# **Barrier Certificates Revisited**

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### ABSTRACT

A barrier certificate can separate the state space of a considered hybrid system (HS) into safe and unsafe parts according to the safety property to be verified. Therefore this notion has been widely used in the verification of HSs. A stronger condition on barrier certificates means that less expressive barrier certificates can be synthesized. On the other hand, synthesizing more expressive barrier certificates often means high complexity. In [9], Kong et al considered how to relax the condition of barrier certificates while still keeping their convexity so that one can synthesize more expressive barrier certificates efficiently using semi-definite programming (SDP). In this paper, we first discuss how to relax the condition of barrier certificates in a general way, while still keeping their convexity. Particularly, one can then utilize different weaker conditions flexibly to synthesize different kinds of barrier certificates with more expressiveness efficiently using **SDP**. These barriers give more opportunities to verify the considered system. We also show how to combine two functions together to form a combined barrier certificate in order to prove a safety property under consideration, whereas neither of them can be used as a barrier certificate separately, even according to any relaxed condition. Another contribution of this paper is that we discuss how to discover certificates from the general relaxed condition by **SDP**. In particular, we focus on how to avoid the unsoundness because of numeric error caused by SDP with symbolic checking.

#### **Categories and Subject Descriptors**

F.3.1 [Specifying and Verifying and Reasoning about Programs]: Invariants

### **General Terms**

Theory

# Keywords

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Inductive invariant, barrier certificate, safety verification, hybrid system, nonlinear system, sum of squares

# 1. INTRODUCTION

Embedded systems make use of computer units to control physical devices so that the behavior of the controlled devices meets expected requirements. They have become ubiguitous in our modern life. How to design correct embedded systems is a grand challenge for computer science and control theory. Model-driven development (MDD) is considered as an effective way of developing correct complex embedded systems, and has been successfully applied in industry [6, 12]. In the framework of MDD, a formal model of the system to be developed is defined at the beginning; then extensive analysis and verification are conducted based on the formal model so that errors can be detected and corrected at the very early stage of the design of the system. Afterwards, model transformation techniques are applied to transform the abstract formal model into lower level models, even into source code. Hybrid systems (HSs) combine discrete mode changes with continuous evolutions specified in the form of differential equations. With mathematically precise semantics, HSs can serve as an appropriate model of embedded systems [14, 2].

In the past, analysis and verification of HSs are mainly done through directly computing reachable sets, either by modelchecking (e.g., [1, 22, 7]) or by decision procedures (e.g., [11]). The basic idea is to partition the state space of a considered system into finite many equivalent classes, or represent to finite many computable sets according to the solutions of the ODEs of the system. Since there is only a very small class of ODEs with closed form solutions, the scalability of these approaches is very restricted, only applicable to very specific linear HSs. To deal with more complicated systems, a deductive method has been recently proposed and successfully applied in practice [17, 18]. The most challenging part of a deductive method is how to discover invariants, which hold at all reachable states of the system. For technical reason, people only consider how to synthesize inductive invariants, which are preserved by all discrete and continuous transitions. In general, a safety property itself is an invariant, but not an inductive invariant. Obviously, an inductive invariant is an approximation of the reachable set, which may be discovered according to the ODEs, rather than their solutions. The basic idea is as follows: first, predefine a property template (linear or non-linear, depending on the property to be verified); then, encode the conditions of a

property to be inductive (discretely and/or continuously) into some constraints on state variables and parameters; finally, find out solutions to the constraints. So, how to define inductiveness conditions and the power of constraint solving are essential in these approaches.

Many approaches have been proposed following the line discussed above. E.g., in [8, 23], the authors independently proposed different approaches for constructing inductive invariants for linear HSs; S. Sankaranarayanan et al presented a computational method to automatically generate algebraic invariants for algebraic HSs in [24, 25], based on the theory of pseudo-ideal over polynomial ring and quantifier elimination; S. Prajna in [19, 20] provided a new notion of inductive invariants called *barrier certificates* for verifying the safety of semi-algebraic HSs with stochastic setting using the technique of sum-of-squares (SOS); while in [17], Platzer and Clarke extended the idea of *barrier certificates* by considering boolean combinations of multiple polynomial inequalities; In [4, 27], S. Gulwani et al investigated how to generate inductive invariants with more expressiveness for semialgebraic HSs by relaxing the inductiveness conditions by considering inductiveness on the boundaries of predefined invariant templates; while in [13], Liu et al considered how to further relax the inductiveness condition given in [4, 27]and first gave a complete method on how to generate semialgebraic invariants for semi-algebraic HSs. In [26], C. Solth at el proposed an approach to constructing global inductive invariant from local differential invariants using optimization technique.

The aforementioned approaches can be classified into two categories: symbolic computation based approaches like [8, 23, 24, 25, 17, 4, 27, 13], and numeric computation based approaches like [19, 20, 26]. In general, the former can synthesize more expressive invariants, but their efficiencies are very low; in contrast, the efficiency of the latter is very high, normally in *polynomial time* as only **SDP** is used, but the expressiveness of synthesized invariants is restrictive. In [9], Kong et al investigated how to synthesize more expressive barrier certificates by proposing *exponential barrier certificate condition*, which is a relaxed inductiveness condition, but still keeps the convexity of *barrier certificates*. Therefore, more expressive barrier certificates can be synthesized efficiently according to their condition still by **SDP**.

In this paper, firstly, following Kong et al's line, in the prerequisite of keeping the convexity of barrier certificates so that **SDP** is still applicable, we discuss how to relax the condition of barrier certificates in a general way. Thus, one can utilize different weaker conditions flexibly to synthesize different kinds of barrier certificates with more expressiveness efficiently, which gives more opportunities to verify the considered system. In addition, we consider how to combine two functions together to form a combined barrier certificate to prove a safety property under consideration, whereas neither of these two functions can be used as a barrier certificate separately, even according to any relaxed conditions. Another contribution of this paper is that we design algorithms to synthesize barrier certificates according to the general relaxed condition by **SDP**. In particular, we focus on how to avoid the unsoundness of our approach caused by numerical errors in **SDP**.

The rest of the paper is organized as follows: Section 2 introduces some basic notions; In Section 3, we discuss how to relax barrier certificate conditions, as well how to combine two functions to form a combined barrier certificate, but neither of them can be used as a barrier certificate separately; Section 4 is devoted to how to synthesize barrier certificates according to relaxed conditions discussed above based on **SDP**; Section 5 provides some case studies as well as experimental results. Finally, we conclude this paper in Section 6.

### 2. PRELIMINARIES

In this section, we first introduce some basic notions, and then explain the basic idea of barrier certificates.

In what follows, we use  $\mathbb{R}$  to stand for the set of reals,  $\mathcal{C}^{\omega}[\mathbb{R}^n]$  for the set of analytic function from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

### **2.1 Basic notions**

An autonomous continuous dynamical system (CDS) is represented by a differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where  $\mathbf{x} \in \mathbb{R}^n$ , and  $\mathbf{f}$  is a vector function, called *field vector*, whose components are in  $\mathcal{C}^{\omega}[\mathbb{R}^n]$ , and satisfy local Lipschitz condition<sup>1</sup>. In the context of HSs, a CDS is normally equipped with a domain  $D \subseteq \mathbb{R}^n$  defining its state space and an initial set of states  $\Xi$ .

In this paper, we use hybrid automata [1] to model HSs, more models of HSs can be found in [28].

DEFINITION 1 (HYBRID AUTOMATA). A hybrid automaton (HA) is a system  $\mathcal{H} \cong (Q, X, f, D, E, G, R, \Xi)$ , where

- $Q = \{q_1, \dots, q_m\}$  is a finite set of discrete states (or modes);
- $X = \{x_1, \ldots, x_n\}$  is a finite set of continuous state variables, with  $\mathbf{x} = (x_1, \ldots, x_n)$  ranging over  $\mathbb{R}^n$ ;
- f: Q → (ℝ<sup>n</sup> → ℝ<sup>n</sup>) assigns to each mode q ∈ Q a locally Lipschitz continuous vector field f<sub>q</sub>;
- D assigns to each mode  $q \in Q$  a mode domain  $D_q \subseteq \mathbb{R}^n$ ;
- $E \subseteq Q \times Q$  is a finite set of discrete transitions;
- G assigns to each transition  $e \in E$  a switching guard  $G_e \subseteq \mathbb{R}^n$ ;
- R assigns to each transition  $e \in E$  a reset function  $R_e$ :  $\mathbb{R}^n \to \mathbb{R}^n$ ;
- $\Xi$  assigns to each  $q \in Q$  a set of initial states  $\Xi_q \subseteq \mathbb{R}^n$ .

For ease of presentation, we make the following assumptions:

• for all  $q \in Q$ ,  $\mathbf{f}_q$  is a polynomial vector function, so it satisfies local Lipschitz condition, and thus the existence and uniqueness of solutions to  $\dot{\mathbf{x}} = \mathbf{f}_q$  is guaranteed;

<sup>&</sup>lt;sup>1</sup>Local Lipschitz condition guarantees the existence and uniqueness of the solution of (1) from any initial  $\mathbf{x}_0$ .

• for all  $q \in Q$  and all  $e \in E$ ,  $\Xi_q$  is a semi-algebraic set,  $D_q$  and  $G_e$  are closed semi-algebraic sets<sup>2</sup>.

Given an HA  $\mathcal{H}$ , a safety requirement S of  $\mathcal{H}$  assigns to each mode  $q \in Q$  a safe region  $S_q \subseteq \mathbb{R}^n$ , i.e.  $S = \bigcup_{q \in Q} (\{q\} \times S_q)$ . Dually,  $S^u = \bigcup_{q \in Q} (\{q\} \times (D_q - S_q))$  is called *unsafe set*. The *reachable set* of  $\mathcal{H}$ , denoted by  $\mathcal{R}_{\mathcal{H}}$ , consists of those  $(q, \mathbf{x})$ for which there exists a finite sequence

$$(q_0, \mathbf{x}_0), (q_1, \mathbf{x}_1), \dots, (q_l, \mathbf{x}_l)$$

s.t.  $(q_0, \mathbf{x}_0) \in \Xi_{\mathcal{H}}, (q_l, \mathbf{x}_l) = (q, \mathbf{x})$ , and for any  $0 \le i \le l-1$ , one of the following two conditions holds:

- (Discrete Jump):  $e = (q_i, q_{i+1}) \in E$ ,  $\mathbf{x}_i \in G_e$  and  $\mathbf{x}_{i+1} = R_e(\mathbf{x}_i)$ ; or
- (Continuous Evolution):  $q_i = q_{i+1}$ , and there exists a  $\delta \ge 0$  s.t. the solution  $\mathbf{x}(\mathbf{x}_i; t)$  to  $\dot{\mathbf{x}} = \mathbf{f}_{q_i}$  satisfies

$$- \mathbf{x}(\mathbf{x}_i; t) \in D_{q_i} \text{ for all } t \in [0, \delta]; \text{ and} - \mathbf{x}(\mathbf{x}_i; \delta) = \mathbf{x}_{i+1}.$$

### 2.2 Barrier certificates

Given an HS  $\mathcal{H}$  and a safety property S (dually, an unsafe set  $S^u$ ), the problem we considered is if  $\mathcal{R}_{\mathcal{H}} \subseteq S$  (dually,  $\mathcal{R}_{\mathcal{H}} \cap S^u = \emptyset$ ). Obviously, it is equivalent to  $\forall q \in Q.\mathcal{R}_{\mathcal{H}} \upharpoonright_q \subseteq$  $S_q$  (dually,  $\forall q \in Q.\mathcal{R}_{\mathcal{H}} \upharpoonright_q \cap S_q^u = \emptyset$ ), where  $\mathcal{R}_{\mathcal{H}} \upharpoonright_q$  stands for all continuous states of  $\mathcal{R}_{\mathcal{H}}$  projecting onto q. For this problem on CDSs, Prajna et al in [19, 20] used the idea of Lyapunov functions for stability analysis in control theory to separate safe states from unsafe states by a barrier function with convexity, called *barrier certificate*. According to their definition, a barrier function  $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n]$  satisfies the following conditions:

- i)  $\varphi(\mathbf{x}) \leq 0$  for any point  $\mathbf{x} \in \Xi_q$ ;
- ii)  $\varphi(\mathbf{x}) > 0$  for any point  $\mathbf{x} \in S_q^u$ ; and
- iii)  $\forall \mathbf{x} \in D_q.\mathcal{L}_{\mathbf{f}_q}\varphi(\mathbf{x}) \leq 0$ , where  $\mathcal{L}_{\mathbf{f}_q}\varphi(\mathbf{x}) = \frac{\partial \varphi}{\partial \mathbf{x}}\mathbf{f}_q(\mathbf{x})$  is the Lie derivative of  $\varphi$  with respect to the vector field  $\mathbf{f}_q$ .

Trivially to see, the existence of a barrier certificate is just a sufficient condition to guarantee the safety property to be verified. Hence, using Prajna et al's approach, one cannot claim the property does not hold if he/she fails to discover a polynomial barrier certificate. Actually, as observed in [9] by Kong et al, if condition iii) is relaxed to the following iii'), one can synthesize barrier certificates with more expressiveness. Certainly, it is more likely to prove a safety property by using a more expressive barrier certificate, as it gives a tighter approximation of the reachable set.

iii')  $\mathcal{L}_{\mathbf{f}_q}\varphi(\mathbf{x}) - \gamma\varphi(\mathbf{x}) \leq 0$ , where  $\gamma$  is a real number.

# 3. REVISITING BARRIER CERTIFICATE CONDITIONS

In this section, we investigate how to relax the condition of barrier certificates in a general way.

# 3.1 Relaxed barrier certificate conditions for CDSs

First of all, we consider how to relax the condition i)-iii) of barrier certificates given in [19, 20] for CDSs in a general way. To the end, we need to have a principle to justify when a relaxed condition of barrier certificates is reasonable. An obvious principle is:

**Principle of Barrier Certificate (PBC):** Given a CDS  $\mathcal{D}$  equipped with an initial set  $\Xi_0$  and an unsafe set  $S^u$ , a barrier certificate should be a real-valued function  $\varphi(\mathbf{x})$  such that  $\varphi(\mathbf{x}) \leq 0$  for any  $\mathbf{x} \in \mathcal{R}_{\mathcal{D}}$ , and  $\varphi(\mathbf{x}) > 0$  for any point  $\mathbf{x} \in S^u$ .

Certainly, if there exists such a function  $\varphi(\mathbf{x})$ , we can assert that  $\mathcal{R}_{\mathcal{D}} \cap S^u = \emptyset$ , and  $\phi(\mathbf{x}) \leq 0$  is an invariant. However, such a principle cannot be effectively checked in general, so we have to strengthen the condition to make it effectively checkable, like in [19, 20, 9]. An interesting problem is with which condition more expressive barrier certificates can be synthesized, but the condition is still effectively checkable and satisfies **PBC**. We answer the problem by the following theorem.

THEOREM 1 (GENERAL BARRIER CONDITION (**GBC**)). Given a CDS  $\mathcal{D}$  equipped with a domain D, an initial set  $\Xi_0$ and an unsafe set  $S^u$ , if there is a function  $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n]$ , a real function  $\psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}]$  such that

$$\forall \mathbf{x} \in \Xi_0.\varphi(\mathbf{x}) \le 0,\tag{2}$$

$$\forall \mathbf{x} \in D.\mathcal{L}_f \varphi(\mathbf{x}) - \psi(\varphi(\mathbf{x})) \le 0, \tag{3}$$

$$\forall \mathbf{x} \in S^u.\varphi(\mathbf{x}) > 0, \tag{4}$$

 $\xi > 0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0$ , where  $\theta(\mathbf{x}(t))$  is the solution of

$$\begin{cases} \theta(\mathbf{x}(0)) \le 0, \\ \mathcal{L}_{\mathbf{f}}\theta(\mathbf{x}) - \psi(\theta(\mathbf{x})) = 0, \end{cases}$$
(5)

then  $\mathcal{R}_{\mathcal{D}} \cap S^u = \emptyset$ .

PROOF. Suppose  $\mathbf{x}_0 \in \Xi_0$  and  $\mathbf{x}(t)$  is the corresponding solution of (1) starting from  $\mathbf{x}_0$ . Our goal is to prove that for any function  $\varphi(\mathbf{x}(t))$  satisfying (2)-(5), then

$$\forall \xi \ge 0.\varphi(\mathbf{x}(\xi)) \le 0. \tag{6}$$

Let  $g(\mathbf{x}) = \mathcal{L}_{\mathbf{f}}\varphi(\mathbf{x}) - \psi(\varphi(\mathbf{x}))$ , then by (3)

$$\forall \mathbf{x} \in \mathbb{R}^n . g(\mathbf{x}) \le 0 \tag{7}$$

Since  $\frac{d\varphi(\mathbf{x}(t))}{dt} = \frac{\partial\varphi}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \frac{\partial\varphi}{\partial \mathbf{x}} f(\mathbf{x}) = \mathcal{L}_f \varphi(\mathbf{x})$ , we have

$$\begin{cases} \frac{d\varphi(\mathbf{x}(t))}{dt} - \psi(\varphi(\mathbf{x}(t))) - g(\mathbf{x}(t)) = 0\\ \varphi(\mathbf{x}(0)) = \varphi(\mathbf{x}_0) \end{cases}$$
(8)

Assume  $\varphi(\mathbf{x}(\xi)) > 0$ , for some  $\xi > 0$ . Let  $\theta(\mathbf{x}(t))$  be a function with

$$\begin{pmatrix} \frac{d\theta(\mathbf{x}(t))}{dt} - \psi(\theta(\mathbf{x}(t))) = 0\\ \theta(\mathbf{x}(0)) = \varphi(\mathbf{x}_0) \end{pmatrix}$$
(9)

Let  $\Theta = \{\xi \mid \varphi(\mathbf{x}(\xi)) > \theta(\mathbf{x}(\xi)), \xi \ge 0\}$ . By (5),  $\forall \xi > 0.\theta(\mathbf{x}(\xi)) \le 0$ .  $\Theta$  is nonempty since the assumption. So there is a number  $\mu$  s.t.  $\mu = \inf(\Theta)$ . Obviously,  $\varphi(\mathbf{x}(t))$ ,  $\theta(\mathbf{x}(t)), g(\mathbf{x}(t)), \frac{d\varphi(\mathbf{x}(t))}{dt}$  and  $\frac{d\theta(\mathbf{x}(t))}{dt}$  are analytic functions

<sup>&</sup>lt;sup>2</sup>A subset  $A \subseteq \mathbb{R}^n$  is called *semi-algebraic* if there is a quantifier-free polynomial formula  $\varphi$  expressed in Tarski's algebra s.t.  $A = \{\mathbf{x} \in \mathbb{R}^n \mid \varphi(\mathbf{x}) \text{ is true}\}.$ 

w.r.t. t. Thus  $\varphi(\mathbf{x}(\mu)) = \theta(\mathbf{x}(\mu))$ . If  $g(\mathbf{x}(\mu)) < 0$ , then  $\frac{d\varphi(\mathbf{x}(t))}{dt}|_{t=\mu} < \frac{d\theta(\mathbf{x}(t))}{dt}|_{t=\mu}$ . Hence,  $\exists \nu.\nu > \mu \land \forall \xi \in (\mu,\nu)$ .  $\frac{d\varphi(\mathbf{x}(t))}{dt}|_{t=\xi} < \frac{d\theta(\mathbf{x}(t))}{dt}|_{t=\xi}$ . Thus,  $\forall \xi \in (\mu,\nu).\varphi(\mathbf{x}(\xi)) < \theta(\mathbf{x}(\xi))$ , which contradicts to the definition of  $\mu$ . So  $g(\mathbf{x}(\mu)) = 0$  and  $\frac{d\varphi(\mathbf{x}(t))}{dt}|_{t=\mu} = \frac{d\theta(\mathbf{x}(t))}{dt}|_{t=\mu}$ . If there is a k > 1 s.t.  $\frac{d^k\varphi(\mathbf{x}(t))}{dt^k}|_{t=\mu} < \frac{d^k\theta(\mathbf{x}(t))}{dt^k}|_{t=\mu}$ , and  $\forall i < k, \frac{d^i\varphi(\mathbf{x}(t))}{dt^i}|_{t=\mu} = \frac{d^i\theta(\mathbf{x}(t))}{dt^k}|_{t=\mu}$ , then there is  $\nu_1 > \mu$  s.t.  $\varphi(\mathbf{x}(\xi)) < \theta(\mathbf{x}(\xi))$  for any  $\xi \in (\mu, \nu_1)$ , which contradicts to the definition of  $\mu$ . If  $\forall k > 1$ .  $\frac{d^k\varphi(\mathbf{x}(t))}{dt^k}|_{t=\mu} = \frac{d^k\theta(\mathbf{x}(t))}{dt^k}|_{t=\mu}$ , then  $\varphi(\mathbf{x}(\xi)) = \theta(\mathbf{x}(\xi))$  for any  $\xi \in \mathbb{R}^+$ , since  $\varphi, \theta$  are analytic functions. So, the claim has been proved. Suppose for some k > 1,  $\frac{d^k\varphi(\mathbf{x}(t))}{dt^k}|_{t=\mu} > \frac{d^k\theta(\mathbf{x}(t))}{dt^k}|_{t=\mu}$ . For all i < k, we simultaneously compute the *i*th derivatives of the two sides of the first formulas of (8) and (9), and obtain  $\frac{d^i\psi(\varphi(\mathbf{x}(t)))}{dt^i}|_{t=\mu} = \frac{d^i\theta(\mathbf{x}(t))}{dt^i}|_{t=\mu} > 0$ . Thus, there is an  $\delta > \mu$  s.t.  $\forall \xi \in (\mu, \delta).g(\mathbf{x}(\xi)) > 0$ , which contradicts to the definition of  $g(\mathbf{x})$ . This completes the proof.  $\Box$ 

From now on, we call  $\varphi$  in Theorem 1 a *barrier certificate* of  $\mathcal{D}$ .

- REMARK 1. The application of Theorem 1 includes the following two steps: i) look for a function  $\psi$  which satisfies condition (5); ii) similar to the work in [9], synthesize barrier certificate according to the resulted conditions of (2)-(4) by instantiating  $\psi$  with the function obtained in the first step.
- All barrier certificates that can be synthesized using the existing approaches can also be synthesized according to these conditions by instantiating ψ to some specific functions satisfying condition (5). For instance, convex condition in [20] and differential invariant in [17] correspond to ψ(φ) = 0, while exponential condition in [9] corresponds to ψ(φ) = αφ, where α ∈ ℝ.

The following lemma indicates that we can find a class of functions  $\psi$  different from existing ones, satisfying condition (5). Thus, from which we can construct a class of relaxed conditions of barrier certificates by **GBC**, that can be used to generate barrier certificates with different expressiveness.

$$\begin{cases} \frac{\partial \theta}{\partial t} - \alpha \theta - \beta \theta^2 = 0, \\ \theta(0) \le 0, \end{cases}$$
(10)

where  $\alpha < 0, \beta \in \mathbb{R}$ , then  $\forall \xi > 0, \theta(\xi) \leq 0$ .

PROOF. If  $\beta \leq 0$ , then from (10) we have

$$\frac{\partial\theta}{\partial t} - \alpha\theta = \beta\theta^2 \le 0 \tag{11}$$

So, the claim is guaranteed by Theorem 1 in [9].

Now, suppose  $\beta > 0$ . Let  $\lambda \in \mathbb{R}$  with  $\beta \lambda = \alpha$ , and  $\theta_0 = \theta(0)$ ,

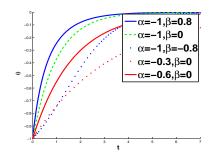


Figure 1: Solutions of (10) with  $\theta_0 = -1$  on different values of  $\alpha, \beta$ .

then

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \alpha \theta + \beta \theta^2 \\ \Rightarrow \quad \frac{\partial \theta}{\alpha \theta + \beta \theta^2} &= \partial t \\ \Rightarrow \quad \frac{d \theta}{\theta (\lambda + \theta)} &= \beta dt \\ \Rightarrow \quad \frac{1}{\lambda} (\frac{d \theta}{\theta} - \frac{d \theta}{\lambda + \theta}) &= \beta dt \end{aligned}$$

$$\Rightarrow ln \frac{\theta}{\lambda + \theta} = \lambda \beta t + c_0 = \alpha t + c_0$$

$$\Rightarrow \frac{\theta}{\lambda + \theta} = e^{\alpha t + c_0}$$

$$\Rightarrow \frac{\theta}{\lambda + \theta} = \frac{\theta_0}{\lambda + \theta_0} e^{\alpha t}$$

$$\Rightarrow \theta = (\frac{1}{1 - \frac{\theta_0}{\lambda + \theta_0}} e^{\alpha t} - 1)\lambda \qquad (12)$$

As  $\theta_0 \leq 0$ ,  $\beta \lambda = \alpha$ ,  $\beta > 0$  and  $\alpha < 0$ , we have  $0 \leq \frac{\theta_0}{\lambda + \theta_0} < 1$ and  $e^{\alpha \xi} \leq 1$ . So,

$$\begin{split} 0 &\leq \frac{\theta_0}{\lambda + \theta_0} e^{\alpha \xi} < 1, \\ \frac{1}{1 - \frac{\theta_0}{\lambda + \theta_0}} e^{\alpha \xi} - 1 \geq 0. \end{split}$$

By  $\beta \lambda = \alpha$ ,  $\beta > 0$  and  $\alpha < 0$ , it follows  $\lambda < 0$ . From (12), we have  $\forall \xi > 0.\theta(\xi) \leq 0$ .  $\Box$ 

REMARK 2. One can flexibly choose different relaxed conditions from the above class by setting different values to  $\alpha$ and  $\beta$  according to the following rules, that is illustrated in Fig. 1:

- if the value of α is smaller, then synthesized barrier certificates by the resulted condition from GBC are more expressive, and vice versa;
- if the value of  $\beta$  is greater, then synthesized barrier certificates are more expressive, and vice versa.

The following example clearly indicates that one can synthesize some interesting barrier certificates with some relaxed conditions from the above class, which cannot be discovered using the existing approaches.

EXAMPLE 1. Consider a CDS  $\mathcal{D}_1$  as follows:

$$\begin{cases} \dot{x_1} = x_1^2 - 2x_1 + x_2, \\ \dot{x_2} = x_1 + x_2^2 - 2x_2, \end{cases}$$
  
with  $\Xi_0 = \{(x_1, x_2) \mid 0.01 - x_1^2 - x_2^2 \ge 0\}, S^u = \{(x_1, x_2) \mid x_1^2 + x_2^2 - 0.25 \ge 0\}$ 

By Theorem 1, we can check that  $\varphi = x_1^2 + x_2^2 - 0.04$  is a barrier certificate w.r.t.  $\psi(\theta) = -\theta + 2\theta^2$  as follows: Let  $g_0 = 0.01 - x_1^2 - x_2^2$ ,  $g_1 = x_1^2 + x_2^2 - 0.25$ . Obviously,  $-\varphi - g_0 = 0.03 > 0$ ,  $\varphi - g_1 = 0.21 > 0$  and  $-\mathcal{L}_f(\varphi) - \varphi + 2\varphi^2 = 2x^4 - 2x^3 + 4x^2y^2 + 2.84x^2 - 4xy + 2y^4 - 2y^3 + 2.84y^2 + 0.0432$ is an **SOS**, so the condition of Theorem 1 is satisfied.

On the other hand, we can show that there is no a barrier certificate  $\varphi$  with

textitdeg( $\varphi$ )  $\leq 2$  that can be synthesized by the condition given in [9]. Assume there is a barrier certificate satisfying the condition of [9] of the form

$$\varphi = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + a_{10}x_1 + a_{01}x_2 + a_{00}$$

w.r.t.  $\psi(\theta) = \alpha \theta$ , where  $\alpha, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00} \in \mathbb{R}$ . Let  $L = -\mathcal{L}_f(\varphi) + \alpha \varphi$ , so L should be **SOS**. From  $\Xi_0$  and  $S^u$ , it follows that not all of  $a_{20}, a_{11}, a_{02}$  are equal to 0. Suppose  $a_{20} \neq 0$ , then L has a monomial  $2a_{20}x_1^3$ . Consider the value of L over the set  $\{(\xi, 0) \mid a_{20}\xi < 0\}$ , it will become negative when  $|\xi|$  becomes large enough. Similarly, we can derive a contradiction in cases when  $a_{11} \neq 0$  and  $a_{02} \neq 0$ . This means that our claim holds.  $\Box$ 

#### **3.2** Combined barrier certificates

Given a CDS  $\mathcal{D}$  equipped with D,  $\Xi_0$  and  $S^u$ , suppose  $\varphi(\mathbf{x})$  is a barrier certificate satisfying Theorem 1 w.r.t. another function  $\psi(\mathbf{x})$ . Clearly,  $\{\mathbf{x} \mid \varphi(\mathbf{x}) \leq 0\}$  is an overapproximation of  $\mathcal{R}_{\mathcal{D}}$ , while  $\{\mathbf{x} \mid \varphi(\mathbf{x}) > 0\}$  is an overapproximation of  $S^u$ . It is very common that in many cases we cannot find such a single barrier certificate to overapproximate the reachable set, but it can be achieved by combining several functions together. We call the combination of these functions a *combined barrier certificate*. Actually, a similar problem on differential invariants has been discussed in [17, 4, 24, 13].

Below, we discuss how to combine two functions together to form a combined barrier certificate. For easing discussion, let's fix the aforementioned CDS  $\mathcal{D}$ .

LEMMA 2. { $\mathbf{x} \mid \chi(\mathbf{x}) \leq 0$ } is an over approximation of  $\mathcal{R}_{\mathcal{D}}$ , if

$$\forall \mathbf{x} \in \Xi_0. \ \chi(\mathbf{x}) \le 0 \tag{13}$$

$$\forall \mathbf{x} \in D. \ \mathcal{L}_f \chi(\mathbf{x}) - \psi(\chi(\mathbf{x})) \le 0$$
(14)

 $\forall \xi. \xi > 0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0, \text{ where } \theta(\mathbf{x}(t)) \text{ is the solution of }$ 

$$\begin{cases} \mathcal{L}_f \theta(\mathbf{x}) - \psi(\theta(\mathbf{x})) = 0, \\ \theta(\mathbf{x}(0)) \le 0, \end{cases}$$
(15)

where  $\chi(\mathbf{x}), \psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n].$ 

PROOF. It can be proved similarly to Theorem 1.  $\Box$ 

LEMMA 3. If there are functions  $\varphi(\mathbf{x}), \chi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n]$  with  $\forall \mathbf{x} \in \Xi_0.\chi(\mathbf{x}) \leq 0, \ \psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}], \ and \ a \ SOS \ polynomial \ \delta^3 \ such \ that$ 

$$\forall \mathbf{x} \in \Xi_0. \ \varphi(\mathbf{x}) \le 0 \tag{16}$$

$$\forall \mathbf{x} \in D. \ \mathcal{L}_f \varphi(\mathbf{x}) - \psi(\varphi(\mathbf{x})) - \delta \chi(\mathbf{x}) \le 0$$
(17)

$$\forall \mathbf{x} \in S^u. \ \varphi(\mathbf{x}) > 0 \tag{18}$$

$$\forall \xi. \xi > 0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0, \text{ where } \theta(\mathbf{x}(t)) \text{ is the solution of} \begin{cases} \mathcal{L}_f \theta(\mathbf{x}) - \psi(\theta(\mathbf{x})) = 0, \\ \theta(\mathbf{x}(0)) \leq 0, \end{cases}$$
(19)

then for every trajectory  $\tau$  of  $\mathcal{D}$ , we have

$$(\forall \xi \ge 0. \chi(\tau(\xi)) \le 0) \Rightarrow (\forall \xi \ge 0. \tau(\xi) \notin S^u).$$

PROOF. We only need to prove  $\forall \xi \geq 0.\varphi(\tau(\xi)) \leq 0$ .

$$\begin{aligned} \forall \mathbf{x} \in \mathbb{R}^{n}. \ \mathcal{L}_{f}\varphi(\mathbf{x}) - \psi(\varphi(\mathbf{x})) - \delta\chi &\leq 0 \\ \Rightarrow \forall \xi \geq 0. \ \frac{\partial\varphi(\tau(t))}{\partial t}|_{t=\xi} - \psi(\varphi(\tau(\xi))) - \delta\chi(\tau(\xi)) \leq 0 \\ & \text{as } \mathcal{L}_{f}\varphi(\mathbf{x}) = \frac{\partial\varphi(\tau(t))}{\partial t} \\ \Rightarrow \forall \xi \geq 0. \ \frac{\partial\varphi(\tau(t))}{\partial t}|_{t=\xi} - \psi(\varphi(\tau(\xi))) \leq 0 \\ & \text{as } \forall \xi \geq 0. \ \chi(\tau(\xi)) \leq 0. \end{aligned}$$

Thus, by Theorem 1, the claim is trivially true.  $\Box$ 

THEOREM 2. Let  $\chi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n]$  satisfy (13)-(15). If there are functions  $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n], \psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}], \text{ and a}$ **SOS** polynomial  $\delta$ , s.t. (16)-(19) hold, then  $\mathcal{R}_{\mathcal{D}} \cap S^u = \emptyset$ .

PROOF. It is straightforward by Lemmas 2&3.  $\Box$ 

We will call the pair  $(\chi, \phi)$  a combined barrier certificate.

Clearly, a single barrier certificate defined in Theorem 1 can be seen as a specific combined barrier certificate by letting  $\chi = 0$ . In addition, actually, it is easy to prove that a combined barrier certificate forms a combined differential invariant.

COROLLARY 1.  $\chi \leq 0 \land \varphi \leq 0$  is a differential invariant (the definition can be found in [17]) of  $\mathcal{D}$ , which can guarantee its safety.

We use the following example to demonstrate the notion of combined barrier certificates gives more power to the verification of CDSs as well as HSs.

EXAMPLE 2. Consider the following CDS  $\mathcal{D}$ 

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - x_1x_2 \\ 2x_1^2 - x_2 \end{bmatrix}$$

<sup>3</sup>That is,  $\delta$  can be represented by  $f_1^2 + \ldots + f_n^2$ , where  $f_1, \ldots, f_n$  are polynomials.

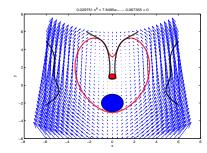


Figure 2: A combined barrier certificate for Example 2

with  $\Xi_0 = {\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + (x_2 + 2) \le 1}$  and  $S^u = {\mathbf{x} \in \mathbb{R}^2 \mid x_2 + (x_2 - 1)^2 \le 0.09}.$ 

To prove its safety, by Theorem 2, we can synthesize a combined barrier certificate  $(\chi, \varphi)$ , see Fig. 2, in which  $\chi(\mathbf{x}) = 0$ is denoted by the red line and  $\varphi(\mathbf{x}) = 0$  is denoted by the black line (their mathematical representations can be found in the appendix). In fact, we can prove  $\chi(\mathbf{x}) \leq 0 \land \varphi(\mathbf{x}) \leq 0$ is indeed a differential invariant according to the definition given in [17], which can guarantee the unsafe set unreachable.

Besides, we can prove that neither of  $\chi(\mathbf{x})$  nor  $\varphi(\mathbf{x})$  is a barrier certificate in the sense of Theorem 1. Furthermore, using the same values of  $\alpha$ ,  $\beta$  and the degree bound as used in synthesizing the combined barrier certificate ( $\chi(\mathbf{x}), \varphi(\mathbf{x})$ ), we cannot obtain any single barrier certificate by Theorem 1.

# 3.3 Relaxed barrier certificate conditions for HSs

As discussed in [9], the principle of the condition of barrier certificates  $\Phi(\mathbf{x})$  for an HS  $\mathcal{H} = (Q, X, f, D, E, G, R, \Xi)$ w.r.t. a given unsafe set  $S^u$  should satisfy the following conditions:

- $\Phi(\mathbf{x})$  consists of a set of functions  $\{\varphi_q(\mathbf{x}) \mid q \in Q\}$ , each  $\varphi_q(\mathbf{x})$  is a barrier certificate for CDS  $\dot{\mathbf{x}} = \mathbf{f}_q$ equipped with the domain  $D_q$ , initial set  $\Xi_q$  and unsafe set  $S_q^u$ ;
- all the discrete transitions starting from every mode  $q \in Q$  have to be taken into account in the barrier certificate condition so that  $\Phi(\mathbf{x})$  can construct a global inductive invariant of  $\mathcal{H}$ .

Based on the discussions about barrier certificate conditions for CDSs as well as the above principle, we can accordingly revisit the condition of barrier certificates for HSs based on the following theorem:

THEOREM 3. Given an HS  $\mathcal{H} = (Q, X, f, D, E, G, R, \Xi)$ and an unsafe set  $S^u$ , if there exists a set of non-negative real numbers  $\{c_e \mid e \in E\}$ , and a set of functions  $\{\varphi_q(\mathbf{x}) \in$ 

$$\mathcal{C}^{\omega}[\mathbb{R}^n] \mid q \in Q\} \cup \{\psi_q(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}] \mid q \in Q\} \ s.t.$$

$$\forall q \in Q \forall \mathbf{x} \in \Xi_q. \varphi_q(\mathbf{x}) \le 0 \tag{20}$$

$$\forall q \in Q \forall \mathbf{x} \in D_q. \mathcal{L}_{\mathbf{f}_q} \varphi_q(\mathbf{x}) - \psi_q(\varphi_q(\mathbf{x})) \le 0 \qquad (21)$$

$$\forall q \in Q \forall \mathbf{x} \in S_q^u . \varphi_q(\mathbf{x}) > 0 \tag{22}$$

$$\forall q \in Q \forall \xi.\xi > 0 \Rightarrow \theta_q(\mathbf{x}(\xi)) \leq 0,$$
  
where  $\theta_q(\mathbf{x}(t))$  is the solution of  
$$\begin{cases} \mathcal{L}_{\mathbf{f}_q} \theta_q(\mathbf{x}) - \psi_q(\theta_q(\mathbf{x})) = 0, \\ \theta_q(\mathbf{x}(0)) \leq 0, \end{cases}$$
(23)

$$\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}' \in R(e)(\mathbf{x}).$$

$$c_e \varphi_{S(e)}(\mathbf{x}) - \varphi_{T(e)}(\mathbf{x}') \ge 0, \qquad (24)$$

then  $\mathcal{R}_{\mathcal{H}} \cap S^u = \emptyset$ , where S(e) and T(e) respectively are the source and target modes of jump e.

Similarly, based on Theorem 2 and Theorem 3, we can revisit the condition of combined barrier certificates for HSs as follows:

THEOREM 4. Given an HS  $\mathcal{H} = (Q, X, f, D, E, G, R, \Xi)$ and an unsafe set  $S^u$ , if there exists a set of non-negative real numbers  $\{c_{e,1}, c_{e,2}, c_{e,3}, c_{e,4} \mid e \in E\}$ , a set of **SOS** polynomials  $\{\delta_q \mid q \in Q\}$ , and a set of functions  $\{\varphi_q(\mathbf{x}), \chi_q(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}^n] \mid q \in Q\} \cup \{\psi_{q,1}(\mathbf{x}), \psi_{q,2}(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}] \mid q \in Q\}$  s.t.

 $\forall q \in Q \forall \mathbf{x} \in \Xi_q. \chi_q(\mathbf{x}) \le 0 \tag{25}$ 

$$\forall q \in Q \forall \mathbf{x} \in D_q. \mathcal{L}_{\mathbf{f}_q} \chi_q(\mathbf{x}) - \psi_{q,1}(\chi_q(\mathbf{x})) \le 0$$
(26)

$$\forall q \in Q \forall \mathbf{x} \in \Xi_q. \varphi_q(\mathbf{x}) \le 0 \tag{27}$$

$$\forall q \in Q \forall \mathbf{x} \in D_q. \mathcal{L}_{\mathbf{f}_q} \varphi_q(\mathbf{x}) - \psi_{q,2}(\varphi_q(\mathbf{x})) - \delta_q \chi_q \le 0 \quad (28)$$

$$\forall q \in Q \forall \mathbf{x} \in S_q^u. \varphi_q(\mathbf{x}) > 0 \tag{29}$$

$$\begin{aligned} \psi \in Q\forall \xi. \xi > 0 \Rightarrow \theta_q(\mathbf{x}(\xi)) \leq 0, \\ where \ \theta_q(\mathbf{x}(t)) \ is \ the \ solution \ of \\ \begin{cases} \mathcal{L}_{\mathbf{f}_q} \theta_q(\mathbf{x}) - \psi_{q,1}(\theta_q(\mathbf{x})) = 0, \\ \theta_q(\mathbf{x}(0)) \leq 0, \end{cases} \end{aligned}$$
(30)

$$\forall q \in Q \forall \xi. \xi > 0 \Rightarrow \theta_q'(\mathbf{x}(\xi)) \le 0,$$

where 
$$\theta'_q(\mathbf{x}(t))$$
 is the solution of

$$\begin{cases} \mathcal{L}_{\mathbf{f}_q} \theta_q(\mathbf{x}) - \psi_{q,2}(\theta'_q(\mathbf{x})) = 0, \\ \theta'_q(\mathbf{x}(0)) \le 0, \end{cases}$$
(31)

 $\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}' \in R(e)(\mathbf{x}).$ 

$$c_{e,1}\varphi_{S(e)}(\mathbf{x}) - \varphi_{T(e)}(\mathbf{x}') \ge 0, \qquad (32)$$

$$\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}' \in R(e)(\mathbf{x}).$$

$$c_{e,2}\varphi_{S(e)}(\mathbf{x}) - \chi_{T(e)}(\mathbf{x}') \ge 0,$$

$$\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}' \in R(e)(\mathbf{x}).$$
(33)

$$c_{e,3}\chi_{S(e)}(\mathbf{x}) - \varphi_{T(e)}(\mathbf{x}') \ge 0,$$
 (34)

$$e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}' \in R(e)(\mathbf{x}).$$

Α

$$c_{e,4}\chi_{S(e)}(\mathbf{x}) - \chi_{T(e)}(\mathbf{x}') \ge 0,$$
 (35)

then  $\mathcal{R}_{\mathcal{H}} \cap S^u = \emptyset$ , where S(e) and T(e) are respectively the source and target modes of the jump e.

### 4. DISCOVERING RELAXED BARRIER CER-TIFICATES BY SDP

Theorems 1&2 (respt. Theorems 3&4) provide relaxed conditions which can guarantee a function (a combination of two functions) to be a (combined) barrier certificate for a CDS (resp. an HS), but these theorems do not provide any constructive method to synthesizing (combined) barrier certificates. In this section, we discuss how to exploit **SDP** techniques [15, 16] to construct (combined) barrier certificates from these relaxed conditions, which is inspired by previous work e.g. [8, 19, 20, 26, 9].

Thus, we briefly review **SDP** first.

### 4.1 SDP

We use  $Sym_n$  to denote the set of  $n \times n$  real symmetric matrices, and deg(f) the highest total degree of f for a given polynomial f.

DEFINITION 2 (POSITIVE SEMIDEFINITE MATRICES). A matrix  $M \in Sym_n$  is called positive semidefinite, denoted by  $M \succeq 0$ , if  $\mathbf{x}^T M \mathbf{x} \ge 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

DEFINITION 3 (INNER PRODUCT). The inner product of two matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}$ , denoted by  $\langle A, B \rangle$ , is defined by  $Tr(A^TB) = \sum_{i,j=1}^n a_{ij}b_{ij}$ .

DEFINITION 4 (SEMIDEFINITE PROGRAMMING (**SDP**)). The standard (primal) and dual forms of a **SDP** are respectively given in the following:

$$p^{*} = \inf_{X \in Sym_{n}} \langle C, X \rangle \quad s.t. \quad X \succeq 0, \quad \langle A_{j}, X \rangle = b_{j} \quad (36)$$
$$(j = 1, \dots, m)$$
$$d^{*} = \sup_{y \in \mathbb{R}^{m}} \mathbf{b}^{T} \mathbf{y} \quad s.t. \quad \sum_{i=1}^{m} y_{j} A_{j} + S = C, \quad S \succeq 0, \quad (37)$$

where  $C, A_1, \ldots, A_m, S \in Sym_n$  and  $\mathbf{b} \in \mathbb{R}^m$ .

There are many efficient algorithms to solve **SDP** such as interior-point method. We present a basic path-following algorithm to solve (36) in Algorithm 1.

DEFINITION 5 (INTERIOR POINT FOR **SDP**).  

$$intF_{p} = \{X : \langle A_{i}, X \rangle = b_{i} \ (i = 1, ..., m), \ X \succ 0\},$$

$$intF_{d} = \left\{ (\mathbf{y}, S) : S = C - \sum_{i=1}^{m} A_{i}y_{i} \succ 0 \right\},$$

$$intF = intF_{p} \times intF_{d}.$$

Obviously,  $\langle C, X \rangle - \mathbf{b}^T \mathbf{y} = \langle X, S \rangle \ge 0$  for all  $(X, \mathbf{y}, S) \in intF$ . Especially, we have  $d^* \le p^*$ . So the soul of interiorpoint method to compute  $p^*$  is to reduce  $\langle X, S \rangle$  incessantly and meanwhile guarantee  $(X, \mathbf{y}, S) \in intF$ .

### 4.2 Symbolic checking

Please be noted that because of the error caused by numeric computation in **SDP**, in particular, a threshold c upon which **SDP** depends, it may happen that the (combined)

### Algorithm 1: Interior\_Point\_Method

**input** : 
$$C, A_j, b_j \ (j = 1, ..., m)$$
 as in (36) and a threshold  $c$ 

output: 
$$p^*$$

- 1 Given a  $(X, \mathbf{y}, S) \in intF$  with  $XS = \mu I$ ;
- /\*  $\mu$  is a positive constant and I is the identity matrix. \*/

2 while  $\mu > c$  do 3  $\mu = \gamma \mu;$ 

/\*  $\gamma$  is a fixed positive constant less than one \*/

4 use **Newton iteration** to solve  $(X, \mathbf{y}, S) \in intF$  with  $XS = \mu I;$ 

5 end

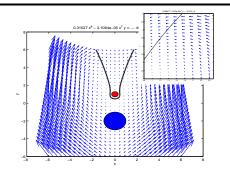


Figure 3: A false barrier certificate of Example 2 due to numeric errors

barrier certificates computed by **SDP** are not real ones, or some real (combined) certificates satisfying the condition cannot be computed or are determined as false ones. For example, considering Example 2, if we encode the condition derived from Theorem 1 as a **SDP**, then call SOSTOOLS<sup>4</sup> [21], and obtain the output is:

"feasratio: 1.0000; pinf: 0; dinf: 0; numerr: 0".

This indicates that the tool does discover a barrier certificate. However, after showing the result in Fig. 3, it is easy to find that the black line in Fig. 3 does not satisfy condition (3), as some vectors cross it into the area which contains unsafe set.

So, we have to take the numerical error into account when using these **SDP** tools. Our experience is:

- larger the size of matrix X is, larger the error due to **SDP**, so it is more likely to obtain a false (combined) barrier certificate;
- higher the degree of undetermined polynomials as predefined templates of barrier certificates is, larger the error due to **SDP**;
- one can synthesize combined barrier certificates with lower degrees by Theorem 2 than by Theorem 1.

It is absolutely necessary to guarantee the soundness of the approaches to the verification of HSs. But the approach based on **SDP** to synthesize (combined) barrier certificates

 $<sup>^4\</sup>mathrm{SOSTOOLS}$  is of version v2.04 with MATLAB R2011b.

according to these relaxed conditions, may be unsound because of the error caused by numeric computation. Below, we advocate to apply symbolic computation techniques to check if the synthesized (combined) barrier certificates are real ones, which is hinted by our previous work [3].

PROBLEM 1. For 
$$f \in \mathbb{R}[\mathbf{x}]$$
, if  $\forall \mathbf{x} \in \mathbb{R}^n . f(\mathbf{x}) \ge 0$ ?

Checking the constraints in Theorems 1&2&3&4 are obviously instances of Problem 1. A lot of work has been done on Problem 1. We choose an exact method based on an improved Cylindrical Algebraic Decomposition(CAD) algorithm [5] for the checking, and call the tool CADpsd in the experiments, which implements the algorithms in [5]. The CADpsd returns **True** when the input polynomial is positive semidefinite and **False** otherwise.

REMARK 3. One may doubt the efficiency of the above symbolic checking since the complexity of CAD is  $O(2^2)$ ) in general, where n is the number of variables. However, please note that Problem 1 is a special case of quantifier elimination. One of the main contributions of [5] is an improved algorithm for solving Problem 1. Although the improved algorithm cannot be proved with a lower complexity theoretically, it has been shown that it does avoid many heavy resultant computation. So, in practice, especially in the case where the number of variables are greater than 2, CADpsd is much faster than any general CAD tool. Please see [5] for details. In our experience, CADpsd can finish checking in few seconds when deg(f) is no larger than 6 and the number of variables in f is less than 5, which is enough for many problems.

### 4.3 Algorithms

We can sketch the basic steps of the algorithm to construct (combined) barrier certificates using **SDP** as follows:

- **Step 1:** predefine parametric polynomial templates with a degree bound as possible candidates of (combined) barrier certificates;
- Step 2: derive constraints on the parameters of these parametric polynomial templates from the considered relaxed barrier condition;
- Step 3: reduce all the constraints on the parameters to a SDP;
- **Step 4** apply some **SDP** solver to solve the resulted **SDP** and obtain instantiations of these parameters.

In the above procedure, for most of the constraints on parameters, we only need to consider how to reduce  $p \ge 0$   $(p \le 0)$  to  $p = \delta$   $(-p = \delta)$ , where p is a polynomial and  $\delta$  is a undetermined **SOS** polynomial. In the literature, there is lot of work on this, please refer to [8, 19, 20, 26, 9, 3] for the detail.

The hardest part is how to reduce the constraints that contain  $\psi$ ,  $\chi$ ,  $\psi_q$ ,  $\psi_{q,i}$ , or  $\chi_q$ , as they may contain the product of two or more parametric polynomials after replacement, which result in non-linear expressions on parameters, that cannot be seen as a **SDP** any more. For instance, let  $\psi = \theta + \theta^2$ , and  $\theta = ax_1 + bx_2$  be a template of barrier certificates. By Theorem 1, the constraint derived from condition (3) will contain expression  $(ax_1 + bx_2) + (ax_1 + bx_2)^2$ , which cannot be reduced to a **SDP** directly.

To address this issue, we explore the iterative approach proposed in [19] which can handle a constraint containing the product of two parametric polynomials. We demonstrate the basic idea of the iterative approach by presenting Algorithm 2 based on which for the following problem.

PROBLEM 2. Suppose  $\Xi_0, S^u, \mathbf{f}, \psi$  are given, where  $\psi$  satisfies (5), our goal is to find a  $\varphi$  which satisfies (2)- (4).

Algorithm 2: Iterative Algorithm for Problem 2					
<b>input</b> : $\Xi_0, S^u, \mathbf{f}, \psi(\theta) = \sum_{i=0}^s a_i \theta^i$ , where $\psi(\theta)$ satisfies					
(5)					
<b>output</b> : $\theta'$ which satisfies (2)-(4)					
1 $\theta' = 0;$					
<b>2</b> $j = 0;$					
3 while $j \leq s$ do					
4 $\psi' = \sum_{i=0}^{j} a_i \theta \theta'^{i-1};$ 5 Use a <b>SDP</b> tool to solve the resulted Problem 2 by					
5 Use a <b>SDP</b> tool to solve the resulted Problem 2 by					
replacing $\psi$ with $\psi'$ ;					
6 Denote the result of the above step by $\theta'$ ;					
$7 \qquad j = j + 1;$					
s end					

# 5. EXPERIMENTAL RESULTS

In this section, we demonstrate our approach by some examples.

EXAMPLE 3 (MODIFY EXAMPLE OF [10]). Consider a CDS  $D_3$  as follows:

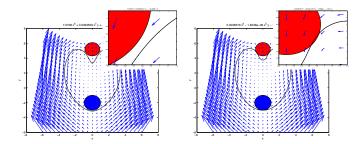
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - x_1x_2 \\ 2x_1^2 - x_2 \end{bmatrix}$$

with  $\Xi_0 = \{ \mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + (x_2 + 2) \le 1 \}$  and  $S^u = \{ \mathbf{x} \in \mathbb{R}^2 \mid x_2 + (x_2 - 5.2)^2 \le 0.81 \}.$ 

No polynomial barrier certificates can be synthesized using the existing approaches for the verification of  $\mathcal{D}_3$ , except for the one in [10] with which a polynomial barrier certificate  $\varphi(\mathbf{x})$  of degree 8 was discovered. By setting  $\alpha = -4$  and  $\beta = 1.5$ , by the corresponding relaxed condition by **GBC**, it is easy to synthesize a polynomial barrier certificate of degree 6, see Fig. 4 (also see the appendix). In contrast, by setting  $\beta = 0$ , the corresponding resulted relaxed condition is degenerated to the case considered in [10]. But unfortunately, we can not synthesize an appropriate barrier certificate from the conditions, see Fig. 5.  $\Box$ 

EXAMPLE 4. Consider the following CDS  $\mathcal{D}_4$ 

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = 2x_1 - x_2 - x_1^2 x_2 - x_1^3 \end{cases}$$



**Figure 4:**  $\alpha = -4, \beta = 1.5$ 

Figure 5:  $\alpha = -4, \beta = 0$ 

with  $\Xi_0 = \{x \in \mathbb{R}^2 \mid (x_1 + 1)^2 + (x_2 - 2)^2 \leq 0.16\}$  and  $S^u = \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \leq 0.04\}$ . Let  $g_0 = 0.16 - (x_1 + 1)^2 - (x_2 - 2)^2$ ,  $g_1 = 0.04 - (x_1 - 1)^2 - x_2^2$ . In order to prove  $\mathcal{R}_{\mathcal{D}_4} \cap S^u = \emptyset$ , according to Theorem 1, using the above procedure, we can obtain the following polynomials:

- $\varphi = \begin{array}{c} -0.91253x_1^2 + 0.40176x_1x_2 + 1.3603x_1 + 0.13922x_2^2 \\ -1.0308x_2 0.27657, \end{array}$
- $\chi = \begin{array}{c} 0.19394x_1^4 + 0.29363x_1^3x_2 0.1696x_1^3 + 0.091674x_1^2x_2^2 \\ -0.2317x_1^2x_2 1.3805x_1^2 + 0.056453x_1x_2^3 0.14904x_1x_2^2 \\ +0.096278x_1x_2 + 1.7932x_1 + 0.070488x_2^4 0.063002x_2^3 \\ +0.48804x_2^2 1.1726x_2 0.38201 \end{array}$
- $$\begin{split} \delta = & 0.1956x_1^4 + 0.23674x_1^3x_2 0.13109x_1^3 + 0.14603x_1^2x_2^2 \\ & -0.16935x_1^2x_2 + 1.0686x_1^2 + 0.35005x_1x_2^3 0.29307x_1x_2^2 \\ & -0.5897x_1x_2 1.8943x_1 + 0.26073x_2^4 0.23047x_2^3 \\ & +0.027813x_2^2 + 0.64131x_2 + 1.7118, \end{split}$$
- $\begin{array}{rl} & & +0.4338x_2^2 + 0.054172x_2 + 0.37428 \\ u_2 = & & 0.45008x_1^4 + 0.0064431x_1^3x_2 0.14066x_1^3 + 0.48519x_1^2x_2^2 \\ & & +0.18081x_1^2x_2 + 0.31882x_1^2 + 0.045636x_1x_2^3 0.030792x_1x_2^2 \\ & & +0.0463x_1x_2 + 0.022898x_1 + 0.3829x_2^4 + 0.24085x_2^3 \\ & & +0.48187x_2^2 + 0.10909x_2 + 0.37734 \end{array}$
- $\begin{array}{ll} u_3 = & 0.5497x_1^4 0.035471x_1^3x_2 + 0.073809x_1^3 + 0.66023x_1^2x_2^2 \\ & -0.085302x_1^2x_2 + 0.34888x_1^2 0.020016x_1x_2^3 + 0.55526x_1x_2^2 \\ & +0.032773x_1x_2 0.10637x_1 + 0.81332x_2^4 0.055596x_2^3 \\ & +0.49761x_2^2 + 0.25765x_2 + 0.93038 \end{array}$

 $\psi_1(\theta) = \psi_2(\theta) = -4\theta + 2\theta^2,$ 

where  $\delta, u_1, u_2, u_3 - \chi - u_1 g_0, -\mathcal{L}_f(\chi) + \psi_1(\chi), -\varphi - u_2 g_0, -\mathcal{L}_f(\varphi) + \psi_2(\varphi) + \delta\chi, \varphi - u_2 g_1$  are positive polynomials.  $\Box$ 

EXAMPLE 5. Consider an HS with two modes in Fig. 6, in which the CDSs at  $q_1$  and  $q_2$  are respectively  $\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x})$ and  $\dot{\mathbf{x}} = \mathbf{f}_2(\mathbf{x})$ , where

$$\mathbf{f}_{1}(\mathbf{x}) = \begin{cases} x_{2} \\ -x_{1} - x_{3} \\ x_{1} + (2x_{2} + 3x_{3})(1 + x_{3}^{2}), \end{cases} \begin{cases} x_{2} \\ -x_{1} - x_{3} \\ -x_{1} - 2x_{2} - 3x_{3}, \end{cases}$$

 $\begin{array}{l} \Xi_{q_1} = \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 0.01 \}, \ \Xi_{q_2} = \emptyset, \ D_{q_1} = \\ x_1^2 + 0.01 x_2^2 + 0.01 x_3^2 \leq 1.01, \ D_{q_2} = x_1^2 + x_2^2 + x_3^2 \geq 0.03 \land \\ x_1^2 \leq 5.1^2, \ g_1 = 0.99 \leq x_1^2 + 0.01 x_2^2 + 0.01 x_3^2 \leq 1.01, \ and \\ g_2 = 0.03 \leq x_1^2 + x_2^2 + x_3^2 \leq 0.05. \ All \ resets \ are \ identity. \end{array}$ 

The proof obligation is to verify  $|x_1| \leq 3.2$  at  $q_2$ . To the end, we synthesize barrier certificates at each mode first (see the appendix), then we need to verify the following five conditions :

$$\begin{cases} c_1 = -\varphi_2 - u_{23}g_{11} - u_{24}g_{12} \ge 0, \\ c_2 = -\chi_2 - u_{21}g_{11} - u_{22}g_{12} \ge 0, \\ c_3 = -\mathcal{L}_f(\chi_2) - 0.2\chi_2 + \chi_2^2 - u_{41}D_2 - u_{41}D_{21} \ge 0, \\ c_4 = -\mathcal{L}_f(\varphi_2) - 0.2\varphi_2 + \varphi_2^2 - \delta_2\chi_2 - u_{51}D_2 - u_{52}D_{21} \ge 0, \\ c_5 = \varphi_2 - U_2 - 0.00001 \ge 0, \end{cases}$$

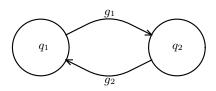


Figure 6: An HS with two modes

	Exp. cond.		Our method		
	Degree	Time(s)	Degree	Time(s)	
				Synthesis	Symb. checking
E.g. 2	×	×	8	36.023	10.766
E.g. 3	8	1.132	6	2.717	0.226
E.g. 4	6	1.516	4	4.658	0.180
E.g. 5	4	1.387	2	4.260	20.472

Table 1: Experimental data.

by **SDP**. In which,  $g_{11} = 1.01 - x_1^2 - 0.01x_2^2 - 0.01x_3^2$ ,  $g_{12} = x_1^2 + 0.01x_2^2 + 0.01x_3^2 - 0.99$ ,  $D_2 = x_1^2 + x_2^2 + x_3^2 - 0.03$ ,  $D_{21} = 26.01 - x_1^2$ ,  $U_2 = x_1^2 - 10.24$ , and  $u_{21}, u_{22}, u_{23}$ ,  $u_{24}, u_{41}, u_{42}, u_{51}, u_{52}, \delta_2$  are **SOS** synthesized in the first step.  $\Box$ 

All the experimental results of all examples given in this paper can be summarised as in Table 1, in which the label  $\times$  means that the corresponding method can not obtain a barrier certificate. All the results listed were computed on a 64-bit Intel(R) Core(TM) i5 CPU 650 @ 3.20GHz with 4GB RAM memory and Ubuntu 12.04 GNU/Linux.

By comparing with the approach reported in [9] (see Table 1), our approach can synthesize more barrier certificates, in particular, with lower degree, but our approach takes more time. However, our approach is still very efficient, typically, symbolic checking can make our approach to avoid unsoundness because of the error due to numeric computation in **SDP**.

### 6. CONCLUDING REMARKS

To summarize, the contributions of this paper include:

- Relaxation of the conditions of barrier certificate in a general way, so that one can utilize weaker conditions flexibly to synthesize various kinds of barrier certificates with more expressiveness, which gives more opportunities to verify the considered system.
- A method to combining two functions together to form a combined barrier certificate in order to prove a safety property under consideration, whereas neither of them can be used as a barrier certificate separately.
- An approach to synthesizing certificates according to the general relaxed conditions by semi-definite programming. In particular, we discussed how to apply

symbolic checking to guarantee the soundness of our approach caused by the error of numeric computation in **SDP**.

• Experimental results demonstrating that our approach can indeed discover more certificates and give more opportunities to verify an HS under consideration.

For future work, we plan to combine more than two functions to form a combined barrier certificate. In particular, we are interested in finding more functions  $\psi$  satisfying condition (5) and establishing a library for them. In addition, it is interesting to investigate how to recover the error caused by the numeric computation in **SDP** by some symbolic computation techniques.

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# APPENDIX A. THE DETAILS OF EXAMPLES

```
The polynomials synthesized in Example 2 are:
\varphi = 0.030317x_1^8 + 6.9115e - 05x_1^7x_2 - 3.6889e - 05x_1^7 + 0.090347x_1^6x_2^2 - 
0.11095x_1^6x_2 - 0.75683x_1^6 - 9.0598e - 05x_1^5x_2^3 - 0.00017438x_1^5x_2^2 -
0.00038237x_1^3x_2^3 - 0.00036159x_1^3x_2^2 - 0.00010184x_1^3x_2 + 9.2214e - 05x_1^3 + 
0.03383x_1^2x_2^6 + 0.33103x_1^2x_2^5 - 2.9864x_1^2x_2^4 + 2.0938x_1^2x_2^3 - 0.12636x_1^2x_2^2 +
0.79519x_1^2x_2 - 0.62237x_1^2 + 1.4962e - 05x_1x_2^7 - 0.00014241x_1x_2^6 +
0.00048485x_1x_2^5 - 0.00065416x_1x_2^4 + 0.00014521x_1x_2^3 + 0.00040002x_1x_2^2 - 0.000400002x_1x_2^2 - 0.00040000x_1x_2^2 - 0.0004000x_1x_2^2 - 0.0004000x_1x_2^2 - 0.00000x_1x_2^2 - 0.00000x_1x_2^2 - 0.0000x_1x_2^2 - 0.0000x
0.00031516x_1x_2 + 6.343e - 05x_1 - 0.0043261x_2^8 + 0.05803x_2^7 - 0.29525x_2^6 +
0.80728x_2^5 - 1.2538x_2^4 + 1.2862x_2^3 - 0.76567x_2^2 + 0.29172x_2 - 0.072688,
\chi = 9.8484x_1^6 + 0.001271x_1^5x_2 + 13.4422x_1^4x_2^2 - 31.2496x_1^4x_2 - 85.8767x_1^4 - 85.8777x_1^4 - 85.87777x_1^4 - 85.87777x_1^4 - 85.8777x_1^4 - 85.8777x_1^4 - 85.87777x_1^4 - 85.87777x_1^4 - 
46.3212x_1^2x_2^2 + 87.5486x_1^2x_2 - 44.1755x_1^2 + 0.0049683x_1x_2^2 - 0.0058767x_1x_2 - 0.005877x_1x_2 - 0.00577x_1x_2 - 0.00577x_1x_2 - 0.00577x_1x_2 - 0.00577x_1x_2 - 0.00577x_1x_2 - 0.00577
0.0020784x_1 + 0.46783x_2^6 - 4.0071x_2^5 + 6.1875x_2^4 + 37.296x_2^3 - 100x_2^2 +
2.0932x_2 - 12.8904
\delta = 0.014034x_1^8 + 3.0608e - 06x_1^7x_2 - 5.813e - 06x_1^7 + 0.0021473x_1^6x_2^2 - 6.0086x_1^2 + 0.0021473x_1^6x_2^2 - 0.0086x_1^2 + 0.0086x_1^2 + 0.0021473x_1^6x_2^2 - 0.0086x_1^2 + 0.0021473x_1^6x_2^2 - 0.0086x_1^2 + 0.0086x_1^
0.013483x_1^6x_2 - 0.0064165x_1^6 + 2.5531e - 06x_1^5x_2^3 + 2.7689e - 05x_1^5x_2^2 - 
1.0371e - 05x_1^5x_2 - 2.9253e - 06x_1^5 + 0.02322x_1^4x_2^4 - 0.008259x_1^4x_2^3 + \\
0.0095319x_1^4x_2^2 + 0.014437x_1^4x_2 + 0.02625x_1^4 + 1.126e - 05x_1^3x_2^5 - 3.9758e - 0.0095319x_1^4x_2^2 + 0.014437x_1^4x_2 + 0.02625x_1^4 + 1.126e - 0.0095319x_1^3x_2^5 - 3.9758e - 0.0095319x_1^4x_2^2 + 0.00953x_1^4x_2^2 + 0.00953x_1^4x_2^2 + 0.00953x_1^4x_2^2 + 0.00953x_1^4x_2^2 + 0.00953x_1^3x_2^5 - 0.00953x_1^3x_2^5 - 0.00953x_1^3x_2^5 + 0.00953x_1^3x_2^5 + 0.00953x_1^3x_2^5 + 0.00953x_1^3x_2^5 + 0.0095x_1^3x_2^5 + 0.0095x_1^5 + 0.0005x_1^5 +
05x_1^3x_2^4 + 3.4426e - 05x_1^3x_2^3 - 1.2647e - 05x_1^3x_2^2 + 8.5785e - 06x_1^3x_2 - \\
5.7495e - 07x_1^3 + 0.00051658x_1^2x_2^6 - 0.010421x_1^2x_2^5 + 0.032928x_1^2x_2^4 - 
0.041062x_1^2x_2^3 + 0.030059x_1^2x_2^2 - 0.030394x_1^2x_2 + 0.013862x_1^2 + 5.2631e - 0.030394x_1^2x_2 + 0.003x_1^2x_2 + 0.003x_1^
07x_1x_2^7 - 4.7302e - 06x_1x_2^6 + 1.3673e - 05x_1x_2^5 - 1.3654e - 05x_1x_2^4 + 
8.3569e - 07x_1x_2^3 - 1.6543e - 06x_1x_2^2 + 1.3272e - 05x_1x_2 - 8.3958e - 06x_1x_2^2 - 1.0543e - 06x_1x_2^2 - 1.0543e - 0.05x_1x_2 - 1.0543e - 0.05x_1x_2 - 
06x_1 + 0.00013121x_2^8 - 0.0015345x_2^7 + 0.0076214x_2^6 - 0.019749x_2^5 + 0.0076214x_2^6 - 0.0019749x_2^5 + 0.0076214x_2^6 - 0.0019749x_2^6 - 0.0019748x_2^6 - 0.0019788x_2^6 - 0.0019788x_2^6 - 0.0019788x_
0.02836x_2^4 - 0.023961x_2^3 + 0.01575x_2^2 - 0.010837x_2 + 0.0044092,
u_1 = 33.1703x_1^4 - 0.0080558x_1^3x_2 - 0.014014x_1^3 + 31.8846x_1^2x_2^2 - 22.7751x_1^2x_2 + 0.01401x_1^2x_2^2 + 0.0140x_1^2x_2^2 + 0.001x_1^2x_2^2 + 0.
33.5594x_1^2 + 0.002164x_1x_2^3 - 0.0059715x_1x_2^2 - 0.037073x_1x_2 + 0.020061x_1 + \\
10.479x_2^4 + 6.0815x_2^3 + 19.5851x_2^2 - 18.8795x_2 + 24.5699,
u_2 = 0.579x_1^8 + 7.0204e - 06x_1^7x_2 - 1.8771e - 05x_1^7 + 0.61572x_1^6x_2^2 - 0.60x_1^7x_2 - 0.60x_1^7x_2 - 0.60x_1^7x_2 - 0.61572x_1^6x_2^2 - 0.61572x_1^6x_2^2 - 0.60x_1^7x_2 - 0.60x_1^7x_2
0.43594x_1^6x_2 + 0.39633x_1^6 + 4.0635e - 06x_1^5x_2^3 + 5.4444e - 06x_1^5x_2^2 - 
9.0679e - 06x_1^5x_2 + 4.1779e - 05x_1^5 + 0.5972x_1^4x_2^4 - 0.446x_1^4x_2^3 + 0.8667x_1^4x_2^2 - 0.446x_1^5x_2^3 + 0.8667x_1^4x_2^2 - 0.446x_1^5x_2^3 + 0.8667x_1^4x_2^2 - 0.446x_1^5x_2^3 + 0.8667x_1^5x_2^3 + 0.8667x_1^5x_2^5 + 0.8667x_1^5x_2^3 + 0.8667x_1^5x_2^5 + 0.867x_1^5x_2^5 + 0.
0.48811x_1^4x_2 + 0.57967x_1^4 + 2.0738e - 06x_1^3x_2^5 + 4.4963e - 06x_1^3x_2^4 - 
3.9037e - 06x_1^3x_2^3 - 1.5008e - 05x_1^3x_2^2 - 6.0762e - 05x_1^3x_2 + 3.4303e - 0.0982e - 0
05x_1^3 + 0.42761x_1^2x_2^6 - 0.20453x_1^2x_2^5 + 0.45199x_1^2x_2^4 - 0.55762x_1^2x_2^3 +
06x_1x_2^6 + 4.0646e - 06x_1x_2^5 - 8.2704e - 06x_1x_2^4 + 1.2324e - 05x_1x_2^3 - 
3.1593e - 06x_1x_2^2 - 8.7493e - 06x_1x_2 + 3.0456e - 06x_1 + 0.18043x_2^8 +
0.1527x_2^7 + 0.11373x_2^6 + 0.090147x_2^5 + 0.40667x_2^4 - 0.26137x_2^3 + 0.68588x_2^2 - 0.26137x_2^3 + 0.68588x_2^2 - 0.26137x_2^3 + 0.68588x_2^2 - 0.26137x_2^3 + 0.68588x_2^3 - 0.26137x_2^3 - 0.26137x_2^3 + 0.68588x_2^3 - 0.26137x_2^3 + 0.2617x_2^3 + 0.2617x_2^3
0.38649x_2 + 0.50807,
u_3 = 0.82691x_1^8 + 6.8463e - 06x_1^7x_2 + 7.824e - 06x_1^7 + 0.66339x_1^6x_2^2 + 0.66338x_1^6x_2^2 + 0.6633x_1^6x_2^2 + 0.6633x_1^6x_2^2 + 0.663x_1^6x_1^6x_1^6x_1
0.69976x_1^6x_2 + 0.71112x_1^6 + 2.8381e - 06x_1^5x_2^3 + 2.1678e - 05x_1^5x_2^2 - 0.69976x_1^6x_2 + 0.71112x_1^6 + 2.8381e - 0.6881e^5x_2^3 + 2.1678e^5x_1^2 + 0.6881e^5x_1^2 + 0.6881e^5x_1^2
1.0011x_1^4x_2^2 + 0.41117x_1^4x_2 + 0.94169x_1^4 + 5.177e - 06x_1^3x_2^5 + 2.4687e - 06x_1^3x
05x_1^3x_2^4 + 4.9193e - 05x_1^3x_2^3 - 8.5264e - 05x_1^3x_2^2 - 9.261e - 05x_1^3x_2 - 9.261e - 0.05x_1^3x_2 - 9.05x_1^3x_2 
0.00018356x_1^3 + 0.55287x_1^2x_2^6 + 0.26734x_1^2x_2^5 + 0.43798x_1^2x_2^4 - 0.42762x_1^2x_2^3 + 0.55287x_1^2x_2^6 + 0.26734x_1^2x_2^5 + 0.43798x_1^2x_2^4 + 0.42762x_1^2x_2^3 + 0.55287x_1^2x_2^6 + 0.26734x_1^2x_2^5 + 0.43798x_1^2x_2^4 + 0.42762x_1^2x_2^3 + 0.55287x_1^2x_2^6 + 0.26734x_1^2x_2^5 + 0.43798x_1^2x_2^4 + 0.42762x_1^2x_2^3 + 0.55287x_1^2x_2^6 + 0.5578x_1^2x_2^6 + 0.5578x_1^2x_1^2 + 0.578x_1^2x_1^2 + 0.578x_1^2x_1^2 + 0.578x_1^2x_1^2 + 0.578
0.90269x_1^2x_2^2 + 0.47337x_1^2x_2 + 0.73082x_1^2 + 6.4342e - 06x_1x_2^7 + 1.7535e - 0.90269x_1^2x_2^2 + 0.47337x_1^2x_2 + 0.73082x_1^2 + 6.4342e - 0.6x_1x_2^7 + 1.7535e - 0.90269x_1^2x_2^2 + 0.90268x_1^2x_2^2 + 0.9028x_1^2x_2^2 + 0.9028x_1^2 + 0.9028x_1^2
05x_1x_2^6 + 3.191e - 05x_1x_2^5 + 2.9182e - 05x_1x_2^4 + 7.677e - 05x_1x_2^3 +
2.0006e - 05x_1x_2^2 - 3.1684e - 05x_1x_2 - 6.6486e - 06x_1 + 0.45107x_2^8 - \\
0.33244x_2 + 0.43254,
\psi_i(\theta) = \psi_2(\theta) = -4\theta + 2\theta^2.
```

The polynomials synthesized in Example 3 are : 
$$\begin{split} \varphi &= 9.8484x_1^6 + 0.001271x_1^5x_2 + 13.4422x_1^4x_2^2 - 31.2496x_1^4x_2 - 85.8767x_1^4 - \\
0.0031705x_1^3x_2^2 - 0.012227x_1^3x_2 - 0.0042103x_1^3 + 5.396x_1^2x_2^4 - 28.4976x_1^2x_2^3 - \\
46.3212x_1^2x_2^2 + 87.5486x_1^2x_2 - 44.1755x_1^2 + 0.0049683x_1x_2^2 - 0.0058767x_1x_2 - \\
0.0020784x_1 + 0.46783x_2^6 - 4.0071x_2^5 + 6.1875x_2^4 + 37.296x_2^3 - 100x_2^2 + \\
2.0932x_2 - 12.8904, \\
\chi &= 0, \delta = 0, u1 = 0, \end{split}$$
 
$$\begin{split} u_2 &= 9.8484x_1^6 + 0.001271x_1^5x_2 + 13.4422x_1^4x_2^2 - 31.2496x_1^4x_2 - 85.8767x_1^4 - \\ 0.0031705x_1^3x_2^2 - 0.012227x_1^3x_2 - 0.0042103x_1^3 + 5.396x_1^2x_2^4 - 28.4976x_1^2x_2^3 - \\ 46.3212x_1^2x_2^2 + 87.5486x_1^2x_2 - 44.1755x_1^2 + 0.0049683x_1x_2^2 - 0.0058767x_1x_2 - \\ 0.0020784x_1 + 0.46783x_2^6 - 4.0071x_2^5 + 6.1875x_2^4 + 37.296x_2^3 - 100x_2^2 + \\ 2.0932x_2 - 12.8904, \\ u_3 &= 9.8484x_1^6 + 0.001271x_1^5x_2 + 13.4422x_1^4x_2^2 - 31.2496x_1^4x_2 - 85.8767x_1^4 - \\ 0.0031705x_1^3x_2^2 - 0.012227x_1^3x_2 - 0.0042103x_1^3 + 5.396x_1^2x_2^4 - 28.4976x_1^2x_2^3 - \\ 46.3212x_1^2x_2^2 + 87.5486x_1^2x_2 - 44.1755x_1^2 + 0.0049683x_1x_2^2 - 0.0058767x_1x_2 - \\ 0.0020784x_1 + 0.46783x_2^6 - 4.0071x_2^5 + 6.1875x_2^4 + 37.296x_3^3 - 100x_2^2 + \\ 2.0932x_2 - 12.8904, \\ \psi_1 &= 0, \psi_2(\theta) = -4\theta + 1.5\theta^2. \end{split}$$

The polynomials synthesized in Example 5 are:  $\varphi_2 = 1.6165x_1^2 - 0.20569x_1x_2 + 0.19824e - 1x_1x_3 + 0.95436e - 5x_1 + 0.54446e - 1x_2^2 + 0.69996e - 3x_2x_3 - 0.16916e - 6x_2 + 0.9101e - 1x_3^2 + 0.1511e - 7x_3 - 9.6424$ 

 $\begin{array}{l} \chi_2 = 0.89818e - 1x_1^2 - 0.82739e - 1x_1x_2 + 0.21192e - 1x_1x_3 - 0.15224e - \\ 8x_1 + 0.54928e - 2x_2^2 + 0.84123e - 2x_2x_3 + 0.1277e - 8x_2 + 0.35173e - \\ 1x_3^2 + 0.27238e - 9x_3 - 5.3973 \end{array}$ 

$$\begin{split} \delta_2 &= 5.5914x_1^2 - 0.21067x_1x_2 - 0.24733e - 1x_1x_3 + 0.87702e - 5x_1 + \\ 0.20573x_2^2 - 0.52174e - 1x_2x_3 + 0.28769e - 6x_2 + 0.22449x_3^2 + 0.87144e - \\ 7x_3 + 0.29484 \end{split}$$

 $\begin{array}{l} u_{21} = 1.5356x_1^2 + 0.13731e - 1x_1x_2 - 0.19249e - 2x_1x_3 - 0.10079e - \\ 6x_1 + 0.66295x_2^2 - 0.64549e - 1x_2x_3 - 0.63485e - 7x_2 + 0.39611x_3^2 - \\ 0.66953e - 8x_3 + 2.6867 \end{array}$ 

 $\begin{array}{l} u_{22}=0.73288x_1^2-0.22775e-2x_1x_2+0.27401e-2x_1x_3-0.51154e-\\ 7x_1+0.59472x_2^2-0.55279e-1x_2x_3-0.48206e-7x_2+0.34978x_3^2-\\ 0.74061e-8x_3+0.60632 \end{array}$ 

 $\begin{array}{l} u_{23}=2.0821x_1^2+0.40593e-1x_1x_2-0.50855e-2x_1x_3-0.8427e-\\ 4x_1+0.61146x_2^2-0.90046e-2x_2x_3-0.83808e-5x_2+0.14389x_3^2-\\ 0.1148e-5x_3+4.5124 \end{array}$ 

 $\begin{array}{l} u_{24} &= 1.0004x_1^2 + 0.1131e - 1x_1x_2 - 0.22779e - 2x_1x_3 - 0.288e - \\ 4x_1 + 0.517x_2^2 - 0.80914e - 2x_2x_3 - 0.11074e - 4x_2 + 0.83205e - 1x_3^2 - \\ 0.82264e - 6x_3 + 0.70099 \end{array}$ 

```
\begin{array}{rcl} u_{41} &=& 0.43056e \, - \, 3x_1^2 \, - \, 0.29796e \, - \, 4x_1x_2 \, + \, 0.10489e \, - \, 3x_1x_3 \, + \\ 0.59287e \, - \, 11x_1 \, + \, 3.8141e \, - \, 6x_2^2 \, + \, 0.95752e \, - \, 5x_2x_3 \, + \, 0.26518e \, - \\ 12x_2 \, + \, 0.39903e \, - \, 4x_3^2 \, + \, 0.51833e \, - \, 12x_3 \, + \, 0.36e \, - \, 2 \end{array}
```

```
\begin{array}{rcl} u_{42} &=& 0.56936e \, - \, 2x_1^2 \, + \, 0.53069e \, - \, 2x_1x_2 \, + \, 0.35737e \, - \, 2x_1x_3 \, - \\ 0.75779e \, - \, 10x_1 \, + \, 0.20039e \, - \, 2x_2^2 \, + \, 0.26891e \, - \, 2x_2x_3 \, + \, 0.29818e \, - \\ 10x_2 \, + \, 0.16505e \, - \, 2x_3^2 \, - \, 0.16159e \, - \, 10x_3 \, + \, 0.52902 \end{array}
```

```
\begin{array}{l} u_{51}=0.28447e-1x_1^2-0.28324e-2x_1x_2-0.37952e-3x_1x_3+0.125e-\\ 6x_1+0.52784e-3x_2^2-0.86183e-4x_2x_3-0.63026e-8x_2+0.88611e-\\ 4x_3^2-0.86257e-9x_3+0.11143e-1 \end{array}
```

```
\begin{array}{l} u_{52}=0.12129x_1^2-0.13405e-1x_1x_2-0.15008e-2x_1x_3-0.82285e-\\ 6x_1+0.47845e-2x_2^2-0.73624e-3x_2x_3-0.20348e-6x_2+0.52816e-\\ 3x_3^2+0.21178e-7x_3+3.5079 \end{array}
```