# Barrier Certificates Revisited 

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#### Abstract

A barrier certificate can separate the state space of a considered hybrid system (HS) into safe and unsafe parts according to the safety property to be verified. Therefore this notion has been widely used in the verification of HSs. A stronger condition on barrier certificates means that less expressive barrier certificates can be synthesized. On the other hand, synthesizing more expressive barrier certificates often means high complexity. In 9, Kong et al considered how to relax the condition of barrier certificates while still keeping their convexity so that one can synthesize more expressive barrier certificates efficiently using semi-definite programming (SDP). In this paper, we first discuss how to relax the condition of barrier certificates in a general way, while still keeping their convexity. Particularly, one can then utilize different weaker conditions flexibly to synthesize different kinds of barrier certificates with more expressiveness efficiently using SDP. These barriers give more opportunities to verify the considered system. We also show how to combine two functions together to form a combined barrier certificate in order to prove a safety property under consideration, whereas neither of them can be used as a barrier certificate separately, even according to any relaxed condition. Another contribution of this paper is that we discuss how to discover certificates from the general relaxed condition by SDP. In particular, we focus on how to avoid the unsoundness because of numeric error caused by SDP with symbolic checking.


## Categories and Subject Descriptors

F.3.1 [Specifying and Verifying and Reasoning about Programs]: Invariants

## General Terms

Theory

## Keywords

Inductive invariant, barrier certificate, safety verification, hybrid system, nonlinear system, sum of squares

## 1. INTRODUCTION

Embedded systems make use of computer units to control physical devices so that the behavior of the controlled devices meets expected requirements. They have become ubiquitous in our modern life. How to design correct embedded systems is a grand challenge for computer science and control theory. Model-driven development (MDD) is considered as an effective way of developing correct complex embedded systems, and has been successfully applied in industry 6, 12 . In the framework of MDD, a formal model of the system to be developed is defined at the beginning; then extensive analysis and verification are conducted based on the formal model so that errors can be detected and corrected at the very early stage of the design of the system. Afterwards, model transformation techniques are applied to transform the abstract formal model into lower level models, even into source code. Hybrid systems (HSs) combine discrete mode changes with continuous evolutions specified in the form of differential equations. With mathematically precise semantics, HSs can serve as an appropriate model of embedded systems 14, 2.

In the past, analysis and verification of HSs are mainly done through directly computing reachable sets, either by modelchecking (e.g., [1, 22, 7]) or by decision procedures (e.g., [11]). The basic idea is to partition the state space of a considered system into finite many equivalent classes, or represent to finite many computable sets according to the solutions of the ODEs of the system. Since there is only a very small class of ODEs with closed form solutions, the scalability of these approaches is very restricted, only applicable to very specific linear HSs. To deal with more complicated systems, a deductive method has been recently proposed and successfully applied in practice 17, 18. The most challenging part of a deductive method is how to discover invariants, which hold at all reachable states of the system. For technical reason, people only consider how to synthesize inductive invariants, which are preserved by all discrete and continuous transitions. In general, a safety property itself is an invariant, but not an inductive invariant. Obviously, an inductive invariant is an approximation of the reachable set, which may be discovered according to the ODEs, rather than their solutions. The basic idea is as follows: first, predefine a property template (linear or non-linear, depending on the property to be verified); then, encode the conditions of a
property to be inductive (discretely and/or continuously) into some constraints on state variables and parameters; finally, find out solutions to the constraints. So, how to define inductiveness conditions and the power of constraint solving are essential in these approaches.

Many approaches have been proposed following the line discussed above. E.g., in [8, 23], the authors independently proposed different approaches for constructing inductive invariants for linear HSs; S. Sankaranarayanan et al presented a computational method to automatically generate algebraic invariants for algebraic HSs in [24, 25], based on the theory of pseudo-ideal over polynomial ring and quantifier elimination; S. Prajna in 19, 20 provided a new notion of inductive invariants called barrier certificates for verifying the safety of semi-algebraic HSs with stochastic setting using the technique of sum-of-squares (SOS); while in 17], Platzer and Clarke extended the idea of barrier certificates by considering boolean combinations of multiple polynomial inequalities; In 44, 27, S. Gulwani et al investigated how to generate inductive invariants with more expressiveness for semialgebraic HSs by relaxing the inductiveness conditions by considering inductiveness on the boundaries of predefined invariant templates; while in [13, Liu et al considered how to further relax the inductiveness condition given in [4, 27] and first gave a complete method on how to generate semialgebraic invariants for semi-algebraic HSs. In [26, C. Solth at el proposed an approach to constructing global inductive invariant from local differential invariants using optimization technique.

The aforementioned approaches can be classified into two categories: symbolic computation based approaches like 8 [23, 24, 25, 17, 4, 27, 13], and numeric computation based approaches like 19, 20, 26. In general, the former can synthesize more expressive invariants, but their efficiencies are very low; in contrast, the efficiency of the latter is very high, normally in polynomial time as only SDP is used, but the expressiveness of synthesized invariants is restrictive. In [9, Kong et al investigated how to synthesize more expressive barrier certificates by proposing exponential barrier certificate condition, which is a relaxed inductiveness condition, but still keeps the convexity of barrier certificates. Therefore, more expressive barrier certificates can be synthesized efficiently according to their condition still by SDP.

In this paper, firstly, following Kong et al's line, in the prerequisite of keeping the convexity of barrier certificates so that SDP is still applicable, we discuss how to relax the condition of barrier certificates in a general way. Thus, one can utilize different weaker conditions flexibly to synthesize different kinds of barrier certificates with more expressiveness efficiently, which gives more opportunities to verify the considered system. In addition, we consider how to combine two functions together to form a combined barrier certificate to prove a safety property under consideration, whereas neither of these two functions can be used as a barrier certificate separately, even according to any relaxed conditions. Another contribution of this paper is that we design algorithms to synthesize barrier certificates according to the general relaxed condition by SDP. In particular, we focus on how to avoid the unsoundness of our approach caused by numerical errors in SDP.

The rest of the paper is organized as follows: Section 2 introduces some basic notions; In Section 3 we discuss how to relax barrier certificate conditions, as well how to combine two functions to form a combined barrier certificate, but neither of them can be used as a barrier certificate separately; Section 4 is devoted to how to synthesize barrier certificates according to relaxed conditions discussed above based on SDP; Section 5 provides some case studies as well as experimental results. Finally, we conclude this paper in Section 6

## 2. PRELIMINARIES

In this section, we first introduce some basic notions, and then explain the basic idea of barrier certificates.

In what follows, we use $\mathbb{R}$ to stand for the set of reals, $\mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$ for the set of analytic function from $\mathbb{R}^{n}$ to $\mathbb{R}$.

### 2.1 Basic notions

An autonomous continuous dynamical system (CDS) is represented by a differential equation of the form

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}) \tag{1}
\end{equation*}
$$

where $\mathbf{x} \in \mathbb{R}^{n}$, and $\mathbf{f}$ is a vector function, called field vector, whose components are in $\mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$, and satisfy local Lipschitz condition I. In the context of HSs, a CDS is normally equipped with a domain $D \subseteq \mathbb{R}^{n}$ defining its state space and an initial set of states $\Xi$.

In this paper, we use hybrid automata 1 to model HSs, more models of HSs can be found in [28].

Definition 1 (Hybrid Automata). A hybrid automaton (HA) is a system $\mathcal{H} \widehat{=}(Q, X, f, D, E, G, R, \Xi)$, where

- $Q=\left\{q_{1}, \ldots, q_{m}\right\}$ is a finite set of discrete states (or modes);
- $X=\left\{x_{1}, \ldots, x_{n}\right\}$ is a finite set of continuous state variables, with $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ ranging over $\mathbb{R}^{n}$;
- $f: Q \rightarrow\left(\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}\right)$ assigns to each mode $q \in Q a$ locally Lipschitz continuous vector field $\mathbf{f}_{q}$;
- $D$ assigns to each mode $q \in Q$ a mode domain $D_{q} \subseteq$ $\mathbb{R}^{n}$;
- $E \subseteq Q \times Q$ is a finite set of discrete transitions;
- $G$ assigns to each transition $e \in E$ a switching guard $G_{e} \subseteq \mathbb{R}^{n}$;
- $R$ assigns to each transition $e \in E$ a reset function $R_{e}$ : $\mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$;
- $\Xi$ assigns to each $q \in Q$ a set of initial states $\Xi_{q} \subseteq \mathbb{R}^{n}$.

For ease of presentation, we make the following assumptions:

- for all $q \in Q, \mathbf{f}_{q}$ is a polynomial vector function, so it satisfies local Lipschitz condition, and thus the existence and uniqueness of solutions to $\dot{\mathbf{x}}=\mathbf{f}_{q}$ is guaranteed;

[^0]- for all $q \in Q$ and all $e \in E, \Xi_{q}$ is a semi-algebraic set, $D_{q}$ and $G_{e}$ are closed semi-algebraic set $\mathbb{Z}^{2}$.

Given an HA $\mathcal{H}$, a safety requirement $S$ of $\mathcal{H}$ assigns to each mode $q \in Q$ a safe region $S_{q} \subseteq \mathbb{R}^{n}$, i.e. $S=\bigcup_{q \in Q}\left(\{q\} \times S_{q}\right)$. Dually, $S^{u}=\bigcup_{q \in Q}\left(\{q\} \times\left(D_{q}-S_{q}\right)\right)$ is called unsafe set. The reachable set of $\mathcal{H}$, denoted by $\mathcal{R}_{\mathcal{H}}$, consists of those $(q, \mathbf{x})$ for which there exists a finite sequence

$$
\left(q_{0}, \mathbf{x}_{0}\right),\left(q_{1}, \mathbf{x}_{1}\right), \ldots,\left(q_{l}, \mathbf{x}_{l}\right)
$$

s.t. $\left(q_{0}, \mathbf{x}_{0}\right) \in \Xi_{\mathcal{H}},\left(q_{l}, \mathbf{x}_{l}\right)=(q, \mathbf{x})$, and for any $0 \leq i \leq l-1$, one of the following two conditions holds:

- (Discrete Jump): $e=\left(q_{i}, q_{i+1}\right) \in E, \mathbf{x}_{i} \in G_{e}$ and $\mathbf{x}_{i+1}=R_{e}\left(\mathbf{x}_{i}\right)$; or
- (Continuous Evolution): $q_{i}=q_{i+1}$, and there exists a $\delta \geq 0$ s.t. the solution $\mathbf{x}\left(\mathbf{x}_{i} ; t\right)$ to $\dot{\mathbf{x}}=\mathbf{f}_{q_{i}}$ satisfies
$-\mathbf{x}\left(\mathbf{x}_{i} ; t\right) \in D_{q_{i}}$ for all $t \in[0, \delta] ;$ and
$-\mathbf{x}\left(\mathbf{x}_{i} ; \delta\right)=\mathbf{x}_{i+1}$.


### 2.2 Barrier certificates

Given an HS $\mathcal{H}$ and a safety property $S$ (dually, an unsafe set $S^{u}$ ), the problem we considered is if $\mathcal{R}_{\mathcal{H}} \subseteq S$ (dually, $\left.\mathcal{R}_{\mathcal{H}} \cap S^{u}=\emptyset\right)$. Obviously, it is equivalent to $\forall q \in Q . \mathcal{R}_{\mathcal{H}} \upharpoonright_{q} \subseteq$ $S_{q}$ (dually, $\forall q \in Q . \mathcal{R}_{\mathcal{H}} \upharpoonright_{q} \cap S_{q}^{u}=\emptyset$ ), where $\mathcal{R}_{\mathcal{H}} \upharpoonright_{q}$ stands for all continuous states of $\mathcal{R}_{\mathcal{H}}$ projecting onto $q$. For this problem on CDSs, Prajna et al in [19, 20 used the idea of Lyapunov functions for stability analysis in control theory to separate safe states from unsafe states by a barrier function with convexity, called barrier certificate. According to their definition, a barrier function $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$ satisfies the following conditions:
i) $\varphi(\mathrm{x}) \leq 0$ for any point $\mathrm{x} \in \Xi_{q}$;
ii) $\varphi(\mathbf{x})>0$ for any point $\mathrm{x} \in S_{q}^{u}$; and
iii) $\forall \mathbf{x} \in D_{q} \cdot \mathcal{L}_{\mathbf{f}_{q}} \varphi(\mathbf{x}) \leq 0$, where $\mathcal{L}_{\mathbf{f}_{q}} \varphi(\mathbf{x})=\frac{\partial \varphi}{\partial \mathbf{x}} \mathbf{f}_{q}(\mathbf{x})$ is the Lie derivative of $\varphi$ with respect to the vector field $\mathbf{f}_{q}$.

Trivially to see, the existence of a barrier certificate is just a sufficient condition to guarantee the safety property to be verified. Hence, using Prajna et al's approach, one cannot claim the property does not hold if he/she fails to discover a polynomial barrier certificate. Actually, as observed in 9 by Kong et al, if condition iii) is relaxed to the following iii'), one can synthesize barrier certificates with more expressiveness. Certainly, it is more likely to prove a safety property by using a more expressive barrier certificate, as it gives a tighter approximation of the reachable set.
iii') $\mathcal{L}_{\mathbf{f}_{q}} \varphi(\mathbf{x})-\gamma \varphi(\mathbf{x}) \leq 0$, where $\gamma$ is a real number.

## 3. REVISITING BARRIER CERTIFICATE CONDITIONS

In this section, we investigate how to relax the condition of barrier certificates in a general way.

[^1]
### 3.1 Relaxed barrier certificate conditions for CDSs

First of all, we consider how to relax the condition i)-iii) of barrier certificates given in 19, 20, for CDSs in a general way. To the end, we need to have a principle to justify when a relaxed condition of barrier certificates is reasonable. An obvious principle is:

Principle of Barrier Certificate (PBC): Given a CDS $\mathcal{D}$ equipped with an initial set $\Xi_{0}$ and an unsafe set $S^{u}$, a barrier certificate should be a real-valued function $\varphi(\mathbf{x})$ such that $\varphi(\mathbf{x}) \leq 0$ for any $\mathbf{x} \in \mathcal{R}_{\mathcal{D}}$, and $\varphi(\mathbf{x})>0$ for any point $\mathbf{x} \in S^{u}$.

Certainly, if there exists such a function $\varphi(\mathbf{x})$, we can assert that $\mathcal{R}_{\mathcal{D}} \cap S^{u}=\emptyset$, and $\phi(\mathbf{x}) \leq 0$ is an invariant. However, such a principle cannot be effectively checked in general, so we have to strengthen the condition to make it effectively checkable, like in [19, 20, 9. An interesting problem is with which condition more expressive barrier certificates can be synthesized, but the condition is still effectively checkable and satisfies PBC. We answer the problem by the following theorem.

Theorem 1 (General Barrier Condition (GBC)). Given a $C D S \mathcal{D}$ equipped with a domain $D$, an initial set $\Xi_{0}$ and an unsafe set $S^{u}$, if there is a function $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$, a real function $\psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}]$ such that

$$
\begin{align*}
& \forall \mathbf{x} \in \Xi_{0} \cdot \varphi(\mathbf{x}) \leq 0  \tag{2}\\
& \forall \mathbf{x} \in D \cdot \mathcal{L}_{f} \varphi(\mathbf{x})-\psi(\varphi(\mathbf{x})) \leq 0  \tag{3}\\
& \forall \mathbf{x} \in S^{u} \cdot \varphi(\mathbf{x})>0  \tag{4}\\
& \xi>0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0, \text { where } \theta(\mathbf{x}(t)) \text { is the solution of } \\
& \qquad\left\{\begin{array}{l}
\theta(\mathbf{x}(0)) \leq 0 \\
\mathcal{L}_{\mathbf{f}} \theta(\mathbf{x})-\psi(\theta(\mathbf{x}))=0
\end{array}\right. \tag{5}
\end{align*}
$$

then $\mathcal{R}_{\mathcal{D}} \cap S^{u}=\emptyset$.

Proof. Suppose $\mathbf{x}_{0} \in \Xi_{0}$ and $\mathbf{x}(t)$ is the corresponding solution of (1) starting from $\mathbf{x}_{0}$. Our goal is to prove that for any function $\varphi(\mathbf{x}(t))$ satisfying (2)-(5), then

$$
\begin{equation*}
\forall \xi \geq 0 . \varphi(\mathbf{x}(\xi)) \leq 0 \tag{6}
\end{equation*}
$$

Let $g(\mathbf{x})=\mathcal{L}_{\mathbf{f}} \varphi(\mathbf{x})-\psi(\varphi(\mathbf{x}))$, then by (3)

$$
\begin{equation*}
\forall \mathbf{x} \in \mathbb{R}^{n} \cdot g(\mathbf{x}) \leq 0 \tag{7}
\end{equation*}
$$

Since $\frac{d \varphi(\mathbf{x}(t))}{d t}=\frac{\partial \varphi}{\partial \mathbf{x}} \frac{d \mathbf{x}}{d t}=\frac{\partial \varphi}{\partial \mathbf{x}} f(\mathbf{x})=\mathcal{L}_{f} \varphi(\mathbf{x})$, we have

$$
\left\{\begin{array}{l}
\frac{d \varphi(\mathbf{x}(t))}{d t}-\psi(\varphi(\mathbf{x}(t)))-g(\mathbf{x}(t))=0  \tag{8}\\
\varphi(\mathbf{x}(0))=\varphi\left(\mathbf{x}_{0}\right)
\end{array}\right.
$$

Assume $\varphi(\mathbf{x}(\xi))>0$, for some $\xi>0$. Let $\theta(\mathbf{x}(t))$ be a function with

$$
\left\{\begin{array}{l}
\frac{d \theta(\mathbf{x}(t))}{d t}-\psi(\theta(\mathbf{x}(t)))=0  \tag{9}\\
\theta(\mathbf{x}(0))=\varphi\left(\mathbf{x}_{0}\right)
\end{array}\right.
$$

Let $\Theta=\{\xi \quad \mid \varphi(\mathbf{x}(\xi))>\theta(\mathbf{x}(\xi)), \xi \geq 0\}$. By (5), $\forall \xi>$ $0 . \theta(\mathbf{x}(\xi)) \leq 0$. $\Theta$ is nonempty since the assumption. So there is a number $\mu$ s.t. $\mu=\inf (\Theta)$. Obviously, $\varphi(\mathbf{x}(t))$, $\theta(\mathbf{x}(t)), g(\mathbf{x}(t)), \frac{d \varphi(\mathbf{x}(t))}{d t}$ and $\frac{d \theta(\mathbf{x}(t))}{d t}$ are analytic functions
w.r.t. $t$. Thus $\varphi(\mathbf{x}(\mu))=\theta(\mathbf{x}(\mu))$. If $g(\mathbf{x}(\mu))<0$, then $\left.\frac{d \varphi(\mathbf{x}(t))}{d t}\right|_{t=\mu}<\left.\frac{d \theta(\mathbf{x}(t))}{d t}\right|_{t=\mu}$. Hence, $\exists \nu . \nu>\mu \wedge \forall \xi \in(\mu, \nu)$. $\left.\frac{d \varphi(\mathbf{x}(t))}{d t}\right|_{t=\xi}<\left.\frac{d \theta(\mathbf{x}(t))}{d t}\right|_{t=\xi}$. Thus, $\forall \xi \in(\mu, \nu) \cdot \varphi(\mathbf{x}(\xi))<$ $\theta(\mathbf{x}(\xi))$, which contradicts to the definition of $\mu$. So $g(\mathbf{x}(\mu))=$ 0 and $\left.\frac{d \varphi(\mathbf{x}(t))}{d t}\right|_{t=\mu}=\left.\frac{d \theta(\mathbf{x}(t))}{d t}\right|_{t=\mu}$. If there is a $k>1$ s.t. $\left.\frac{d^{k} \varphi(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}<\left.\frac{d^{k} \theta(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}$, and $\forall i<k,\left.\frac{d^{i} \varphi(\mathbf{x}(t))}{d t^{i}}\right|_{t=\mu}=$ $\left.\frac{d^{i} \theta(\mathbf{x}(t))}{d t^{i}}\right|_{t=\mu}$, then there is $\nu_{1}>\mu$ s.t. $\varphi(\mathbf{x}(\xi))<\theta(\mathbf{x}(\xi))$ for any $\xi \in\left(\mu, \nu_{1}\right)$, which contradicts to the definition of $\mu$. If $\forall k>1$. $\left.\frac{d^{k} \varphi(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}=\left.\frac{d^{k} \theta(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}$, then $\varphi(\mathbf{x}(\xi))=$ $\theta(\mathbf{x}(\xi))$ for any $\xi \in \mathbb{R}^{+}$, since $\varphi, \theta$ are analytic functions. So, the claim has been proved. Suppose for some $k>1$, $\left.\frac{d^{k} \varphi(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}>\left.\frac{d^{k} \theta(\mathbf{x}(t))}{d t^{k}}\right|_{t=\mu}$ and $\forall i<\left.k \cdot \frac{d^{i} \varphi(\mathbf{x}(t))}{d t^{i}}\right|_{t=\mu}=$ $\left.\frac{d^{i} \theta(\mathbf{x}(t))}{d t^{i}}\right|_{t=\mu}$. For all $i<k$, we simultaneously compute the $i$ th derivatives of the two sides of the first formulas of (8) and (9), and obtain $\left.\frac{d^{2} \psi(\varphi(\mathbf{x}(t)))}{d t^{i}}\right|_{t=\mu}=\left.\frac{d^{2} \psi(\theta(\mathbf{x}(t)))}{d t^{i}}\right|_{t=\mu}$, $\left.\frac{d^{i} g(\mathbf{x}(t))}{d t^{i}}\right|_{t=\mu}=0$ for $i<k-1$, and $\left.\frac{d^{k-1} g(\mathbf{x}(t))}{d t^{k-1}}\right|_{t=\mu}>0$. Thus, there is an $\delta>\mu$ s.t. $\forall \xi \in(\mu, \delta) . g(\mathbf{x}(\xi))>0$, which contradicts to the definition of $g(\mathbf{x})$. This completes the proof.

From now on, we call $\varphi$ in Theorem 1 a barrier certificate of D.

Remark 1. - The application of Theorem 1 includes the following two steps: i) look for a function $\psi$ which satisfies condition (5); ii) similar to the work in [9], synthesize barrier certificate according to the resulted conditions of (2)-(4) by instantiating $\psi$ with the function obtained in the first step.

- All barrier certificates that can be synthesized using the existing approaches can also be synthesized according to these conditions by instantiating $\psi$ to some specific functions satisfying condition (5). For instance, convex condition in [20] and differential invariant in [17] correspond to $\psi(\varphi)=0$, while exponential condition in [9] corresponds to $\psi(\varphi)=\alpha \varphi$, where $\alpha \in \mathbb{R}$.

The following lemma indicates that we can find a class of functions $\psi$ different from existing ones, satisfying condition (55). Thus, from which we can construct a class of relaxed conditions of barrier certificates by GBC, that can be used to generate barrier certificates with different expressiveness.

## Lemma 1. If

$$
\left\{\begin{array}{l}
\frac{\partial \theta}{\partial t}-\alpha \theta-\beta \theta^{2}=0  \tag{10}\\
\theta(0) \leq 0
\end{array}\right.
$$

where $\alpha<0, \beta \in \mathbb{R}$, then $\forall \xi>0 . \theta(\xi) \leq 0$.

Proof. If $\beta \leq 0$, then from (10) we have

$$
\begin{equation*}
\frac{\partial \theta}{\partial t}-\alpha \theta=\beta \theta^{2} \leq 0 \tag{11}
\end{equation*}
$$

So, the claim is guaranteed by Theorem 1 in 9.
Now, suppose $\beta>0$. Let $\lambda \in \mathbb{R}$ with $\beta \lambda=\alpha$, and $\theta_{0}=\theta(0)$,


Figure 1: Solutions of (10) with $\theta_{0}=-1$ on different values of $\alpha, \beta$.
then

$$
\begin{align*}
& \frac{\partial \theta}{\partial t}=\alpha \theta+\beta \theta^{2} \\
\Rightarrow & \frac{\partial \theta}{\alpha \theta+\beta \theta^{2}}=\partial t \\
\Rightarrow & \frac{d \theta}{\theta(\lambda+\theta)}=\beta d t \\
\Rightarrow & \frac{1}{\lambda}\left(\frac{d \theta}{\theta}-\frac{d \theta}{\lambda+\theta}\right)=\beta d t \\
\Rightarrow & \ln \frac{\theta}{\lambda+\theta}=\lambda \beta t+c_{0}=\alpha t+c_{0} \\
\Rightarrow & \frac{\theta}{\lambda+\theta}=e^{\alpha t+c_{0}} \\
\Rightarrow & \frac{\theta}{\lambda+\theta}=\frac{\theta_{0}}{\lambda+\theta_{0}} e^{\alpha t} \\
\Rightarrow & \theta=\left(\frac{1}{1-\frac{\theta_{0}}{\lambda+\theta_{0}} e^{\alpha t}}-1\right) \lambda \tag{12}
\end{align*}
$$

As $\theta_{0} \leq 0, \beta \lambda=\alpha, \beta>0$ and $\alpha<0$, we have $0 \leq \frac{\theta_{0}}{\lambda+\theta_{0}}<1$ and $e^{\alpha \xi} \leq 1$. So,

$$
\begin{array}{r}
0 \leq \frac{\theta_{0}}{\lambda+\theta_{0}} e^{\alpha \xi}<1, \\
\frac{1}{1-\frac{\theta_{0}}{\lambda+\theta_{0}} e^{\alpha \xi}}-1 \geq 0 .
\end{array}
$$

By $\beta \lambda=\alpha, \beta>0$ and $\alpha<0$, it follows $\lambda<0$. From (12), we have $\forall \xi>0 . \theta(\xi) \leq 0$.

Remark 2. One can flexibly choose different relaxed conditions from the above class by setting different values to $\alpha$ and $\beta$ according to the following rules, that is illustrated in Fig. [1:

- if the value of $\alpha$ is smaller, then synthesized barrier certificates by the resulted condition from $\boldsymbol{G B C}$ are more expressive, and vice versa;
- if the value of $\beta$ is greater, then synthesized barrier certificates are more expressive, and vice versa.

The following example clearly indicates that one can synthesize some interesting barrier certificates with some relaxed
conditions from the above class, which cannot be discovered using the existing approaches.

Example 1. Consider a $C D S \mathcal{Z}_{1}$ as follows:

$$
\left\{\begin{array}{l}
\dot{x_{1}}=x_{1}^{2}-2 x_{1}+x_{2}, \\
\dot{x_{2}}=x_{1}+x_{2}^{2}-2 x_{2},
\end{array}\right.
$$

with $\Xi_{0}=\left\{\left(x_{1}, x_{2}\right) \mid 0.01-x_{1}^{2}-x_{2}^{2} \geq 0\right\}, S^{u}=\left\{\left(x_{1}, x_{2}\right) \mid\right.$ $\left.x_{1}^{2}+x_{2}^{2}-0.25 \geq 0\right\}$

By Theorem 11, we can check that $\varphi=x_{1}^{2}+x_{2}^{2}-0.04$ is a barrier certificate w.r.t. $\psi(\theta)=-\theta+2 \theta^{2}$ as follows: Let $g_{0}=0.01-x_{1}^{2}-x_{2}^{2}, g_{1}=x_{1}^{2}+x_{2}^{2}-0.25$. Obviously, $-\varphi-g_{0}=$ $0.03>0, \varphi-g_{1}=0.21>0$ and $-\mathcal{L}_{f}(\varphi)-\varphi+2 \varphi^{2}=$ $2 x^{4}-2 x^{3}+4 x^{2} y^{2}+2.84 x^{2}-4 x y+2 y^{4}-2 y^{3}+2.84 y^{2}+0.0432$ is an SOS, so the condition of Theorem 1 is satisfied.

On the other hand, we can show that there is no a barrier certificate $\varphi$ with
textitdeg $(\varphi) \leq 2$ that can be synthesized by the condition given in [9]. Assume there is is a barrier certificate satisfying the condition of [9] of the form

$$
\varphi=a_{20} x_{1}^{2}+a_{11} x_{1} x_{2}+a_{02} x_{2}^{2}+a_{10} x_{1}+a_{01} x_{2}+a_{00}
$$

w.r.t. $\psi(\theta)=\alpha \theta$, where $\alpha, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00} \in \mathbb{R}$. Let $L=-\mathcal{L}_{f}(\varphi)+\alpha \varphi$, so $L$ should be $\boldsymbol{S O S}$. From $\Xi_{0}$ and $S^{u}$, it follows that not all of $a_{20}, a_{11}, a_{02}$ are equal to 0 . Suppose $a_{20} \neq 0$, then $L$ has a monomial $2 a_{20} x_{1}^{3}$. Consider the value of $L$ over the set $\left\{(\xi, 0) \mid a_{20} \xi<0\right\}$, it will become negative when $|\xi|$ becomes large enough. Similarly, we can derive a contradiction in cases when $a_{11} \neq 0$ and $a_{02} \neq 0$. This means that our claim holds.

### 3.2 Combined barrier certificates

Given a CDS $\mathcal{D}$ equipped with $D, \Xi_{0}$ and $S^{u}$, suppose $\varphi(\mathbf{x})$ is a barrier certificate satisfying Theorem 1 w.r.t. another function $\psi(\mathbf{x})$. Clearly, $\{\mathbf{x} \mid \varphi(\mathbf{x}) \leq 0\}$ is an overapproximation of $\mathcal{R}_{\mathcal{D}}$, while $\{\mathbf{x} \mid \varphi(\mathbf{x})>0\}$ is an overapproximation of $S^{u}$. It is very common that in many cases we cannot find such a single barrier certificate to overapproximate the reachable set, but it can be achieved by combining several functions together. We call the combination of these functions a combined barrier certificate. Actually, a similar problem on differential invariants has been discussed in [17, 4, 24, 13].

Below, we discuss how to combine two functions together to form a combined barrier certificate. For easing discussion, let's fix the aforementioned CDS $\mathcal{D}$.

Lemma 2. $\{\mathbf{x} \mid \chi(\mathbf{x}) \leq 0\}$ is an over approximation of $\mathcal{R}_{\mathcal{D}}$, if

$$
\begin{equation*}
\forall \mathbf{x} \in \Xi_{0} \cdot \chi(\mathbf{x}) \leq 0 \tag{13}
\end{equation*}
$$

$$
\begin{equation*}
\forall \mathbf{x} \in D . \mathcal{L}_{f} \chi(\mathbf{x})-\psi(\chi(\mathbf{x})) \leq 0 \tag{14}
\end{equation*}
$$

$\forall \xi . \xi>0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0$, where $\theta(\mathbf{x}(t))$ is the solution of

$$
\left\{\begin{array}{l}
\mathcal{L}_{f} \theta(\mathbf{x})-\psi(\theta(\mathbf{x}))=0  \tag{15}\\
\theta(\mathbf{x}(0)) \leq 0
\end{array}\right.
$$

where $\chi(\mathbf{x}), \psi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$.

Proof. It can be proved similarly to Theorem 1

Lemma 3. If there are functions $\varphi(\mathbf{x}), \chi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$ with $\forall \mathrm{x} \in \Xi_{0} \cdot \chi(\mathrm{x}) \leq 0, \psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}]$, and a SOS polynomial $\delta$ ${ }^{3}$ such that

$$
\begin{align*}
& \forall \mathbf{x} \in \Xi_{0} \cdot \varphi(\mathbf{x}) \leq 0  \tag{16}\\
& \forall \mathbf{x} \in D \cdot \mathcal{L}_{f} \varphi(\mathbf{x})-\psi(\varphi(\mathbf{x}))-\delta \chi(\mathbf{x}) \leq 0  \tag{17}\\
& \forall \mathbf{x} \in S^{u} \cdot \varphi(\mathbf{x})>0 \tag{18}
\end{align*}
$$

$\forall \xi . \xi>0 \Rightarrow \theta(\mathbf{x}(\xi)) \leq 0$, where $\theta(\mathbf{x}(t))$ is the solution of

$$
\left\{\begin{array}{l}
\mathcal{L}_{f} \theta(\mathbf{x})-\psi(\theta(\mathbf{x}))=0  \tag{19}\\
\theta(\mathbf{x}(0)) \leq 0
\end{array}\right.
$$

then for every trajectory $\tau$ of $\mathcal{D}$, we have

$$
(\forall \xi \geq 0 \cdot \chi(\tau(\xi)) \leq 0) \Rightarrow\left(\forall \xi \geq 0 . \tau(\xi) \notin S^{u}\right)
$$

Proof. We only need to prove $\forall \xi \geq 0 . \varphi(\tau(\xi)) \leq 0$.

$$
\begin{gathered}
\forall \mathbf{x} \in \mathbb{R}^{n} \cdot \mathcal{L}_{f} \varphi(\mathbf{x})-\psi(\varphi(\mathbf{x}))-\delta \chi \leq 0 \\
\Rightarrow \forall \xi \geq\left. 0 \cdot \frac{\partial \varphi(\tau(t))}{\partial t}\right|_{t=\xi}-\psi(\varphi(\tau(\xi)))-\delta \chi(\tau(\xi)) \leq 0 \\
\quad \text { as } \mathcal{L}_{f} \varphi(\mathbf{x})=\frac{\partial \varphi(\tau(t))}{\partial t} \\
\Rightarrow \forall \xi \geq 0 .\left.\frac{\partial \varphi(\tau(t))}{\partial t}\right|_{t=\xi}-\psi(\varphi(\tau(\xi))) \leq 0 \\
\quad \text { as } \forall \xi \geq 0 . \chi(\tau(\xi)) \leq 0 .
\end{gathered}
$$

Thus, by Theorem the claim is trivially true.

Theorem 2. Let $\chi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right]$ satisfy (13)-(15). If there are functions $\varphi(\mathbf{x}) \in \mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right], \psi(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}]$, and a SOS polynomial $\delta$, s.t. (16)-(19) hold, then $\mathcal{R}_{\mathcal{D}} \cap S^{u}=\emptyset$.

Proof. It is straightforward by Lemmas $2 \mathbb{2} 3$

We will call the pair $(\chi, \phi)$ a combined barrier certificate.
Clearly, a single barrier certificate defined in Theorem 1 can be seen as a specific combined barrier certificate by letting $\chi=0$. In addition, actually, it is easy to prove that a combined barrier certificate forms a combined differential invariant.

Corollary 1. $\chi \leq 0 \wedge \varphi \leq 0$ is a differential invariant (the definition can be found in [17]) of $\mathcal{D}$, which can guarantee its safety.

We use the following example to demonstrate the notion of combined barrier certificates gives more power to the verification of CDSs as well as HSs.

Example 2. Consider the following $C D S \mathcal{D}$

$$
\left[\begin{array}{l}
\dot{x_{1}} \\
\dot{x_{2}}
\end{array}\right]=\left[\begin{array}{c}
2 x_{1}-x_{1} x_{2} \\
2 x_{1}^{2}-x_{2}
\end{array}\right]
$$

[^2] $f_{1}, \ldots, f_{n}$ are polynomials.


Figure 2: A combined barrier certificate for Example 2
with $\Xi_{0}=\left\{\mathbf{x} \in \mathbb{R}^{2} \mid x_{1}^{2}+\left(x_{2}+2\right) \leq 1\right\}$ and $S^{u}=\{\mathbf{x} \in$ $\left.\mathbb{R}^{2} \mid x_{2}+\left(x_{2}-1\right)^{2} \leq 0.09\right\}$.

To prove its safety, by Theorem圆 we can synthesize a combined barrier certificate $(\chi, \varphi)$, see Fig. 囩, in which $\chi(\mathbf{x})=0$ is denoted by the red line and $\varphi(\mathbf{x})=0$ is denoted by the black line (their mathematical representations can be found in the appendix). In fact, we can prove $\chi(\mathbf{x}) \leq 0 \wedge \varphi(\mathbf{x}) \leq 0$ is indeed a differential invariant according to the definition given in [17], which can guarantee the unsafe set unreachable.

Besides, we can prove that neither of $\chi(\mathbf{x})$ nor $\varphi(\mathbf{x})$ is a barrier certificate in the sense of Theorem 1]. Furthermore, using the same values of $\alpha, \beta$ and the degree bound as used in synthesizing the combined barrier certificate $(\chi(\mathbf{x}), \varphi(\mathbf{x}))$, we cannot obtain any single barrier certificate by Theorem 1 .

### 3.3 Relaxed barrier certificate conditions for HSs

As discussed in 9, the principle of the condition of barrier certificates $\Phi(\mathbf{x})$ for an HS $\mathcal{H}=(Q, X, f, D, E, G, R, \Xi)$ w.r.t. a given unsafe set $S^{u}$ should satisfy the following conditions:

- $\Phi(\mathbf{x})$ consists of a set of functions $\left\{\varphi_{q}(\mathbf{x}) \mid q \in Q\right\}$, each $\varphi_{q}(\mathbf{x})$ is a barrier certificate for $\operatorname{CDS} \dot{\mathbf{x}}=\mathbf{f}_{q}$ equipped with the domain $D_{q}$, initial set $\Xi_{q}$ and unsafe set $S_{q}^{u}$;
- all the discrete transitions starting from every mode $q \in Q$ have to be taken into account in the barrier certificate condition so that $\Phi(\mathbf{x})$ can construct a global inductive invariant of $\mathcal{H}$.

Based on the discussions about barrier certificate conditions for CDSs as well as the above principle, we can accordingly revisit the condition of barrier certificates for HSs based on the following theorem:

Theorem 3. Given an $H S \mathcal{H}=(Q, X, f, D, E, G, R, \Xi)$ and an unsafe set $S^{u}$, if there exists a set of non-negative real numbers $\left\{c_{e} \mid e \in E\right\}$, and a set of functions $\left\{\varphi_{q}(\mathbf{x}) \in\right.$

$$
\begin{align*}
& \left.\mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right] \mid q \in Q\right\} \cup\left\{\psi_{q}(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}] \mid q \in Q\right\} \text { s.t. } \\
& \forall q \in Q \forall \mathbf{x} \in \Xi_{q} \cdot \varphi_{q}(\mathbf{x}) \leq 0  \tag{20}\\
& \forall q \in Q \forall \mathbf{x} \in D_{q} \cdot \mathcal{L}_{\mathbf{f}_{q}} \varphi_{q}(\mathbf{x})-\psi_{q}\left(\varphi_{q}(\mathbf{x})\right) \leq 0  \tag{21}\\
& \forall q \in Q \forall \mathbf{x} \in S_{q}^{u} \cdot \varphi_{q}(\mathbf{x})>0  \tag{22}\\
& \forall q \in Q \forall \xi \cdot \xi>0 \Rightarrow \theta_{q}(\mathbf{x}(\xi)) \leq 0 \\
& \text { where } \theta_{q}(\mathbf{x}(t)) \text { is the solution of } \\
& \quad\left\{\begin{array}{l}
\mathcal{L}_{\mathbf{f}_{q}} \theta_{q}(\mathbf{x})-\psi_{q}\left(\theta_{q}(\mathbf{x})\right)=0 \\
\theta_{q}(\mathbf{x}(0)) \leq 0, \\
\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}^{\prime} \in R(e)(\mathbf{x}) \\
c_{e} \varphi_{S(e)}(\mathbf{x})-\varphi_{T(e)}\left(\mathbf{x}^{\prime}\right) \geq 0
\end{array}\right. \tag{23}
\end{align*}
$$

then $\mathcal{R}_{\mathcal{H}} \cap S^{u}=\emptyset$, where $S(e)$ and $T(e)$ respectively are the source and target modes of jump e.

Similarly, based on Theorem 2 and Theorem 3 we can revisit the condition of combined barrier certificates for HSs as follows:

Theorem 4. Given an $H S \mathcal{H}=(Q, X, f, D, E, G, R, \Xi)$ and an unsafe set $S^{u}$, if there exists a set of non-negative real numbers $\left\{c_{e, 1}, c_{e, 2}, c_{e, 3}, c_{e, 4} \mid e \in E\right\}$, a set of $\boldsymbol{S O S}$ polynomials $\left\{\delta_{q} \mid q \in Q\right\}$, and a set of functions $\left\{\varphi_{q}(\mathbf{x}), \chi_{q}(\mathbf{x}) \in\right.$ $\left.\mathcal{C}^{\omega}\left[\mathbb{R}^{n}\right] \mid q \in Q\right\} \cup\left\{\psi_{q, 1}(\mathbf{x}), \psi_{q, 2}(\mathbf{x}) \in \mathcal{C}^{\omega}[\mathbb{R}] \mid q \in Q\right\}$ s.t.

$$
\begin{align*}
& \forall q \in Q \forall \mathbf{x} \in \Xi_{q} \cdot \chi_{q}(\mathbf{x}) \leq 0  \tag{25}\\
& \forall q \in Q \forall \mathbf{x} \in D_{q} \cdot \mathcal{L}_{\mathbf{f}_{q}} \chi_{q}(\mathbf{x})-\psi_{q, 1}\left(\chi_{q}(\mathbf{x})\right) \leq 0  \tag{26}\\
& \forall q \in Q \forall \mathbf{x} \in \Xi_{q} \cdot \varphi_{q}(\mathbf{x}) \leq 0  \tag{27}\\
& \forall q \in Q \forall \mathbf{x} \in D_{q} \cdot \mathcal{L}_{\mathbf{f}_{q}} \varphi_{q}(\mathbf{x})-\psi_{q, 2}\left(\varphi_{q}(\mathbf{x})\right)-\delta_{q} \chi_{q} \leq 0  \tag{28}\\
& \forall q \in Q \forall \mathbf{x} \in S_{q}^{u} \cdot \varphi_{q}(\mathbf{x})>0  \tag{29}\\
& \forall q \in Q \forall \xi \cdot \xi>0 \Rightarrow \theta_{q}(\mathbf{x}(\xi)) \leq 0, \\
& \quad \text { where } \theta_{q}(\mathbf{x}(t)) \text { is the solution of } \\
&  \tag{30}\\
& \quad\left\{\begin{array}{l}
\mathcal{L}_{\mathbf{f}_{q}} \theta_{q}(\mathbf{x})-\psi_{q, 1}\left(\theta_{q}(\mathbf{x})\right)=0, \\
\theta_{q}(\mathbf{x}(0)) \leq 0,
\end{array}\right. \\
& \forall q \in Q \forall \xi \cdot \xi>0 \Rightarrow \theta_{q}^{\prime}(\mathbf{x}(\xi)) \leq 0, \\
& \quad \text { where } \theta_{q}^{\prime}(\mathbf{x}(t)) \text { is the solution of }  \tag{31}\\
& \left\{\begin{array}{l}
\mathcal{L}_{\mathbf{f}_{q}} \theta_{q}(\mathbf{x})-\psi_{q, 2}\left(\theta_{q}^{\prime}(\mathbf{x})\right)=0, \\
\theta_{q}^{\prime}(\mathbf{x}(0)) \leq 0,
\end{array}\right. \\
& \forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}^{\prime} \in R(e)(\mathbf{x})  \tag{32}\\
& \quad c_{e, 1} \varphi_{S(e)}(\mathbf{x})-\varphi_{T(e)}\left(\mathbf{x}^{\prime}\right) \geq 0
\end{align*}
$$

$\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}^{\prime} \in R(e)(\mathbf{x})$.

$$
\begin{equation*}
c_{e, 2} \varphi_{S(e)}(\mathbf{x})-\chi_{T(e)}\left(\mathbf{x}^{\prime}\right) \geq 0, \tag{33}
\end{equation*}
$$

$\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}^{\prime} \in R(e)(\mathbf{x})$.

$$
\begin{equation*}
c_{e, 3} \chi_{S(e)}(\mathbf{x})-\varphi_{T(e)}\left(\mathbf{x}^{\prime}\right) \geq 0 \tag{34}
\end{equation*}
$$

$$
\forall e \in E \forall \mathbf{x} \in G(e) \forall \mathbf{x}^{\prime} \in R(e)(\mathbf{x})
$$

$$
\begin{equation*}
c_{e, 4} \chi_{S(e)}(\mathbf{x})-\chi_{T(e)}\left(\mathbf{x}^{\prime}\right) \geq 0 \tag{35}
\end{equation*}
$$

then $\mathcal{R}_{\mathcal{H}} \cap S^{u}=\emptyset$, where $S(e)$ and $T(e)$ are respectively the source and target modes of the jump e.

## 4. DISCOVERING RELAXED BARRIER CERTIFICATES BY SDP

Theorems $1 \mathbb{1} 2$ (respt. Theorems 3 $\sqrt{2}$ 4) provide relaxed conditions which can guarantee a function (a combination of two functions) to be a (combined) barrier certificate for a CDS (resp. an HS), but these theorems do not provide any constructive method to synthesizing (combined) barrier certificates. In this section, we discuss how to exploit SDP techniques [15, [16] to construct (combined) barrier certificates from these relaxed conditions, which is inspired by previous work e.g. [8, 19, 20, 26, 9].

Thus, we briefly review SDP first.

### 4.1 SDP

We use $S y m_{n}$ to denote the set of $n \times n$ real symmetric matrices, and $\operatorname{deg}(f)$ the highest total degree of $f$ for a given polynomial $f$.

Definition 2 (Positive semidefinite matrices). A matrix $M \in$ Sym $_{n}$ is called positive semidefinite, denoted by $M \succeq 0$, if $\mathbf{x}^{T} M \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^{n}$.

Definition 3 (InNer product). The inner product of two matrices $A=\left(a_{i j}\right), B=\left(b_{i j}\right) \in \mathbb{R}^{n \times n}$, denoted by $\langle A, B\rangle$, is defined by $\operatorname{Tr}\left(A^{T} B\right)=\sum_{i, j=1}^{n} a_{i j} b_{i j}$.

Definition 4 (Semidefinite programming (SDP)). The standard (primal) and dual forms of a SDP are respectively given in the following:

$$
\begin{align*}
p^{*}= & \inf _{X \in S y m_{n}}\langle C, X\rangle \text { s.t. } X \succeq 0,\left\langle A_{j}, X\right\rangle=b_{j}  \tag{36}\\
& (j=1, \ldots, m) \\
d^{*}= & \sup _{y \in \mathbb{R}^{m}} \mathbf{b}^{T} \mathbf{y} \text { s.t. } \sum_{j=1}^{m} y_{j} A_{j}+S=C, S \succeq 0, \tag{37}
\end{align*}
$$

where $C, A_{1}, \ldots, A_{m}, S \in$ Sym $_{n}$ and $\mathbf{b} \in \mathbb{R}^{m}$.

There are many efficient algorithms to solve SDP such as interior-point method. We present a basic path-following algorithm to solve (36) in Algorithm 1

## Definition 5 (Interior point for SDP).

$$
\begin{aligned}
i n t F_{p} & =\left\{X:\left\langle A_{i}, X\right\rangle=b_{i}(i=1, \ldots, m), X \succ 0\right\} \\
i n t F_{d} & =\left\{(\mathbf{y}, S): S=C-\sum_{i=1}^{m} A_{i} y_{i} \succ 0\right\} \\
i n t F & =i n t F_{p} \times i n t F_{d} .
\end{aligned}
$$

Obviously, $\langle C, X\rangle-\mathbf{b}^{T} \mathbf{y}=\langle X, S\rangle \geq 0$ for all $(X, \mathbf{y}, S) \in$ $i n t F$. Especially, we have $d^{*} \leq p^{*}$. So the soul of interiorpoint method to compute $p^{*}$ is to reduce $\langle X, S\rangle$ incessantly and meanwhile guarantee $(X, \mathbf{y}, S) \in \operatorname{int} F$.

### 4.2 Symbolic checking

Please be noted that because of the error caused by numeric computation in SDP, in particular, a threshold $c$ upon which SDP depends, it may happen that the (combined)

```
Algorithm 1: Interior_Point_Method
input : \(C, A_{j}, b_{j}(j=1, \ldots, m)\) as in (36) and a
            threshold \(c\)
output: \(p^{*}\)
Given a \((X, \mathbf{y}, S) \in \operatorname{int} F\) with \(X S=\mu I\);
/* \(\mu\) is a positive constant and \(I\) is the identity
        matrix.
                            */
while \(\mu>c\) do
        \(\mu=\gamma \mu\);
        /* \(\gamma\) is a fixed positive constant less than one
            */
        use Newton iteration to solve \((X, \mathbf{y}, S) \in\) int \(F\) with
        \(X S=\mu I ;\)
5 end
```



Figure 3: A false barrier certificate of Example 2 due to numeric errors
barrier certificates computed by SDP are not real ones, or some real (combined) certificates satisfying the condition cannot be computed or are determined as false ones. For example, considering Example 2 if we encode the condition derived from Theorem 1 as a SDP, then call SOSTOOLS [21, and obtain the output is:
" feasratio: 1.0000; pinf: 0 ; dinf: 0 ; numerr: 0 ".
This indicates that the tool does discover a barrier certificate. However, after showing the result in Fig. 33 it is easy to find that the black line in Fig. 3 does not satisfy condition (3), as some vectors cross it into the area which contains unsafe set.

So, we have to take the numerical error into account when using these SDP tools. Our experience is:

- larger the size of matrix $X$ is, larger the error due to SDP, so it is more likely to obtain a false (combined) barrier certificate;
- higher the degree of undetermined polynomials as predefined templates of barrier certificates is, larger the error due to SDP;
- one can synthesize combined barrier certificates with lower degrees by Theorem 2 than by Theorem 1

It is absolutely necessary to guarantee the soundness of the approaches to the verification of HSs. But the approach based on SDP to synthesize (combined) barrier certificates

[^3]according to these relaxed conditions, may be unsound because of the error caused by numeric computation. Below, we advocate to apply symbolic computation techniques to check if the synthesized (combined) barrier certificates are real ones, which is hinted by our previous work 3.

## Problem 1. For $f \in \mathbb{R}[\mathbf{x}]$, if $\forall \mathbf{x} \in \mathbb{R}^{n} . f(\mathbf{x}) \geq 0$ ?

Checking the constraints in Theorems $1 \& 2 \sqrt{3} \sqrt[4]{ }$ are obviously instances of Problem 1 A lot of work has been done on Problem 1 We choose an exact method based on an improved Cylindrical Algebraic Decomposition(CAD) algorithm [5] for the checking, and call the tool CADpsd in the experiments, which implements the algorithms in [5]. The CADpsd returns True when the input polynomial is positive semidefinite and False otherwise.

Remark 3. One may doubt the efficiency of the above symbolic checking since the complexity of CAD is $O\left(2^{2^{n}}\right)$ in general, where $n$ is the number of variables. However, please note that Problem 1 is a special case of quantifier elimination. One of the main contributions of [5] is an improved algorithm for solving Problem 11 Although the improved algorithm cannot be proved with a lower complexity theoretically, it has been shown that it does avoid many heavy resultant computation. So, in practice, especially in the case where the number of variables are greater than 2, CADpsd is much faster than any general CAD tool. Please see [5] for details. In our experience, CADpsd can finish checking in few seconds when $\operatorname{deg}(f)$ is no larger than 6 and the number of variables in $f$ is less than 5 , which is enough for many problems.

### 4.3 Algorithms

We can sketch the basic steps of the algorithm to construct (combined) barrier certificates using SDP as follows:

Step 1: predefine parametric polynomial templates with a degree bound as possible candidates of (combined) barrier certificates;

Step 2: derive constraints on the parameters of these parametric polynomial templates from the considered relaxed barrier condition;

Step 3: reduce all the constraints on the parameters to a SDP;

Step 4 apply some SDP solver to solve the resulted SDP and obtain instantiations of these parameters.

In the above procedure, for most of the constraints on parameters, we only need to consider how to reduce $p \geq 0$ $(p \leq 0)$ to $p=\delta(-p=\delta)$, where $p$ is a polynomial and $\delta$ is a undetermined SOS polynomial. In the literature, there is lot of work on this, please refer to [8, 19, 20, 26, 9, 3] for the detail.

The hardest part is how to reduce the constraints that contain $\psi, \chi, \psi_{q}, \psi_{q, i}$, or $\chi_{q}$, as they may contain the product of two or more parametric polynomials after replacement, which result in non-linear expressions on parameters, that cannot be seen as a SDP any more. For instance, let
$\psi=\theta+\theta^{2}$, and $\theta=a x_{1}+b x_{2}$ be a template of barrier certificates. By Theorem 1 the constraint derived from condition (3) will contain expression $\left(a x_{1}+b x_{2}\right)+\left(a x_{1}+b x_{2}\right)^{2}$, which cannot be reduced to a SDP directly.

To address this issue, we explore the iterative approach proposed in [19] which can handle a constraint containing the product of two parametric polynomials. We demonstrate the basic idea of the iterative approach by presenting Algorithm 2 based on which for the following problem.

Problem 2. Suppose $\Xi_{0}, S^{u}, \mathbf{f}, \psi$ are given, where $\psi$ satisfies (5), our goal is to find $a \varphi$ which satisfies (2)- (4).

```
Algorithm 2: Iterative Algorithm for Problem 2]
input : \(\Xi_{0}, S^{u}, \mathbf{f}, \psi(\theta)=\sum_{i=0}^{s} a_{i} \theta^{i}\), where \(\psi(\theta)\) satisfies
            (5)
output: \(\theta^{\prime}\) which satisfies (2)-(4)
\(\theta^{\prime}=0\);
\(j=0\);
while \(j \leq s\) do
    \(\psi^{\prime}=\sum_{i=0}^{j} a_{i} \theta \theta^{\prime i-1}\);
    Use a SDP tool to solve the resulted Problem 2 by
    replacing \(\psi\) with \(\psi^{\prime}\);
    Denote the result of the above step by \(\theta^{\prime}\);
    \(j=j+1\);
end
```


## 5. EXPERIMENTAL RESULTS

In this section, we demonstrate our approach by some examples.

Example 3 (modify example of 10). Consider a $C D S$ $\mathcal{T}_{3}$ as follows:

$$
\left[\begin{array}{l}
\dot{x_{1}} \\
\dot{x_{2}}
\end{array}\right]=\left[\begin{array}{c}
2 x_{1}-x_{1} x_{2} \\
2 x_{1}^{2}-x_{2}
\end{array}\right]
$$

with $\Xi_{0}=\left\{\mathbf{x} \in \mathbb{R}^{2} \mid x_{1}^{2}+\left(x_{2}+2\right) \leq 1\right\}$ and $S^{u}=\{\mathbf{x} \in$ $\left.\mathbb{R}^{2} \mid x_{2}+\left(x_{2}-5.2\right)^{2} \leq 0.81\right\}$.

No polynomial barrier certificates can be synthesized using the existing approaches for the verification of $\mathcal{D}_{3}$, except for the one in [10] with which a polynomial barrier certificate $\varphi(\mathbf{x})$ of degree 8 was discovered. By setting $\alpha=-4$ and $\beta=1.5$, by the corresponding relaxed condition by $\boldsymbol{G B C}$, it is easy to synthesize a polynomial barrier certificate of degree 6, see Fig. 4 (also see the appendix). In contrast, by setting $\beta=0$, the corresponding resulted relaxed condition is degenerated to the case considered in [10]. But unfortunately, we can not synthesize an appropriate barrier certificate from the conditions, see Fig. 5

Example 4. Consider the following $C D S \mathcal{D}_{4}$

$$
\left\{\begin{array}{l}
\dot{x_{1}}=x_{2} \\
\dot{x_{2}}=2 x_{1}-x_{2}-x_{1}^{2} x_{2}-x_{1}^{3}
\end{array}\right.
$$



Figure 4: $\alpha=-4, \beta=1.5$


Figure 5: $\alpha=-4, \beta=0$
with $\Xi_{0}=\left\{x \in \mathbb{R}^{2} \mid\left(x_{1}+1\right)^{2}+\left(x_{2}-2\right)^{2} \leq 0.16\right\}$ and $S^{u}=\left\{x \in \mathbb{R}^{2} \mid\left(x_{1}-1\right)^{2}+x_{2}^{2} \leq 0.04\right\}$. Let $g_{0}=0.16-$ $\left(x_{1}+1\right)^{2}-\left(x_{2}-2\right)^{2}, g_{1}=0.04-\left(x_{1}-1\right)^{2}-x_{2}^{2}$. In order to prove $\mathcal{R}_{\mathcal{L}}{ }^{\square} \cap S^{u}=\emptyset$, according to Theorem [1 using the above procedure, we can obtain the following polynomials:

$$
\begin{aligned}
& \varphi=\quad-0.91253 x_{1}^{2}+0.40176 x_{1} x_{2}+1.3603 x_{1}+0.13922 x_{2}^{2} \\
& -1.0308 x_{2}-0.27657 \text {, } \\
& \chi=\quad 0.19394 x_{1}^{4}+0.29363 x_{1}^{3} x_{2}-0.1696 x_{1}^{3}+0.091674 x_{1}^{2} x_{2}^{2} \\
& -0.2317 x_{1}^{2} x_{2}-1.3805 x_{1}^{2}+0.056453 x_{1} x_{2}^{3}-0.14904 x_{1} x_{2}^{2} \\
& +0.096278 x_{1} x_{2}+1.7932 x_{1}+0.070488 x_{2}^{4}-0.063002 x_{2}^{3} \\
& +0.48804 x_{2}^{2}-1.1726 x_{2}-0.38201 \\
& \delta=\quad 0.1956 x_{1}^{4}+0.23674 x_{1}^{3} x_{2}-0.13109 x_{1}^{3}+0.14603 x_{1}^{2} x_{2}^{2} \\
& -0.16935 x_{1}^{2} x_{2}+1.0686 x_{1}^{2}+0.35005 x_{1} x_{2}^{3}-0.29307 x_{1} x_{2}^{2} \\
& -0.5897 x_{1} x_{2}-1.8943 x_{1}+0.26073 x_{2}^{4}-0.23047 x_{2}^{3} \\
& +0.027813 x_{2}^{2}+0.64131 x_{2}+1.7118 \text {, } \\
& u_{1}=\quad 0.47292 x_{1}^{4}+0.03761 x_{1}^{3} x_{2}-0.15676 x_{1}^{3}+0.45935 x_{1}^{2} x_{2}^{2} \\
& +0.13126 x_{1}^{2} x_{2}+0.26007 x_{1}^{2}+0.0766 x_{1} x_{2}^{3}-0.02395 x_{1} x_{2}^{2} \\
& +0.045239 x_{1} x_{2}+0.068505 x_{1}+0.33983 x_{2}^{4}+0.17729 x_{2}^{3} \\
& +0.4338 x_{2}^{2}+0.054172 x_{2}+0.37428 \\
& u_{2}=\quad 0.45008 x_{1}^{4}+0.0064431 x_{1}^{3} x_{2}-0.14066 x_{1}^{3}+0.48519 x_{1}^{2} x_{2}^{2} \\
& +0.18081 x_{1}^{2} x_{2}+0.31882 x_{1}^{2}+0.045636 x_{1} x_{2}^{3}-0.030792 x_{1} x_{2}^{2} \\
& +0.0463 x_{1} x_{2}+0.022898 x_{1}+0.3829 x_{2}^{4}+0.24085 x_{2}^{3} \\
& +0.48187 x_{2}^{2}+0.10909 x_{2}+0.37734 \\
& u_{3}=\quad 0.5497 x_{1}^{4}-0.035471 x_{1}^{3} x_{2}+0.073809 x_{1}^{3}+0.66023 x_{1}^{2} x_{2}^{2} \\
& -0.085302 x_{1}^{2} x_{2}+0.34888 x_{1}^{2}-0.020016 x_{1} x_{2}^{3}+0.55526 x_{1} x_{2}^{2} \\
& +0.032773 x_{1} x_{2}-0.10637 x_{1}+0.81332 x_{2}^{4}-0.055596 x_{2}^{3} \\
& +0.49761 x_{2}^{2}+0.25765 x_{2}+0.93038 \\
& \psi_{1}(\theta)=\psi_{2}(\theta)=-4 \theta+2 \theta^{2}, \\
& \text { where } \delta, u_{1}, u_{2}, u_{3}-\chi-u_{1} g_{0},-\mathcal{L}_{f}(\chi)+\psi_{1}(\chi),-\varphi-u_{2} g_{0} \text {, } \\
& -\mathcal{L}_{f}(\varphi)+\psi_{2}(\varphi)+\delta \chi, \varphi-u_{2} g_{1} \text { are positive polynomials. }
\end{aligned}
$$

Example 5. Consider an $H S$ with two modes in Fig. 6, in which the CDSs at $q_{1}$ and $q_{2}$ are respectively $\dot{\mathbf{x}}=\mathbf{f}_{1}(\mathbf{x})$ and $\dot{\mathbf{x}}=\mathbf{f}_{2}(\mathbf{x})$, where
$\mathbf{f}_{1}(\mathbf{x})=\left\{\begin{array}{l}x_{2} \\ -x_{1}-x_{3} \\ x_{1}+\left(2 x_{2}+3 x_{3}\right)\left(1+x_{3}^{2}\right),\end{array} \mathbf{f}_{2}(\mathbf{x})=\left\{\begin{array}{l}x_{2} \\ -x_{1}-x_{3} \\ -x_{1}-2 x_{2}-3 x_{3},\end{array}\right.\right.$
$\Xi_{q_{1}}=\left\{\mathbf{x} \in \mathbb{R}^{3} \mid x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \leq 0.01\right\}, \Xi_{q_{2}}=\emptyset, D_{q_{1}}=$ $x_{1}^{2}+0.01 x_{2}^{2}+0.01 x_{3}^{2} \leq 1.01, D_{q_{2}}=x_{1}^{2}+x_{2}^{2}+x_{3}^{2} \geq 0.03 \wedge$ $x_{1}^{2} \leq 5.1^{2}, g_{1}=0.99 \leq x_{1}^{2}+0.01 x_{2}^{2}+0.01 x_{3}^{2} \leq 1.01$, and $g_{2}=0.03 \leq x_{1}^{2}+x_{2}^{2}+\bar{x}_{3}^{2} \leq 0.05$. All resets are identity.

The proof obligation is to verify $\left|x_{1}\right| \leq 3.2$ at $q_{2}$. To the end, we synthesize barrier certificates at each mode first (see the appendix), then we need to verify the following five conditions :

$$
\left\{\begin{array}{l}
c_{1}=-\varphi_{2}-u_{23} g_{11}-u_{24} g_{12} \geq 0, \\
c_{2}=-\chi_{2}-u_{21} g_{11}-u_{22} g_{12} \geq 0, \\
c_{3}=-\mathcal{L}_{f}\left(\chi_{2}\right)-0.2 \chi_{2}+\chi_{2}^{2}-u_{41} D_{2}-u_{41} D_{21} \geq 0, \\
c_{4}=-\mathcal{L}_{f}\left(\varphi_{2}\right)-0.2 \varphi_{2}+\varphi_{2}^{2}-\delta_{2} \chi_{2}-u_{51} D_{2}-u_{52} D_{21} \geq 0, \\
c_{5}=\varphi_{2}-U_{2}-0.00001 \geq 0,
\end{array}\right.
$$



Figure 6: An HS with two modes

|  | Exp. cond. |  | Our method |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Degree | Time(s) | Degree | Time(s) |  |
|  |  |  |  |  | Synthesis |
| E.g. [2 | $\times$ | $\times$ | 8 | 36.023 | Symb. checking |
| E.g. [3 | 8 | 1.132 | 6 | 2.717 | 0.766 |
| E.g. [] | 6 | 1.516 | 4 | 4.658 | 0.180 |
| E.g. [5] | 4 | 1.387 | 2 | 4.260 | 20.472 |

Table 1: Experimental data.
by SDP. In which, $g_{11}=1.01-x_{1}^{2}-0.01 x_{2}^{2}-0.01 x_{3}^{2}$, $g_{12}=x_{1}^{2}+0.01 x_{2}^{2}+0.01 x_{3}^{2}-0.99, D_{2}=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}-0.03$, $D_{21}=26.01-x_{1}^{2}, U_{2}=x_{1}^{2}-10.24$, and $u_{21}, u_{22}, u_{23}$, $u_{24}, u_{41}, u_{42}, u_{51}, u_{52}, \delta_{2}$ are SOS synthesized in the first step.

All the experimental results of all examples given in this paper can be summarised as in Table 1 in which the label $\times$ means that the corresponding method can not obtain a barrier certificate. All the results listed were computed on a 64-bit Intel(R) Core(TM) i5 CPU 650 @ 3.20 GHz with 4GB RAM memory and Ubuntu 12.04 GNU/Linux.

By comparing with the approach reported in 9 (see Table (1), our approach can synthesize more barrier certificates, in particular, with lower degree, but our approach takes more time. However, our approach is still very efficient, typically, symbolic checking can make our approach to avoid unsoundness because of the error due to numeric computation in SDP.

## 6. CONCLUDING REMARKS

To summarize, the contributions of this paper include:

- Relaxation of the conditions of barrier certificate in a general way, so that one can utilize weaker conditions flexibly to synthesize various kinds of barrier certificates with more expressiveness, which gives more opportunities to verify the considered system.
- A method to combining two functions together to form a combined barrier certificate in order to prove a safety property under consideration, whereas neither of them can be used as a barrier certificate separately.
- An approach to synthesizing certificates according to the general relaxed conditions by semi-definite programming. In particular, we discussed how to apply
symbolic checking to guarantee the soundness of our approach caused by the error of numeric computation in SDP.
- Experimental results demonstrating that our approach can indeed discover more certificates and give more opportunities to verify an HS under consideration.

For future work, we plan to combine more than two functions to form a combined barrier certificate. In particular, we are interested in finding more functions $\psi$ satisfying condition (5) and establishing a library for them. In addition, it is interesting to investigate how to recover the error caused by the numeric computation in SDP by some symbolic computation techniques.

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## APPENDIX

## A. THE DETAILS OF EXAMPLES

The polynomials synthesized in Example 2 are: $\varphi=0.030317 x_{1}^{8}+6.9115 e-05 x_{1}^{7} x_{2}-3.6889 e-05 x_{1}^{7}+0.090347 x_{1}^{6} x_{2}^{2}-$ $0.11095 x_{1}^{6} x_{2}-0.75683 x_{1}^{6}-9.0598 e-05 x_{1}^{5} x_{2}^{3}-0.00017438 x_{1}^{5} x_{2}^{2}-$ $5.4845 e-05 x_{1}^{5} x_{2}+7.291 e-05 x_{1}^{5}-0.30715 x_{1}^{4} x_{2}^{4}+1.0445 x_{1}^{4} x_{2}^{3}-$ $1.5458 x_{1}^{4} x_{2}^{2}+0.57141 x_{1}^{4} x_{2}-0.26344 x_{1}^{4}-6.3369 e-05 x_{1}^{3} x_{2}^{5}+0.00010503 x_{1}^{3} x_{2}^{4}+$ $0.00038237 x_{1}^{3} x_{2}^{3}-0.00036159 x_{1}^{3} x_{2}^{2}-0.00010184 x_{1}^{3} x_{2}+9.2214 e-05 x_{1}^{3}+$ $0.03383 x_{1}^{2} x_{2}^{6}+0.33103 x_{1}^{2} x_{2}^{5}-2.9864 x_{1}^{2} x_{2}^{4}+2.0938 x_{1}^{2} x_{2}^{3}-0.12636 x_{1}^{2} x_{2}^{2}+$ $0.79519 x_{1}^{2} x_{2}-0.62237 x_{1}^{2}+1.4962 e-05 x_{1} x_{2}^{7}-0.00014241 x_{1} x_{2}^{6}+$ $0.00048485 x_{1} x_{2}^{5}-0.00065416 x_{1} x_{2}^{4}+0.00014521 x_{1} x_{2}^{3}+0.00040002 x_{1} x_{2}^{2}-$ $0.00031516 x_{1} x_{2}+6.343 e-05 x_{1}-0.0043261 x_{2}^{8}+0.05803 x_{2}^{7}-0.29525 x_{2}^{6}+$ $0.80728 x_{2}^{5}-1.2538 x_{2}^{4}+1.2862 x_{2}^{3}-0.76567 x_{2}^{2}+0.29172 x_{2}-0.072688$, $\chi=9.8484 x_{1}^{6}+0.001271 x_{1}^{5} x_{2}+13.4422 x_{1}^{4} x_{2}^{2}-31.2496 x_{1}^{4} x_{2}-85.8767 x_{1}^{4}-$ $0.0031705 x_{1}^{3} x_{2}^{2}-0.012227 x_{1}^{3} x_{2}-0.0042103 x_{1}^{3}+5.396 x_{1}^{2} x_{2}^{4}-28.4976 x_{1}^{2} x_{2}^{3}-$ $46.3212 x_{1}^{2} x_{2}^{2}+87.5486 x_{1}^{2} x_{2}-44.1755 x_{1}^{2}+0.0049683 x_{1} x_{2}^{2}-0.0058767 x_{1} x_{2}-$ $0.0020784 x_{1}+0.46783 x_{2}^{6}-4.0071 x_{2}^{5}+6.1875 x_{2}^{4}+37.296 x_{2}^{3}-100 x_{2}^{2}+$ $2.0932 x_{2}-12.8904$,
$\delta=0.014034 x_{1}^{8}+3.0608 e-06 x_{1}^{7} x_{2}-5.813 e-06 x_{1}^{7}+0.0021473 x_{1}^{6} x_{2}^{2}-$ $0.013483 x_{1}^{6} x_{2}-0.0064165 x_{1}^{6}+2.5531 e-06 x_{1}^{5} x_{2}^{3}+2.7689 e-05 x_{1}^{5} x_{2}^{2}-$ $1.0371 e-05 x_{1}^{5} x_{2}-2.9253 e-06 x_{1}^{5}+0.02322 x_{1}^{4} x_{2}^{4}-0.008259 x_{1}^{4} x_{2}^{3}+$ $0.0095319 x_{1}^{4} x_{2}^{2}+0.014437 x_{1}^{4} x_{2}+0.02625 x_{1}^{4}+1.126 e-05 x_{1}^{3} x_{2}^{5}-3.9758 e-$ $05 x_{1}^{3} x_{2}^{4}+3.4426 e-05 x_{1}^{3} x_{2}^{3}-1.2647 e-05 x_{1}^{3} x_{2}^{2}+8.5785 e-06 x_{1}^{3} x_{2}-$ $5.7495 e-07 x_{1}^{3}+0.00051658 x_{1}^{2} x_{2}^{6}-0.010421 x_{1}^{2} x_{2}^{5}+0.032928 x_{1}^{2} x_{2}^{4}-$ $0.041062 x_{1}^{2} x_{2}^{3}+0.030059 x_{1}^{2} x_{2}^{2}-0.030394 x_{1}^{2} x_{2}+0.013862 x_{1}^{2}+5.2631 e-$ $07 x_{1} x_{2}^{7}-4.7302 e-06 x_{1} x_{2}^{6}+1.3673 e-05 x_{1} x_{2}^{5}-1.3654 e-05 x_{1} x_{2}^{4}+$ $8.3569 e-07 x_{1} x_{2}^{3}-1.6543 e-06 x_{1} x_{2}^{2}+1.3272 e-05 x_{1} x_{2}-8.3958 e-$ $06 x_{1}+0.00013121 x_{2}^{8}-0.0015345 x_{2}^{7}+0.0076214 x_{2}^{6}-0.019749 x_{2}^{5}+$ $0.02836 x_{2}^{4}-0.023961 x_{2}^{3}+0.01575 x_{2}^{2}-0.010837 x_{2}+0.0044092$,
$u_{1}=33.1703 x_{1}^{4}-0.0080558 x_{1}^{3} x_{2}-0.014014 x_{1}^{3}+31.8846 x_{1}^{2} x_{2}^{2}-22.7751 x_{1}^{2} x_{2}+$ $33.5594 x_{1}^{2}+0.002164 x_{1} x_{2}^{3}-0.0059715 x_{1} x_{2}^{2}-0.037073 x_{1} x_{2}+0.020061 x_{1}+$ $10.479 x_{2}^{4}+6.0815 x_{2}^{3}+19.5851 x_{2}^{2}-18.8795 x_{2}+24.5699$, $u_{2}=0.579 x_{1}^{8}+7.0204 e-06 x_{1}^{7} x_{2}-1.8771 e-05 x_{1}^{7}+0.61572 x_{1}^{6} x_{2}^{2}-$ $0.43594 x_{1}^{6} x_{2}+0.39633 x_{1}^{6}+4.0635 e-06 x_{1}^{5} x_{2}^{3}+5.4444 e-06 x_{1}^{5} x_{2}^{2}-$ $9.0679 e-06 x_{1}^{5} x_{2}+4.1779 e-05 x_{1}^{5}+0.5972 x_{1}^{4} x_{2}^{4}-0.446 x_{1}^{4} x_{2}^{3}+0.8667 x_{1}^{4} x_{2}^{2}-$ $0.48811 x_{1}^{4} x_{2}+0.57967 x_{1}^{4}+2.0738 e-06 x_{1}^{3} x_{2}^{5}+4.4963 e-06 x_{1}^{3} x_{2}^{4}-$ $3.9037 e-06 x_{1}^{3} x_{2}^{3}-1.5008 e-05 x_{1}^{3} x_{2}^{2}-6.0762 e-05 x_{1}^{3} x_{2}+3.4303 e-$ $05 x_{1}^{3}+0.42761 x_{1}^{2} x_{2}^{6}-0.20453 x_{1}^{2} x_{2}^{5}+0.45199 x_{1}^{2} x_{2}^{4}-0.55762 x_{1}^{2} x_{2}^{3}+$ $0.80255 x_{1}^{2} x_{2}^{2}-0.18571 x_{1}^{2} x_{2}+0.36852 x_{1}^{2}+5.1943 e-06 x_{1} x_{2}^{7}-5.229 e-$ $06 x_{1} x_{2}^{6}+4.0646 e-06 x_{1} x_{2}^{5}-8.2704 e-06 x_{1} x_{2}^{4}+1.2324 e-05 x_{1} x_{2}^{3}-$ $3.1593 e-06 x_{1} x_{2}^{2}-8.7493 e-06 x_{1} x_{2}+3.0456 e-06 x_{1}+0.18043 x_{2}^{8}+$ $0.1527 x_{2}^{7}+0.11373 x_{2}^{6}+0.090147 x_{2}^{5}+0.40667 x_{2}^{4}-0.26137 x_{2}^{3}+0.68588 x_{2}^{2}-$ $0.38649 x_{2}+0.50807$,
$u_{3}=0.82691 x_{1}^{8}+6.8463 e-06 x_{1}^{7} x_{2}+7.824 e-06 x_{1}^{7}+0.66339 x_{1}^{6} x_{2}^{2}+$ $0.69976 x_{1}^{6} x_{2}+0.71112 x_{1}^{6}+2.8381 e-06 x_{1}^{5} x_{2}^{3}+2.1678 e-05 x_{1}^{5} x_{2}^{2}-$ $2.269 e-05 x_{1}^{5} x_{2}-2.7155 e-05 x_{1}^{5}+0.67426 x_{1}^{4} x_{2}^{4}+0.31337 x_{1}^{4} x_{2}^{3}+$ $1.0011 x_{1}^{4} x_{2}^{2}+0.41117 x_{1}^{4} x_{2}+0.94169 x_{1}^{4}+5.177 e-06 x_{1}^{3} x_{2}^{5}+2.4687 e-$ $05 x_{1}^{3} x_{2}^{4}+4.9193 e-05 x_{1}^{3} x_{2}^{3}-8.5264 e-05 x_{1}^{3} x_{2}^{2}-9.261 e-05 x_{1}^{3} x_{2}-$ $0.00018356 x_{1}^{3}+0.55287 x_{1}^{2} x_{2}^{6}+0.26734 x_{1}^{2} x_{2}^{5}+0.43798 x_{1}^{2} x_{2}^{4}-0.42762 x_{1}^{2} x_{2}^{3}+$ $0.90269 x_{1}^{2} x_{2}^{2}+0.47337 x_{1}^{2} x_{2}+0.73082 x_{1}^{2}+6.4342 e-06 x_{1} x_{2}^{7}+1.7535 e-$ $05 x_{1} x_{2}^{6}+3.191 e-05 x_{1} x_{2}^{5}+2.9182 e-05 x_{1} x_{2}^{4}+7.677 e-05 x_{1} x_{2}^{3}+$ $2.0006 e-05 x_{1} x_{2}^{2}-3.1684 e-05 x_{1} x_{2}-6.6486 e-06 x_{1}+0.45107 x_{2}^{8}-$ $0.15576 x_{2}^{7}+0.12208 x_{2}^{6}-0.29031 x_{2}^{5}+0.39853 x_{2}^{4}-0.57126 x_{2}^{3}+0.29347 x_{2}^{2}-$ $0.33244 x_{2}+0.43254$,
$\psi_{i}(\theta)=\psi_{2}(\theta)=-4 \theta+2 \theta^{2}$.
The polynomials synthesized in Example 3 are :
$\varphi=9.8484 x_{1}^{6}+0.001271 x_{1}^{5} x_{2}+13.4422 x_{1}^{4} x_{2}^{2}-31.2496 x_{1}^{4} x_{2}-85.8767 x_{1}^{4}-$ $0.0031705 x_{1}^{3} x_{2}^{2}-0.012227 x_{1}^{3} x_{2}-0.0042103 x_{1}^{3}+5.396 x_{1}^{2} x_{2}^{4}-28.4976 x_{1}^{2} x_{2}^{3}-$ $46.3212 x_{1}^{2} x_{2}^{2}+87.5486 x_{1}^{2} x_{2}-44.1755 x_{1}^{2}+0.0049683 x_{1} x_{2}^{2}-0.0058767 x_{1} x_{2}-$ $0.0020784 x_{1}+0.46783 x_{2}^{6}-4.0071 x_{2}^{5}+6.1875 x_{2}^{4}+37.296 x_{2}^{3}-100 x_{2}^{2}+$ $2.0932 x_{2}-12.8904$,
$\chi=0, \delta=0, u 1=0$,
$u_{2}=9.8484 x_{1}^{6}+0.001271 x_{1}^{5} x_{2}+13.4422 x_{1}^{4} x_{2}^{2}-31.2496 x_{1}^{4} x_{2}-85.8767 x_{1}^{4}-$ $0.0031705 x_{1}^{3} x_{2}^{2}-0.012227 x_{1}^{3} x_{2}-0.0042103 x_{1}^{3}+5.396 x_{1}^{2} x_{2}^{4}-28.4976 x_{1}^{2} x_{2}^{3}-$ $46.3212 x_{1}^{2} x_{2}^{2}+87.5486 x_{1}^{2} x_{2}-44.1755 x_{1}^{2}+0.0049683 x_{1} x_{2}^{2}-0.0058767 x_{1} x_{2}-$ $0.0020784 x_{1}+0.46783 x_{2}^{6}-4.0071 x_{2}^{5}+6.1875 x_{2}^{4}+37.296 x_{2}^{3}-100 x_{2}^{2}+$ $2.0932 x_{2}-12.8904$,
$u_{3}=9.8484 x_{1}^{6}+0.001271 x_{1}^{5} x_{2}+13.4422 x_{1}^{4} x_{2}^{2}-31.2496 x_{1}^{4} x_{2}-85.8767 x_{1}^{4}-$ $0.0031705 x_{1}^{3} x_{2}^{2}-0.012227 x_{1}^{3} x_{2}-0.0042103 x_{1}^{3}+5.396 x_{1}^{2} x_{2}^{4}-28.4976 x_{1}^{2} x_{2}^{3}-$ $46.3212 x_{1}^{2} x_{2}^{2}+87.5486 x_{1}^{2} x_{2}-44.1755 x_{1}^{2}+0.0049683 x_{1} x_{2}^{2}-0.0058767 x_{1} x_{2}-$ $0.0020784 x_{1}+0.46783 x_{2}^{6}-4.0071 x_{2}^{5}+6.1875 x_{2}^{4}+37.296 x_{2}^{3}-100 x_{2}^{2}+$ $2.0932 x_{2}-12.8904$,
$\psi_{1}=0, \psi_{2}(\theta)=-4 \theta+1.5 \theta^{2}$.
The polynomials synthesized in Example 5are: $\varphi_{2}=1.6165 x_{1}^{2}-$ $0.20569 x_{1} x_{2}+0.19824 e-1 x_{1} x_{3}+0.95436 e-5 x_{1}+0.54446 e-1 x_{2}^{2}+$ $0.69996 e-3 x_{2} x_{3}-0.16916 e-6 x_{2}+0.9101 e-1 x_{3}^{2}+0.1511 e-7 x_{3}-$ 9.6424
$\chi_{2}=0.89818 e-1 x_{1}^{2}-0.82739 e-1 x_{1} x_{2}+0.21192 e-1 x_{1} x_{3}-0.15224 e-$ $8 x_{1}+0.54928 e-2 x_{2}^{2}+0.84123 e-2 x_{2} x_{3}+0.1277 e-8 x_{2}+0.35173 e-$ $1 x_{3}^{2}+0.27238 e-9 x_{3}-5.3973$
$\delta_{2}=5.5914 x_{1}^{2}-0.21067 x_{1} x_{2}-0.24733 e-1 x_{1} x_{3}+0.87702 e-5 x_{1}+$ $0.20573 x_{2}^{2}-0.52174 e-1 x_{2} x_{3}+0.28769 e-6 x_{2}+0.22449 x_{3}^{2}+0.87144 e-$ $7 x_{3}+0.29484$
$u_{21}=1.5356 x_{1}^{2}+0.13731 e-1 x_{1} x_{2}-0.19249 e-2 x_{1} x_{3}-0.10079 e-$ $6 x_{1}+0.66295 x_{2}^{2}-0.64549 e-1 x_{2} x_{3}-0.63485 e-7 x_{2}+0.39611 x_{3}^{2}-$ $0.66953 e-8 x_{3}+2.6867$
$u_{22}=0.73288 x_{1}^{2}-0.22775 e-2 x_{1} x_{2}+0.27401 e-2 x_{1} x_{3}-0.51154 e-$ $7 x_{1}+0.59472 x_{2}^{2}-0.55279 e-1 x_{2} x_{3}-0.48206 e-7 x_{2}+0.34978 x_{3}^{2}-$ $0.74061 e-8 x_{3}+0.60632$
$u_{23}=2.0821 x_{1}^{2}+0.40593 e-1 x_{1} x_{2}-0.50855 e-2 x_{1} x_{3}-0.8427 e-$ $4 x_{1}+0.61146 x_{2}^{2}-0.90046 e-2 x_{2} x_{3}-0.83808 e-5 x_{2}+0.14389 x_{3}^{2}-$ $0.1148 e-5 x_{3}+4.5124$
$u_{24}=1.0004 x_{1}^{2}+0.1131 e-1 x_{1} x_{2}-0.22779 e-2 x_{1} x_{3}-0.288 e-$ $4 x_{1}+0.517 x_{2}^{2}-0.80914 e-2 x_{2} x_{3}-0.11074 e-4 x_{2}+0.83205 e-1 x_{3}^{2}-$ $0.82264 e-6 x_{3}+0.70099$
$u_{41}=0.43056 e-3 x_{1}^{2}-0.29796 e-4 x_{1} x_{2}+0.10489 e-3 x_{1} x_{3}+$ $0.59287 e-11 x_{1}+3.8141 e-6 x_{2}^{2}+0.95752 e-5 x_{2} x_{3}+0.26518 e-$ $12 x_{2}+0.39903 e-4 x_{3}^{2}+0.51833 e-12 x_{3}+0.36 e-2$
$u_{42}=0.56936 e-2 x_{1}^{2}+0.53069 e-2 x_{1} x_{2}+0.35737 e-2 x_{1} x_{3}-$ $0.75779 e-10 x_{1}+0.20039 e-2 x_{2}^{2}+0.26891 e-2 x_{2} x_{3}+0.29818 e-$ $10 x_{2}+0.16505 e-2 x_{3}^{2}-0.16159 e-10 x_{3}+0.52902$
$u_{51}=0.28447 e-1 x_{1}^{2}-0.28324 e-2 x_{1} x_{2}-0.37952 e-3 x_{1} x_{3}+0.125 e-$ $6 x_{1}+0.52784 e-3 x_{2}^{2}-0.86183 e-4 x_{2} x_{3}-0.63026 e-8 x_{2}+0.88611 e-$ $4 x_{3}^{2}-0.86257 e-9 x_{3}+0.11143 e-1$
$u_{52}=0.12129 x_{1}^{2}-0.13405 e-1 x_{1} x_{2}-0.15008 e-2 x_{1} x_{3}-0.82285 e-$ $6 x_{1}+0.47845 e-2 x_{2}^{2}-0.73624 e-3 x_{2} x_{3}-0.20348 e-6 x_{2}+0.52816 e-$ $3 x_{3}^{2}+0.21178 e-7 x_{3}+3.5079$


[^0]:    ${ }^{1}$ Local Lipschitz condition guarantees the existence and uniqueness of the solution of (1) from any initial $\mathbf{x}_{0}$.

[^1]:    ${ }^{2} \mathrm{~A}$ subset $A \subseteq \mathbb{R}^{n}$ is called semi-algebraic if there is a quantifier-free $\overline{\text { polynomial formula } \varphi \text { expressed in Tarski's }}$ algebra s.t. $A=\left\{\mathbf{x} \in \mathbb{R}^{n} \mid \varphi(\mathbf{x})\right.$ is true $\}$.

[^2]:    ${ }^{3}$ That is, $\delta$ can be represented by $f_{1}^{2}+\ldots+f_{n}^{2}$, where

[^3]:    ${ }^{4}$ SOSTOOLS is of version v2.04 with MATLAB R2011b.

