q^{(n)})$; while the target map $\tau^{(n)}: \mathcal{E}^{(n)} \rightarrow \mathcal{J}^{(n)}$ is the projection given by the prolonged action $(X, U, P, Q^{(n)})$, where

$$P = \hat{P}(p, \psi_0^{(1)})$$

as per (2.3) and

$$Q^{(n)} = \hat{Q}^{(n)}(p, q^{(n)}, \psi_0^{(n+2)}) = \left\{ \hat{Q}_{ijk}(p, q^{(n)}, \psi_0^{(i+j+k+2)}) : 0 \leq i + j + k \leq n \right\}.$$

The diffeomorphism pseudo-group $\mathcal{D} = \text{Diff}(\mathbb{R}^2)$ has a prolonged action on $\mathcal{J}^{(n)}$ and two dual prolonged actions on $\mathcal{E}^{(n)}$. Given a point transformation $(X, U) = \varphi(x, u) \in \mathcal{D}$ we define the prolonged actions

$$\varphi^{(n)}: \mathcal{J}^{(n)} \rightarrow \mathcal{J}^{(n)}, \quad \varphi_L^{(n)}: \mathcal{E}^{(n)} \rightarrow \mathcal{E}^{(n)}, \quad \varphi_R^{(n)}: \mathcal{E}^{(n)} \rightarrow \mathcal{E}^{(n)}$$

according to [28]:

$$\begin{aligned} \varphi^{(n)}: (x, u, p, q^{(n)}) &\mapsto (X, U, \hat{P}(p, \varphi_0^{(1)}), \hat{Q}^{(n)}(p, q^{(n)}, \varphi_0^{(n+2)})); \\ \varphi_L^{(n)}: (x, u, p, q^{(n)}, \psi^{(n)}) &\mapsto (x, u, p, q^{(n)}, (\varphi \circ \psi)^{(n)}); \\ \varphi_R^{(n)}: (x, u, p, q^{(n)}, \psi^{(n)}) &\mapsto (X, U, \hat{P}(p, \varphi_0^{(1)}), \hat{Q}^{(n)}(p, q^{(n)}, \varphi_0^{(n+2)}), (\psi \circ \varphi^{-1})^{(n)}). \end{aligned} \tag{3.1}$$

From the above definitions, it follows immediately that the source projection $\sigma^{(n)}$ is φ_L -invariant and φ_R -equivariant:

$$\sigma^{(n)} \circ \varphi_L^{(n)} = \sigma^{(n)}, \quad \sigma^{(n)} \circ \varphi_R^{(n)} = \varphi^{(n)} \circ \sigma^{(n)};$$

and that, dually, the target projection is φ_R -invariant and φ_L -equivariant:

$$\tau^{(n)} \circ \varphi_R^{(n)} = \tau^{(n)}, \quad \tau^{(n)} \circ \varphi_L^{(n)} = \varphi^{(n)} \circ \tau^{(n)}. \tag{3.2}$$

A coframe on $\mathcal{J}^{(\infty)}$ is given by the basic horizontal one-forms

$$dx, \quad du, \quad dp, \tag{3.3a}$$

and the contact one-forms

$$\theta_{ijk} = dq_{ijk} - q_{i+1,j,k} dx - q_{i,j+1,k} du - q_{i,j,k+1} dp, \quad i, j, k \geq 0. \tag{3.3b}$$

The standard coframe on $\mathcal{D}^{(\infty)}$ is spanned by dx, du together with the group forms

$$\Upsilon_{ij} = dX_{ij} - X_{i+1,j} dx - X_{i,j+1} du, \quad \Psi_{ij} = dU_{ij} - U_{i+1,j} dx - U_{i,j+1} du, \quad i, j \geq 0, \quad (3.4)$$

while a right-invariant coframe on $\mathcal{D}^{(\infty)}$ consists of the one-forms

$$\omega^x = X_x dx + X_u du, \quad \omega^u = U_x dx + U_u du \quad (3.5)$$

and the Maurer–Cartan one-forms

$$\mu_{ij} = \mu_{X^i U^j}, \quad \nu_{ij} = \nu_{X^i U^j}, \quad i, j \geq 0. \quad (3.6)$$

The latter are defined, implicitly, by taking formal derivatives of the relations

$$\Upsilon = \mu, \quad \Psi = \nu,$$

with respect to x, u , and then solving for the μ_{ij}, ν_{ij} . The first few relations that result are shown below:

$$\begin{aligned} \Upsilon_x &= X_x \mu_X + U_x \mu_U, & \Psi_x &= X_x \nu_X + U_x \nu_U, \\ \Upsilon_u &= X_u \mu_X + U_u \mu_U, & \Psi_u &= X_u \nu_X + U_u \nu_U, \\ \Upsilon_{xx}^1 &= X_x^2 \mu_{XX} + 2X_x U_x \mu_{UX} + U_x^2 \mu_{UU} + X_{xx} \mu_X + U_{xx} \mu_U, \\ \Upsilon_{ux}^1 &= X_x X_u \mu_{XX} + (X_x U_u + X_u U_x) \mu_{UX} + U_x U_u \mu_{UU} + X_{ux} \mu_X + U_{ux} \mu_U, \\ \Upsilon_{uu}^1 &= X_u^2 \mu_{XX} + 2X_u U_u \mu_{UX} + U_u^2 \mu_{UU} + X_{uu} \mu_X + U_{uu} \mu_U, \\ &\vdots \end{aligned}$$

with the coefficients of the higher-order relations given by the multi-variate Fàa-di-Bruno polynomials, [33]. The precise coordinate expression of the Maurer–Cartan forms is derived in [28], but these are not necessary for the symbolic implementation of the moving frame method.

To construct an invariant coframe on $\mathcal{E}^{(\infty)}$ we note that the space of differential forms $\Omega^* = \Omega^*(\mathcal{E}^{(\infty)})$ on the infinite-order lifted bundle decomposes into

$$\Omega^* = \oplus_{k,l} \Omega^{k,l},$$

where k indicates the number of jet forms (3.3) and l the number of group forms (3.4). Let $\Omega_J^* = \oplus_k \Omega^{k,0}$ denote the subspace of jet forms, and define the projection map $\pi_J: \Omega^* \rightarrow \Omega_J^*$ onto the jet component.

Definition 3.1. The *lift* transformation $\lambda: \Omega^*(\mathcal{J}^\infty) \rightarrow \Omega_J^*$ is defined by

$$\lambda = \pi_J \circ (\tau^{(\infty)})^*. \quad (3.7)$$

Thus, by construction, the lift of a jet form on $\mathcal{J}^{(\infty)}$ is an invariant jet form defined on $\mathcal{E}^{(\infty)}$. In particular, we have

$$\omega^x = \lambda(dx), \quad \omega^u = \lambda(du).$$

We also introduce the invariant one-forms

$$\begin{aligned}
\omega^p &= \boldsymbol{\lambda}(dp) = \pi_J(dP) \\
&= (pX_u + X_x)^{-2} \left\{ (X_x U_u - X_u U_x) dp + \right. \\
&\quad + \left(p^2 (U_{ux} X_u - X_{ux} U_u) + p (U_{xx} X_u - X_{xx} U_u + U_{ux} X_x - X_{ux} U_x) \right) + \\
&\quad + (U_{xx} X_x - X_{xx} U_x) dx \\
&\quad + \left(p^2 (U_{uu} X_u - X_{uu} U_u) + p (U_{uu} X_x - X_{uu} U_x + U_{ux} X_u - X_{ux} U_u) \right) + \\
&\quad \left. + (U_{ux} X_x - X_{ux} U_x) du \right\}, \\
\vartheta_{0,0,0} &= \boldsymbol{\lambda}(\theta_{0,0,0}) = \frac{X_x U_u - U_x X_u}{(pX_u + X_x)^3} (dq - q_x dx - q_u du - q_p dp),
\end{aligned} \tag{3.8}$$

and more generally,

$$\vartheta_{ijk} = \boldsymbol{\lambda}(\theta_{ijk}).$$

Next, we introduce the infinitesimal generator

$$\begin{aligned}
\mathbf{v} &= \xi^1(x, u) \frac{\partial}{\partial x} + \xi^2(x, u) \frac{\partial}{\partial u} + \xi^3(x, u, p) \frac{\partial}{\partial p} + \phi(x, u, p, q) \frac{\partial}{\partial q} + \sum_{i+j+k \geq 1} \phi^{ijk} \frac{\partial}{\partial q_{ijk}} \\
\mathbf{v} &= \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + [\eta_x + p(\eta_u - \xi_x) - p^2 \xi_u] \frac{\partial}{\partial p} + [\eta_{xx} + q(\eta_u - 2\xi_x) \\
&\quad + p(2\eta_{xu} - \xi_{xx}) - 3pq\xi_u + p^2(\eta_{uu} - 2\xi_{xu}) - p^3 \xi_{uu}] \frac{\partial}{\partial q} + \sum_{i+j+k \geq 1} \phi^{ijk} \frac{\partial}{\partial q_{ijk}}
\end{aligned} \tag{3.9}$$

of the $\text{Diff}(\mathbb{R}^2)$ action on $\mathcal{J}^{(\infty)}$ obtained by prolonging (2.2). The coefficients ϕ^{ijk} are defined recursively by the usual prolongation formula

$$\phi^{ijk} = D_x^i D_u^j D_p^k (\phi - \xi^1 q_x - \xi^2 q_u - \xi^3 q_p) + \xi^1 q_{ij,k+1} + \xi^2 q_{i,j+1,k} + \xi^3 q_{i,j,k+1}, \tag{3.10}$$

where

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + \sum_{i,j,k \geq 0} q_{i+1,jk} \frac{\partial}{\partial q_{ijk}}, \quad D_u = \frac{\partial}{\partial u} + \sum_{i,j,k \geq 0} q_{i,j+1,k} \frac{\partial}{\partial q_{ijk}}, \\
D_p &= \frac{\partial}{\partial p} + \sum_{i,j,k \geq 0} q_{ij,k+1} \frac{\partial}{\partial q_{ijk}},
\end{aligned}$$

are the total derivative operators on $\mathcal{J}^{(\infty)}$.

We now extend of the definition of the lift map (3.7) to the vector field jet coordinates ξ_{ij} and η_{ij} following [28, Section 5]. For this, let

$$\mathbf{v} = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \tag{3.11}$$

be an infinitesimal generator of $\mathcal{D} = \text{Diff}(\mathbb{R}^2)$. Then the lift of (3.11) is the right-invariant vector field

$$\boldsymbol{\lambda}(\mathbf{v}) = \sum_{i,j \geq 0} \left[\mathbb{D}_x^i \mathbb{D}_u^j \xi(X, U) \frac{\partial}{\partial X_{ij}} + \mathbb{D}_x^i \mathbb{D}_u^j \eta(X, U) \frac{\partial}{\partial U_{ij}} \right]$$

tangent to the source fibers of $\mathcal{D}^{(\infty)}$, where

$$\mathbb{D}_x = X_x D_X + U_x D_U, \quad \mathbb{D}_u = X_u D_X + U_u D_U.$$

Let $j_\infty \mathbf{v} \in J^\infty T\mathbb{R}^2$ denotes the infinite jet of (3.11), then the lift of a section $\zeta \in (J^\infty T\mathbb{R}^2)^*$ in the dual bundle to the vector field jet bundle $J^\infty T\mathbb{R}^2$ is defined by the equality

$$\langle \lambda(\zeta); \lambda(\mathbf{v}) \rangle \big|_{\phi^{(\infty)}} = \langle \zeta; j_\infty \mathbf{v} \rangle \big|_{(X,U)} \quad \text{whenever} \quad \begin{aligned} \phi^{(\infty)} &\in \mathcal{D}^{(\infty)}, \\ (X, U) &= \tau(\phi). \end{aligned} \quad (3.12)$$

Since each vector field jet coordinate functions $\xi_{x^i u^j} = \xi_{ij}$, $\eta_{x^i u^j} = \eta_{ij}$ can be viewed as sections of $(J^\infty T\mathbb{R}^2)^*$ we have from (3.12) the following defining equalities

$$\lambda(\xi_{ij}) := \mu_{ij}, \quad \lambda(\eta_{ij}) := \nu_{ij}, \quad i, j \geq 0, \quad (3.13)$$

where μ_{ij} , ν_{ij} are the Maurer–Cartan forms introduced in (3.13).

We also note that the lift of the source variables, called *lifted invariants*, gives the target variables:

$$X = \lambda(x), \quad U = \lambda(u), \quad P = \lambda(p), \quad Q_{ijk} = \lambda(q_{ijk}). \quad (3.14)$$

Proposition 3.2. The *universal recurrence relations* for the lifted invariants are

$$\begin{aligned} dX &= \omega^x + \mu, \\ dU &= \omega^u + \nu, \end{aligned} \quad (3.15a)$$

$$\begin{aligned} dP &= \omega^p + \nu_X + P(\nu_U - \mu_X) - P^2 \mu_U, \\ dQ_{ijk} &= Q_{i+1,jk} \omega^x + Q_{i,j+1,k} \omega^u + Q_{ij,k+1} \omega^p + \vartheta_{ijk} + \lambda(\phi^{ijk}), \end{aligned} \quad (3.15b)$$

where $\lambda(\phi^{ijk})$ is the lift of the prolonged vector field coefficient (3.10).

By the prolongation formula (3.10) and (3.9), the vector field coefficient ϕ^{ijk} is well-defined linear combination of the vector field jet coordinates ξ_{ij} , η_{ij} with polynomial coefficients in p and q_{ijk} . Thus, by virtue of (3.13) and (3.14) the correction term $\lambda(\phi^{ijk})$ in (3.15b) is a certain linear combination of the Maurer–Cartan forms μ_{ij} , ν_{ij} whose coefficients depend polynomially on the lifted invariants P , Q_{ijk} .

Finally, in our analysis we will need to consider the structure equations of the Maurer–Cartan forms (3.13).

Proposition 3.3. The *Maurer–Cartan structure equations* of the diffeomorphism pseudo-group (2.1) are, [31],

$$\begin{aligned} d\mu_{ij} &= \sum_{\substack{(0,0) \leq (k,\ell) \leq (i,j) \\ (k,\ell) \neq (i,j)}} \binom{i}{k} \binom{j}{\ell} (\mu_{k+1,\ell} \wedge \mu_{i-k,j-\ell} + \mu_{k,\ell+1} \wedge \nu_{i-k,j-\ell}), \\ &\quad - \mu_{i+1,j} \wedge \omega^x - \mu_{i,j+1} \wedge \omega^u, \\ d\nu_{ij} &= \sum_{\substack{(0,0) \leq (k,\ell) \leq (i,j) \\ (k,\ell) \neq (i,j)}} \binom{i}{k} \binom{j}{\ell} (\nu_{k+1,\ell} \wedge \mu_{i-k,j-\ell} + \nu_{k,\ell+1} \wedge \nu_{i-k,j-\ell}) \\ &\quad - \nu_{i+1,j} \wedge \omega^x - \nu_{i,j+1} \wedge \omega^u. \end{aligned} \quad (3.16)$$

4 The equivalence problem

With all the tools in hand, we can now delve into the point equivalence problem of second-order ODEs and prove Theorem 2.4. We begin by setting the equivalence problem within the geometrical framework of the previous section.

4.1 The direct, infinite-dimensional formulation

Given a smoothly defined ODE $q = f(x, u, p)$, let $\mathcal{E}_f^{(n)} \rightarrow N^3$ denote the bundle over N^3 given by the pullback of $\mathcal{E}^{(n)}$ by $f^{(n)}$. The canonical bundle coordinates on $\mathcal{E}_f^{(n)}$ are $x, u, p, X, U, \psi_0^{(n+2)}$, which means that

$$\dim \mathcal{E}_f^{(n)} = 3 + (n+3)(n+4).$$

The corresponding embedding $\mathcal{E}_f^{(n)} \hookrightarrow \mathcal{E}^{(n)}$ is given by

$$(x, u, p, X, U, \psi_0^{(n+2)}) \mapsto (x, u, p, f^{(n)}(x, u, p), X, U, \psi_0^{(n+2)}).$$

It follows that $\mathcal{E}_f^{(n)}$ is no longer a groupoid, but merely a fibre bundle. The corresponding restrictions of the source and target projections to $\mathcal{E}_f^{(n)}$, denoted $\sigma_f^{(n)}: \mathcal{E}_f^{(n)} \rightarrow N^3$ and $\tau_f^{(n)}: \mathcal{E}_f^{(n)} \rightarrow \mathcal{J}^{(n)}$, respectively, are given by

$$\begin{aligned} \sigma_f^{(n)}: (x, u, p, X, U, \psi_0^{(n+2)}) &\mapsto (x, u, p), \\ \tau_f^{(n)}: (x, u, p, X, U, \psi_0^{(n+2)}) &\mapsto (X, U, \hat{P}(p, \psi_0^{(1)}), \hat{Q}^{(n)}(p, f^{(n)}(x, u, p), \psi_0^{(n+2)})). \end{aligned}$$

Since the mapping

$$(p, \psi^{(n)}) \mapsto (X, U, \hat{P}(p, \psi_0^{(1)}))$$

has constant rank, the rank of $\tau_f^{(n)}$ is constant if and only if the ODE is regular as per Definition 2.2. Also, by (3.2), the restricted target map, $\tau_f^{(n)}$, is invariant with respect to point transformations. Hence, if a given ODE is classified by n th order jets, then the $\text{img } \tau_f^{(n)} \subset \mathcal{J}^{(n)}$ serves as a signature manifold for the ODE; that is, two ODEs are locally point-equivalent if and only if their signatures overlap on an open set.

The contact invariant one-forms ϑ_{ijk} in (3.15b) are essential when working with the invariant variational bicomplex, [16, 37], or studying geometric submanifold flows, [20, 26]. Since the restriction of $\mathcal{E}^{(n)}$ to $\mathcal{E}_f^{(n)}$ annihilates the one-forms ϑ_{ijk} , the equivalence problem must be formulated in terms of the invariant $\mathcal{E}_f^{(\infty)}$ -coframe

$$\omega^x, \quad \omega^u, \quad \omega^p, \quad \mu_{ij}, \quad \nu_{ij}. \quad (4.1)$$

The structure equations for this coframe consist of (3.16) as well as

$$\begin{aligned} d\omega^x &= -d\mu = \mu_X \wedge \omega^x + \mu_U \wedge \omega^u, \\ d\omega^u &= -d\nu = \nu_X \wedge \omega^x + \nu_U \wedge \omega^u, \\ d\omega^p &= (\nu_U - \mu_X - 2P\mu_U) \wedge \omega^p + (\nu_{UX} + P(\nu_{UU} - \mu_{UX}) - P^2\mu_{UU}) \wedge \omega^u \\ &\quad + (\nu_{XX} + P(\nu_{UX} - \mu_{XX}) - P^2\mu_{UX}) \wedge \omega^x. \end{aligned} \quad (4.2)$$

The latter are obtained by computing the exterior derivative of (3.15a), and taking into account the Maurer–Cartan structure equations (3.16).

On $\mathcal{E}_f^{(\infty)}$, the universal recurrence relations (3.15b) must be considered modulo the contact one-forms $\vartheta = \{\vartheta_{ijk}\}$. These relations then express the one-forms dQ_{ijk} as invariant linear combinations of the coframe (4.1). So in effect, the present setting can be considered as an infinite-dimensional overdetermined equivalence problem; see [25, p. 297] for a discussion.

4.2 The universal reduction

The direct formulation of the equivalence problem outlined in the preceding subsection suffers from an essential difficulty stemming from the fact that the lifted invariants depend on an unbounded number of pseudo-group variables. Indeed, there is no upper bound on $\text{rank } \tau_f^{(n)}$ as $n \rightarrow \infty$, and so, apriori, it is not even possible to assert that an IC bound exists.

To overcome this difficulty, we introduce a partial moving frame [27, 39] for the action of the point transformation pseudo-group on $\mathcal{J}^{(\infty)}$. In effect, this moves the equivalence problem from an infinite-dimensional setting to an 8-dimensional principal bundle. After this *universal* reduction, which is valid for all smoothly defined ODEs, the invariant classification proceeds using the usual method of reduction of structure, [12, 25], albeit with a certain amount of branching.

To illustrate the normalization procedure, we first consider the order zero normalization in some details, and then pass to the description of the full normalization. Considering the recurrence relations (3.15a) and

$$\begin{aligned} dQ &= Q_X \omega^x + Q_U \omega^u + Q_P \omega^p + \lambda(\phi) \\ &= Q_X \omega^x + Q_U \omega^u + Q_P \omega^p + \nu_{XX} + Q(\nu_U - 2\mu_X) + P(2\nu_{XU} - \mu_{XX}) \\ &\quad - 3PQ\mu_U + P^2(\nu_{UU} - 2\mu_{XU}) - P^3\mu_{UU}, \end{aligned}$$

we see that it is possible to normalize

$$X, U, P, Q \rightarrow 0 \tag{4.3}$$

as their exterior derivatives involve the linearly independent Maurer–Cartan forms μ , ν , ν_X , ν_{XX} . The result is the system of equations

$$0 = \omega^x + \mu, \quad 0 = \omega^u + \nu, \quad 0 = \omega^p + \nu_X, \quad 0 = Q_P \omega^p + Q_U \omega^u + Q_X \omega^x + \nu_{XX},$$

which can be solved for the partially normalized Maurer–Cartan forms:

$$\mu = -\omega^x, \quad \nu = -\omega^u, \quad \nu_X = -\omega^p, \quad \nu_{XX} = -(Q_P \omega^p + Q_U \omega^u + Q_X \omega^x).$$

Remark 4.1. After substituting (4.3) in (3.5) and (3.8) and normalizing the pseudo-group jets, we recover the usual G -structure formulation of the equivalence problem [25],

$$\begin{pmatrix} \omega^u \\ \omega^p \\ \omega^x \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_1/a_4 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \begin{pmatrix} du - p dx \\ dp - q dx \\ dx \end{pmatrix}, \tag{4.4}$$

where

$$a_1 = U_u, \quad a_2 = \frac{\widehat{D}_x(U_u)}{\widehat{D}_x(X)}, \quad a_3 = X_u, \quad a_4 = \widehat{D}_x(X) = pX_u + X_x.$$

Continuing the normalization procedure, order by order, let $\Xi \subset \mathcal{J}^{(\infty)}$ be the submanifold defined by

$$\begin{aligned} x = u = p = 0, \\ q_{ij0} = q_{ij1} = 0, \quad i, j \geq 0, \\ q_{0j2} = q_{1j2} = q_{0j3} = q_{1j3} = 0, \quad j \geq 0, \end{aligned} \tag{4.5}$$

and let

$$\tilde{\mathcal{E}} = (\boldsymbol{\tau}^{(\infty)})^{-1}(\Xi) \subset \mathcal{E}^{(\infty)}$$

denote the lift of Ξ ; that is, $\tilde{\mathcal{E}}$ is defined by the equations

$$\begin{aligned} X = U = P = 0, \\ Q_{ij0} = Q_{ij1} = 0, \quad i, j \geq 0, \\ Q_{0j2} = Q_{1j2} = Q_{0j3} = Q_{1j3} = 0, \quad j \geq 0. \end{aligned} \tag{4.6}$$

Let $H \subset \text{SL}_3\mathbb{R}$ be the 5-dimensional subgroup

$$H = \left\{ \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{pmatrix} : a_1 b_2 c_3 = 1 \right\}$$

of fractional linear transformations that preserve the origin $(x, u) = (0, 0)$. The proof of the next two Propositions is presented in Appendix B.

Proposition 4.2. The submanifold $\Xi \subset \mathcal{J}^{(\infty)}$ is a global³ cross-section for the action of $\text{Diff}(\mathbb{R}^2)$ on $\mathcal{J}^{(\infty)}$.

Proposition 4.3. We have $H = \mathcal{G}_\Xi$; that is, H is the subgroup of $\text{SL}_3\mathbb{R}$ that preserves Ξ .

In light of Proposition 4.2, we can utilize Ξ as a partially normalizing cross-section for the point-equivalence problem, [27]. Given a smoothly defined ODE $q = f(x, u, p)$, let $\tilde{\boldsymbol{\sigma}}_f: \tilde{\mathcal{E}}_f \rightarrow N^3$ denote the pullback bundle of $\tilde{\boldsymbol{\sigma}}: \tilde{\mathcal{E}} \rightarrow \mathcal{J}^{(\infty)}$ via $f^{(\infty)}: N^3 \rightarrow \mathcal{J}^{(\infty)}$. By Proposition 4.3, this pullback is a reduction of structure from $\boldsymbol{\sigma}_f: \mathcal{E}_f^{(\infty)} \rightarrow \mathcal{J}^{(\infty)}$ to the principal H -bundle, $\tilde{\boldsymbol{\sigma}}_f: \tilde{\mathcal{E}}_f \rightarrow N^3$.

In the sequel, we use the tilde decoration to denote the pullback to $\tilde{\mathcal{E}}_f$, and refer to the quantities

$$\tilde{Q}_{ijk} = \widehat{Q}_{ijk}(p, q^{(i+j+k)}, \psi_0^{(i+j+k+2)}) \Big|_{\tilde{\mathcal{E}}_f}$$

as *universal invariants*. The universal invariants are, in fact, the components of $\tilde{\boldsymbol{\tau}}_f: \tilde{\mathcal{E}}_f \rightarrow \Xi$; the latter obtained by imposing the normalizations (4.6). As such, the universal invariants are functions of $3 + 5 = 8$ variables and are H -equivariant with respect to the restricted left action (3.1).

³This means that any jet $(x, u, p, q^{(\infty)}) \in \mathcal{J}^{(\infty)}$ can be mapped to a point in Ξ under the action of $\text{Diff}(\mathbb{R}^2)$.

As is shown in Appendix B, the following one-forms

$$\mu, \quad \nu, \quad \nu_X, \quad \nu_{XX}, \quad \mu_{ij}, \quad i+j \geq 2, \quad \nu_{ij}, \quad i+j \geq 3,$$

are normalized by (4.6). In particular, applying (4.6) to the universal recurrence formulas for

$$dQ_{P^k}, \quad dQ_{P^k X}, \quad dQ_{P^k U}, \quad k = 0, 1, 2, 3,$$

yields following relations:

$$\begin{aligned} \tilde{\mu} &= -\tilde{\omega}^x, & \tilde{\nu} &= -\tilde{\omega}^u, & \tilde{\nu}_X &= -\tilde{\omega}^p, \\ \tilde{\nu}_{X^2} &= \tilde{\nu}_{X^3} = \tilde{\nu}_{X^2 U} = \tilde{\mu}_{X^4} = 0, & \tilde{\mu}_{X^2} &= 2\tilde{\nu}_{XU}, \\ \tilde{\mu}_{XU} &= \frac{1}{2}\tilde{\nu}_{UU}, & \tilde{\mu}_{U^2} &= \frac{1}{6}\tilde{Q}_{P^4}\tilde{\omega}^p, & 3\tilde{\mu}_{X^2 U} &= 6\tilde{\nu}_{XU^2} = \tilde{Q}_{P^2 X^2}\tilde{\omega}^x, \\ 6\tilde{\mu}_{XU^2} &= 3\tilde{\nu}_{U^3} = \tilde{Q}_{P^3 X^2}\tilde{\omega}^x + \tilde{Q}_{XP^4}\tilde{\omega}^p, & 6\tilde{\mu}_{U^3} &= -\tilde{Q}_{P^4}\tilde{\nu}_{XU} + \tilde{Q}_{P^4 U}\tilde{\omega}^p. \end{aligned} \quad (4.7)$$

The expressions (4.7) and subsequent formulas were obtained by importing the universal recurrence relations (3.15a) into MATHEMATICA. Substituting the above relations into (3.16) and (4.2) yields the structure equations for $\tilde{\mathcal{E}}_f$, namely:

$$\begin{aligned} d\tilde{\omega}^x &= \tilde{\mu}_X \wedge \tilde{\omega}^x + \tilde{\mu}_U \wedge \tilde{\omega}^u, \\ d\tilde{\omega}^u &= \tilde{\nu}_U \wedge \tilde{\omega}^u + \tilde{\omega}^x \wedge \tilde{\omega}^p, \\ d\tilde{\omega}^p &= \tilde{\nu}_{UX} \wedge \tilde{\omega}^u + (\tilde{\nu}_U - \tilde{\mu}_X) \wedge \tilde{\omega}^p, \\ d\tilde{\mu}_X &= -2\tilde{\nu}_{UX} \wedge \tilde{\omega}^x - \frac{1}{2}\tilde{\nu}_{UU} \wedge \tilde{\omega}^u - \tilde{\mu}_U \wedge \tilde{\omega}^p, \\ d\tilde{\mu}_U &= -\frac{1}{2}\tilde{\nu}_{UU} \wedge \tilde{\omega}^x + (\tilde{\mu}_X - \tilde{\nu}_U) \wedge \tilde{\mu}_U + \frac{1}{6}\tilde{Q}_{P^4}\tilde{\omega}^u \wedge \tilde{\omega}^p, \\ d\tilde{\nu}_U &= -\tilde{\nu}_{UX} \wedge \tilde{\omega}^x + \tilde{\mu}_U \wedge \tilde{\omega}^p - \tilde{\nu}_{UU} \wedge \tilde{\omega}^u, \\ d\tilde{\nu}_{UU} &= 2\tilde{\nu}_{UX} \wedge \tilde{\mu}_U + \tilde{\nu}_{UU} \wedge \tilde{\nu}_U + \frac{1}{3}\tilde{Q}_{P^4 X}\tilde{\omega}^u \wedge \tilde{\omega}^p + \frac{1}{3}\tilde{Q}_{P^3 X^2}\tilde{\omega}^u \wedge \tilde{\omega}^x, \\ d\tilde{\nu}_{UX} &= \tilde{\nu}_{UX} \wedge \tilde{\mu}_X - \frac{1}{2}\tilde{\nu}_{UU} \wedge \tilde{\omega}^p + \frac{1}{6}\tilde{Q}_{P^2 X^2}\tilde{\omega}^u \wedge \tilde{\omega}^x. \end{aligned} \quad (4.8)$$

We now define the *reduced rank sequence*

$$\tilde{\varrho}_n := \text{rank } \tilde{\tau}_f^{(n)}, \quad n \geq 0. \quad (4.9)$$

Proposition 4.4. The following are equivalent:

- (i) $q = f(x, u, p)$ is a regular ODE;
- (ii) for every n , the reduced rank $\tilde{\varrho}_n$ is constant;
- (iii) the invariant coframe

$$\tilde{\omega}^x, \quad \tilde{\omega}^u, \quad \tilde{\omega}^p, \quad \tilde{\mu}_X, \quad \tilde{\mu}_U, \quad \tilde{\nu}_U, \quad \tilde{\nu}_{XU}, \quad \tilde{\nu}_{UU} \quad (4.10)$$

on $\tilde{\mathcal{E}}_f$ is fully regular [25, Definition 8.14].

Proof. The equivalence of (i) and (ii) follows from the equivariance property (3.2) of the target projection. The equivalence of (ii) and (iii) follows from the structure equations (4.8) and from the universal recurrence relations (3.15b). \square

In light of the above remarks and Proposition 4.4 we observe that

$$\tilde{\varrho}_0 = \tilde{\varrho}_1 = \tilde{\varrho}_2 = \tilde{\varrho}_3 = 0, \quad \text{and} \quad \tilde{\varrho}_4 \leq \tilde{\varrho}_5 \leq \tilde{\varrho}_6 \leq \dots \leq 8.$$

It follows that the reduced rank sequence stabilizes at a sufficiently high order n . Indeed, the following is true.

Proposition 4.5. The IC order can be characterized as the smallest integer $n \geq 4$ such that $\tilde{\varrho}_{n-1} = \tilde{\varrho}_n$.

Proof. See Proposition 8.18 of [25]. \square

Remark 4.6. As alluded in Remark 4.1, the equivariant moving frame formalism offers an alternative approach to the G -structure formulation of the equivalence problem. The 8-dimensional coframe (4.10) and the structure equations (4.8) obtained after carrying out the universal normalizations (4.6) can also be found using Cartan's equivalence algorithm, [4, 12, 23, 25]. At this stage, the equivalence problem splits into different branches according to the values of \tilde{Q}_{P^4} and $\tilde{Q}_{P^2X^2}$. As advocated by Gardner, [7, 25], the different scenarios could be analyzed symbolically using the structure equations (4.8) and the identity $d^2 = 0$ for the exterior derivative. But as mentioned in [25], one has to be careful as one might be led down spurious branches of the equivalence problem owing to unexpected normalizations or cancellations due to the explicit forms of the coframe. In the equivariant formalism we dispense ourself from these computations and issues by exploiting the recurrence relations (3.15b).

Another benefit of the moving frame formalism is the possibility of determining the order of an invariant without knowing its coordinate expression which is not something that can be easily done within Cartan's framework. According to [30, Lemma 7.4], once the pseudo-group action becomes free at order n , in other word all the pseudo-group parameters of the n th prolonged action can be normalized, then the normalization of a lifted invariant Q_{ijk} of order $i + j + k \leq n$ is an invariant of order $i + j + k$.

4.3 The fundamental branching

The universal reduction (4.6) leads to the normalization of all the Maurer–Cartan forms (3.13) except for

$$\tilde{\mu}_X, \quad \tilde{\mu}_U, \quad \tilde{\nu}_U, \quad \tilde{\nu}_{UU}, \quad \tilde{\nu}_{XU}. \quad (4.11)$$

To proceed further, the value of the universal invariants

$$\tilde{Q}_{P^{k+4}U^jX^i}, \quad \tilde{Q}_{P^3U^jX^{i+2}}, \quad \tilde{Q}_{P^2U^jX^{i+2}}, \quad i, j, k \geq 0, \quad (4.12)$$

(recall that these are restrictions of the lifted invariants Q_{ijk} to the universal normalizing cross-section (4.6)) must be analyzed in more details order by order. Up to order 6, the non-trivial universal invariants (4.12) are

$$\begin{aligned} n = 4: & \quad \tilde{Q}_{P^4}, \quad \tilde{Q}_{P^2X^2}, \\ n = 5: & \quad \tilde{Q}_{P^5}, \quad \tilde{Q}_{P^4U}, \quad \tilde{Q}_{P^4X}, \quad \tilde{Q}_{P^3X^2}, \quad \tilde{Q}_{P^2UX^2}, \quad \tilde{Q}_{P^2X^3}, \\ n = 6: & \quad \tilde{Q}_{P^6}, \quad \tilde{Q}_{P^5U}, \quad \tilde{Q}_{P^5X}, \quad \tilde{Q}_{P^4U^2}, \quad \tilde{Q}_{P^4UX}, \quad \tilde{Q}_{P^4X^2}, \quad \tilde{Q}_{P^3UX^2}, \\ & \quad \tilde{Q}_{P^3X^3}, \quad \tilde{Q}_{P^2U^2X^2}, \quad \tilde{Q}_{P^2UX^3}, \quad \tilde{Q}_{P^2X^4}. \end{aligned} \quad (4.13)$$

Writing the recurrence relations for the invariants (4.13) of order ≤ 5 we obtain

$$\begin{aligned}
d\tilde{Q}_{P^4} &= \tilde{Q}_{P^5} \tilde{\omega}^p + \tilde{Q}_{P^4U} \tilde{\omega}^u + \tilde{Q}_{P^4X} \tilde{\omega}^x + \tilde{Q}_{P^4}(2\tilde{\mu}_X - 3\tilde{\nu}_U), \\
d\tilde{Q}_{P^2X^2} &= \tilde{Q}_{P^3X^2} \tilde{\omega}^p + \tilde{Q}_{P^2UX^2} \tilde{\omega}^u + \tilde{Q}_{P^2X^3} \tilde{\omega}^x - \tilde{Q}_{P^2X^2}(\tilde{\nu}_U + 2\tilde{\mu}_X), \\
d\tilde{Q}_{P^5} &= \tilde{Q}_{P^6} \tilde{\omega}^p + \tilde{Q}_{P^5U} \tilde{\omega}^u + \tilde{Q}_{P^5X} \tilde{\omega}^x + 5\tilde{Q}_{P^4} \tilde{\mu}_U + \tilde{Q}_{P^5}(3\tilde{\mu}_X - 4\tilde{\nu}_U), \\
d\tilde{Q}_{P^4X} &= (\tilde{Q}_{P^5X} + \tilde{Q}_{P^4U}) \tilde{\omega}^p + \tilde{Q}_{P^4UX} \tilde{\omega}^u + \tilde{Q}_{P^4X^2} \tilde{\omega}^x + \tilde{Q}_{P^4} \tilde{\nu}_{UX} \\
&\quad + \tilde{Q}_{P^4X}(\tilde{\mu}_X - 3\tilde{\nu}_U), \\
d\tilde{Q}_{P^4U} &= \tilde{Q}_{P^5U} \tilde{\omega}^p + \tilde{Q}_{P^4U^2} \tilde{\omega}^u + \tilde{Q}_{P^4UX} \tilde{\omega}^x - 2\tilde{Q}_{P^4} \tilde{\nu}_{UU} - \tilde{Q}_{P^5} \tilde{\nu}_{UX} - \tilde{Q}_{P^4X} \tilde{\mu}_U \\
&\quad + \tilde{Q}_{P^4U}(2\tilde{\mu}_X - 4\tilde{\nu}_U), \\
d\tilde{Q}_{P^3X^2} &= \tilde{Q}_{P^4X^2} \tilde{\omega}^p + \tilde{Q}_{P^3UX^2} \tilde{\omega}^u + (\tilde{Q}_{P^3X^3} - 2\tilde{Q}_{P^2UX^2}) \tilde{\omega}^x - \tilde{Q}_{P^2X^2} \tilde{\mu}_U \\
&\quad - \tilde{Q}_{P^3X^2}(2\tilde{\nu}_U + \tilde{\mu}_X), \\
d\tilde{Q}_{P^2UX^2} &= \tilde{Q}_{P^3UX^2} \tilde{\omega}^p + \tilde{Q}_{P^2U^2X^2} \tilde{\omega}^u + \tilde{Q}_{P^2UX^3} \tilde{\omega}^x - 2\tilde{Q}_{P^2X^2} \tilde{\nu}_{UU} - \tilde{Q}_{P^3X^2} \tilde{\nu}_{UX} \\
&\quad - \tilde{Q}_{P^2X^3} \tilde{\mu}_U - 2\tilde{Q}_{P^2UX^2}(\tilde{\nu}_U + \tilde{\mu}_X), \\
d\tilde{Q}_{P^2X^3} &= (\tilde{Q}_{P^3X^3} - \tilde{Q}_{P^2UX^2}) \tilde{\omega}^p + \tilde{Q}_{P^2UX^3} \tilde{\omega}^u + \tilde{Q}_{P^2X^4} \tilde{\omega}^x - 5\tilde{Q}_{P^2X^2} \tilde{\nu}_{UX} \\
&\quad - \tilde{Q}_{P^2X^3}(\tilde{\nu}_U + 3\tilde{\mu}_X).
\end{aligned} \tag{4.14}$$

Considering the first two recurrence relations in (4.14), and concentrating on the correction terms involving the partially normalized Maurer–Cartan forms $\tilde{\mu}_X$, $\tilde{\nu}_U$, we notice that the values of \tilde{Q}_{P^4} and $\tilde{Q}_{P^2X^2}$ will govern the next possible normalizations. For example, if $\tilde{Q}_{P^4} \equiv \tilde{Q}_{P^2X^2} \equiv 0$ then the correction terms vanish and \tilde{Q}_{P^4} , $\tilde{Q}_{P^2X^2}$ are genuine invariants that cannot be normalized. In this case, higher order universal invariants have to be considered in order to normalize the remaining pseudo-group parameters. In total, there are 4 different cases splitting the equivalence problem into 4 branches:

$$\begin{array}{ll}
\text{I)} \quad \tilde{Q}_{P^4} \equiv 0 \text{ and } \tilde{Q}_{P^2X^2} \equiv 0, & \text{III)} \quad \tilde{Q}_{P^4} \equiv 0 \text{ and } \tilde{Q}_{P^2X^2} \neq 0, \\
\text{II)} \quad \tilde{Q}_{P^4} \neq 0 \text{ and } \tilde{Q}_{P^2X^2} \equiv 0, & \text{IV)} \quad \tilde{Q}_{P^4} \neq 0 \text{ and } \tilde{Q}_{P^2X^2} \neq 0.
\end{array}$$

Branch **I** corresponds to the equivalence class of linearizable differential equations discussed in the introduction. For this class of equations we have $\tilde{\varrho}_4 = 0$, and so the IC order equals 4.

As for case **IV**, we see from the recurrence relations (4.14) that the Maurer–Cartan forms (4.11) can be normalized by setting

$$\tilde{Q}_{P^4}, \tilde{Q}_{P^2X^2} \rightarrow 1, \quad \tilde{Q}_{P^5}, \tilde{Q}_{P^4U}, \tilde{Q}_{P^4X} \rightarrow 0.$$

This uses up all of the remaining H -freedom and produces a genuine moving frame. The algebra of absolute differential invariants is then generated by the remaining invariants (4.13) of order 5 and 6. The reduced ranks are

$$\tilde{\varrho}_4 = 2, \quad \tilde{\varrho}_5 \geq 5.$$

Hence, the “worst-case scenario”, as far as the IC order is concerned, is

$$(\tilde{\varrho}_5, \tilde{\varrho}_6, \tilde{\varrho}_7, \tilde{\varrho}_8, \tilde{\varrho}_9) = (5, 6, 7, 8, 8);$$

and as a consequence, the highest IC order achievable is 9. This bound will be attained if, post-normalization, the remaining 5th order invariants are constant, and the signature manifold is parametrized by three invariants of order 6, 7, 8, respectively, with the

higher order invariants obtained by differentiating the order 6 invariant. We do not push the analysis further as we will show that cases **II** and **III** contain equations with invariant classification order equal to 10. Since our goal is to find the branch(es) with highest classification order, we will focus on those branches. Indeed, in the sequel we consider case **III** in detail, as branches **II** and **III** are dual to each other. This duality was first observed by Cartan in his study of projective connections, [4]. For completeness, the duality among second-order ordinary differential equations is presented in Appendix A.

4.4 Case III

From now on, we assume that $\tilde{Q}_{P^2X^2} \neq 0$ and $\tilde{Q}_{P^4} \equiv 0$. With these assumptions, we will show that the reduced rank sequence obeys

$$\tilde{\varrho}_4 = 1, \quad \tilde{\varrho}_5 = 4, \quad \tilde{\varrho}_6 \geq 5.$$

Hence, for this class of ODEs there exists a genuine moving frame formulated in term of 6th order jets. For class **III** equations, the “worst case scenario” is the rank sequence

$$(\tilde{\varrho}_4, \tilde{\varrho}_5, \tilde{\varrho}_6, \tilde{\varrho}_7, \tilde{\varrho}_8, \tilde{\varrho}_9, \tilde{\varrho}_{10}) = (1, 4, 5, 6, 7, 8, 8);$$

which makes an IC order of 10 a possibility.

By Proposition 4.7, below, the class **III** has two branches, which we label **III.1** and **III.2**. For sub-case **III.1** we will show that $\tilde{\varrho}_6 = 5$ implies $\tilde{\varrho}_7 = 5$, which means that the IC order is 7. The other possibility is that $\tilde{\varrho}_6 \geq 6$, but this means that the IC order is ≤ 9 . Hence, sub-case **III.1** can be ruled out.

Finally, for sub-case **III.2** we will show that there is essentially one type of configuration of invariant values that gives the rank sequence $(\tilde{\varrho}_4, \tilde{\varrho}_5, \tilde{\varrho}_6, \tilde{\varrho}_7, \tilde{\varrho}_8, \tilde{\varrho}_9, \tilde{\varrho}_{10}) = (1, 4, 5, 6, 7, 8, 8)$. We will derive this configuration, and in the subsequent section integrate the corresponding structure equations.

Under the non-degeneracy assumption $\tilde{Q}_{P^2X^2} \neq 0$ it is possible to normalize

$$\tilde{Q}_{P^2X^2} \rightarrow 1, \quad \tilde{Q}_{P^3X^2}, \tilde{Q}_{P^2UX^2}, \tilde{Q}_{P^2X^3} \rightarrow 0, \quad (4.15)$$

which consequently normalizes the Maurer–Cartan forms $\tilde{\nu}_U, \tilde{\nu}_U, \tilde{\nu}_{UU}, \tilde{\nu}_{UX}$ to certain linear combinations of $\tilde{\omega}^x, \tilde{\omega}^u, \tilde{\omega}^p, \tilde{\mu}_X$. Henceforth, to avoid confusion, we use the “check” \check{Q}_{ijk} decoration to indicate the invariants and one-forms obtained via additional normalization of the universal invariants as per (4.15). Furthermore, the recurrence relation for $\tilde{Q}_{P^4} \equiv 0$, forces the following fifth-order invariants:

$$\check{Q}_{P^5} \equiv \check{Q}_{P^4U} \equiv \check{Q}_{P^4X} \equiv 0 \quad (4.16)$$

to be identically equal to zero. Combining (4.15) and (4.16) we conclude that all universal 5th order invariants can be normalized to a constant, and that to normalize the remaining Maurer–Cartan form $\tilde{\mu}_X$ we must consider universal invariants of order 6. At order 6, the constraints (4.16) force the invariants

$$\check{Q}_{P^6} \equiv \check{Q}_{P^5U} \equiv \check{Q}_{P^5X} \equiv \check{Q}_{P^4U^2} \equiv \check{Q}_{P^4UX} \equiv \check{Q}_{P^4X^2} \equiv 0 \quad (4.17)$$

to be identically zero, and more generally,

$$\check{Q}_{P^{4+i}U^jX^k} \equiv 0, \quad i, j, k \geq 0. \quad (4.18)$$

Hence from (4.17) we conclude that the remaining non-constant universal invariants of order 6 are

$$\check{Q}_{P^3UX^2}, \quad \check{Q}_{P^3X^3}, \quad \check{Q}_{P^2U^2X^2}, \quad \check{Q}_{P^2UX^3}, \quad \check{Q}_{P^2X^4}.$$

Proposition 4.7. The invariants $\check{Q}_{P^2X^4}$ and $\check{Q}_{P^3X^3}$ cannot simultaneously be equal to zero.

Proof. Considering the recurrence relations for $\check{Q}_{P^3X^3}$, $\check{Q}_{P^2UX^3}$, $\check{Q}_{P^2X^4}$ we have

$$\begin{aligned} d\check{Q}_{P^3X^3} &= \frac{3}{4}\check{Q}_{P^3UX^2}\check{\omega}^p + \left(\check{Q}_{P^3UX^3} - \frac{9}{4}\check{Q}_{P^2U^2X^2}\right)\check{\omega}^u + \left(\check{Q}_{P^3X^4} - \frac{9}{4}\check{Q}_{P^2UX^3}\right)\check{\omega}^x \\ &\quad + 2\check{Q}_{P^3X^3}\check{\mu}_X, \\ d\check{Q}_{P^2UX^2} &= \left(\check{Q}_{P^3UX^3} - \check{Q}_{P^2U^2X^2} - \frac{1}{5}\check{Q}_{P^3X^3}^2\right)\check{\omega}^p + \left(\check{Q}_{P^2U^2X^3} - \check{Q}_{P^3UX^2}\check{Q}_{P^2X^4}\right. \\ &\quad \left.- \frac{1}{5}\check{Q}_{P^3X^3}\check{Q}_{P^2UX^3}\right)\check{\omega}^u + \left(\check{Q}_{P^2UX^4} - \frac{5}{6} - \frac{6}{5}\check{Q}_{P^3X^3}\check{Q}_{P^2X^4}\right)\check{\omega}^x + \check{Q}_{P^2UX^3}\check{\mu}_X, \\ d\check{Q}_{P^2X^4} &= \check{Q}_{P^3X^4}\check{\omega}^p + \check{Q}_{P^2UX^4}\check{\omega}^u + \check{Q}_{P^2X^5}\check{\omega}^x - 2\check{Q}_{P^2X^4}\check{\mu}_X. \end{aligned}$$

Assuming $\check{Q}_{P^2X^4} \equiv \check{Q}_{P^3X^3} \equiv 0$, the syzygy

$$D_X(\check{Q}_{P^3X^3}) + \frac{9}{4}\check{Q}_{P^2UX^3} = \check{Q}_{P^3X^4} = D_P(\check{Q}_{P^2X^4})$$

forces $\check{Q}_{P^2UX^3} \equiv 0$, which when combined with the syzygy

$$D_X(\check{Q}_{P^2UX^3}) + \frac{5}{6} + \frac{6}{5}\check{Q}_{P^3X^3}\check{Q}_{P^2X^4} = \check{Q}_{P^2UX^4} = D_U(\check{Q}_{P^2X^4}),$$

leads to the contradiction $0 = 5/6$. \square

By virtue of Proposition 4.7, two sub-cases must be considered:

$$\text{III.1)} \quad \check{Q}_{P^3X^3} \neq 0, \quad \text{III.2)} \quad \check{Q}_{P^2X^4} \neq 0.$$

4.4.1 Sub-case III.1

Assuming $\check{Q}_{P^3X^3} \neq 0$, we can set

$$\check{Q}_{P^3X^3} \rightarrow 1$$

and normalize the Maurer–Cartan form $\check{\mu}_X$. After normalization, the remaining sixth-order invariants

$$\check{Q}_{P^3UX^2}, \quad \check{Q}_{P^2U^2X^2}, \quad \check{Q}_{P^2UX^3}, \quad \check{Q}_{P^2X^4}, \quad (4.19)$$

are genuine invariants in the sense that they do not depend on pseudo-group parameters. In an attempt to minimize the rank, we assume that the functions (4.19) are constant:

$$\check{Q}_{P^3UX^2} \equiv C_1, \quad \check{Q}_{P^2U^2X^2} \equiv C_2, \quad \check{Q}_{P^2UX^3} \equiv C_3, \quad \check{Q}_{P^2X^4} \equiv C_4. \quad (4.20)$$

Combining (4.17) with (4.20), it follows from a careful analysis of the recurrence relations that all seventh-order invariants are constant which in turns forces all higher-order invariants to also be constant. On the other hand, if the invariants (4.19) are not constant, the IC order is ≤ 9 .

4.4.2 Sub-case III.2

We now assume that $\check{Q}_{P^2X^4} \neq 0$, and set

$$\check{Q}_{P^2X^4} \rightarrow 1 \quad (4.21)$$

to normalize $\check{\mu}_X$ and obtain a genuine moving frame. From now on, for the sake of notational convenience, we omit writing the check decoration and simply use Q_{ijk} to denote the absolute differential invariants obtained by normalizing the universal invariants using (4.15) and (4.21). Likewise, the invariant coframe on N^3 will be written simply as $\omega^x, \omega^u, \omega^p$ and the dual derivative operators as D_X, D_U, D_P .

At order 6, we are left with the absolute differential invariants

$$Q_{P^3UX^2}, \quad Q_{P^3X^3}, \quad Q_{P^2U^2X^2}, \quad Q_{P^2UX^3}. \quad (4.22)$$

Once more, in an attempt to minimize the rank, we assume that the invariants (4.22) are constant:

$$Q_{P^3UX^2} \equiv C_1, \quad Q_{P^3X^3} \equiv C_2, \quad Q_{P^2U^2X^2} \equiv C_3, \quad Q_{P^2UX^3} \equiv C_4. \quad (4.23)$$

The seventh-order invariant $Q_{P^2X^5}$ plays an important role in the following considerations. To single out this invariant, let us set

$$I_7 = Q_{P^2X^5}.$$

Considering the recurrence relations of the invariants (4.23) we obtain a collection of constraints on the seventh-order invariants:

$$\begin{aligned} 0 &= dC_1 = \frac{5C_1Q_{P^3X^4}}{2}\omega^p + \left(Q_{P^3U^2X^2} - C_1C_2 + \frac{5C_1Q_{P^2UX^4}}{2}\right)\omega^u \\ &\quad + \left(Q_{P^3UX^3} - C_2^2 - 2C_3 + \frac{5I_7C_1}{2}\right)\omega^x, \\ 0 &= dC_2 = \left(C_2Q_{P^3X^4} + \frac{3C_1}{4}\right)\omega^p + \left(Q_{P^3UX^3} + C_2Q_{P^2UX^4} - \frac{9C_3}{4}\right)\omega^u \\ &\quad + \left(Q_{P^3X^4} + I_7C_2 - \frac{9C_4}{4}\right)\omega^x, \\ 0 &= dC_3 = \left(Q_{P^3U^2X^2} + 2C_3Q_{P^3X^4} - \frac{2C_1C_2}{5}\right)\omega^p + \left(Q_{P^2U^3X^2} + 2C_3Q_{P^2UX^4} \right. \\ &\quad \left. - \frac{12C_1C_4}{5}\right)\omega^u + \left(Q_{P^2U^2X^3} + 2I_7C_3 - 2C_2C_4 - \frac{2C_1}{5}\right)\omega^x, \\ 0 &= dC_4 = \left(Q_{P^3UX^3} - C_3 + \frac{C_4Q_{P^3X^4}}{2} - \frac{C_2^2}{5}\right)\omega^p + \left(Q_{P^2U^2X^3} - C_1 + \frac{C_4Q_{P^2UX^4}}{2} \right. \\ &\quad \left. - \frac{C_2C_4}{5}\right)\omega^u + \left(Q_{P^2UX^4} + \frac{I_7C_4}{2} - \frac{6C_2}{5} - \frac{5}{6}\right)\omega^x. \end{aligned} \quad (4.24)$$

If the invariant I_7 is constant, (4.18) and (4.24) imply that all seventh-order invariants are constant. Similarly, all higher order invariants are constant. On the other hand, when I_7 is a non-constant invariant, the constraints (4.24) yield

$$C_1 = C_2 = C_3 = C_4 = 0.$$

Which in turn, implies, together with (4.18), that all seventh-order invariants are identically equal to zero except for

$$Q_{P^2X^5} = I_7 \quad \text{and} \quad Q_{P^2UX^4} = \frac{5}{6}.$$

Taking the exterior derivative of I_7 we obtain

$$dI_7 = I_8 \omega^x - \frac{5}{4} I_7 \omega^u + \frac{5}{6} \omega^p, \quad (4.25)$$

where $I_8 = D_X(I_7)$ is the only new (functionally independent) invariant of order 8. Then, differentiating I_8 with respect to D_X we find the only new invariant of order 9:

$$I_9 = D_X(I_8) = D_X^2(I_7).$$

Generically, the invariants I_7, I_8, I_9 , are functionally independent, and the structure of the invariant signature manifold is completely determined by the functional relation

$$I_{10} = D_X(I_9) = \phi(I_7, I_8, I_9).$$

Modulo duality, all branches of the equivalence problem have now been considered, and we can safely conclude that 10 is an upper bound on the IC order.

5 The maximal IC order class

5.1 Abstract existence

To terminate the proof of Theorem 2.4, we must show that there exists a class of differential equations satisfying the invariant constraints imposed in sub-case **III.2**. To do so, we need the structure equations of the invariant one-forms $\omega^x, \omega^u, \omega^p$.

These equations are obtained symbolically by substituting the Maurer–Cartan form normalizations

$$\begin{aligned} \mu &= -\omega^x, & \nu &= -\omega^u, & \nu_X &= -\omega^p, \\ -2\mu_X &= \nu_U = -I_7 \omega^x - \frac{5}{6} \omega^u, & \mu_U &= \nu_{XX} = 0, & \nu_{UX} &= \frac{1}{5} \omega^x, \end{aligned}$$

obtained by solving the recurrence relations for the phantom invariants, into the structure equations (4.2). The result is

$$\begin{aligned} d\omega^x &= \frac{5}{12} \omega^u \wedge \omega^x, \\ d\omega^u &= \omega^x \wedge \omega^p + I_7 \omega^u \wedge \omega^x, \\ d\omega^p &= \frac{3}{2} I_7 \omega^p \wedge \omega^x + \frac{5}{4} \omega^p \wedge \omega^u + \frac{1}{5} \omega^x \wedge \omega^u. \end{aligned} \quad (5.1)$$

If the invariant I_7 were constant, we could apply Cartan’s Integration Theorem, [3], to conclude the existence of differential equations solving the integration problem. But since this is not the case we must use the following generalization of Cartan’s integration theorem [2].

Theorem 5.1. Let $\omega^1, \dots, \omega^\ell$ be a coframe with structure equations

$$d\omega^i = \sum_{1 \leq j < k \leq \ell} C_{jk}^i(I^a) \omega^j \wedge \omega^k,$$

such that the structure coefficients are functions of I^a , $a = 1, \dots, s$ and

$$dI^a = \sum_{i=1}^{\ell} \left(F_i^a(I^b) + \sum_{\alpha=1}^r A_{i\alpha}^a(I^b) J^\alpha \right) \omega^i, \quad a, b = 1, \dots, s.$$

Assuming

- the identity $d^2 = 0$ holds,
- the functions $C_{jk}^i, F_i^a, A_{i\alpha}^a$ are real analytic,
- the tableau $A(I^b) = (A_{i\alpha}^a(I^b))$ has rank r and is involutive with Cartan characters $s_1 \geq s_2 \geq \dots \geq s_q > s_{q+1} = 0$ for all values of (I^b) ,

modulo a diffeomorphism, the general real-solution exists and depends on s_q functions of q variables. Moreover, (I^a) and (J^α) can be arbitrarily specified at a point.

Applying Theorem 5.1 to (4.25) and (5.1) we conclude that there exists a family of second-order ordinary differential equations depending on one function of one variable that solves the integration problem. Hence, by duality we conclude there are two families of equations, both depending on one function of one variable, that achieve the maximal classification order of Theorem 2.4.

5.2 Explicit integration

In this section we integrate the structure equations (5.1) to obtain an explicit representation of one of the two families the differential equations satisfying the maximal classification order of Theorem 2.4.

Proposition 5.2. Let (α, β, γ) be a local coordinate system on $\mathbb{R}^3 \setminus \{\alpha = 0\}$, then the one-forms

$$\omega^x = \alpha d\gamma, \quad \omega^u = \frac{12}{5} \frac{d\alpha}{\alpha} - \beta d\gamma, \quad \omega^p = \frac{d\beta}{\alpha} + \frac{12I_7}{5} \frac{d\alpha}{\alpha} + \Gamma(\alpha, \beta, \gamma) d\gamma, \quad (5.2)$$

where $\Gamma(\alpha, \beta, \gamma)$ is a solution of the of linear partial differential equations

$$\Gamma_\alpha + \frac{3}{\alpha} \Gamma = \frac{12I_{7,\gamma}}{5\alpha} + \frac{18I_7^2}{5} - \frac{3I_7\beta}{\alpha} - \frac{12}{25}, \quad \Gamma_\beta = \frac{3I_7}{2} - \frac{5\beta}{4\alpha}, \quad (5.3)$$

satisfy the structure equations (5.1).

Substituting (5.2) into (4.25) we obtain the differential equations

$$I_{7,\beta} = \frac{5}{6\alpha}, \quad \alpha I_{7,\alpha} = -I_7, \quad I_8 = \frac{I_{7,\gamma}}{\alpha} - \frac{5\beta I_7}{4\alpha} - \frac{5\Gamma}{6\alpha}.$$

Integrating the first two equations, we deduce that

$$I_7 = \frac{5\beta}{6\alpha} + \frac{h(\gamma)}{\alpha}, \quad (5.4)$$

where $h(\gamma)$ is an arbitrary (analytic) function, while the third equation defines I_8 in terms of I_7 and Γ . The coframe (5.2) is not uniquely defined. The degree of freedom is given by the infinite-dimensional Lie pseudo-group

$$\begin{aligned}\bar{\gamma} &= \sigma(\gamma), & \bar{\alpha} &= \frac{\alpha}{\sigma'}, & \bar{\beta} &= \frac{\beta}{\sigma'} - \frac{12\sigma''}{5(\sigma')^2}, \\ \bar{\Gamma} &= \frac{\Gamma}{\sigma'} + \frac{12\bar{I}_7\sigma''}{5(\sigma')^2} + \frac{\beta\sigma''}{\alpha(\sigma')^2} + \frac{12\sigma'''}{5\alpha(\sigma')^2} - \frac{24(\sigma'')^2}{5\alpha(\sigma')^3},\end{aligned}\tag{5.5}$$

where $\bar{\gamma} = \sigma(\gamma)$ is a local diffeomorphism of the real line. Under the pseudo-group action (5.5), the invariant (5.4) transforms according to

$$\bar{I}_7 = \frac{5\beta}{6\alpha} - \frac{2\sigma''}{\alpha\sigma'} + \frac{\sigma'\bar{h}}{\alpha}, \quad \text{with} \quad \bar{h} = h \circ \sigma.$$

By choosing $\sigma(\gamma)$ such that

$$2\sigma'' = (\sigma')^2\bar{h},$$

we can assume $h(\gamma) = 0$ in (5.4). Doing so and solving the differential equations (5.3) we find that

$$\Gamma(\alpha, \beta, \gamma) = -\frac{3\alpha}{25} + \frac{g(\gamma)}{\alpha^3},$$

where $g(\gamma)$ is an arbitrary (analytic) function which cannot be removed by some change of variables.

Proposition 5.3. The one-forms

$$\omega^x = \alpha d\gamma, \quad \omega^u = \frac{12}{5}\frac{d\alpha}{\alpha} - \beta d\gamma, \quad \omega^p = \frac{d\beta}{\alpha} + \frac{2\beta}{\alpha^2}d\alpha + \left(\frac{g(\gamma)}{\alpha^3} - \frac{3\alpha}{25}\right)d\gamma \tag{5.6}$$

satisfy the structure equations (5.1) with

$$I_7(\alpha, \beta, \gamma) = \frac{5\beta}{6\alpha}. \tag{5.7}$$

Proposition 5.4. Let $\omega^x, \omega^u, \omega^p$ be given by (5.6). Then there exist functions a_1, a_2, a_3, a_4 and a change of variables

$$\alpha = \alpha(x, u, p), \quad \beta = \beta(x, u, p), \quad \gamma = \gamma(x, u, p) \tag{5.8}$$

such that (4.4) holds.

Proof. The functions a_1, a_2, a_3, a_4 and the change of variables (5.8) are not unique. Assuming $u > 0$, we can choose

$$a_1 = \frac{6}{5\alpha^2}, \quad a_2 = a_3 = 0, \quad a_4 = \alpha,$$

and

$$\alpha = \sqrt{u}, \quad \beta = \frac{6p}{5u}, \quad \gamma = x.$$

Then

$$\omega^x = u^{1/2}dx, \quad \omega^u = \frac{6}{5u}[du - p dx], \quad \omega^p = \frac{6}{5u^{3/2}}\left[dp - \left(\frac{u^2}{10} - \frac{5g(x)}{6}\right)dx\right]. \tag{5.9}$$

□

We deduce from ω^p in (5.9) that the second-order ordinary differential equations

$$u_{xx} = \frac{u^2}{10} - \frac{5g(x)}{6}, \quad (5.10)$$

is one of the 2 families of differential equations with maximal classification order, provided I_7, I_8, I_9 are functionally independent. A straightforward scaling transformation gives the form shown in (2.7). We note that the coordinate expressions for the dual equations to (5.10) are very difficult to obtain.

Finally, for the differential equations (5.10), the coordinate expressions of the invariant (5.7) and the derivatives dual to $\omega^x, \omega^u, \omega^p$ are

$$I_7 = \frac{p}{u^{3/2}}, \quad (5.11)$$

and

$$D_X = \frac{1}{u^{1/2}} \left[\frac{\partial}{\partial x} + p \frac{\partial}{\partial u} + \left(\frac{u^2}{10} - \frac{5g(x)}{6} \right) \frac{\partial}{\partial p} \right], \quad D_U = \frac{5u}{6} \frac{\partial}{\partial u}, \quad D_P = \frac{5u^{3/2}}{6} \frac{\partial}{\partial p},$$

respectively. Hence, differentiating (5.11) twice with respect to D_X we deduce that the invariant signature manifold can be parametrized by the invariants

$$I_7 = \frac{p}{u^{3/2}}, \quad I_8 = \frac{g(x)}{u^2}, \quad I_9 = \frac{(g'(x))^4}{g^5(x)} = 256 \left(D_X(g(x)^{-1/4}) \right)^4.$$

The above invariants are independent if and only if I_9 is non-constant, which gives the inequality in (2.7). If this inequality holds, then the invariant classification of an equation in this class is completely determined by the functional relationship between I_9 and the invariant

$$I_{10} = \frac{g(x)g''(x)}{(g'(x))^2}.$$

Remark 5.5. Now one can understand the necessity of (2.7) in Theorem 2.4. Indeed, when I_9 is constant, which is equivalent to the requirement that

$$D_x^2[g(x)^{-1/4}] = 0,$$

the IC order is 9 and the coframe (5.9) admits a 1-dimensional symmetry group, [25, Theorem 8.22].

A Cartan's duality for second-order ODEs

In his study of projective connections, [4], Cartan mentions that there exists a notion of duality among second-order ordinary differential equations. Modern accounts can be found in [5, 23], but for completeness, we summarize the construction below.

For second-order ordinary differential equations, a solution to an initial-value problem may be represented by a two-parameter family

$$\Phi(x, u, \bar{x}, \bar{u}) = 0, \quad (A.1)$$

where the parameters \bar{x}, \bar{u} correspond to the initial values

$$\bar{x} = u(0), \quad \bar{u} = u_x(0).$$

Differentiating (A.1) twice with respect to x we obtain the equations

$$\Phi = \Phi_x + u_x \Phi_u = \Phi_{xx} + 2u_x \Phi_{xu} + u_x^2 \Phi_{uu} + u_{xx} \Phi_u = 0, \quad (\text{A.2})$$

which leads to the second-order differential equation

$$u_{xx} = q(x, u, u_x) \quad (\text{A.3})$$

when the parameters \bar{x} and \bar{u} are eliminated from (A.2). The dual equation to (A.3) is obtained from (A.1) by interchanging the roles of (x, u) and (\bar{x}, \bar{u}) . In other words, let (x, u) be parameters and $\bar{u} = \bar{u}(\bar{x})$ a function of the independent variable \bar{x} . Then, differentiating (A.1) with respect to \bar{x} twice and getting rid of (x, u) from the equations obtained yields the *dual equation*

$$\bar{u}_{\bar{x}\bar{x}} = \bar{q}(\bar{x}, \bar{u}, \bar{u}_{\bar{x}}). \quad (\text{A.4})$$

Let us now consider the consequences of the contact transformation that sends (A.3) to (A.4) on the point equivalence problem. For the equations

$$\Phi(x, u, \bar{x}, \bar{u}) = 0, \quad \Phi_x(x, u, \bar{x}, \bar{u}) + p \Phi_u(x, u, \bar{x}, \bar{u}) = 0,$$

to determine $\bar{x}, \bar{u}, \bar{p}$ in terms of x, u, p we must impose

$$0 \neq \Delta = \det \begin{pmatrix} 0 & \Phi_{\bar{x}} & \Phi_{\bar{u}} \\ \Phi_x & \Phi_{x\bar{x}} & \Phi_{x\bar{u}} \\ \Phi_u & \Phi_{u\bar{x}} & \Phi_{u\bar{u}} \end{pmatrix} = -\Phi_u \Phi_{\bar{u}} (\Phi_{x\bar{x}} + p \Phi_{u\bar{x}} + \bar{p} \Phi_{x\bar{u}} + p \bar{p} \Phi_{u\bar{u}}),$$

which in particular requires

$$\Phi_u \neq 0, \quad \Phi_{\bar{u}} \neq 0.$$

Now, let

$$\theta = du - p dx, \quad \bar{\theta} = d\bar{u} - \bar{p} d\bar{x}, \quad \theta_1 = dp - q dx, \quad \bar{\theta}_1 = d\bar{p} - \bar{q} d\bar{x}.$$

Taking the exterior derivative of (A.1) we deduce that

$$\theta = -\frac{\Phi_{\bar{u}}}{\Phi_u} \bar{\theta}.$$

On the other hand, the exterior derivative of $\Phi_x + p \Phi_u = 0$ yields

$$\theta_1 = \frac{\Delta}{\Phi_{\bar{u}} \Phi_u^2} d\bar{x} \mod \bar{\theta},$$

while the exterior derivative of $\Phi_{\bar{x}} + \bar{p} \Phi_{\bar{u}} = 0$ gives

$$dx = \frac{\Phi_{\bar{u}}^2 \Phi_u}{\Delta} \bar{\theta}_1 \mod \bar{\theta}.$$

In matrix form,

$$\begin{pmatrix} \theta \\ dx \\ \theta_1 \end{pmatrix} = \begin{pmatrix} -a_1 & 0 & 0 \\ a_2 & a_1/a_4 & 0 \\ a_3 & 0 & a_4 \end{pmatrix} \begin{pmatrix} \bar{\theta} \\ \bar{\theta}_1 \\ d\bar{x} \end{pmatrix}, \quad \text{where} \quad a_1 = \frac{\Phi_{\bar{u}}}{\Phi_u}, \quad a_4 = \frac{\Delta}{\Phi_{\bar{u}} \Phi_u^2}.$$

We note that, up to a sign in the first entry, the 3×3 matrix made of the functions a_1, a_2, a_3, a_4 is an element of the structure group (4.4). Hence, under the contact transformation $(x, u, p) \leftrightarrow (\bar{x}, \bar{u}, \bar{p})$, the lifted coframe (3.5) and (3.8) undergoes the transformation

$$\omega^x \leftrightarrow \omega^{\bar{p}}, \quad \omega^p \leftrightarrow \omega^{\bar{x}}, \quad \omega^u \leftrightarrow -\omega^{\bar{u}}. \quad (\text{A.5})$$

To understand the duality between cases **II** and **III** of the equivalence problem in Section 4.3 we consider the structure equations (4.8) obtained once the normalizations (4.6) are done. Focusing on the structure equations

$$d\tilde{\mu}_U = \cdots + \frac{1}{6}\tilde{Q}_{P^4}\tilde{\omega}^u \wedge \tilde{\omega}^p, \quad d\tilde{\nu}_{UX} = \cdots + \frac{1}{6}\tilde{Q}_{P^2X^2}\tilde{\omega}^u \wedge \tilde{\omega}^x,$$

we observe that under the coframe transformation (A.5) the role of the universal invariants $\tilde{Q}_{P^4}, \tilde{Q}_{P^2X^2}$ is interchanged, which is exactly what happens when switching between cases **II** and **III**.

B Proofs of Propositions 4.2 and 4.3

We start by proving Proposition 4.2. Let us first consider local transversality. First, we observe that $dX, dU, dP, dQ_{ijk}, \mu_{ij}, \nu_{ij}$ and $\omega^x, \omega^u, \omega^p, \vartheta_{ijk}, \mu_{ij}, \nu_{ij}$ are two choices of right-invariant coframes on $\mathcal{E}^{(\infty)}$. Then, we note that the left-action of $\text{Diff}(\mathbb{R}^2)$ in (3.1) generates the source distribution of $\sigma: \mathcal{E}^{(\infty)} \rightarrow \mathcal{J}^{(\infty)}$ given by

$$\ker\{dx, du, dp, \theta_{ijk}\} = \ker\{\omega^x, \omega^u, \omega^p, \vartheta_{ijk}\},$$

while the right-action generates the target distribution of $\tau: \mathcal{E}^{(\infty)} \rightarrow \mathcal{J}^{(\infty)}$, given by $\ker\{dX, dU, dP, dQ_{ijk}\}$. Hence, our goal is to show that the tangent space to $\tilde{\mathcal{E}}$, defined by equations (4.6), is transverse to the source distribution on $\mathcal{E}^{(\infty)}$.

According to (3.9),

$$\phi(x, u, p, q) = \eta_{xx} + q(\eta_u - 2\xi_x) + p(2\eta_{xu} - \xi_{xx}) - 3pq\xi_u + p^2(\eta_{uu} - 2\xi_{xu}) - p^3\xi_{uu}.$$

Writing

$$\begin{aligned} \xi_{ij} &= \xi_{x^i u^j}, & \xi^{(n)} &= \{\xi_{ij} : 0 \leq i + j \leq n\}, \\ \eta_{ij} &= \eta_{x^i u^j}, & \eta^{(n)} &= \{\eta_{ij} : 0 \leq i + j \leq n\}, \end{aligned}$$

the prolongation formula (3.10) yields, when $p = 0$,

$$\begin{aligned} \phi^{000} &\equiv \eta_{xx} && \text{mod } p, \xi^{(1)}, \eta^{(1)}; \\ \phi^{001} &\equiv 2\eta_{xu} - \xi_{xx} && \text{mod } p, \xi^{(1)}, \eta^{(1)}; \\ \phi^{002} &\equiv 2\eta_{uu} - 4\xi_{xu} && \text{mod } p, \xi^{(1)}, \eta^{(1)}; \\ 3\phi^{0,j-2,2} - 2\phi^{1,j-3,3} &\equiv 6\eta_{0j} && \text{mod } \xi^{(j-1)}, \eta^{(j-1)}, j \geq 3; \\ 4\phi^{0,j-1,1} - \phi^{1,j-2,2} &\equiv 6\eta_{1j} && \text{mod } \xi^{(j)}, \eta^{(j)}, j \geq 2; \\ \phi^{i-2,j0} &\equiv \eta_{ij} && \text{mod } p, \xi^{(i+j-1)}, \eta^{(i+j-1)}, i \geq 2, j \geq 0; \\ \phi^{0,j-2,3} &\equiv -6\xi_{0j} && \text{mod } \xi^{(j-1)}, \eta^{(j-1)}, j \geq 2; \\ \phi^{1,j-2,3} &\equiv -6\xi_{1j} && \text{mod } \xi^{(j)}, \eta^{(j)}, j \geq 2; \end{aligned}$$

$$\begin{aligned}
\phi^{0,j,1} - \phi^{1,j-1,2} &\equiv 3\xi_{2j} && \text{mod } \xi^{(j+1)}, \eta^{(j+1)}, j \geq 1; \\
\phi^{i-2,j,1} - 2\phi^{i-3,j+1,0} &\equiv \xi_{ij} && \text{mod } \xi^{(i+j-1)}, \eta^{(i+j-1)}, i \geq 3, j \geq 2.
\end{aligned}$$

By the universal recurrence formulas (3.15), when $P = 0$ we have, modulo $\omega^x, \omega^u, \omega^p, \vartheta_{ijk}$,

$$\begin{aligned}
dX &\equiv \mu; \\
dU &\equiv \nu; \\
dP &\equiv \nu_X; \\
dQ_{000} &\equiv \nu_{XX} && \text{mod } \mu^{(1)}, \nu^{(1)}; \\
dQ_{001} &\equiv 2\nu_{XU} - \mu_{XX} && \text{mod } \mu^{(1)}, \nu^{(1)}; \\
dQ_{002} &\equiv 2\eta_{UU} - 4\mu_{XU} && \text{mod } \mu^{(1)}, \nu^{(1)}; \\
3dQ_{0,j-2,2} - 2dQ_{1,j-3,3} &\equiv 6\nu_{0j} && \text{mod } \mu^{(j-1)}, \nu^{(j-1)}, j \geq 3; \\
4dQ_{0,j-1,1} - dQ_{1,j-2,2} &\equiv 6\nu_{1j} && \text{mod } \mu^{(j)}, \nu^{(j)}, j \geq 2; \\
dQ_{i-2,j,0} &\equiv \nu_{ij} && \text{mod } \mu^{(i+j-1)}, \nu^{(i+j-1)}, i \geq 2, j \geq 0; \\
dQ_{0,j-2,3} &\equiv -6\mu_{0j} && \text{mod } \mu^{(j-1)}, \nu^{(j-1)}, j \geq 2; \\
dQ_{1,j-2,3} &\equiv -6\mu_{1j} && \text{mod } \mu^{(j)}, \nu^{(j)}, j \geq 2; \\
dQ_{0,j,1} - dQ_{1,j-1,2} &\equiv 3\mu_{2j} && \text{mod } \mu^{(j+1)}, \nu^{(j+1)}, j \geq 1; \\
dQ_{i-2,j,1} - 2dQ_{i-3,j+1,0} &\equiv \mu_{ij} && \text{mod } \mu^{(i+j+1)}, \nu^{(i+j+1)}, i \geq 3, j \geq 2.
\end{aligned} \tag{B.1}$$

It follows that $\omega^x, \omega^u, \omega^p, \nu_U, \mu_X, \mu_U, \nu_{XU}, \nu_{UU}, \vartheta_{ijk}$ form a basis of $T^*\tilde{\mathcal{E}}$. As a consequence, since, $\omega^x, \omega^u, \omega^p, \vartheta_{ijk}$ are linearly independent on $\tilde{\mathcal{E}}$, the latter is transverse to the source distribution.

Now we prove surjectivity. Let $J = \{(i, j, k) : k < 2 \text{ or } (i < 2 \text{ and } k < 4)\}$ be the indicated set of indices. We have to show the consistency of the following equations:

$$\hat{P}(p, X_x, X_u, U_x, U_u) = 0$$

which is equivalent to

$$pU_u + U_x = 0;$$

and

$$\hat{Q}_{ijk}(p, q^{(i+j+k)}, X_0^{(i+j+k+2)}, U_0^{(i+j+k+2)}) = 0, \quad (i, j, k) \in J.$$

Since \mathcal{G} is transitive on $N^3 = \mathcal{J}^1(\mathbb{R}, \mathbb{R})$, no generality is lost if we set $p = 0$ above; that is, it suffices to demonstrate the consistency of the equations

$$0 = \hat{Q}_{ijk}(0, q^{(i+j+k)}, X_0^{(i+j+k+2)}, U_0^{(i+j+k+2)}), \quad (i, j, k) \in J.$$

Furthermore, the above equations can be greatly simplified by restricting to the sub-pseudo-group of \mathcal{G} defined by

$$X_x = U_u = 1, \quad U_x = X_u = 0. \tag{B.2}$$

At $p = 0$ the transformation law for $Q_{ijk}, k < 4$, is, modulo (B.2) and $X^{(i+j+1)}, U^{(i+j+1)}$, affine to leading order

$$Q_{ij0} \equiv q_{ij0} + U_{i+2,j},$$

$$\begin{aligned}
Q_{ij1} &\equiv q_{ij1} + 2U_{i+1,j+1} - X_{i+2,j}, \\
Q_{ij2} &\equiv q_{ij2} + 2U_{i,j+2} - 4X_{i+1,j+1}, \\
Q_{ij3} &\equiv q_{ij3} - 6X_{i,j+2}.
\end{aligned}$$

We now impose relations for which $(i, j, k) \in J$. Setting $Q_{i,0,0} = Q_{i,j+1,0} = Q_{i+1,j,1} = 0$ fixes $U_{i+2,0}, U_{i+2,j+1}, X_{i+3,j}$. Setting $Q_{0j3} = Q_{1j3} = 0$ fixes $X_{0,j+2}, X_{1,j+2}$. Fixing $Q_{0,j+1,2} = Q_{1,j+1,2} = 0$ determines $U_{0,j+3}, U_{1,j+3}$, and then setting $Q_{0,j+2,1} = 0$ specifies $X_{2,j+2}$. Consequently, the remaining relations are

$$\begin{aligned}
Q_{001} &\equiv q_{001} + 2U_{11} - X_{20}, \\
Q_{011} &\equiv q_{011} + 2U_{12} - X_{21}, \\
Q_{102} &\equiv q_{102} + 2U_{12} - 4X_{21}, \\
Q_{002} &\equiv q_{002} + 2U_{02} - 4X_{11}, \\
Q_{012} &\equiv q_{012} + 2U_{03} - 4X_{12}.
\end{aligned}$$

Setting the left-hand side to zero, these relations fix $X_{12}, X_{21}, U_{12}, X_{20}, X_{11}$ leaving U_{11} and U_{02} as free variables.

We now consider the proof of Proposition 4.3. Let

$$\mathcal{E}_{\Xi} = \sigma^{-1}(\Xi) \cap \tau^{-1}(\Xi) \subset \tilde{\mathcal{E}}$$

be the subgroupoid of transformations that preserve Ξ , and let \mathcal{E}_H be the subgroupoid corresponding to the action of H on $\mathcal{J}^{(\infty)}$. It is straightforward to check that H preserves Ξ ; that is, $\mathcal{E}_H \subset \mathcal{E}_{\Xi}$. Equations (B.1) also demonstrate that $\nu_U, \mu_X, \mu_U, \nu_{XU}, \nu_{UU}$ form a basis of one-forms for the source fibres of $\sigma: \mathcal{E}_{\Xi} \rightarrow \Xi$. Since H is 5-dimensional, it follows that $\mathcal{E}_H = \mathcal{E}_{\Xi}$ by dimensional exhaustion, as was to be shown.

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