# Proving Inequalities and Solving Global Optimization Problems via Simplified CAD Projection

Jingjun Han, \* Zhi Jin, Bican Xia

LMAM & School of Mathematical Sciences, Peking University, Beijing 100871, China

#### Abstract

Let  $\mathbf{x}_n = (x_1, \ldots, x_n)$  and  $f \in \mathbb{R}[\mathbf{x}_n, k]$ . The problem of finding all  $k_0$  such that  $f(\mathbf{x}_n, k_0) \ge 0$  on  $\mathbb{R}^n$  is considered in this paper, which obviously takes as a special case the problem of computing the global infimum or proving the semi-definiteness of a polynomial. For solving the problems, we propose a simplified Brown's CAD projection operator, Nproj, of which the projection scale is always no larger than that of Brown's. For many problems, the scale is much smaller than that of Brown's. As a result, the lifting phase is also simplified. Some new algorithms based on Nproj for solving those problems are designed and proved to be correct. Comparison to some existing tools on some examples is reported to illustrate the effectiveness of our new algorithms.

Key words: CAD projection, global optimization, semi-definiteness, polynomials.

### 1. Introduction

Let  $\mathbb{R}$  be the field of real numbers and  $x_n = (x_1, \ldots, x_n)$  be *n* ordered variables. Consider the following three well-known problems.

**Problem 1.** For  $f \in \mathbb{R}[x_n]$ , prove or disprove  $f(x_n) \ge 0$  on  $\mathbb{R}^n$ .

**Problem 2.** For  $f \in \mathbb{R}[\boldsymbol{x}_n]$ , find the global infimum inf  $f(\mathbb{R}^n)$ .

**Problem 3.** For  $f \in \mathbb{R}[\boldsymbol{x}_n, k]$ , find all  $k_0 \in \mathbb{R}$  such that  $f(\boldsymbol{x}_n, k_0) \ge 0$  on  $\mathbb{R}^n$ .

A lot of work has been done for Problem 1 since Hilbert (1888). For related classical results, see for example, Bernstein (1915); Artin (1927); Pólya (1928); Hardy et al. (1952);

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<sup>\*</sup> Corresponding author

*Email addresses:* hanjingjunfdfz@gmail.com (Jingjun Han,), j2i5nzhi@sina.com (Zhi Jin,), xbc@math.pku.edu.cn (Bican Xia).

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Motzkin (1952); Motzkin et al. (1969); Berg et al. (1982). In recent years, many other methods have been proposed. See for example, Putinar (1993); Schweighofer (2005); Yang (2005); Scheiderer (2009); Yao (2010); Castle (2011); Xu et al. (2012).

Problem 2 can be regarded as a generalization of Problem 1. Various methods based on different principles have been proposed for solving Problem 2, including methods based on Gröbner base (Hägglöf et al., 1995; Hanzon et al., 2003), semi-definite programming or SOS based methods (Parrilo, 2000; Lasserre, 2001; Jibetean et al., 2005; Nie et al., 2006; Hà et al., 2009; Guo et al., 2010), and methods based on Wu's method (Xiao et al., 2011). Semi-definite programming returns numerical solutions, which, in some cases, may be larger than the supremum. Some methods need additional assumptions, for example, that the polynomial can attain the infimum (Nie et al., 2006) or the zero of the set of the first partial derivatives is zero-dimensional (Hägglöf et al., 1995). Safey El Din (2008) provided a certified algorithm based on the topology property of generalized critical values (Kurdyka et al., 2000) to solve problem 2. The algorithm was designed to compute critical values and asymptotic critical values based on Gröbner basis computation. The algorithm has been implemented in the RAGlib package of Maple.

Problem 3 is more general. It is a typical problem of quantifier elimination (QE) on real closed fields. Algorithms of single exponential complexity to solve Problem 3 in the case of integer coefficients were given in (Grigor'ev et al., 1988; Renegar, 1992; Heintz et al., 1993; Basu et al., 1996, 2006). They are all based on computation of critical values and have not lead to efficient implementations. Theoretically, it is feasible to apply general quantifier elimination methods (Collins, 1975, 1998; Collins et al., 1991; Dolzmann et al., 1999) to solve Problem 3. Since the problem of QE is inherently doubly exponential in the number of variables (Fischer et al., 1974; Davenport et al., 1991; Brown et al., 2007), general tools for QE are not the best choice in practice for special problems.

The original algorithm of Cylindrical Algebraic Decomposition (CAD) (Collins, 1975) for QE is not efficient since the algorithm process of CAD projection phase involves a large amount of resultant calculation and the lifting phase needs to choose a sample point in every cell. Hence a lot of work tries to improve the CAD projection. A well known improvement is Hong's projection operator which is applicable in all cases (Hong, 1990). For many problems, a smaller projection operator given by McCallum (1988, 1998), with an improvement by Brown (2001), is more efficient. Strzeboński (2000) proposed an algorithm called Generic Cylindrical Algebraic Decomposition(GCAD) for solving systems of strict polynomial inequalities, which made use of the so-called generic projection, the same projection operator as that proposed by Brown (2001). Based on Wu's principle of finite kernel (Wu, 1998, 2003), Yang proposed without proof the successive resultant method (Yang, 2001; Yang et al., 2008) to solve the global optimization problem involving polynomials and square-roots, in which Brown's projection is used in the projection phase and only sample points from the highest dimensional cells need to be chosen in the lifting phase. McCallum (1993) once pointed out that, in order to prove a polynomial inequality, only those sample points from the highest dimensional cells need to be chosen. Xiao (2009) proved that, in terms of the Brown projection, at least one sample point can be taken from every highest dimensional cell via the Open CAD lifting.

In this paper, we consider how to improve the CAD based methods for solving Problems 1, 2 and 3. We propose a simplified Brown's CAD projection operator, Nproj, of which the projection scale is always no larger than that of Brown's. Some new algorithms based on Nproj for solving those problems are designed and proved to be correct. Some examples that could not be solved by existing CAD based tools have been solved by our tool.

The structure of this paper is as follows. Section 2 shows by a simple example our main idea of designing new projection operators. Section 3 introduces basic definitions, lemmas and concepts of CAD and Brown's projection. Section 4 proves the correctness of the successive resultant method proposed in Yang (2001). In Section 5 and Section 6, our new projection operator Nproj is introduced and some new complete algorithms based on Nproj are proposed for solving the above three problems. The correctness of our algorithms are proved. The last section includes several examples which demonstrate the process and effectiveness of our algorithms.

#### 2. Main idea

First, let us show the comparison of our new operator Nproj and Brown's projection operator on the following simple example. Formal description and proofs of our algorithms are given in subsequent sections.

Example 2.1. Prove or disprove

$$\forall (x, y, z) \in \mathbb{R}^3 (f(x, y, z) \ge 0)$$

where

$$f(x, y, z) = 4z^4 - 4z^2y^2 - 4z^2 + 4y^2x^4 + 4x^2y^4 + 8x^2y^2 + 5y^4 + 6y^2 + 4x^4 + 4x^2 + 1.$$

We solve this example by a CAD based method. First we apply Brown's operator and take the following steps:

Step 1.

$$f_1 := \operatorname{Res}(\operatorname{sqrfree}(f), \frac{\partial}{\partial z} \operatorname{sqrfree}(f), z) = 1048576g_1^3g_2h_1^2h_2^2,$$

where

$$g_1 = y^2 + 1, \ g_2 = 4x^4 + 4x^2y^2 + 4x^2 + 5y^2 + 1, \ h_1 = x^2 + 1, \ h_2 = x^2 + y^2,$$

"Res" means the Sylvester resultant and "sqrfree" means "squarefree" that is defined in Definition 15.

Step 2.

$$f_{2} := \operatorname{Res}(\operatorname{sqrfree}(f_{1}), \frac{\partial}{\partial y}\operatorname{sqrfree}(f_{1}), y)$$
  
=  $\operatorname{Res}(g_{1}g_{2}h_{1}h_{2}, \frac{\partial(g_{1}g_{2}h_{1}h_{2})}{\partial y}, y)$   
=  $16384(x^{2}+1)^{15}(x-1)^{12}(x+1)^{12}(2x^{2}+1)^{2}(4x^{2}+5)^{2}x^{2}.$ 

Actually, computing  $f_2$  is equivalent to computing the following 6 resultants.

- (a)  $\operatorname{Res}(g_i, \frac{\partial}{\partial y}g_i, y)$  (i = 1, 2),(b)  $\operatorname{Res}(h_2, \frac{\partial}{\partial y}h_2, y),$
- (c)  $\operatorname{Res}(g_1, g_2, y), \operatorname{Res}(g_i, h_2, y)$  (i = 1, 2).

Step 3. By real root isolation of  $f_2 = 0$ , choose 4 sample points of  $x: x_1 = -2, x_2 = -\frac{1}{2}$ ,  $x_3 = \frac{1}{2}, x_4 = 2$ . At the lifting phase, we first get 4 sample points of (x, y) for  $f_1(x_i, y) \neq \tilde{0}$ :  $(-2,0), (-\frac{1}{2},0), (\frac{1}{2},0), (2,0).$  Then get 4 sample points of (x,y,z) for  $f(x_i,y_i,z) \neq 0$ :  $(-2,0,0), (-\frac{1}{2},0,0), (\frac{1}{2},0,0), (2,0,0).$ 

Step 4. Finally we should check that whether or not  $f(x, y, z) \ge 0$  at all the 4 sample points. Because  $f(x, y, z) \ge 0$  at all the sample points, the answer is

$$\forall (x, y, z) \in \mathbb{R}^3 (f(x, y, z) \ge 0).$$

Now, we apply our new projection operator to the problem. Step 1 is the same as above. According to our algorithm, at Step 2, we need only to compute the following 3 resultants.

(a)  $\operatorname{Res}(g_i, \frac{\partial}{\partial y}g_i, y), \quad (i = 1, 2)$ (b)  $\operatorname{Res}(h_2, \frac{\partial}{\partial y}h_2, y),$ 

That gives a polynomial (after squarefree)  $f'_2 = x(x^2+1)(2x^2+1)(4x^2+5)$ .

At Step 3, by real root isolation of  $f'_2 = 0$ , we choose  $x_1 = -1$  and  $x_2 = 1$  as sample points for x. At lifting phase, compute 2 sample points (-1,0), (1,0) of (x,y) and verify that  $g_1g_2 \ge 0$  at the two points. Then compute 2 sample points (-1,0,0), (1,0,0) of (x, y, z).

At Step 4, check whether or not  $f(x, y, z) \ge 0$  at all the 2 sample points. Because  $f(x, y, z) \ge 0$  at all the sample points, the answer is

$$\forall (x, y, z) \in \mathbb{R}^3 (f(x, y, z) \ge 0).$$

For this example, our new projection operator Nproj avoids computing 3 resultants compared to Brown's operator. In general, for a polynomial  $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ , Nproj first computes  $f_1 = \text{Res}(\text{sqrfree}(f), \frac{\partial}{\partial x_n} \text{sqrfree}(f), x_n)$  as other CAD based meth-ods do. Then, divides the irreducible factors of  $f_1$  into two groups:  $L_1$  and  $L_2$ , where  $L_1$  contains all factors with odd multiplicities and  $L_2$  contains all factors with even multiplicities. Compared to Brown's projection, at the next level of projection, neither the resultants of those polynomial pairs of which one is from  $L_1$  and the other from  $L_2$  nor the resultants of the polynomial pairs in  $L_1$  are to be computed. Therefore, the scale of Nproj is no larger than that of Brown's. For a wide class of problems (see for example Remark 28), especially when  $n \geq 3$ , the scale of Nproj is much smaller than that of Brown's. Based on the new operator, we obtain a new algorithm Proineq (see Section 5 for details) to prove or disprove a polynomial to be positive semi-definite.

The main idea behind our method is that Lemma 12 provides a condition from which it can be derived that, (roughly speaking) to show that a polynomial  $f(x_1, ..., x_n)$  is positive semi-definite (as a polynomial in  $x_n$  whose coefficients are given parametrically as polynomials in  $x_1, ..., x_{n-1}$ ) throughout a region U in (n-1)-space it suffices that the even multiplicity factors are sign-invariant in U (typical CAD) and the odd factors are semi-definite in U (a weaker condition than sign-invariance). Please see Theorems 34 and 35 in Section 5 for details.

#### 3. Preliminaries

In this paper, if not specified, for a positive integer  $n, a_n, b_n$  and  $\mathbf{0}_n$  denote the points  $(a_1,\ldots,a_n) \in \mathbb{R}^n, (b_1,\ldots,b_n) \in \mathbb{R}^n, \text{ and } (0,0,\ldots,0) \in \mathbb{R}^n, \text{ respectively.}$ 

**Definition 1.** For  $a_n, b_n \in \mathbb{R}^n$ , the Euclidean distance of  $a_n$  and  $b_n$  is defined by

$$\rho(\boldsymbol{a}_n, \boldsymbol{b}_n) := \sqrt{\sum_{i=1}^n (a_i - b_i)^2}.$$

**Definition 2.** For  $a_n \in \mathbb{R}^n$ , let  $B_{a_n}(r)$  be the open ball which centered in  $a_n$  with radius r, that is

$$B_{\boldsymbol{a}_n}(r) := \{ \boldsymbol{b}_n \in \mathbb{R}^n \mid \rho(\boldsymbol{a}_n, \boldsymbol{b}_n) < r \}$$

**Definition 3.** Let  $f \in \mathbb{R}[x_n]$ , the set of real zeros of f is denoted by Zero(f). Let L be a subset of  $\mathbb{R}[x_n]$ . Define

$$\operatorname{Zero}(L) = \{ \boldsymbol{a}_n \in \mathbb{R}^n | \forall f \in L, f(\boldsymbol{a}_n) = 0 \}.$$

The elements of  $\operatorname{Zero}(L)$  are the common real zeros of L. If  $L = \{f_1, \ldots, f_m\}$ ,  $\operatorname{Zero}(L)$  is also denoted by  $\operatorname{Zero}(f_1, \ldots, f_m)$ .

**Definition 4.** The *level* of  $f \in \mathbb{R}[\mathbf{x}_n]$  is the largest j such that  $\deg(f, x_j) > 0$  where  $\deg(f, x_j)$  is the degree of f with respect to  $x_j$ . The *level* of  $f \in \mathbb{R}[\mathbf{x}_n, k]$  is the largest j such that  $\deg(f, x_j) > 0$  and the level is zero if all  $x_i$ s do not appear in f.

**Definition 5.** For a polynomial set  $L \subseteq \mathbb{R}[\boldsymbol{x}_n]$  or  $L \subseteq \mathbb{R}[\boldsymbol{x}_n, k]$ ,  $L^i$  is the set of polynomials in L of level i.

**Definition 6.** For  $a_n, b_n \in \mathbb{R}^n$ , we denote by  $a_n b_n$  the segment  $a_n \to b_n$ . For *m* points  $a_n^1, \ldots, a_n^m$ , we denote by  $a_n^1 \to a_n^2 \to \cdots \to a_n^m$  the broken line through  $a_n^1, \ldots, a_n^m$  in turn.

The following two lemmas are well-known results.

**Lemma 7.** Let  $f, g \in \mathbb{R}[x_n]$ , there exist nonzero  $p, q \in \mathbb{R}[x_n]$  such that  $pf + qg = \text{Res}(f, g, x_n)$  with  $\deg(p, x_n) < \deg(g, x_n)$  and  $\deg(q, x_n) < \deg(f, x_n)$ , where  $\text{Res}(f, g, x_n)$  is the resultant of f and g with respect to  $x_n$ .

**Proof.** See, for example (Cox et al., 2005).  $\Box$ 

**Lemma 8.** Let  $f(\boldsymbol{x}_n) \in \mathbb{R}[\boldsymbol{x}_n]$  and r be a real positive number. If  $f(\boldsymbol{a}_n) = 0$  for all  $\boldsymbol{a}_n \in B_{\boldsymbol{0}_n}(r)$ , then  $f(\boldsymbol{x}_n) \equiv 0$ .

**Proof.** See, for example (Marshall, 2008).  $\Box$ 

**Lemma 9.** For  $f, g \in \mathbb{R}[\boldsymbol{x}_n]$ , if f and g are coprime in  $\mathbb{R}[\boldsymbol{x}_n]$ , then after any linear invertible transform, f and g are still coprime in  $\mathbb{R}[\boldsymbol{x}_n]$ , namely for  $A \in GL_n(\mathbb{R}), B_n \in \mathbb{R}^n$ ,  $\boldsymbol{x}_n^{*T} = A\boldsymbol{x}_n^T + B_n^T$ , then  $gcd(f(\boldsymbol{x}_n^*), g(\boldsymbol{x}_n^*)) = 1$  in  $\mathbb{R}[\boldsymbol{x}_n]$ .

**Proof.** If  $gcd(f(\boldsymbol{x}_n^*), g(\boldsymbol{x}_n^*)) = h(\boldsymbol{x}_n)$  and h is not a constant. Then  $h(A^{-1}(\boldsymbol{x}_n - B_n^T))$  is a non-trivial common divisor of f and g in  $\mathbb{R}[\boldsymbol{x}_n]$ , which is a contradiction.  $\Box$ 

**Lemma 10.** Suppose  $f, g \in \mathbb{R}[\boldsymbol{x}_n]$  and gcd(f,g) = 1 in  $\mathbb{R}[\boldsymbol{x}_n]$ . For any  $\boldsymbol{a}_{n-1} \in \mathbb{R}^{n-1}$ and r > 0, there exists  $\boldsymbol{a}'_{n-1} \in \mathbb{R}^{n-1}$  such that  $\rho(\boldsymbol{a}_{n-1}, \boldsymbol{a}'_{n-1}) < r$  and for all  $a'_n \in \mathbb{R}$ ,  $(\boldsymbol{a}'_{n-1}, a'_n) \notin Zero(f, g)$ . **Proof.** Otherwise, there exist  $a_{n-1}^0 = (a_1^0, \ldots, a_{n-1}^0) \in \mathbb{R}^{n-1}$  and  $r_0 > 0$ , such that for any  $a_{n-1}^1 = (a_1^1, \ldots, a_{n-1}^1)$  satisfying  $\rho(a_{n-1}^0, a_{n-1}^1) < r_0$ , there exists an  $a_n^1 \in \mathbb{R}$  such that  $f(a_{n-1}^1, a_n^1) = g(a_{n-1}^1, a_n^1) = 0$ . Thus  $\operatorname{Res}(f, g, x_n) = 0$  at every point of  $B_{a_{n-1}^0}(r_0)$ . From Lemma 8, we get that  $\operatorname{Res}(f, g, x_n) \equiv 0$ , meaning  $\operatorname{gcd}(f, g)$  is non-trivial, which is impossible.  $\Box$ 

**Definition 11.** Let  $f(x) = c_l x^l + \dots + c_0 \in \mathbb{R}[x]$  with  $c_l \neq 0$ . The discriminant of f(x) is

discrim
$$(f, x) = c_l^{2l-2} \prod_{i < j} (z_i - z_j)^2$$
,

where  $z_i$  (i = 1, ..., l) are the complex roots of the equation f(x) = 0.

The following well-known equation shows the relationship between discrim(f, x) and  $\operatorname{Res}(f, \frac{\partial}{\partial x}f, x)$ ,

$$c_l$$
discrim $(f, x) = (-1)^{\frac{l(l-1)}{2}} \operatorname{Res}(f, \frac{\partial}{\partial x} f, x).$ 

Suppose the coefficients of f are given parametrically as polynomials in  $\boldsymbol{x}_n$ . If the *leading* coefficient  $lc(f, x) = c_l \neq 0$ , the discriminant of  $f(\boldsymbol{x}_n, x)$  can be written as

$$\operatorname{discrim}(f,x) = (-1)^{\frac{l(l-1)}{2}} \begin{pmatrix} 1 & c_{l-1} & c_{l-2} & \cdots & c_j & \cdots \\ 0 & c_l & c_{l-1} & \cdots & c_{j+1} & \cdots \\ 0 & 0 & c_l & \cdots & c_{j+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ l & (l-1)c_{l-1} & (l-2)c_{l-2} & \cdots & jc_j & \cdots \\ 0 & lc_l & (l-1)c_{l-1} & \cdots & (j+1)c_{j+1} & \cdots \\ 0 & 0 & lc_l & \cdots & (j+2)c_{j+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

If  $c_l = c_{l-1} = 0$  at point  $a_n$ , from the above expression, discrim(f, x) = 0 at this point.

**Lemma 12.** (Weiss, 1963) Let  $f(x) \in \mathbb{R}[x]$  be a monic squarefree polynomial of degree l, the sign of its discriminant is  $(-1)^{\frac{l-r}{2}}$ , where r is the number of its real roots.

It is clear that the conclusion of the above lemma still holds when lc(f, x) is positive.

**Lemma 13.** Given a polynomial  $f(\boldsymbol{x}_n, x_{n+1}) \in \mathbb{R}[\boldsymbol{x}_n, x_{n+1}]$ , say

$$f(\boldsymbol{x}_n, x_{n+1}) = \sum_{i=0}^{l} c_i x_{n+1}^i, c_l \neq 0,$$

where  $c_i(i = 0, ..., l)$  is a polynomial in  $\mathbf{x}_n$ . Let U be an open set in  $\mathbb{R}^n$ . If  $f(\mathbf{x}_n, x_{n+1}) \ge 0$  on  $U \times \mathbb{R}$ , then l is even and

$$(-1)^{\frac{1}{2}}\operatorname{discrim}(f, x_{n+1}) \ge 0$$
 and  $\operatorname{lc}(f, x_{n+1}) \ge 0$  for all  $a_n \in U$ .

**Proof.** Since f is positive semi-definite for any given  $a_n \in U$ ,  $lc(f, x_{n+1})$  is positive semi-definite on U and l is even. If  $c_l > 0$  at  $a_n$  and  $f(a_n, x_{n+1})$  is squarefree, then

$$(-1)^{\frac{1}{2}}\operatorname{discrim}(f(\boldsymbol{x}_n, x_{n+1}), x_{n+1}) \mid_{\boldsymbol{x}_n = \boldsymbol{a}_n} = (-1)^{\frac{1}{2}}\operatorname{discrim}(f(\boldsymbol{a}_n, x_{n+1}), x_{n+1}) > 0$$

by Lemma 12. Otherwise, either  $c_l = 0$  at  $\boldsymbol{a}_n$  which suggests  $c_{l-1} = 0$  at  $\boldsymbol{a}_n$ , or  $c_l > 0$  at  $\boldsymbol{a}_n$  and  $f(\boldsymbol{a}_n, x_{n+1})$  is not squarefree. In both cases we can deduce

$$(-1)^{\frac{1}{2}}\operatorname{discrim}(f(\boldsymbol{x}_n, x_{n+1}), x_{n+1}) = 0$$

at  $a_n$ . That completes the proof.  $\Box$ 

Before we go further, we would like to give a remark on the coefficient ring of polynomials.

**Remark 14.** Although most of the theorems of this paper are valid for  $\mathbb{R}[x_n]$ , we restrict ourselves to  $\mathbb{Z}[x_n]$  when we design algorithms because they need effective factorization and real root isolation. Actually, suppose  $\mathcal{R}$  is a subring of  $\mathbb{R}$  and takes  $\mathbb{Z}$  as a subring. If  $\mathcal{R}[x_n]$  admits effective factorization and  $\mathcal{R}[x]$  admits effective real root isolation, all the algorithms in this paper are effective. Two examples of such rings are  $\mathbb{Q}$  and the field of real algebraic numbers. In the following, we use  $\mathcal{R}$  to denote such a ring.

**Definition 15.** Suppose  $h \in \mathcal{R}[\boldsymbol{x}_n]$  can be factorized in  $\mathcal{R}[\boldsymbol{x}_n]$  as:

$$h = ah_1^{i_1} h_2^{i_2} \dots h_m^{i_m},$$

where  $a \in \mathcal{R}$ ,  $h_i (i = 1, ..., m)$  are pairwise different irreducible monic polynomials (under a suitable ordering) with degree greater than or equal to one in  $\mathcal{R}[\boldsymbol{x}_n]$ . Define

$$\operatorname{sqrfree}(h) = \prod_{i=1}^{m} h_i$$

If h is a constant, let sqrfree(h) = 1.

**Lemma 16.** Given a real polynomial f with real parameters, say

$$f(\mathbf{c}, x) = c_m x^m + c_{m-1} x^{m-1} + \ldots + c_0$$

where  $\mathbf{c} = (c_m, \ldots, c_0)$  is real parameter. Let  $R(\mathbf{c}) = \operatorname{sqrfree}(\operatorname{Res}(f, f', x))$ . If  $\mathbf{s_1}$  and  $\mathbf{s_2}$  are two points in the same connected component of parameter space  $R(\mathbf{c}) \neq 0$ , then  $f(\mathbf{s_1}, x)$  and  $f(\mathbf{s_2}, x)$  have the same number of real roots  $y_1(\mathbf{s_1}) < y_2(\mathbf{s_1}) < \ldots < y_d(\mathbf{s_1})$  and  $y_1(\mathbf{s_2}) < y_2(\mathbf{s_2}) < \ldots < y_d(\mathbf{s_2})$ . Moreover,  $y_i(\mathbf{c})(i = 1, \ldots, d)$  is continuous in the connected component.

**Proof.** See for example Yang et al. (2008).  $\Box$ 

In the following, we introduce some basic concepts and results of CAD. The reader is referred to Collins (1975), Hong (1990), McCallum (1988, 1998), Brown (2001) and Xiao (2009) for a detailed discussion on the properties of CAD and Open CAD.

**Definition 17.** (Collins, 1975; McCallum, 1988) An *n*-variate polynomial  $f(\boldsymbol{x}_{n-1}, x_n)$  over the reals is said to be *delineable* on a subset S (usually connected) of  $\mathbb{R}^{n-1}$  if (1) the portion of the real variety of f that lies in the cylinder  $S \times \mathbb{R}$  over S consists of the union of the graphs of some  $t \geq 0$  continuous functions  $\theta_1 < \cdots < \theta_t$  from S to  $\mathbb{R}$ ; and (2) there exist integers  $m_1, \ldots, m_t \geq 1$  such that for every  $a \in S$ , the multiplicity of the root  $\theta_i(a)$  of  $f(a, x_n)$  (considered as a polynomial in  $x_n$  alone) is  $m_i$ .

**Definition 18.** (Collins, 1975; McCallum, 1988) In the above definition, the  $\theta_i$  are called the real root functions of f on S, the graphs of the  $\theta_i$  are called the f-sections over S, and the regions between successive f-sections are called f-sectors.

**Theorem 19.** (McCallum, 1988, 1998) Let  $f(\mathbf{x}_n, \mathbf{x}_{n+1})$  be a polynomial in  $\mathbb{R}[\mathbf{x}_n, \mathbf{x}_{n+1}]$ of positive degree and discrim $(f, \mathbf{x}_{n+1})$  is a nonzero polynomial. Let S be a connected submanifold of  $\mathbb{R}^n$  on which f is degree-invariant and does not vanish identically, and in which discrim $(f, \mathbf{x}_{n+1})$  is order-invariant. Then f is analytic delineable on S and is order-invariant in each f-section over S.

Based on this theorem, McCallum proposed the projection operator MCproj, which consists of the discriminant of f and all coefficients of f.

**Theorem 20.** (Brown, 2001) Let  $f(\mathbf{x}_n, x_{n+1})$  be an (n+1)-variate polynomial of positive degree m in the variable  $x_{n+1}$  with discrim $(f, x_{n+1}) \neq 0$ . Let S be a connected submanifold of  $\mathbb{R}^n$  where discrim $(f, x_{n+1})$  is order-invariant, the leading coefficient of f is sign-invariant, and such that f vanishes identically at no point in S. f is degreeinvariant on S.

Based on this theorem, Brown obtained a reduced McCallum projection in which only leading coefficients and discriminants appear.

**Definition 21.** (Brown, 2001) Given a polynomial  $f \in \mathcal{R}[x_n]$  of level n, the Brown projection operator for f is

$$Bproj(f, x_n) = Res(sqrfree(f), \frac{\partial(sqrfree(f))}{\partial x_n}, x_n).$$

If L is a polynomial set and the level of any polynomial in L is n, then

$$Bproj(L, x_n) = \bigcup_{f \in L} \{ \text{Res}(\text{sqrfree}(f), \frac{\partial(\text{sqrfree}(f))}{\partial x_n}, x_n) \} \bigcup_{\substack{\bigcup_{f, g \in L, f \neq g}}} \{ \text{Res}(\text{sqrfree}(f), \text{sqrfree}(g), x_n) \}.$$

Algorithm 1. Bprojection (Brown, 2001) Input: A polynomial  $f(\boldsymbol{x}_n) \in \mathbb{Z}[\boldsymbol{x}_n]$ . Output: A projection factor set F. 1:  $F := \{f(\boldsymbol{x}_n)\};$ 2: for i from n downto 2 do 3:  $F := F \bigcup \{Bproj(F^i, x_i)\}; (F^i \text{ is the set of polynomials in } F \text{ of level } i).$ 4: end for 5: return F Open CAD is a modified CAD construction algorithm, which was named in Rong Xiao's Ph.D. thesis (Xiao, 2009). In fact, Open CAD is similar to the Generic Cylindrical Algebraic Decomposition (GCAD) proposed by Strzeboński (2000) and was used in DISCOVERER (Xia, 2000) for real root classification. For convenience, we describe the framework of the Open CAD here.

For a polynomial  $f(\boldsymbol{x}_n) \in \mathcal{R}[\boldsymbol{x}_n]$ , an Open CAD defined by  $f(\boldsymbol{x}_n)$  is a set of rational sample points in  $\mathbb{R}^n$  obtained through the following three phases: (1) Projection. Use the Brown projection operator (Algorithm 1) on  $f(\boldsymbol{x}_n)$ ; (2) Base. Choose one rational point in each of the open intervals defined by the real roots of  $F^1$  (see Algorithm 1); (3) Lifting. Substitute each sample point of  $\mathbb{R}^{i-1}$  for  $\boldsymbol{x}_{i-1}$  in  $F^i$  and then, by the same method as Base phase, choose rational sample points for  $F^i(\boldsymbol{x}_i)$ .

#### 4. The Successive Resultant Method

The Successive Resultant Method (SRes) was introduced in (Yang, 2001) without a proof. The method can be used for solving Problem 2 of this paper, i.e., problem of global optimization.

For a polynomial  $f(\boldsymbol{x}_n)$  in Problem 2, the SRes method first applies Algorithm 1 on polynomial  $f(\boldsymbol{x}_n) - K$  to get a polynomial g(K). Suppose g(K) has m distinct real roots  $k_i(1 \leq i \leq m)$ . Then computes m + 1 rational numbers  $p_i(0 \leq i \leq m)$  such that  $k_i \in (p_{i-1}, p_i)$ . Finally, substitutes each  $p_i$  in turn for K in  $f(\boldsymbol{x}_n) - K$  to check if  $f(\boldsymbol{x}_n) - p_i \geq 0$  holds for all  $\boldsymbol{x}_n$ . If  $p_j$  is the first such that  $f(\boldsymbol{x}_n) - p_j \geq 0$  does not hold, then  $k_j$  is the infimum (let  $k_0 = -\infty, k_{m+1} = +\infty$ ). To check if  $f(\boldsymbol{x}_n) - p_i \geq 0$  holds for all  $\boldsymbol{x}_n$ , the SRes method applies Brown's projection on  $f(\boldsymbol{x}_n) - p_i$  and choose sample points by Open CAD in the lifting phase.

The SRes method is formally described as Algorithm 2 and we prove its correctness in the rest part of this section.

Algorithm 2. SRes (Successive Resultant Method, Yang (2001); Yang et al. (2008)) Input: A squarefree polynomial  $f \in \mathbb{Z}[\boldsymbol{x}_n]$ .

- **Output:** The supremum of  $k \in \mathbb{R}$ , such that  $\forall a_n \in \mathbb{R}^n$ ,  $f(a_n) \ge k$ . If there doesn't exist such k, then returns  $-\infty$ .
- 1: g := f k (g is viewed as a polynomial in  $k \prec x_1 \prec \cdots \prec x_n$ );
- 2: F := Bprojection(g) (F<sup>i</sup> is the set of polynomials in F of level i. Here F<sup>i</sup> has no more than one polynomial, we denote this polynomial by F<sub>i</sub>.);
- 3:  $C_0 :=$ an Open CAD of  $\mathbb{R}$  defined by  $F_0(k)$  (Suppose  $C_0 = \bigcup_{i=0}^m \{p_i\}, p_i \in (k_i, k_{i+1}),$ where  $k_i \ (1 \le i \le m)$  are the real roots of  $F_0$  and  $k_0 = -\infty, k_{m+1} = +\infty$ .);
- 4: for l from 0 to m do
- 5: for i from 1 to n do
- 6:  $C_{li}:=$  an Open CAD of  $\mathbb{R}^i$  defined by  $\bigcup_{j=1}^i F_j(\boldsymbol{x}_j, p_l);$
- 7: end for
- 8: **if** there exists a sample point  $a_n$  in  $C_{ln}$  such that  $f(a_n) p_l < 0$  then
- 9: return  $k_l$
- 10: end if
- 11: end for
- 12: return  $-\infty$

**Remark 22.** If  $g(\boldsymbol{x}_n) \geq 0$  for all  $\boldsymbol{x}_n \in \mathbb{R}^n$ , Algorithm 2 can also be applied to compute  $\inf\{\frac{f(\boldsymbol{x}_n)}{g(\boldsymbol{x}_n)} | \boldsymbol{x}_n \in \mathbb{R}^n\}$ . We just need to replace F := Bprojection(f - k) of Line 2 by F := Bprojection(f - kg). The proof of the correctness is the same.

The following lemma can be inferred from the results of (McCallum, 1998) and (Brown, 2001), *i.e.*, f is delineable over the maximal connected regions defined by  $\text{Bproj}(f, x_n) \neq 0$ . We give a new proof here.

**Lemma 23.** (McCallum, 1998; Brown, 2001) Let  $F_i, F_{i-1}$  be as in Algorithm 2. Let U be a connected component of  $F_{i-1} \neq 0$  in  $\mathbb{R}^i$  and  $y_i^1(\gamma) < y_i^2(\gamma) < \cdots < y_i^m(\gamma)$  be all real roots of  $F_i(\gamma, x_i) = 0$  for any given  $\gamma \in U$ . Then for all  $\alpha, \beta \in U$ ,  $\alpha \times (y_i^{j-1}(\alpha), y_i^j(\alpha))$  and  $\beta \times (y_i^{j-1}(\beta), y_i^j(\beta))(j = 2, 3, \dots, m)$  are in the same connected component of  $F_i \neq 0$  in  $\mathbb{R}^{i+1}$ .

**Proof.** For  $\alpha \in U$ , let  $\varepsilon = \min_{2 \leq i \leq m} |y_i(\alpha) - y_{i-1}(\alpha)|$ , by Lemma 16,  $\exists \delta > 0$ , such that  $\forall \alpha' \in B(\alpha, \delta), \max_{1 \leq i \leq m} |y_i(\alpha) - y_i(\alpha')| < \frac{\varepsilon}{6}$ .

Consider segment  $(\alpha, \frac{y_{j-1}(\alpha)+y_j(\alpha)}{2}) \to (\alpha', \frac{y_{j-1}(\alpha')+y_j(\alpha')}{2})$  where  $\alpha' \in B(\alpha, \delta)$ . For any point  $(\alpha'', y)$  on the segment, we have

$$\begin{aligned} | y - y_s(\alpha'') | \\ &= | (y_s(\alpha'') - y_s(\alpha)) + (y_s(\alpha) - \frac{y_{j-1}(\alpha) + y_j(\alpha)}{2}) + (\frac{y_{j-1}(\alpha) + y_j(\alpha)}{2} - y) | \\ &\geq | y_s(\alpha) - \frac{y_{j-1}(\alpha) + y_j(\alpha)}{2} | - | y_s(\alpha'') - y_s(\alpha) | - | \frac{y_{j-1}(\alpha) + y_j(\alpha)}{2} - y | \\ &\geq \frac{\varepsilon}{2} - \frac{\varepsilon}{6} - | \frac{y_{j-1}(\alpha) + y_j(\alpha)}{2} - \frac{y_{j-1}(\alpha') + y_j(\alpha')}{2} | \\ &\geq \frac{\varepsilon}{2} - \frac{\varepsilon}{6} - \frac{\varepsilon}{6} \\ &> 0 \end{aligned}$$

So the points satisfying  $F_i = 0$  are not on the segment.

*. ...* 

Therefore, for any points  $r_1 \in (y_{j-1}(\alpha), y_j(\alpha)), r_2 \in (y_{j-1}(\alpha'), y_j(\alpha')), (\alpha' \in B_{\alpha}(\delta)),$ the points satisfying  $F_i = 0$  are not on the broken line  $(\alpha, r_1) \to (\alpha, \frac{y_{j-1}(\alpha)+y_j(\alpha)}{2}) \to (\alpha', \frac{y_{j-1}(\alpha')+y_j(\alpha')}{2}) \to (\alpha', r_2).$ 

Hence we know that for any  $\alpha \in U$ , there exists  $\delta > 0$  such that for any point  $\alpha' \in B_{\alpha}(\delta)$  and  $2 \leq s \leq m$ ,  $\alpha \times (y_{s-1}(\alpha), y_s(\alpha))$  and  $\alpha' \times (y_{s-1}(\alpha'), y_s(\alpha'))$  are in the same connected component of  $F_i \neq 0$  in  $\mathbb{R}^{i+1}$ . For all  $\alpha, \beta \in U$ , there exists a path  $\gamma : [0,1] \to U$  that connects  $\alpha$  and  $\beta$ . Due to the compactness of the path, there are finitely many open sets  $B_{\alpha_t}(\delta_t)$  covering  $\gamma([0,1])$  with  $\alpha_t \in \gamma([0,1]), \forall \alpha' \in B_{\alpha_t}(\delta_t), 2 \leq j \leq m, \alpha \times (y_{j-1}(\alpha), y_j(\alpha))$  and  $\alpha' \times (y_{j-1}(\alpha'), y_j(\alpha'))$  are in the same connected component of  $F_i \neq 0$ . Since the union of these open sets are connected, the lemma is proved.  $\Box$ 

**Remark 24.** By the above Lemma, in Algorithm 2, for any two points  $p_l, p'_l \in (k_l, k_{l+1})$ , their corresponding sample points obtained through the Open CAD lifting phase are in the same connected component of  $F_n \neq 0$  in  $\mathbb{R}^{n+1}$ . Since at least one sample point can be taken from every highest dimensional cell via the Open CAD lifting phase, the set

of the corresponding sample points of  $p_l$  obtained through the Open CAD lifting phase,  $C_{ln}$  in Algorithm 2, contains at least one point from every connected component U of  $F_n(\boldsymbol{x}_n, k) \neq 0$ , in which  $U \cap (\mathbb{R}^n \times (k_{l-1}, k_l)) \neq \emptyset$ .

**Theorem 25.** The Successive Resultant Method is correct.

**Proof.** Let notations be as in Algorithm 2. If there exists a  $k' \in (k_i, k_{i+1})$ , such that  $F_n(\boldsymbol{x}_n, k') \geq 0$  for all  $\boldsymbol{x}_n \in \mathbb{R}^n$ , then by Lemma 23, for any  $k \in (k_i, k_{i+1})$ ,  $F_n(\boldsymbol{x}_n, k) \geq 0$  for all  $\boldsymbol{x}_n \in \mathbb{R}^n$  (since their corresponding sample points obtained through the Open CAD lifting phase are in the same connected component of  $F_n(\boldsymbol{x}_n, k) \neq 0$  in  $\mathbb{R}^{n+1}$ ). Therefore, for any  $k \in [k_i, k_{i+1}]$ ,  $F_n(\boldsymbol{x}_n, k) \geq 0$  for all  $\boldsymbol{x}_n \in \mathbb{R}^n$ . The global optimum k will be found by checking whether  $\forall \boldsymbol{a}_n \in \mathbb{R}^n, F_n(\boldsymbol{a}_n, p_i) \geq 0$  holds where  $p_i$  is the sample point of  $(k_i, k_{i+1})$ . Since Algorithm 2 ensures that at least one point is chosen from every connected component of  $F_n(\boldsymbol{x}_n, p_i) \neq 0$  in  $\mathbb{R}^n$ , the theorem is proved.  $\Box$ 

## 5. Solving Problem 1 via simplified CAD projection

To improve the efficiency of CAD based methods for solving Problem 1, *i.e.*, proving or disproving  $f(x_n) \ge 0$ , we propose a new projection operator called Nproj. The operator has been illustrated by a simple example in Section 2. In this section, we give a formal description of our method for solving Problem 1 based on Nproj and prove its correctness.

#### 5.1. Notations

**Definition 26.** Suppose  $h \in \mathcal{R}[\boldsymbol{x}_n]$  can be factorized in  $\mathcal{R}[\boldsymbol{x}_n]$  as:

$$h = a l_1^{2j_1 - 1} \dots l_t^{2j_t - 1} h_1^{2i_1} \dots h_m^{2i_m},$$

where  $a \in \mathcal{R}$ ,  $h_i(i = 1, ..., m)$  and  $l_j(i = 1, ..., t)$  are pairwise different irreducible monic polynomials (under a suitable ordering) with degree greater than or equal to one in  $\mathcal{R}[\boldsymbol{x}_n]$ . Define

$$sqrfree_1(h) = \{l_i, i = 1, 2, ..., t\},\$$
  
 $sqrfree_2(h) = \{h_i, i = 1, 2, ..., m\}$ 

If h is a constant, let  $\operatorname{sqrfree}_1(h) = \{1\}$ ,  $\operatorname{sqrfree}_2(h) = \{1\}$ .

**Definition 27.** Suppose  $f \in \mathcal{R}[\boldsymbol{x}_n]$  is a polynomial of level *n*. Define

$$\begin{aligned} &\operatorname{Oc}(f, x_n) = \operatorname{sqrfree}_1(\operatorname{lc}(f, x_n)), \ \operatorname{Od}(f, x_n) = \operatorname{sqrfree}_1(\operatorname{discrim}(f, x_n)), \\ &\operatorname{Ec}(f, x_n) = \operatorname{sqrfree}_2(\operatorname{lc}(f, x_n)), \ \operatorname{Ed}(f, x_n) = \operatorname{sqrfree}_2(\operatorname{discrim}(f, x_n)), \\ &\operatorname{Ocd}(f, x_n) = \operatorname{Oc}(f, x_n) \cup \operatorname{Od}(f, x_n), \ \operatorname{Ecd}(f, x_n) = \operatorname{Ec}(f, x_n) \cup \operatorname{Ed}(f, x_n). \end{aligned}$$

The secondary and principal parts of the new projection are defined as

$$\begin{split} \mathtt{Nproj}_1(f,x_n) &= \mathrm{Ocd}(f,x_n),\\ \mathtt{Nproj}_2(f,x_n) &= \{\prod_{g\in \mathrm{Ecd}(f,x_n)\backslash \mathrm{Ocd}(f,x_n)} g\}. \end{split}$$

If L is a set of polynomials of level n, define

$$\operatorname{Nproj}_1(L, x_n) = \bigcup_{g \in L} \operatorname{Ocd}(g, x_n),$$

$$\operatorname{Nproj}_2(L, x_n) = \bigcup_{g \in L} \{ \prod_{h \in \operatorname{Ecd}(g, x_n) \setminus \operatorname{Nproj}_1(L, x_n)} h \}.$$

5.2. Algorithm

By Theorem 35 (see Section 5.3 for details), the task of proving  $f(\boldsymbol{x}_n) \geq 0$  on  $\mathbb{R}^n$  can be accomplished by (1) computing sample points of  $\operatorname{Nproj}_2(f, x_n) \neq 0$  in  $\mathbb{R}^{n-1}$  and checking  $f(\alpha, x_n) \geq 0$  on  $\mathbb{R}$  for all sample points  $\alpha$ ; and (2) proving all the polynomials in  $\operatorname{Nproj}_1(f, x_n)$  are positive semi-definite on  $\mathbb{R}^{n-1}$ . For (1), typical CAD based methods, *e.g.*, Open CAD, can be applied. For (2), we can call this procedure recursively. Now the idea of our algorithm **Proineq** is clear and is formally described here.

```
Algorithm 3. Proineq
Input: A polynomial f(\boldsymbol{x}_n) \in \mathbb{Z}[\boldsymbol{x}_n] (monic under a suitable ordering)
Output: Whether or not f(\boldsymbol{x}_n) \geq 0 on \mathbb{R}^n
 1: if f is a constant then
 2:
        if f \ge 0 then return true
         else return false
 3:
         end if
 4:
 5: else
        if f is reducible in \mathbb{Z}[x_n] then
 6:
 7:
             for g in sqrfree<sub>1</sub>(f) do
                 if Proineq(g) = false then return false
 8:
                 end if
 9:
             end for
10:
         end if
11:
         L_1 := \operatorname{Nproj}_1(f, x_n)
12:
         L_2 := \operatorname{Bprojection}(\operatorname{Nproj}_2(f, x_n)) \bigcup \{f(x_n)\}
13:
         for g in L_1 do
14:
             if Proineq(g) = false then return false
15:
             end if
16:
         end for
17:
         for i from 1 to n do
18:
             C_i := An Open CAD of \mathbb{R}^i defined by \bigcup_{j=1}^i L_2^j (If i = n - 1, we require that
19:
    for any sample point a_{n-1} in C_{n-1}, a_{n-1} \notin \bigcup_{h \in L_1} \operatorname{Zero}(h)
         end for
20:
         if there exists an a_n \in C_n such that f(a_n) < 0 then return false
21:
22:
         end if
23:
        return true
24: end if
```

To give the readers a picture of how our new projection operator is different from existing CAD projection operators, we give Algorithm 4 here, which returns all possible polynomials that may appear in the projection phase of Algorithm 3.

**Remark 28.** For polynomial  $P(x_1, \ldots, x_{n-1}, x_n) = p(x_1, \ldots, x_{n-1}, x_n^2)$  (deg $(P, x_n) \ge 2, n \ge 2$ ), the resultant of P and  $P'_{x_n}$  with respect to  $x_n$  is (may differ from a constant)

$$p(x_1,\ldots,x_{n-1},0)$$
Res $(p,p'_{x_n},x_n)^2$ .

### Algorithm 4. Nproj

**Input:** A polynomial  $f(\boldsymbol{x}_n) \in \mathbb{Z}[\boldsymbol{x}_n]$ .

- **Output:** Two projection factor sets containing all possible polynomials that may appear in the projection phase of Algorithm 3.
- 1:  $L_1 := \operatorname{sqrfree}_1(f);$
- 2:  $L_2 := \{\};$
- 3: for i from n downto 2 do
- 4:  $L_2 := L_2 \bigcup \operatorname{Nproj}_2(L_1^i, x_i) \bigcup \bigcup_{g \in L_2^i} \operatorname{Bproj}(g, x_i)$ ; (Recall that  $L^i$  is the set of polynomials in L of level i.)
- 5:  $L_1 := L_1 \bigcup \operatorname{Nproj}_1(L_1^i, x_i);$
- 6: end for
- 7: return  $(L_1, L_2)$ .

If  $p(x_1, \ldots, x_{n-1}, 0)$  is not a square, the set  $Nproj_1(P, x_n)$  is not empty and thus the scale of Nproj(P) is smaller than that of Bprojection(P).

If for any polynomial  $f \in \mathbb{Z}[x_1, \ldots, x_n]$ , the iterated discriminants of f always have odd factors and are reducible (for generic f or for most polynomials, it is quite likely), then for  $n \geq 3$ , the the scale of Nproj(f) is always strictly smaller than that of Bprojection(f).

# 5.3. The correctness of Algorithm Proineq

**Theorem 29.** Let  $f(\mathbf{x}_n)$  and  $g(\mathbf{x}_n)$  be coprime in  $\mathbb{R}[\mathbf{x}_n]$ . For any connected open set U in  $\mathbb{R}^n$ , the open set  $V = U \setminus \text{Zero}(f, g)$  is also connected.

This theorem plays an important role in our proof. It can be proved by the fact that closed and bounded semi-algebraic set is semi-algebraically triangulable (Bochnak, 1998) and Alexander duality. Here we give an elementary proof.

**Proof.** For any two points  $\alpha$ ,  $\beta$  in V, we only need to prove that there exists a path  $\gamma(t) : [0,1] \to V$  such that  $\gamma(0) = \alpha, \gamma(1) = \beta$ . Choose a path  $\gamma_U$  that connects  $\alpha$  and  $\beta$  in U. Notice that U is an open set, so for any  $X_n \in \gamma_U$ , there exists  $\delta_{X_n} > 0$  such that  $U \supset B_{X_n}(\delta_{X_n})$ . Since  $\gamma_U$  is compact and  $\bigcup B_{X_n}(\delta_{X_n})$  is an open covering of  $\gamma_U$ , there exists an  $m \in \mathbb{N}$ , such that  $\bigcup_{k=1}^{m} B_{X_n^k}(\delta_{X_n^k}) \supset \gamma_U$  and  $\alpha \in B_{X_n^1}(\delta_{X_n^1}), \beta \in B_{X_n^m}(\delta_{X_n^m}), B_{X_n^i}(\delta_{X_n^i}) \bigcap B_{X_n^{i+1}}(\delta_{X_n^{i+1}}) \neq \emptyset$   $(i = 1, 2, \ldots, m-1)$ . Now we only need to prove that for every k,  $B_{X_n^k}(\delta_{X_n^k}) \setminus \operatorname{Zero}(f, g)$  is connected. If this is the case, we can find k paths  $\gamma_1, \gamma_2, \ldots, \gamma_k$  with  $\gamma_1(0) = \alpha, \gamma_1(1) \in B_{X_n^1}(\delta_{X_n^1}) \bigcap B_{X_n^2}(\delta_{X_n^2}), \gamma_{i+1}(0) = \gamma_i(1), \gamma_{i+1}(1) \in B_{X_n^i}(\delta_{X_n^i}) \bigcap B_{X_n^{i+1}}(\delta_{X_{n+1}^{i+1}})$   $(i = 1, 2, \ldots, m-1), \gamma_m(1) = \beta$ . Let  $\gamma$  be the path:  $[0, 1] \to U$  which satisfies  $\gamma([\frac{j-1}{m}, \frac{j}{m}]) = \gamma_j([0, 1])$   $(j = 1, 2, \ldots, m)$ , then  $\gamma$  is the path as desired. Choose  $a, b \in B_{X_n^k}(\delta_{X_n^k}) \setminus \operatorname{Zero}(f, g)$ . There exists an affine coordinate transformation T

Choose  $a, b \in B_{X_n^k}(\delta_{X_n^k}) \setminus \text{Zero}(f, g)$ . There exists an affine coordinate transformation Tsuch that  $T(B_{X_n^k}(\delta_{X_n^k})) = B_{\mathbf{0}_n}(1)$  and  $\overrightarrow{T(a)T(b)}$  and  $\overrightarrow{(\mathbf{0}_{n-1}, 1)}$  are parallel. Thus the first n-1 coordinates of T(a) and T(b) are the same. Let  $T(a) = (Y_{n-1}, a'), T(b) = (Y_{n-1}, b')$ . Without loss of generality, we assume that a' > b'.

In the new coordinate, f and g become T(f) and T(g), respectively.  $B_{\mathbf{0}_n}(1)$  is an open set and  $T(a), T(b) \notin \operatorname{Zero}(T(f), T(g))$ , so there exists r > 0 such that the cylinder  $B_{Y_{n-1}}(r) \times [b', a'] \subseteq B_{\mathbf{0}_n}(1), B_{T(a)}(r) \cap \operatorname{Zero}(T(f), T(g)) = \emptyset$  and  $B_{T(b)}(r) \cap \operatorname{Zero}(T(f), T(g)) = \emptyset$ . By Lemma 9, T(f) and T(g) are coprime in  $\mathbb{R}[\mathbf{x}_n]$ . So by Lemma 10,

there exists  $X'_{n-1} \in B_{Y_{n-1}}(r)$ , such that for any  $x_n \in \mathbb{R}$ ,  $(X'_{n-1}, x_n) \notin \text{Zero}(T(f), T(g))$ . Thus the broken line  $T(a) \to (X'_{n-1}, a') \to (X'_{n-1}, b') \to T(b)$  is a path that connects T(a) and T(b) in  $B_{\mathbf{0}_n}(1) \setminus \text{Zero}(T(f), T(g))$ . The theorem is proved.  $\Box$ 

**Proposition 30.** Suppose  $U \subseteq \mathbb{R}^n$  is a connected open set,  $f, g \in \mathbb{R}[\boldsymbol{x}_n]$ , gcd(f,g) = 1in  $\mathbb{R}[\boldsymbol{x}_n]$  and for all  $X_n \in U$ ,  $f(X_n)g(X_n) \ge 0$ . Then either  $f(X_n) \ge 0$ ,  $g(X_n) \ge 0$  for all  $X_n \in U$  or  $f(X_n) \le 0$ ,  $g(X_n) \le 0$  for all  $X_n \in U$ . Similarly, if for all  $X_n \in U$ ,  $f(X_n)g(X_n) \le 0$ , then either  $f(X_n) \ge 0$ ,  $g(X_n) \le 0$  for all  $X_n \in U$  or  $f(X_n) \le$  $0, g(X_n) \ge 0$  for all  $X_n \in U$ .

**Proof.** If not, there exist  $X_n^1, X_n^2 \in U$ , such that  $f(X_n^1) \leq 0, g(X_n^1) \leq 0$  and  $f(X_n^2) \geq 0$ ,  $g(X_n^2) \geq 0$ . By Theorem 29,  $U \setminus \text{Zero}(f, g)$  is connected. So we can choose a path  $\gamma$  that connects  $X_n^1$  with  $X_n^2$  and  $\gamma \bigcap \text{Zero}(f, g) = \emptyset$ . Consider the sign of f + g on  $\gamma$ . Since the sign is different at  $X_n^1$  and  $X_n^2$ , by Mean Value Theorem we know there exists  $X_n^3$  on  $\gamma$  such that  $f(X_n^3) + g(X_n^3) = 0$ . From the condition we know that  $f(\boldsymbol{x}_n)g(\boldsymbol{x}_n) \geq 0$ , hence  $X_n^3 \in \text{Zero}(f, g)$ , which contradicts the choice of  $\gamma$ .

The second part of the proposition can be proved similarly.  $\Box$ 

The following proposition is an easy corollary of Proposition 30.

**Proposition 31.** Let  $f \in \mathcal{R}[\mathbf{x}_n]$  be a monic (under a suitable ordering) polynomial of level n, the necessary and sufficient condition for  $f(\mathbf{x}_n)$  to be positive semi-definite on  $\mathbb{R}^n$  is, for any polynomial  $g \in \text{sqrfree}_1(f)$ , g is positive semi-definite on  $\mathbb{R}^n$ .

**Proposition 32.** Suppose  $f \in \mathbb{R}[\mathbf{x}_n]$  is a non-zero squarefree polynomial and U is a connected open set of  $\mathbb{R}^n$ . If  $f(\mathbf{x}_n)$  is semi-definite on U, then  $U \setminus \text{Zero}(f)$  is also a connected open set.

**Proof.** Without loss of generality, we assume  $f(\boldsymbol{x}_n) \geq 0$  on U. Since f is non-zero, we only need to consider the case that the level of f is non-zero. Let i > 0 be the level of f and consider f as a polynomial of  $x_i$ . Because  $f(\boldsymbol{x}_n) \geq 0$  on U, we know  $\operatorname{Zero}(f) \bigcap U = \operatorname{Zero}(f, f'_{x_i}) \bigcap U$ . Otherwise, we may assume there exists a point  $X_n^0 = (x_1^0, \ldots, x_n^0) \in U$  such that  $f(X_n^0) = 0$  and  $f'_{x_i}(X_n^0) > 0$ . Thus, there exists r such that  $\forall X_n \in B_{X_n^0}(r)$ ,  $f'_{x_i}(X_n) > 0$ . Let  $F(x_i) = f(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)$ . The Taylor series of F at point  $x_i^0$  is

$$F(x_i) = F(x_i^0) + (x_i - x_i^0)F'_{x_i}(x_i^0 + \theta(x_i - x_i^0)),$$

where  $\theta \in (0, 1)$ . Let  $x_i^0 > x_i^1 > x_i^0 - r$ , then  $F(x_i^1) < 0$ , which contradicts the definition of F.

If f is irreducible in  $\mathbb{R}[\boldsymbol{x}_n]$ , f and  $f'_{x_i}$  are coprime in  $\mathbb{R}[\boldsymbol{x}_n]$ . Thus  $U \setminus \text{Zero}(f, f'_{x_i})$  is connected by Theorem 29. So  $U \setminus \text{Zero}(f)$  is a connected open set.

If f is reducible in  $\mathbb{R}[\boldsymbol{x}_n]$ , let  $f = a \prod_{t=1}^{j} f_t$ , in which  $a \in \mathbb{R}$  and all  $f_t(t = 1, \dots, j)$  are irreducible monic polynomials (under a suitable ordering) in  $\mathbb{R}[\boldsymbol{x}_n]$ , then  $U \setminus \text{Zero}(f) = U \setminus \bigcup_{t=1}^{j} \text{Zero}(f_t)$  is a connected open set. The proposition is proved.  $\Box$ 

**Theorem 33.** Given a positive integer  $n \geq 2$ . Let  $f \in \mathcal{R}[\boldsymbol{x}_n]$  be a non-zero squarefree polynomial and U be a connected component of  $\operatorname{Nproj}_2(f, x_n) \neq 0$  in  $\mathbb{R}^{n-1}$ . If the polynomials in  $\operatorname{Nproj}_1(f, x_n)$  are semi-definite on U, then f is delineable on  $V = U \setminus \bigcup_{h \in \operatorname{Nproj}_1(f, x_n)} \operatorname{Zero}(h)$ .

**Proof.** According to Theorem 19 and Theorem 20, f is delineable over the connected component of  $\operatorname{Res}(f, f'_{x_n}, x_n) \neq 0$ . By Proposition 32,  $V = U \setminus \bigcup_{h \in \operatorname{Nproj}_1(f, x_n)} \operatorname{Zero}(h)$  is a connected open set. Thus, f is delineable on V.  $\Box$ 

**Theorem 34.** Given a positive integer  $n \ge 2$ . Let  $f \in \mathcal{R}[X_n]$  be a squarefree polynomial of level n and U a connected open set of  $\operatorname{Nproj}_2(f, x_n) \neq 0$  in  $\mathbb{R}^{n-1}$ . The necessary and sufficient condition for  $f(\mathbf{x}_n)$  to be semi-definite on  $U \times \mathbb{R}$  is the following two conditions hold.

(1) The polynomials in  $Nproj_1(f, x_n)$  are semi-definite on U;

(2) There exists a point  $\alpha \in U \setminus \bigcup_{h \in \operatorname{Nproj}_1(f, x_n)} \operatorname{Zero}(h)$ ,  $f(\alpha, x_n)$  is semi-definite on  $\mathbb{R}$ .

**Proof.**  $\Longrightarrow$ : By Lemma 13, discrim $(f, x_n)$  is semi-definite on U. Thus by Proposition 30, the polynomials in Nproj<sub>1</sub> $(f, x_n)$  are semi-definite on U. It is obvious that  $f(\alpha, x_n)$  is semi-definite on  $\mathbb{R}$ .

 $\Leftarrow$ : If the polynomials in Nproj<sub>1</sub>( $f, x_n$ ) are semi-definite on U, by Theorem 33, f is delineable on the connected open set  $V = U \setminus \bigcup_{h \in Nproj_1(f, x_n)} \operatorname{Zero}(h)$ . From that  $f(\alpha, x_n)$  is semi-definite on  $\mathbb{R}$ , we know that  $f(\boldsymbol{x}_n)$  is semi-definite on  $U \times \mathbb{R}$ .  $\Box$ 

The following theorem is an easy corollary of the above theorem.

**Theorem 35.** Given a positive integer  $n \geq 2$ . Let  $f \in \mathcal{R}[\boldsymbol{x}_n]$  be a squarefree monic (under a suitable ordering) polynomial of level n, the necessary and sufficient condition for  $f(\boldsymbol{x}_n)$  to be positive semi-definite on  $\mathbb{R}^n$  is the following two conditions hold. (1) The polynomials in  $\text{Nproj}_1(f, x_n)$  are positive semi-definite on  $\mathbb{R}^{n-1}$ ;

(2) For every connected components U of  $\operatorname{Nproj}_2(f, x_n) \neq 0$ , there exists a point  $\alpha \in U$ , and  $\alpha$  is not a zero of any polynomial in  $\operatorname{Nproj}_1(f, x_n)$ , such that  $f(\alpha, x_n) \geq 0$  on  $\mathbb{R}$ .

Theorem 36. Algorithm 3 is correct.

**Proof.** By Proposition 31, we only need to consider the case that  $f(\boldsymbol{x}_n)$  is irreducible in  $\mathbb{Z}[\boldsymbol{x}_n]$ . When n = 1, it is obvious that Algorithm 3 is correct.

We prove the theorem by induction on the level of f. Now, suppose that Algorithm 3 is correct for every polynomial h of level less than or equal to n-1. If f is positive semi-definite on  $\mathbb{R}^n$ , by Theorem 35, the polynomials in  $\operatorname{Nproj}_1(f, x_n)$  are positive semi-definite on  $\mathbb{R}^{n-1}$ . By induction, Proineq returns true for all these polynomials. Since f is positive semi-definite,  $f(X_n^1) \geq 0$  for all sample points obtained in  $\operatorname{Proineq}(f)$ . Thus  $\operatorname{Proineq}(f)$  returns true. If f is not positive semi-definite, by Theorem 35, there are two possible cases.

(1) There exists at least one polynomial in Nproj<sub>1</sub> $(f, x_n)$  which is not positive semidefinite on  $\mathbb{R}^{n-1}$ . Since the level of this polynomial is less than n, for this case, by induction, Algorithm 3 returns false. (2) There exists a connected open set U of  $\operatorname{Nproj}_2(f, x_n) \neq 0$ , a point  $\alpha \in U$  where  $\alpha$  is not a zero of any polynomial in  $\operatorname{Nproj}_1(f, x_n)$  and a point  $a \in \mathbb{R}$  such that  $f(\alpha, a) < 0$ . By (1), we can assume the polynomials in  $\operatorname{Nproj}_1(f, x_n)$  are positive semi-definite on  $\mathbb{R}^{n-1}$ . So, by Theorem 33, f is delineable on  $V = U \setminus \bigcup_{h \in \operatorname{Nproj}_1(f, x_n)} \operatorname{Zero}(h)$ . Thus, for any  $\beta \in U$ , there exists a point  $b \in \mathbb{R}$  such that  $f(\beta, b) < 0$ . By the lifting property of Open CAD, in Algorithm 3, there exists a sample point  $X_{n-1}^0 \in C_{n-1}$  with  $X_{n-1}^0 \in V$ . Thus there exists  $c \in \mathbb{R}$  such that  $(X_{n-1}^0, c) \in C_n$ ,  $f(X_{n-1}^0, c) < 0$ . Algorithm 3 returns false in this case.  $\Box$ 

#### 6. Solving Problems 2 and 3

# 6.1. Problem 3

Recall that Problem 3 is: For  $f \in \mathbb{R}[\boldsymbol{x}_n, k]$ , find all  $k_0 \in \mathbb{R}$  such that  $f(\boldsymbol{x}_n, k_0) \geq 0$  on  $\mathbb{R}^n$ . Since this is a typical QE problem, any CAD based methods can be applied. Under a suitable ordering on variables,  $e.g., k \prec x_1 \prec \cdots \prec x_n$ , by CAD projection, one can obtain a polynomial in k, say g(k). Assume  $k_1 < \cdots < k_m$  are the real roots of g(k) and  $p_j \in (k_{j-1}, k_j)(1 \leq j \leq m+1)$  are rational sample points in the m+1 intervals where  $k_0 = -\infty, k_{m+1} = +\infty$ . Then checking whether or not  $f(\boldsymbol{x}_n, k_i) \geq 0(1 \leq i \leq m)$  and  $f(\boldsymbol{x}_n, p_j) \geq 0(1 \leq j \leq m+1)$  on  $\mathbb{R}^n$  will give the answer. Namely, if there exist  $p_j$  such that  $f(\boldsymbol{x}_n, p_j) \geq 0$  then  $(k_{j-1}, k_j)$  should be output. If  $f(\boldsymbol{x}_n, k_i) \geq 0$  for some  $k_i, \{k_i\}$  should be output.

Thus, a natural idea for improving efficiency is to apply the new projection operator Nproj instead of Brown's projection in the above procedure. In this subsection, we first show by an example why Nproj cannot be applied directly to Problem 3. Then we propose an algorithm based on Nproj for solving Problem 3 and prove its correctness.

**Example 6.1.** Find all  $k \in \mathbb{R}$  such that

$$(\forall x, y \in \mathbb{R}) f(x, y, k) = x^2 + y^2 - k^2 \ge 0.$$

If we apply Nproj directly (with an ordering  $k \prec x \prec y$ ), we will get

$$Nproj(f) = (\{f(x, y, k), x - k, x + k, 1\}, \{1\}).$$

Because  $L_2 = \{1\}$ , there is only one sample point with respect to k, say  $k_0 = 0$ . Substituting  $k_0$  for k in f(x, y, k), we check whether  $(\forall x, y \in \mathbb{R})x^2 + y^2 \ge 0$ . This is obviously true. So, it leads to a wrong result:  $(\forall k, x, y \in \mathbb{R})x^2 + y^2 - k^2 \ge 0$ .

The reason for the error is that (x - k)(x + k) will be a square if k = 0. The point k = 0 can be found by computing the resultant Res(x - k, x + k, x) which is avoided by Nproj since  $x - k \in L_1$  and  $x + k \in L_1$ .

This example indicates that, if we use Nproj to solve Problem 3, we have to consider some "bad" values of k at which some odd factors of  $\operatorname{sqrfree}_1(f)$  may become some new even factors. In the following, we first show that such "bad" values of k are finite and propose an algorithm for computing all possible "bad" values. Then we give an algorithm for solving Problem 3, which handles the "bad" values and the "good" values of k obtained by Nproj separately. **Definition 37.** Let  $f(\boldsymbol{x}_n, k) \in \mathbb{Z}[\boldsymbol{x}_n, k]$  and  $(L_1, L_2) = \operatorname{Nproj}(f(\boldsymbol{x}_n, k))$  with the ordering  $k \prec x_1 \prec \cdots \prec x_n$ . If  $\alpha \in \mathbb{R}$  satisfying that

- (1) there exist two different polynomials  $g_1, g_2 \in \text{sqrfree}_1(f(\boldsymbol{x}_n, k))$  such that  $g_1|_{k=\alpha}$ and  $g_2|_{k=\alpha}$  have non-trivial common factors in  $\mathbb{R}[\boldsymbol{x}_n]$ ; or
- (2) there exist an  $i(2 \le i \le n)$ , a polynomial  $g \in L_1^i$  and two different polynomials  $g_1, g_2 \in \operatorname{Nproj}_1(g, x_i)$  such that  $g_1|_{k=\alpha}$  and  $g_2|_{k=\alpha}$  have non-trivial common factors in  $\mathbb{R}[\boldsymbol{x}_n]$ ; or
- (3) there exists a polynomial  $g \in L_1$  such that  $g|_{k=\alpha}$  has non-trivial square factors in  $\mathbb{R}[\boldsymbol{x}_n]$ ,

then  $\alpha$  is called a *bad value* of k. The set of all the bad values is denoted by Bad(f, k).

For two coprime multivariate polynomials with parametric coefficients, the problem of finding all parameter values such that the two polynomials have non-trivial common factors at those parameter values is very interesting. We believe that there should have existed some work on this problem. However, we do not find such work in the literature. So, we use an algorithm in (Qian, 2013). The detail of improvements on the algorithm is omitted.

# Algorithm 5. BK

**Input:** Two coprime polynomials  $f(\boldsymbol{x}_n, k), g(\boldsymbol{x}_n, k) \in \mathbb{Z}[\boldsymbol{x}_n, k]$  and k. **Output:** B, a finite set of polynomials in k.

1:  $B := \emptyset$ ; 2:  $r := \operatorname{Res}(f, g, k)$ ; Let S be the set of all irreducible factors of r. 3: Let  $X = \operatorname{indets}(S)$ ; (X is the set of variables appearing in S) 4: while  $X \neq \emptyset$  do 5: Choose a variable  $x \in X$  such that the cardinal number of 6:  $T = \{p \in S \mid x \text{ appears in } p\}$  is the biggest; 7:  $h := \operatorname{Res}(f, g, x)$ ; 8:  $B := B \cup \{q(k) \mid q(k) \text{ is irreducible and divides } h\}$ ; 9:  $S := S \setminus T$ ;  $X := \operatorname{indets}(S)$ ; 10: end while 11: return B

It is not hard to prove the following lemmas.

**Lemma 38.** (*Qian, 2013*)  $\mathsf{BK}(f, g, k) \supseteq \{\alpha \in \mathbb{R} | \gcd(f(\boldsymbol{x}_n, \alpha), g(\boldsymbol{x}_n, \alpha)) \text{ is non-trivial} \}.$ 

# Lemma 39. Let notations be as in Algorithm 6.

- (1) The first two outputs,  $L_1$  and  $L_2$ , are the same as  $\operatorname{Nproj}(f(\boldsymbol{x}_n, k))$  with the ordering  $k \prec x_1 \prec \cdots \prec x_n$ .
- (2)  $\bigcup_{h \in B} \operatorname{Zero}(h) \supseteq \operatorname{Bad}(f)$ . Thus,  $\operatorname{Bad}(f,k)$  is finite.
- (3) If  $k_0$  is not a bad value and  $f(\boldsymbol{x}_n, k_0) \ge 0$  on  $\mathbb{R}^n$ , then for any  $h \in \operatorname{sqrfree}_1(f)$ ,  $h(\boldsymbol{x}_n, k_0)$  is semi-definite on  $\mathbb{R}^n$ .

**Proof.** (1) and (2) are obvious. For (3), because  $k_0 \notin \text{Bad}(f)$ ,  $g_1(\boldsymbol{x}_n, k_0)$  and  $g_2(\boldsymbol{x}_n, k_0)$  are coprime in  $\mathbb{Z}[\boldsymbol{x}_n]$  for any  $g_1 \neq g_2 \in \text{sqrfree}_1(f)$ . Since  $f(\boldsymbol{x}_n, k_0) \geq 0$ , by Proposition 30, for any  $h \in \text{sqrfree}_1(f(\boldsymbol{x}_n, k))$ ,  $h(\boldsymbol{x}_n, k_0)$  is semi-definite on  $\mathbb{R}^n$ .  $\Box$ 

#### Algorithm 6. NKproj

**Input:** A polynomial  $f(\boldsymbol{x}_n, k) \in \mathbb{Z}[\boldsymbol{x}_n]$  and an ordering  $k \prec x_1 \prec \cdots \prec x_n$ . **Output:** Two projection factor sets as in Algorithm 4 and a set of polynomials in k. 1:  $L_1 := \operatorname{sqrfree}_1(f); L_2 := \{\}; B := \emptyset;$ 2: for i from n down to 1 do  $L_2 := L_2 \bigcup \operatorname{Nproj}_2(L_1^i, x_i) \bigcup \cup_{g \in L_2^i} \operatorname{Bproj}(g, x_i);$ 3: for  $h \in L_1^i$  do 4: 
$$\begin{split} L_{1h} &:= \operatorname{Nproj}_1(h, x_i); \\ B &:= B \cup \operatorname{BK}(h, \frac{\partial}{\partial x_i}h, k); \end{split}$$
5:6:  $B := B \bigcup \cup_{h_1 \neq h_2 \in L_{1h}} \mathsf{BK}(h_1, h_2, k);$ 7:  $L_1 := L_1 \cup L_{1h};$ 8: end for 9: 10: end for 11: return  $(L_1, L_2, B)$ .

**Lemma 40.** Let  $f(\boldsymbol{x}_n, k) \in \mathbb{Z}[\boldsymbol{x}_n, k]$  and  $(L_1, L_2, B) = \operatorname{NKproj}(f)$ . Suppose  $\operatorname{Zero}(L_1^0 \cup L_2^0) = \bigcup_{i=1}^m \{k_i\}$  with  $k_1 < \cdots < k_m$ ,  $k_0 = -\infty$ ,  $k_{m+1} = +\infty$  and, for every  $l(1 \leq l \leq m+1)$ ,  $p_l \in (k_{l-1}, k_l) \setminus \bigcup_{h \in B} \operatorname{Zero}(h)$ . Denote by  $C_{li}$  an Open CAD of  $\mathbb{R}^i$  defined by  $\bigcup_{j=1}^i L_1^j \mid_{k=p_l} \bigcup \bigcup_{j=1}^j L_2^j \mid_{k=p_l}$ . If there exists an  $l(1 \leq l \leq m+1)$ , such that (1)  $\forall X_n \in C_{ln}$ ,  $f(X_n, p_l) \geq 0$ ; and (2)  $\forall i(0 \leq i \leq n-1) \forall g \in L_1^i \forall X_i^1, X_i^2 \in C_{li}$ ,  $g(X_i^1, p_l)g(X_i^2, p_l) \geq 0$ , then for any  $0 \leq i \leq n$  and  $g_i(\boldsymbol{x}_i, k)$  in  $L_1^i$ ,  $g_i(\boldsymbol{x}_i, k)$  is semi-definite on  $\mathbb{R}^i \times (k_{l-1}, k_l)$ .

**Proof.** We prove it by induction on *i*. When i = 0, the conclusion is obvious. When i = 1, by Theorem 34, it is also true. Assume the conclusion is true when  $i = j - 1(j \ge 2)$ . For any polynomial  $g_j(\boldsymbol{x}_j, k)$  in  $L_1^j$ , notice that  $\operatorname{Nproj}_1(g_j) \subseteq L_1^{j-1}$ ,  $\operatorname{Nproj}_2(g_j) \subseteq L_2^{j-1}$ . By the assumption of induction, we know that every polynomial in  $\operatorname{Nproj}_1(g_j)$  is semidefinite on  $\mathbb{R}^{j-1} \times (k_{l-1}, k_l)$ . By Theorem 34,  $g_j(\boldsymbol{x}_j, k)$  is semi-definite on  $\mathbb{R}^j \times (k_{l-1}, k_l)$ . That finishes the induction.  $\Box$ 

**Theorem 41.** The output of Algorithm 7,  $FK_f$ , is  $\{\alpha \in \mathbb{R} | \forall X_n \in \mathbb{R}^n, f(X_n, \alpha) \ge 0\}$ .

**Proof.** Denote  $\{\alpha \in \mathbb{R} | \forall X_n \in \mathbb{R}^n, f(X_n, \alpha) \ge 0\}$  by  $K_f$ .

We first prove that  $FK_f \subseteq K_f$ . Suppose  $(k_{l-1}, k_l) \subseteq FK_f$ . Since  $\operatorname{sqrfree}_1(f) \in L_1^n$ , the semi-definiteness of f on  $\mathbb{R}^n \times (k_{l-1}, k_l)$  follows from Lemma 40. Because we check the positive definiteness of f on sample points,  $(k_{l-1}, k_l) \subseteq K_f$ .

We then prove that  $K_f \subseteq FK_f$ . It is sufficient to prove that if there exists  $k' \in (k_{l-1}, k_l) \setminus \bigcup_{h \in B} \operatorname{Zero}(h)$  such that  $\forall X_n \in \mathbb{R}^n, f(X_n, k') \ge 0$ , then  $(k_{l-1}, k_l) \in FK_f$ .

It is obviously true when n = 1. When  $n \ge 2$ , for any  $g_n \in \text{sqrfree}_1(f)$ , since  $f(\boldsymbol{x}_n, k')$  is semi-definite and  $k' \notin \bigcup_{h \in B} \text{Zero}(h)$ ,  $g_n(\boldsymbol{x}_n, k')$  is semi-definite by Lemma 39. For any

$$g_{n-1}(\boldsymbol{x}_{n-1},k) \in \operatorname{Nproj}_1(g_n(\boldsymbol{x}_n,k),x_n) = \operatorname{Oc}(g_n,x_n) \cup \operatorname{Od}(g_n,x_n),$$

we have

 $\operatorname{sqrfree}_1(g_{n-1}(\boldsymbol{x}_{n-1},k')) \subseteq \operatorname{Nproj}_1(g_n(\boldsymbol{x}_n,k'),x_n)$ 

because  $k' \notin \bigcup_{h \in B} \operatorname{Zero}(h)$ . By Theorem 35,  $g_{n-1}(\boldsymbol{x}_{n-1},k')$  is semi-definite on  $\mathbb{R}^{n-1}$ . Hence, for any polynomial  $g_{n-1}(\boldsymbol{x}_{n-1},k)$  in  $L_1^{n-1}$ ,  $g_{n-1}(\boldsymbol{x}_{n-1},k')$  is semi-definite. In a Algorithm 7. Findk **Input:** A polynomial  $f(\boldsymbol{x}_n, k) \in \mathbb{Z}[\boldsymbol{x}_n, k]$ . **Output:** A set  $FK_f$ . 1:  $FK_f := \{\};$ 2:  $(L_1, L_2, B) := \mathsf{NKproj}(f)$ ; (with an ordering  $k \prec x_1 \prec \cdots \prec x_n$ ) 3: Suppose  $\operatorname{Zero}(L_1^0 \cup L_2^0) = \bigcup_{i=1}^m \{k_i\}$  with  $k_1 < \cdots < k_m$ . Let  $k_0 = -\infty$ ,  $k_{m+1} = +\infty$ . 4: for l from 1 to m + 1 do Choose a sample point  $p_l \in (k_{l-1}, k_l) \setminus \bigcup_{h \in B} \operatorname{Zero}(h);$ 5: 6: v := 1;for i from 1 to n do 7:  $C_{li}$ :=an Open CAD of  $\mathbb{R}^i$  defined by  $\bigcup_{j=1}^i L_1^j |_{k=p_l} \bigcup \bigcup_{j=1}^i L_2^j |_{k=p_l}$ ; 8: if i = n and there exists  $X_n \in C_{ln}$  such that  $f(X_n, p_l) < 0$  then 9: v := 0: 10: else 11: if there exist  $X_i^1, X_i^2 \in C_{li}$  and  $g \in L_1^i$  such that  $g(X_i^1, p_l)g(X_i^2, p_l) < 0$ 12:then v := 0;13:break 14:end if 15:end if 16:end for 17:if v = 1 then 18: $FK_f := FK_f \bigcup (k_{l-1}, k_l);$ 19:20:end if 21: end for 22: for  $\alpha$  in  $\{k_1, \ldots, k_m\} \cup \bigcup_{h \in B} \operatorname{Zero}(h) \setminus FK_f$  do if  $Proineq(f(\boldsymbol{x}_n, \alpha)) =$ true then 23: 24: $FK_f := FK_f \bigcup \{\alpha\};$ end if 25:26: end for 27: return  $FK_f$ 

similar way, we know that for any  $1 \leq j \leq n-1$  and any polynomial  $g_j(\boldsymbol{x}_j, k)$  in  $L_1^j$ ,  $g_j(\boldsymbol{x}_j, k')$  is semi-definite on  $\mathbb{R}^j$ . Therefore, for any  $0 \leq i \leq n$  and any polynomial  $g_i(\boldsymbol{x}_i, k)$  in  $L_1^i$ ,  $g_i(\boldsymbol{x}_i, k)$  is semi-definite on  $\mathbb{R}^i \times (k_{l-1}, k_l)$  by Lemma 40. Hence, no matter what point  $p_l \in (k_{l-1}, k_l)$  is chosen as the sample point of this open interval,  $(k_{l-1}, k_l)$  will be in the output of Algorithm 7, *i.e.*,  $(k_{l-1}, k_l) \in FK_f$ . The proof is completed.  $\Box$ 

# 6.2. Problem 2

For solving the global optimum problem (Problem 2), we only need to modify the algorithm Findk a little and get the algorithm Findinf.

**Theorem 42.** The output of Algorithm 8 is the global infimum  $\inf f(\mathbb{R}^n)$ .

**Proof.** We only need to prove that if there exists  $k' \in (k_{l-1}, k_l)$  such that  $f(\boldsymbol{x}_n) \geq k'$  on  $\mathbb{R}^n$ , then  $f(\boldsymbol{x}_n) \geq k_l$  on  $\mathbb{R}^n$ .

Algorithm 8. Findinf **Input:** A squarefree polynomial  $f \in \mathbb{Z}[\boldsymbol{x}_n]$ . **Output:**  $k \in \mathbb{R}$  such that  $k = \inf_{\boldsymbol{x}_n \in \mathbb{R}^n} f(\boldsymbol{x}_n)$ . 1:  $(LI_1, LI_2) := \operatorname{Nproj}(f(\boldsymbol{x}_n) - k)$  (with an ordering  $k \prec x_1 \prec \cdots \prec x_n$ ) 2: Suppose  $\operatorname{Zero}(LI_1^0 \cup LI_2^0) = \bigcup_{i=1}^m \{k_i\}$  with  $k_1 < \cdots < k_m$ . Let  $k_0 = -\infty$ ,  $k_{m+1} = +\infty$ . 3: for l from 1 to m + 1 do Choose a sample point  $p_l$  of  $(k_{l-1}, k_l)$ 4: v := 1;5: 6: for i from 1 to n do  $C_{li}$ :=an Open CAD of  $\mathbb{R}^i$  defined by  $\cup_{j=1}^i LI_1^j |_{k=p_l} \bigcup \cup_{j=1}^i LI_2^j |_{k=p_l}$ ; if i = n and there exists  $X_n \in C_{ln}$  such that  $f(X_n) - p_l < 0$  then 7:8: v := 0;9: 10: else if there exist  $X_i^1, X_i^2 \in C_{li}, g \in LI_1^i$  such that  $g(X_i^1, p_l)g(X_i^2, p_l) < 0$  then 11: 12: v := 0;break 13:end if 14:end if 15:end for 16:17:if v = 0 then return  $k_{l-1}$ 18:end if 19: 20: end for

The result is obviously true when n = 1. When  $n \ge 2$ , we can find a "good" value  $k'' \in (k_{l-1}, k') \setminus \text{Bad}(f - k, k)$  because the bad values are finite according to Lemma 39. Since  $f(\boldsymbol{x}_n) \ge \bar{k}$  for  $\bar{k} \in (k_{l-1}, k')$ ,  $f(\boldsymbol{x}_n) - k'' \ge 0$ . Then, by Lemma 39 (3),  $h(\boldsymbol{x}_n, k'')$  is semi-definite on  $\mathbb{R}^n$  for any  $h \in \text{sqrfree}_1(f(\boldsymbol{x}_n) - k)$ . In a similar way, we know that for any  $1 \le j \le n - 1$  and any polynomial  $g_j(\boldsymbol{x}_j, k)$  in  $LI_1^j, g_j(\boldsymbol{x}_j, k'')$  is semi-definite on  $\mathbb{R}^j$ . Therefore, for any  $0 \le i \le n$  and any polynomial  $g_i(\boldsymbol{x}_i, k)$  in  $LI_1^i, g_i(\boldsymbol{x}_i, k)$  is semi-definite on  $\mathbb{R}^i \times (k_{l-1}, k_l)$  by Lemma 40. Hence,  $f(\boldsymbol{x}_n) - k$  is positive semi-definite on  $\mathbb{R}^n \times (k_{l-1}, k_l)$  by Theorem 34.  $\Box$ 

**Remark 43.** For  $f, g \in \mathbb{R}[\boldsymbol{x}_n]$ , if  $g(\boldsymbol{x}_n) \geq 0$  on  $\mathbb{R}^n$ , Algorithm Findinf can also be applied to compute  $\inf\{\frac{f(\boldsymbol{x}_n)}{g(\boldsymbol{x}_n)} | \boldsymbol{x}_n \in \mathbb{R}^n\}$ . We just need to replace  $(LI_1, LI_2) := \operatorname{Nproj}(f(\boldsymbol{x}_n) - k)$  of Line 1 by  $(LI_1, LI_2) := \operatorname{Nproj}(f(\boldsymbol{x}_n) - kg(\boldsymbol{x}_n))$ .

## 7. Examples

We haven't made any complexity analysis on our new algorithms. We believe that the complexity is still doubly exponential but we do not know how to prove it yet. In this section, we report the performance of Algorithms Findinf, Findk and Proineq on several non-trivial examples. Since our main contribution is an improvement on the CAD projection for solving those three special problems, we only make some comparison with other CAD based tools on these examples. Algorithm Findinf will be compared with the algorithm SRes. The program Proineq we implemented using Maple will be compared with the function PartialCylindricalAlgebraicDecomposition (PCAD) of RegularChains package in Maple15, function FindInstance in Mathematica9, and QEPCAD B.

Because we do not have Mathematica and QEPCAD B installed in our computer, we ask others' help. So the computations were performed on different computers. FindInstance (FI) was performed on a laptop with Inter Core(TM) i5-3317U 1.70GHz CPU and 4GB RAM. QEPCAD B (QEPCAD) was performed on a laptop with Intel(R) Core(TM) i5 3.20GHz CPU and 4GB RAM. The other computations were performed on a laptop with Inter Core2 2.10GHz CPU and 2GB RAM.

We show the different results of projection of Algorithm Findinf and Algorithm SRes by Example 7.1.

**Example 7.1.** Compute  $\inf_{x,y,z \in \mathbb{R}} G(x, y, z)$ , where

$$G = \frac{(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1)}{(xyz)^2 - xyz + 1}.$$

Let  $f = (x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1), g = (xyz)^2 - xyz + 1$ . Since  $g \ge 0$  for any  $x, y, z \in \mathbb{R}$ , this problem can be solved either by Algorithm Findinf or by Algorithm SRes.

If we apply Algorithm Findinf, after Nproj(f - kg) with an ordering  $k \prec z \prec y \prec x$ , we will get a polynomial in k,

$$P = (k - \frac{2}{3})(k^2 + 6k - 3)(k - \frac{279}{256})k(k - 1)(k - \frac{3}{4})(k - \frac{9}{16})(k - 9),$$

which has 9 distinct real roots. After sampling and checking signs, we finally know that the maximum k is the real root of  $k^2 + 6k - 3$  in  $(\frac{1}{4}, \frac{1}{2})$ .

If we apply Algorithm SRes, after Bprojection(f - kg) with an ordering  $k \prec z \prec y \prec x$ , we will get a polynomial in k,

$$F = \frac{1}{614656} (614656k^4 - 4409856k^3 + 11013408k^2 - 11477376k + 4021893) (k^4 - 294k^3 + 1425k^2 - 2277k + 1089)(k - \frac{9}{4}) \cdot P,$$

which has 14 distinct real roots. After sampling and checking signs, we finally know that the maximum k is the real root of  $k^2 + 6k - 3$  in  $(\frac{1}{4}, \frac{1}{2})$ .

Obviously, the scale of projection with the new projection is smaller. The polynomial in k calculated through the successive resultant method has three extraneous factors.

**Example 7.2.** (Han, 2011) Prove

$$F(\boldsymbol{x}_n, n) = \prod_{i=1}^n (x_i^2 + n - 1) - n^{n-2} (\sum_{i=1}^n x_i)^2 \ge 0 \text{ on } \mathbb{R}^n.$$

When n = 3, 4, 5, 6, 7, we compared Proineq, FI, PCAD, QEPCAD in the following table. Hereafter >3000 means either the running time is over 3000 seconds or the software is failure to get an answer.

$n \neq \text{Time}(s)$	Proineq	FI	PCAD	QEPCAD
3	0.063	0.015	0.078	0.020
4	0.422	0.062	0.250	0.024
5	0.875	2.312	2.282	0.372
6	4.188	>3000	>3000	>3000
7	>3000	>3000	>3000	>3000

When  $n = 3, 4, 5, \overline{6}$ , we compared the number of polynomials in the projection sets of Bproj with Nproj(under the same ordering) as well as the number of sample points need to be chosen through the lifting phase under these two projection operators.

$\overline{n}$	Bprojection		Nproj	
	# polys	# points	# polys	# points
3	11	4	8	3
4	22	10	12	3
5	88	36	18	5
6	Unknown	Unknown	32	15

**Example 7.3.** Decide the nonnegativity of G(n, k)

$$G(n,k) = \left(\sum_{i=1}^{n} x_i^2\right)^2 - k \sum_{i=1}^{n} x_i^3 x_{i+1},$$

where  $x_{n+1} = x_1$ .

In the following table, (T) means that the corresponding program outputs  $G(n,k) \ge 0$ on  $\mathbb{R}^n$ . (F) means the converse.

$(n,k) \neq \text{Time}(s)$	Proineq	FI	PCAD	QEPCAD
(3,3)	0.047(T)	0.031(T)	0.078(T)	0.032(T)
(4, 3)	0.171(T)	284.484(T)	0.891(T)	196.996(T)
(5, 3)	244.188(T)	>3000	>3000	>3000
(6, 3)	>3000	>3000	>3000	>3000
$(4, k_1)$	13.782(F)	5638.656(F)	24.656(F)	>3000

where  $k_1 = \frac{227912108939855024517609}{75557863725914323419136}$ . By Algorithm Findk, for the case n = 4, we can find the maximum value of k satisfying the following inequality

$$(\forall (x_1, x_2, x_3, x_4) \in \mathbb{R}^4) \quad G(4, k) = (\sum_{i=1}^4 x_i^2)^2 - k \sum_{i=1}^4 x_i^3 x_{i+1} \ge 0$$

is the real root in  $(3, \frac{7}{2})$  of

$$800000k^8 - 29520000k^6 + 311367675k^4 - 422100992k^2 - 5183373312 = 0.$$

n	Bprojection		Proineq	
	# polys	# points	# polys	# points
3	5	10	4	5
4	6	4	5	2
5	Unknown	Unknown	16	20

The following example was once studied by Parrilo (2000).

# Example 7.4.

$\forall X_{3m+2} \in \mathbb{R}^{3m+2}, B(x) = \left(\sum_{i=1}^{3m+2} x_i^2\right)^2 - 2\sum_{i=1}^{3m+2} x_i^2 \sum_{j=1}^m x_{i+3j+1}^2 \ge \frac{1}{2} \sum_{i=1}^m x_i^2 \sum_{j=1}^m x_{j+1}^2 = \frac{1}{2} \sum_{i=1}^m x_{i+1}^2 = \frac{1}{2} \sum_{i=1}^m x_{i+1}^2 = \frac{1}{2} \sum_{j=1}^m x_{j+1}^2 = \frac{1}{2} \sum_{i=1}^m x_{i+1}^2 =$					0,	
	$3m + 2 \neq \text{Time(s)}$	Proineq	FI	PCAD	QEPCAD	
	5	0.297	0.109	0.265	0.104	
	8	27.218	>3000	>3000	>3000	
	11	>3000	>3000	>3000	>3000	
3m + 2	Bprojection			Proine	q	
	# polys	# p	oints	# poly	s	# points
5	13	96		10		88
8	Unknown	Unk	nown	27		6720

The above examples demonstrate that in terms of proving inequalities, among CAD based methods, Algorithm **Proineq** is efficient and our new algorithms can work out some examples which could not be solved by other existing general CAD tools.

Further improvements on the projection and lifting phase are our ongoing work.

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## References

Arnon D, Collins G E, McCallum S: Cylindrical algebraic decomposition I: The basic algorithm. SIAM Journal of Computing, 13(04): 865–877, 1984.

- Artin E: Über die Zerlegung definiter Funktionen in Quadrate, Abh. Math. Sem. Hamburg Univ. 5 (1927), 100–115.
- Basu S, Pollack R, Roy M F: Complexity of computing semi-algebraic descriptions of the connected components of a semi-algebraic set. Proceedings of the 1998 international symposium on Symbolic and algebraic computation. ACM, 1998: 25–29.
- Basu S, Pollack R D, Roy M F: Algorithms in real algebraic geometry. Springer, 2006.
- Berg C, Maserick P H: Polynomially positive definite sequences. Math Ann, 1982, 259: 487–495.
- Bernstein S: Sur la représentation des polynômes positif. Soobshch Har'k Mat Obshch, 1915, 2: 227–228.
- Bochnak J, Coste M, Roy M F: Real algebraic geometry. Berlin: Springer, 1998.
- Brown C W: Improved projection for cylindrical algebraic decomposition. J.Symbolic Computation, 2001,32: 447–465.
- Brown C W, Davenport J H: The complexity of quantifier elimination and cylindrical algebraic decomposition, Proceedings of the 2007 international symposium on Symbolic and algebraic computation. ACM, 2007: 54–60.
- Castle M, Powers V, Reznick B: Pólyas theorem with zeros. J Symbolic Comput, 2011, 46: 1039–1048.
- Collins G E: Quantifier Elimination for the Elementary Theory of Real Closed Fields by Cylindrical Algebraic Decomposition. Lect. Notes Comput. Sci. 33, 134–183, 1975.
- Collins G E: Quantifier Elimination by Cylindrical Algebraic Decomposition–Twenty Years of Progress. In Quantifier Elimination and Cylindrical Algebraic Decomposition (Caviness B F and Johnson J R Eds.). New York: Springer-Verlag, 8–23, 1998.
- Collins G E and Hong H: Partial Cylindrical Algebraic Decomposition for Quantifier Elimination. J. Symb. Comput. 12, 299–328, 1991.
- Cox D A, Little J, O'shea D: Using algebraic geometry. Springer Verlag, 2005.
- Davenport J H, Heintz J: Real quantifier elimination is doubly exponential. Journal of Symbolic Computation, 1988, 5(1): 29–35.
- Dolzmann A, Sturm T, Weispfenning V: Real quantifier elimination in practice. In: Algorithmic Algebra and Number Theory (B. H. Matzat et al. eds.), 221–247, Springer Berlin Heidelberg, 1999.
- M. J. Fischer and M. O. Rabin: Super-exponential complexity of Pressburger arithmetic. Complexity of Computation (AMS-SIAM Proceedings), 7: 27–41, 1974.
- Grigor'ev D Y, Vorobjov N N: Solving systems of polynomial inequalities in subexponential time. Journal of symbolic computation, 1988, 5(1): 37–64.
- Guo F, Safey El Din M, Zhi L: Global optimization of polynomials using generalized critical values and sums of squares. In: Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation. 107–114, 2010.
- Hà H V, Pham T S: Solving polynomial optimization problems via the truncated tangency variety and sums of squares. J. Pure Appl. Algebra 213 (11), 2167–2176, 2009. http://dx.doi.org/10.1016/j.jpaa.2009.03.014.
- Hägglöf, Lindberg P O, Stevenson L: Computing global minima to polynomial optimization problems using Gröbner bases. J Global Optimization, 1995, 7: 115–125.
- Han J J: An Introduction to the Proving of Elementary Inequalities. Harbin: Harbin Institute of Technology Press, 2011, 221.(in Chinese)
- Hanzon B, Jibetean D: Global minimization of a multivariate polynomial using matrix methods. J Global Optimization, 2003, 27: 1–23.

- Hardy G H, Littlewood J E, Pólya G: Inequalities. 2nd ed. Cambridge: Cambridge University Press, 1952.
- Heintz J, Roy M F, Solerno P: On the theoretical and practical complexity of the existential theory of reals. The Computer Journal, 1993, 36(5): 427–431.
- Hilbert D: Uber die Darstellung definiter Formen als Summe von Formenquadraten. Math Ann, 1888, 32: 342–350. http://dx.doi.org/10.1007/BF01443605.
- Hong H: An improvement of the projection operator in cylindrical algebraic decomposition. In: Proceedings of the International Symposium on Symbolic and Algebraic Computation, 261–264, 1990.
- Jibetean D and Laurent M: Semidefinite approximations for global unconstrained polynomial optimization. SIAM J. on Optimization, 16(2):490–514, 2005.
- Kurdyka K, Orro P, Simon S: Semialgebraic Sard theorem for generalized critical values. Journal of differential geometry, 2000, 56(1): 67–92.
- Lasserre J: Global optimization with polynomials and the problem of moments. SIAM J. on Optimization, 11(3), 796–817 (2001).
- Marshall M: Positive polynomials and sums of squares. American Mathematical Soc., 3, 2008.
- McCallum S: An improved projection operation for cylindrical algebraic decomposition of three-dimensional space. J.Symbolic Computation., 1988, 5:141–161.
- McCallum S: An improved projection operation for cylindrical algebraic decomposition. In Caviness, B., Johnson, J. eds, Quantifier Elimination and Cylindrical Algebraic Decomposition, Texts and Monographs in Symbolic Computation. Vienna, Springer-Verlag, 242–268, 1998.
- McCallum S: Solving polynomial strict inequalities using cylindrical algebraic decomposition. Comput. J., 36, 432–438, 1993.
- Motzkin T S: Copositive quadratic forms. Natl Bur Stand Rep, 1952, 1818: 11-22.
- Motzkin T S, Strauss E G: Divisors of polynomials and power series with positive coeffcients. Pacific J Math, 1969, 29: 641–652.
- Nie J, Demmel J, Sturmfels B: Minimizing polynomials via sum of squares over the gradient ideal. Math. Program. 106 (3, Ser. A), 587–606, 2006. http://dx.doi.org/ 10.1007/s10107-005-0672-6.
- Parrilo P A: Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. Dissertation (Ph.D.), California Institute of Technology, 2000. http://resolver.caltech.edu/CaltechETD:etd-05062004-055516.
- Pólya G: Uber positive Darstellung von Polynomen. Vierteljahrschrift Naturforsch. Ges. Zürich 73, 1928, 141–145.
- Putinar M: Positive polynomials on compact semi-algebraic sets. Indiana Univ Math J, 1993, 42: 969–984.
- Qian X: Improvements on a simplified CAD projection operator with application to global optimization. Master thesis, Peking University, Beijing, 2013.
- Renegar J: On the computational complexity and geometry of the first-order theory of the reals. Part I: Introduction. Preliminaries. The geometry of semi-algebraic sets. The decision problem for the existential theory of the reals. Journal of symbolic computation, 1992, 13(3): 255–299.
- Safey El Din M: Computing the global optimum of a multivariate polynomial over the reals. ISSAC 2008 Proceedings, D. Jeffrey (eds), Austria (Hagenberg), 2008, 71–78.

- Scheiderer C: Positivity and Sums of Squares: A Guide to Recent Results, Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), IMA Volumes Math. Appl. 149, New York-Berlin-Heidelberg: Springer-Verlag, 2009, 271–324.
- Schweighofer M: Certificates for nonnegativity of polynomials with zeros on compact semi-algebraic sets. Manuscripta Math, 2005, 117: 407–428.
- Strzeboński A: Solving systems of strict polynomial inequalities. J. Symbolic Comput. 29, 471–480, 2000.
- Weiss E: Algebraic Number Theory. New York: McGrawHill, 1963: 166.
- Wu W T: On Global-Optimization Problems, Proc. ASCM'98, Lanzhou: Lanzhou University Press, 1998: 135–138.
- Wu W T: Mathematics Mechanization, Beijing: Science Press, 2003(in Chinese).
- Xia B: DISCOVERER: A tool for solving problems involving polynomial inequalities. In: Proc. ATCM'2000 (W-C Yang, et al. eds.), 472–481, ATCM Inc., Blacksburg, USA, Dec, 2000.
- Xiao R: Parametric Polynomial Systems Solving. Dissertation (Ph.D.), Peking University, 2009.
- Xiao S J, Zeng G X: Algorithms for computing the global infimum and minimum of a polynomial function (in Chinese). Sci Sin Math, 2011, 41(9): 759–788, doi:10.1360/012010-977.
- Xu J, Yao Y: Pólya's method and the successive difference substitution method (in Chinese). Sci Sin Math, 2012, 42(3): 203–213, doi:10.1360/012011-684.
- Yang L: Symbolic algorithm for global optimization and principle of finite kernel. In: Mathematics and Mathematical Mechanization (Lin D. ed.) (in Chinese), 210–220, 2001.
- Yang L: Solving Harder Problems with Lesser Mathematics. Proceedings of the 10th Asian Technology Conference in Mathematics, Blacksburg: ATCM Inc, 2005: 37–46.
- Yang L, Xia B C: Automated Proving and Discovering on Inequalities, Beijing: Science Press, 2008: 131–138 (in Chinese).
- Yao Y: Infinite product convergence of column stochastic mean matrix and machine decision for positive semi-definite forms (in Chinese). Sci Sin Math, 2010, 53(3): 251–264.