# Polynomial Reduction and Super Congruences 

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#### Abstract

Based on a reduction processing, we rewrite a hypergeometric term as the sum of the difference of a hypergeometric term and a reduced hypergeometric term (the reduced part, in short). We show that when the initial hypergeometric term has a certain kind of symmetry, the reduced part contains only odd or even powers. As applications, we derived two infinite families of super-congruences.


## 1 Introduction

In recent years, many super congruences involving combinatorial sequences are discovered, see for example, Sun [16]. The standard methods for proving these congruences include combinatorial identities [18, Gauss sums [5], symbolic computation [14 et al.

We are interested in the following super congruence conjectured by van

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$$
\sum_{k=0}^{\frac{p-1}{2}}(-1)^{k}(4 k+1)\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{3} \equiv(-1)^{\frac{p-1}{2}} p \quad\left(\bmod p^{3}\right)
$$

where $p$ is a odd prime and $(a)_{k}=a(a+1) \cdots(a+k-1)$ is the rising factorial. This congruence was proved by Mortenson [13] Zudilin [21] and Long [12] by different methods. Sun [17] proved a stronger version for prime $p \geq 5$

$$
\sum_{k=0}^{\frac{p-1}{2}}(-1)^{k}(4 k+1)\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{3} \equiv(-1)^{\frac{p-1}{2}} p+p^{3} E_{p-3} \quad\left(\bmod p^{4}\right)
$$

where $E_{n}$ is the $n$-th Euler number defined by

$$
\frac{2}{e^{x}+e^{-x}}=\sum_{n=0}^{\infty} E_{n} \frac{x^{n}}{n!}
$$

A similar congruence was given by van Hamme [19] for $p \equiv 1(\bmod 4)$ :

$$
\sum_{k=0}^{\frac{p-1}{2}}(4 k+1)\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{4} \equiv p \quad\left(\bmod p^{3}\right)
$$

Long [12] showed that in fact the above congruence holds for arbitrary odd prime modulo $p^{4}$. Motivated by these two congruences, Guo [8] proposed the following conjectures (corrected version).

Conjecture 1.1 - For any odd prime $p$, positive integer $r$ and odd integer $m$, there exists an integer $a_{m . p}$ such that

$$
\begin{equation*}
\sum_{k=0}^{\frac{p^{r}-1}{2}}(-1)^{k}(4 k+1)^{m}\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{3} \equiv a_{m, p} p^{r}(-1)^{\frac{(p-1) r}{2}} \quad\left(\bmod p^{r+2}\right) \tag{1.1}
\end{equation*}
$$

- For any odd prime $p>(m-1) / 2$, positive integer $r$ and odd integer $m$, there exists an integer $b_{m, p}$ such that

$$
\begin{equation*}
\sum_{k=0}^{\frac{p^{r}-1}{2}}(4 k+1)^{m}\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{4} \equiv b_{m, p} p^{r} \quad\left(\bmod p^{r+3}\right) \tag{1.2}
\end{equation*}
$$

Liu [11] and Wang [20] confirmed the conjectures for $r=1$ and some initial values $m$. Jana and Kalita [10] and Guo [9 confirmed (1.1) for $m=3$ and $r \geq 1$. We will prove a stronger version of (1.1) for the case of $r=1$ and arbitrary odd $m$ and a weaker version of (1.2) for the case of $r=1$ and arbitrary odd $m$ by a reduction process.

Recall that a hypergeometric term $t_{k}$ is a function of $k$ such that $t_{k+1} / t_{k}$ is a rational function of $k$. Our basic idea is to rewrite the product of a polynomial $f(k)$ in $k$ and a hypergeometric term $t_{k}$ as

$$
f(k) t_{k}=\Delta\left(g(k) t_{k}\right)+h(k) t_{k}=\left(g(k+1) t_{k+1}-g(k) t_{k}\right)+h(k) t_{k},
$$

where $g(k), h(k)$ are polynomials in $k$ such that the degree of $h(k)$ is bounded. To this aim, we construct $x(k)$ such that $\Delta x(k) t_{k}$ equals the product of $t_{k}$ and a polynomial $u(k)$ and that $f(k)$ and $u(k)$ has the same leading term. Then we have

$$
f(k) t_{k}-\Delta x(k) t_{k}=(f(k)-u(k)) t_{k}
$$

is the product of $t_{k}$ and a polynomial of degree less than $f(k)$. We call such a reduction process one reduction step. Continuing this reduction process, we finally obtain a polynomial $h(k)$ with bounded degree. We will show that for $t_{k}=\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{r}, r=3,4$ and an arbitrary polynomial of form $(4 k+1)^{m}$ with $m$ odd, the reduced polynomial $h(k)$ can be taken as $(4 k+1)$. This enables us to reduce the congruences (1.1) and (1.2) to the special case of $m=1$, which is known for $r=1$.

We notice that Pirastu-Strehl [15] and Abramov [1, 2] gave the minimal decomposition when $t_{k}$ is a rational function, Abramov-Petkovšek [3, 4] gave the minimal decomposition when $t_{k}$ is a hypergeometric term, and Chen-Huang-Kauers-Li [6] applied the reduction to give an efficient creative telescoping algorithm. These algorithms concern a general hypergeometric term. While we focus on a kind of special hypergeometric term so that the reduced part $h(k) t_{k}$ has a nice form.

The paper is organized as follows. In Section 2, we consider the reduction process for a general hypergeometric term $t_{k}$. Then in Section 3 we consider those $t_{k}$ with the property $a(k)$ is a shift of $-b(k)$, where $t_{k+1} / t_{k}=a(k) / b(k)$. As an application, we prove a stronger version of (1.1) for the case $r=1$. Finally, we consider the case of $a(k)$ is a shift of $b(k)$, which corresponds to (1.2). In this case, we show that there is a rational number $b_{m}$ instead of an integer such that (1.2) holds when $r=1$.

## 2 The Difference Space and Polynomial Reduction

Let $K$ be a field and $K[k]$ be the ring of polynomials in $k$ with coefficients in $K$. Let $t_{k}$ be a hypergeometric term. Suppose that

$$
\frac{t_{k+1}}{t_{k}}=\frac{a(k)}{b(k)},
$$

where $a(k), b(k) \in K[k]$. It is straightforward to verify that

$$
\begin{equation*}
\Delta_{k}\left(b(k-1) x(k) t_{k}\right)=(a(k) x(k+1)-b(k-1) x(k)) t_{k} . \tag{2.1}
\end{equation*}
$$

We thus define the difference space corresponding to $a(k)$ and $b(k)$ to be

$$
S_{a, b}=\{a(k) x(k+1)-b(k-1) x(k): x(k) \in K[k]\} .
$$

We see that for $f(k) \in S_{a, b}$, we have $f(k) t_{k}=\Delta_{k}\left(p(k) t_{k}\right)$ for a certain polynomial $p(k) \in K[k]$.

Let $\mathbb{N}, \mathbb{Z}$ denote the set of nonnegative integers and the set of integers, respectively. Given $a(k), b(k) \in K[k]$, we denote

$$
\begin{gather*}
u(k)=a(k)-b(k-1),  \tag{2.2}\\
d=\max \{\operatorname{deg} u(k), \operatorname{deg} a(k)-1\}, \tag{2.3}
\end{gather*}
$$

and

$$
\begin{equation*}
m_{0}=-\operatorname{lc} u(k) / \operatorname{lc} a(k), \tag{2.4}
\end{equation*}
$$

where lc $p(k)$ denotes the leading coefficient of $p(k)$.
We first introduce the concept of degeneration.

Definition 2.1 Let $a(k), b(k) \in K[k]$ and $u(k), m_{0}$ be given by (2.2) and (2.4). If

$$
\operatorname{deg} u(k)=\operatorname{deg} a(k)-1 \quad \text { and } \quad m_{0} \in \mathbb{N},
$$

we say that the pair $(a(k), b(k))$ is degenerated.

We will see that the degeneration is closely related to the degrees of the elements in $S_{a, b}$.

Lemma 2.2 Let $a(k), b(k) \in K[k]$ and $d, m_{0}$ be given by (2.3) and (2.4). For any polynomial $x(k) \in K[k]$, let

$$
p(k)=a(k) x(k+1)-b(k-1) x(k)
$$

If $(a(k), b(k))$ is degenerated and $\operatorname{deg} x(k)=m_{0}$, then $\operatorname{deg} p(k)<d+m_{0}$; Otherwise, $\operatorname{deg} p(k)=d+\operatorname{deg} x(k)$.

Proof. Notice that

$$
p(k)=u(k) x(k)+a(k)(x(k+1)-x(k))
$$

If the leading terms of $u(k) x(k)$ and $a(k)(x(k+1)-x(k))$ do not cancel, the degree of $p(k)$ is $d+\operatorname{deg} x(k)$. Otherwise, we have $\operatorname{deg} u(k)=\operatorname{deg} a(k)-1$ and

$$
\operatorname{lc} u(k)+\operatorname{lc} a(k) \cdot \operatorname{deg} x(k)=0
$$

i.e., $\operatorname{deg} x(k)=m_{0}$.

It is clear that $S_{a, b}$ is a subspace of $K[k]$, but is not a sub-ring of $K[k]$ in general. Let $[p(k)]=p(k)+S_{a, b}$ denote the coset of a polynomial $p(k)$. We see that the quotient space $K[k] / S_{a, b}$ is finite dimensional.

Theorem 2.3 Let $a(k), b(k) \in K[k]$ and $d, m_{0}$ be given by (2.3) and (2.4). We have
$K[k] / S_{a, b}= \begin{cases}\left\langle\left[k^{0}\right],\left[k^{1}\right], \ldots,\left[k^{d-1}\right],\left[k^{d+m_{0}}\right]\right\rangle, & \text { if }(a(k), b(k)) \text { is degenerated }, \\ \left\langle\left[k^{0}\right],\left[k^{1}\right], \ldots,\left[k^{d-1}\right]\right\rangle, & \text { otherwise } .\end{cases}$

Proof. For any nonnegative integer $s$, let

$$
p_{s}(k)=a(k)(k+1)^{s}-b(k-1) k^{s}
$$

We first consider the case when the pair $(a(k), b(k))$ is not degenerated. By Lemma 2.2, we have

$$
\operatorname{deg} p_{s}(k)=d+s, \quad \forall s \geq 0
$$

Suppose that $p(k)$ is a polynomial of degree $m \geq d$. Then

$$
\begin{equation*}
p^{\prime}(k)=p(k)-\frac{\operatorname{lc} p(k)}{\operatorname{lc} p_{m-d}(k)} p_{m-d}(k) \tag{2.5}
\end{equation*}
$$

is a polynomial of degree less than $m$ and $p(k) \in\left[p^{\prime}(k)\right]$. By induction on $m$, we derive that for any polynomial $p(k)$ of degree $\geq d$, there exists a polynomial $\tilde{p}(k)$ of degree $<d$ such that $p(k) \in[\tilde{p}(k)]$. Therefore,

$$
K[k] / S_{a, b}=\left\langle\left[k^{0}\right],\left[k^{1}\right], \ldots,\left[k^{d-1}\right]\right\rangle .
$$

Now assume that $(a(k), b(k))$ is degenerated. By Lemma 2.2,

$$
\operatorname{deg} p_{s}(k)=d+s, \quad \forall s \neq m_{0} \quad \text { and } \quad \operatorname{deg} p_{m_{0}}(k)<d+m_{0} .
$$

The above reduction process (2.5) works well except for the polynomials $p(k)$ of degree $d+m_{0}$. But in this case,

$$
p(k)-\operatorname{lc} p(k) \cdot k^{d+m_{0}}
$$

is a polynomial of degree less than $d+m_{0}$. Then the reduction process continues until the degree is less than $d$. We thus derive that

$$
K[k] / S_{a, b}=\left\langle\left[k^{0}\right],\left[k^{1}\right], \ldots,\left[k^{d-1}\right],\left[k^{d+m_{0}}\right]\right\rangle,
$$

completing the proof.

Example 2.1 Let $n$ be a positive integer and

$$
t_{k}=(-n)_{k} / k!
$$

where $(\alpha)_{k}=\alpha(\alpha+1) \cdots(\alpha+k-1)$ is the raising factorial. Then

$$
a(k)=k-n, \quad b(k)=k+1,
$$

and

$$
S_{a, b}=\{(k-n) \cdot x(k+1)-k \cdot x(k): x(k) \in K[k]\} .
$$

We have

$$
K[k] / S_{a, b}=\left\langle\left[k^{n}\right]\right\rangle
$$

is of dimension one.

## 3 The case when $a(k)=-b(k+\alpha)$

In this section, we consider the case when $a(k)=-b(k+\alpha)$ and $b(k)$ has a symmetric property. We will show that in this case, the reduction process maintains the symmetric property. Notice that in this case

$$
u(k)=a(k)-b(k-1)=-b(k+\alpha)-b(k-1)
$$

has the same degree as $a(k)$, the pair $(a(k), b(k))$ is not degenerated.
We first consider the relation between the symmetric property and the expansion of a polynomial.

Lemma 3.1 Let $p(k) \in K[k]$ and $\beta \in K$. Then the following two statements are equivalent.
(1) $p(\beta+k)=p(\beta-k)(p(\beta+k)=-p(\beta-k)$, respectively $)$.
(2) $p(k)$ is the linear combination of $(k-\beta)^{2 i}, i=0,1, \ldots\left((k-\beta)^{2 i+1}, i=\right.$ $0,1, \ldots$, respectively).

Proof. Suppose that

$$
p(\beta+k)=\sum_{i} c_{i} k^{i} .
$$

Then

$$
p(\beta-k)=\sum_{i} c_{i}(-k)^{i} .
$$

Therefore,

$$
p(\beta+k)=p(\beta-k) \Longleftrightarrow c_{2 i+1}=0, i=0,1, \ldots .
$$

The case of $p(\beta+k)=-p(\beta-k)$ can be proved in a similar way.
Now we are ready to state the main theorem.
Theorem 3.2 Let $a(k), b(k) \in K[k]$ such that

$$
a(k)=-b(k+\alpha) \quad \text { and } \quad b(\beta+k)= \pm b(\beta-k),
$$

for some $\alpha, \beta \in K$. Then for any non-negative integer $m$, we have

$$
\left[(k+\gamma)^{2 m}\right] \in\left\langle\left[(k+\gamma)^{2 i}\right]: 0 \leq 2 i<\operatorname{deg} a(k)\right\rangle
$$

and

$$
\left[(k+\gamma)^{2 m+1}\right] \in\left\langle\left[(k+\gamma)^{2 i+1}\right]: 0 \leq 2 i+1<\operatorname{deg} a(k)\right\rangle
$$

where

$$
\begin{equation*}
\gamma=-\beta+\frac{\alpha-1}{2} \tag{3.1}
\end{equation*}
$$

Proof. We only prove the case of $b(\beta+k)=b(\beta-k)$. The case of $b(\beta+k)=$ $-b(\beta-k)$ can be proved in a similar way. By Lemma 3.1, we may assume that

$$
b(k)=b_{r}(k-\beta)^{r}+b_{r-2}(k-\beta)^{r-2}+\cdots+b_{0}
$$

where $r=\operatorname{deg} a(k)=\operatorname{deg} b(k)$ is even and $b_{r}, b_{r-2}, \ldots, b_{0} \in K$ are the coefficients.

Since $(a(k), b(k))$ is not degenerated, taking

$$
\begin{equation*}
x(k)=x_{s}(k)=-\frac{1}{2}\left(k+\gamma-\frac{1}{2}\right)^{s} \tag{3.2}
\end{equation*}
$$

in Lemma 2.2, we derive that

$$
\begin{equation*}
p_{s}(k)=a(k) x_{s}(k+1)-b(k-1) x_{s}(k) \tag{3.3}
\end{equation*}
$$

is a polynomial of degree $s+r$. More explicitly, we have

$$
p_{s}(k)=\frac{1}{2}\left(b(k+\alpha)\left(k+\gamma+\frac{1}{2}\right)^{s}+b(k-1)\left(k+\gamma-\frac{1}{2}\right)^{s}\right)
$$

is a polynomial with leading term $b_{r} k^{s+r}$.
Notice that

$$
p_{s}(-\gamma+k)=\frac{1}{2}\left(b(k+\alpha-\gamma)\left(k+\frac{1}{2}\right)^{s}+b(k-\gamma-1)\left(k-\frac{1}{2}\right)^{s}\right)
$$

and

$$
\begin{aligned}
p_{s}(-\gamma-k) & =\frac{1}{2}\left(b(-k+\alpha-\gamma)\left(-k+\frac{1}{2}\right)^{s}+b(-k-\gamma-1)\left(-k-\frac{1}{2}\right)^{s}\right) \\
& =\frac{(-1)^{s}}{2}\left(b(-k+\alpha-\gamma)\left(k-\frac{1}{2}\right)^{s}+b(-k-\gamma-1)\left(k+\frac{1}{2}\right)^{s}\right)
\end{aligned}
$$

Since $b(\beta+k)=b(\beta-k)$, i.e., $b(k)=b(2 \beta-k)$, we deduce that

$$
\begin{aligned}
& p_{s}(-\gamma-k) \\
& =\frac{(-1)^{s}}{2}\left(b(2 \beta+k-\alpha+\gamma)\left(k-\frac{1}{2}\right)^{s}+b(2 \beta+k+\gamma+1)\left(k+\frac{1}{2}\right)^{s}\right) .
\end{aligned}
$$

By the relation (3.1), we derive that

$$
p_{s}(-\gamma-k)=(-1)^{s} p_{s}(-\gamma+k) .
$$

Suppose that $p(k)$ is a linear combination of the even powers of $(k+\gamma)$ and $\operatorname{deg} p(k) \geq r$. By Lemma 3.1, we have $p(-\gamma-k)=p(-\gamma+k)$ and thus

$$
p^{\prime}(k)=p(k)-\frac{\operatorname{lc} p(k)}{b_{r}} \cdot p_{\operatorname{deg} p(k)-r}(k)
$$

also satisfies $p^{\prime}(-\gamma-k)=p^{\prime}(-\gamma+k)$ since $\operatorname{deg} p(k)$ and $r$ are both even. It is clear that $p(k) \in\left[p^{\prime}(k)\right]$ and the degree of $p^{\prime}(k)$ is less than the degree of $p(k)$. Continuing this reduction process, we finally derive that $p(k) \in[\tilde{p}(k)]$ for some polynomial $\tilde{p}(k)$ with degree $<r$ and satisfying $\tilde{p}(-\gamma-k)=\tilde{p}(-\gamma+k)$. Therefore,

$$
[p(k)] \in\left\langle\left[(k+\gamma)^{2 i}\right]: 0 \leq 2 i<r\right\rangle .
$$

Suppose that $p(k)$ is a linear combination of the odd powers of $(k+\gamma)$ and $\operatorname{deg} p(k) \geq r$. Then we have $p(-\gamma-k)=-p(-\gamma+k)$ and thus

$$
p^{\prime}(k)=p(k)-\frac{\operatorname{cc} p(k)}{b_{r}} \cdot p_{\operatorname{deg} p(k)-r}(k)
$$

also satisfies $p^{\prime}(-\gamma-k)=-p^{\prime}(-\gamma+k)$. Continuing this reduction process, we finally derive that

$$
[p(k)] \in\left\langle\left[(k+\gamma)^{2 i+1}\right]: 0 \leq 2 i+1<r\right\rangle .
$$

This completes the proof.
We may further require to express $\left[(k+\gamma)^{m}\right]$ as an integral linear combination of $\left[(k+\gamma)^{i}\right], 0 \leq i<r$ when $b(k)=(k+1)^{r}$.

Theorem 3.3 Let

$$
t_{k}=(-1)^{k}\left(\frac{(\alpha)_{k}}{k!}\right)^{r}
$$

where $r$ is a positive integer and $\alpha$ is a rational number with denominator $D$. Then for any positive integer $m$, there exist integers $a_{0}, \ldots, a_{r-1}$ and $a$ polynomial $x(k) \in \mathbb{Z}[k]$ such that

$$
(2 D k+D \alpha)^{m} t_{k}=\sum_{i=0}^{r-1} a_{i}(2 D k+D \alpha)^{i} t_{k}+\Delta_{k}\left(2^{r-1}(D k)^{r} x(2 D k) t_{k}\right)
$$

Moreover, $a_{i}=0$ if $i \not \equiv m(\bmod 2)$.

Proof. We have

$$
\frac{t_{k+1}}{t_{k}}=\frac{-(k+\alpha)^{r}}{(k+1)^{r}}
$$

Let

$$
a(k)=-(k+\alpha)^{r} \quad \text { and } \quad b(k)=(k+1)^{r} .
$$

We see that it is the case of $\beta=-1$ and $\gamma=\alpha / 2$ of Theorem 3.2. From (2.1), we derive that

$$
\begin{equation*}
\Delta_{k}\left(k^{r} x_{s}(k) t_{k}\right)=p_{s}(k) t_{k} \tag{3.4}
\end{equation*}
$$

where $x_{s}(k)$ and $p_{s}(k)$ are given by (3.2) and (3.3) respectively. Multiplying $(2 D)^{s+r}$ on both sides, we obtain

$$
\begin{equation*}
\Delta_{k}\left(2^{r-1}(D k)^{r} \tilde{x}_{s}(2 D k) t_{k}\right)=\tilde{p}_{s}\left(k^{\prime}\right) t_{k}, \tag{3.5}
\end{equation*}
$$

where $k^{\prime}=2 D k+D \alpha$,

$$
\begin{equation*}
\tilde{x}_{s}(k)=-(k+D \alpha-D)^{s}, \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{p}_{s}(k)=\frac{1}{2}\left((k+D \alpha)^{r}(k+D)^{s}+(k-D \alpha)^{r}(k-D)^{s}\right) . \tag{3.7}
\end{equation*}
$$

Notice that $\tilde{x}_{s}(k), \tilde{p}_{s}(k) \in \mathbb{Z}[k]$ and $\tilde{p}_{s}(k)$ is a monic polynomial of degree $s+r$. Moreover, $\tilde{p}_{s}(k)$ contains only even powers of $k$ or only odd powers of $k$. Using $\tilde{p}_{s}(k)$ to do the reduction (2.5), we derive that there exists integers $c_{m}, c_{m-2}, \ldots$ such that

$$
p(k)=k^{m}-c_{m} \tilde{p}_{m-r}(k)-c_{m-2} \tilde{p}_{m-r-2}(k)-\cdots
$$

becomes a polynomial of degree less than $r$. Clearly, $p(k) \in \mathbb{Z}[k]$. Replacing $k$ by $k^{\prime}$ and multiplying $t_{k}$, we derive that
$\left(k^{\prime}\right)^{m} t_{k}=p\left(k^{\prime}\right) t_{k}+\Delta_{k}\left(2^{r-1}(D k)^{r}\left(c_{m} \tilde{x}_{m-r}(2 D k)+c_{m-2} \tilde{x}_{m-r-2}(2 D k)+\cdots\right) t_{k}\right)$, completing the proof.

As an application, we confirm Conjecture 6 of [11].

## Theorem 3.4 Let

$$
S_{m}=\sum_{k=0}^{\frac{p-1}{2}}(-1)^{k}(4 k+1)^{m}\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{3} .
$$

For any positive odd integer $m$, there exist integers $a_{m}$ and $c_{m}$ such that

$$
S_{m} \equiv a_{m}\left(p(-1)^{\frac{p-1}{2}}+p^{3} E_{p-3}\right)+p^{3} c_{m} \quad\left(\bmod p^{4}\right)
$$

holds for any prime $p \geq 5$.

Proof. Taking $r=3$ and $\alpha=1 / 2$ in Theorem 3.3, there exist an integer $a_{m}$ and a polynomial $q_{m}(k) \in \mathbb{Z}[k]$ such that

$$
(4 k+1)^{m} t_{k}-a_{m}(4 k+1) t_{k}=\Delta_{k}\left(32 k^{3} q_{m}(4 k) t_{k}\right),
$$

where $t_{k}=(-1)^{k}\left(\frac{1}{2}\right)_{k}^{3} /(1)_{k}^{3}$. Summing over $k$ from 0 to $\frac{p-1}{2}$, we derive that

$$
S_{m}-a_{m} S_{1}=32 \omega^{3} q_{m}(4 \omega)(-1)^{\omega}\left(\frac{(1 / 2)_{\omega}}{(1)_{\omega}}\right)^{3}
$$

where $\omega=\frac{p+1}{2}$. Noting that

$$
\frac{(1 / 2)_{\omega}}{(1)_{\omega}}=p \frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{2 i-1}{2 i}
$$

and

$$
\frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{2 i-1}{2 i}=\frac{1}{p+1} \prod_{i=1}^{\frac{p-1}{2}} \frac{p-2 i}{2 i} \equiv(-1)^{\frac{p-1}{2}} \quad(\bmod p),
$$

we have

$$
\left(\frac{(1 / 2)_{\omega}}{(1)_{\omega}}\right)^{3} \equiv p^{3}(-1)^{\frac{p-1}{2}} \quad\left(\bmod p^{4}\right)
$$

Hence

$$
S_{m}-a_{m} S_{1} \equiv-32 p^{3} \omega^{3} q_{m}(4 \omega) \quad\left(\bmod p^{4}\right)
$$

Let $c_{m}=-4 q_{m}(2)$. We then have

$$
S_{m} \equiv a_{m} S_{1}+p^{3} c_{m} \quad\left(\bmod p^{4}\right)
$$

Sun [17] proved that for any prime $p \geq 5$,

$$
S_{1} \equiv(-1)^{\frac{p-1}{2}} p+p^{3} E_{p-3} \quad\left(\bmod p^{4}\right)
$$

Therefore,

$$
S_{m} \equiv a_{m}\left(p(-1)^{\frac{p-1}{2}}+p^{3} E_{p-3}\right)+p^{3} c_{m} \quad\left(\bmod p^{4}\right)
$$

Remark 1. The coefficient $a_{m}$ and the polynomial $q_{m}(k)$ can be computed by the extended Zeilberger's algorithm [7].

## 4 The case when $a(k)=b(k+\alpha)$

We first give a criterion on the degeneration of $(a(k), b(k))$.
Lemma 4.1 Let $a(k), b(k) \in K[k]$ such that $a(k)=b(k+\alpha)$. Suppose that $-(\alpha+1) \operatorname{deg} a(k) \notin \mathbb{N}$. Then $(a(k), b(k))$ is not degenerated.

Proof. Let $r=\operatorname{deg} a(k)=\operatorname{deg} b(k)$ and

$$
u(k)=a(k)-b(k-1)=b(k+\alpha)-b(k-1) .
$$

It is clear that the coefficient of $k^{r}$ in $u(k)$ is 0 and the coefficient of $k^{r-1}$ in $u(k)$ is lc $b(k) \cdot(\alpha+1) r$. Since $(\alpha+1) r \neq 0$, we derive that $\operatorname{deg} u(k)=r-1$. Thus,

$$
-\operatorname{lc} u(k) / \operatorname{lc} a(k)=-\operatorname{lc} u(k) / \operatorname{lc} b(k)=-(\alpha+1) r .
$$

Since $-(\alpha+1) r \notin \mathbb{N}$, the pair $(a(k), b(k))$ is not degenerated.
When $a(k)$ is a shift of $b(k)$, we have a result similar to Theorem 3.2,
Theorem 4.2 Let $a(k), b(k) \in K[k]$ such that

$$
a(k)=b(k+\alpha) \quad \text { and } \quad b(\beta+k)= \pm b(\beta-k),
$$

for some $\alpha, \beta \in K$. Assume further that $-(\alpha+1) \operatorname{deg} a(k) \notin \mathbb{N}$. Then for any non-negative integer $m$, we have

$$
(k+\gamma)^{2 m} \in\left\langle\left[(k+\gamma)^{2 i}\right]: 0 \leq 2 i<\operatorname{deg} a(k)-1\right\rangle
$$

and

$$
(k+\gamma)^{2 m+1} \in\left\langle\left[(k+\gamma)^{2 i+1}\right]: 0 \leq 2 i+1<\operatorname{deg} a(k)-1\right\rangle,
$$

where

$$
\gamma=-\beta+\frac{\alpha-1}{2}
$$

Proof. The proof is parallel to the proof of Theorem 3.2, Instead of (3.2), we take

$$
x(k)=x_{s}(k)=\left(k+\gamma-\frac{1}{2}\right)^{s}
$$

in Lemma 2.2. By Lemma 4.1, $(a(k), b(k))$ is not degenerated and

$$
\operatorname{deg}(a(k)-b(k-1))=\operatorname{deg} a(k)-1 .
$$

Hence the polynomial

$$
p_{s}(k)=a(k) x_{s}(k+1)-b(k-1) x_{s}(k)
$$

satisfies

$$
\operatorname{deg} p_{s}(k)=s+\operatorname{deg} a(k)-1 .
$$

Moreover, we have

$$
p_{s}(-\gamma-k)=(-1)^{s+1} p_{s}(-\gamma+k),
$$

so that the reduction process maintains the symmetric property. Therefore, the reduction process continues until the degree is less than $\operatorname{deg} a(k)-1$.

Similar to Theorem 3.3, we have the following result.

## Theorem 4.3 Let

$$
t_{k}=\left(\frac{(\alpha)_{k}}{k!}\right)^{r}
$$

where $r$ is a positive integer and $\alpha$ is a rational number with denominator $D$. Suppose that $-\alpha r \notin \mathbb{N}$. Then for any positive integer $m$, there exist integers $a_{0}, \ldots, a_{r-2}$ and a polynomial $x(k) \in \mathbb{Z}[k]$ such that
$(2 D k+D \alpha)^{m} t_{k}=\frac{1}{C_{m}} \sum_{i=0}^{r-2} a_{i}(2 D k+D \alpha)^{i} t_{k}+\frac{1}{C_{m}} \Delta_{k}\left(2^{r-1}(D k)^{r} x(2 D k) t_{k}\right)$,
where

$$
C_{m}=\prod_{0 \leq 2 i \leq m-r+1}((\alpha r+m-r+1-2 i) \cdot D) .
$$

Moreover, $a_{i}=0$ if $i \not \equiv m(\bmod 2)$.

Proof. The proof is parallel to the proof of Theorem 3.3, Instead of (3.6) and (3.7), we take

$$
\begin{equation*}
\tilde{x}_{s}(k)=(k+D \alpha-D)^{s} \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{p}_{s}(k)=\frac{1}{2}\left((k+D \alpha)^{r}(k+D)^{s}-(k-D \alpha)^{r}(k-D)^{s}\right), \tag{4.2}
\end{equation*}
$$

so that (3.5) still holds. It is clear that $\tilde{x}_{s}(k), \tilde{p}_{s}(k) \in \mathbb{Z}[k]$. But in this case, $\tilde{p}_{s}(k)$ is not monic. The leading term of $\tilde{p}_{s}(k)$ is

$$
(\alpha r+s) D \cdot k^{s+r-1}
$$

Now let us consider the reduction process. Let $p(k) \in \mathbb{Z}[k]$ be a polynomial of degree $\ell \geq r-1$. Assume further that $p(k)$ contains only even powers of $k$ or only odd powers of $k$. Setting

$$
\begin{aligned}
p^{\prime}(k) & =\operatorname{lc} \tilde{p}_{\ell-r+1}(k) \cdot p(k)-\operatorname{lc} p(k) \cdot \tilde{p}_{\ell-r+1}(k) \\
& =(\alpha r+\ell-r+1) D \cdot p(k)-\operatorname{lc} p(k) \cdot \tilde{p}_{\ell-r+1}(k),
\end{aligned}
$$

we see that $p^{\prime}(k) \in \mathbb{Z}[k]$ and $\operatorname{deg} p^{\prime}(k)<\ell$. Since $\tilde{p}_{\ell-r+1}(k)$ contains only even powers of $k$ or only odd powers of $k$, so does $p^{\prime}(k)$. Therefore, $\operatorname{deg} p^{\prime}(k) \leq \ell-2$.

Continuing this reduction process until $\ell<r-1$, we finally obtain that there exist integers $c_{m}, c_{m-2}, \ldots$ such that

$$
C_{m} k^{m}-c_{m} \tilde{p}_{m-r+1}(k)-c_{m-2} \tilde{p}_{m-r-1}(k)-\cdots,
$$

is a polynomial of degree less than $r-1$ and with integral coefficients, where $C_{m}$ is the product of the leading coefficient of $\tilde{p}_{m-r+1}(k), \tilde{p}_{m-r-1}(k), \ldots$

$$
C_{m}=\prod_{0 \leq 2 i \leq m-r+1}((\alpha r+m-r+1-2 i) D),
$$

as desired.
For the special case of $t_{k}=(1 / 2)_{k}^{4} /(1)_{k}^{4}$, we may further reduce the factor $C_{m}$.

Lemma 4.4 Let $m$ be a positive integer and

$$
t_{k}=\frac{(1 / 2)_{k}^{4}}{(1)_{k}^{4}}
$$

- If $m$ is odd, then there exist an integer $c$ and a polynomial $x(k) \in \mathbb{Z}[k]$ such that

$$
(4 k+1)^{m} t_{k}=\frac{c}{C_{m}^{\prime}}(4 k+1) t_{k}+\frac{1}{C_{m}^{\prime}} \Delta_{k}\left(32 k^{4} x(4 k) t_{k}\right),
$$

where $C_{m}^{\prime}=\left(\frac{m-1}{2}\right)!$.

- If $m$ is even, then there exist integers $c, c^{\prime}$ and a polynomial $x(k) \in \mathbb{Z}[k]$ such that

$$
(4 k+1)^{m} t_{k}=\frac{1}{C_{m}^{\prime}}\left(c+(4 k+1)^{2} c^{\prime}\right) t_{k}+\frac{1}{C_{m}^{\prime}} \Delta_{k}\left(64 k^{4} x(4 k) t_{k}\right),
$$

where $C_{m}^{\prime}=(m-1)!$ !.

Proof. This is the special case of Theorem 4.3 for $\alpha=1 / 2$ and $r=4$. Therefore, $D=2$ and $\alpha r-r+1=-1$.

We need only to show that the coefficients of $\tilde{p}_{s}(k)$ given by (4.2) is divisible by 2 when $s$ is odd and is divisible by 4 when $s$ is even. Then we may replace $\tilde{x}_{s}(k)$ given by (4.1) by $\tilde{x}_{s}(k) / 4$ and $\tilde{x}_{s}(k) / 2$ so that the leading coefficient of $\tilde{p}_{s}(k)$ is reduced. Correspondingly, the product $C_{m}$ of the leading coefficients becomes

$$
\prod_{0 \leq 2 i \leq m-3} \frac{1}{2} \operatorname{lc} \tilde{p}_{m-3-2 i}(k)=\prod_{0 \leq 2 i \leq m-3}(m-1-2 i)=(m-1)!!, \quad m \text { even },
$$

and

$$
\prod_{0 \leq 2 i \leq m-3} \frac{1}{4} \operatorname{lc} \tilde{p}_{m-3-2 i}(k)=\prod_{0 \leq 2 i \leq m-3} \frac{m-1-2 i}{2}=\left(\frac{m-1}{2}\right)!, \quad m \text { odd. }
$$

Notice that

$$
\tilde{p}_{s}(k)=\frac{1}{2}\left((k+1)^{4}(k+2)^{s}-(k-1)^{4}(k-2)^{s}\right) .
$$

The coefficient of $k^{j}$ is

$$
\frac{1-(-1)^{s-j}}{2} \sum_{0 \leq \ell \leq 4,0 \leq j-\ell \leq s}\binom{4}{\ell}\binom{s}{j-\ell} 2^{s-j+\ell} .
$$

If $j-\ell<s$, the corresponding summand is divisible by 2 . If $j-\ell=s$ and $\ell$ is even, then $(-1)^{s-j}=1$ and the coefficient is 0 . Otherwise, $\ell=1$ or $\ell=3$, and thus $4 \left\lvert\,\binom{ 4}{\ell}\right.$. Therefore, the coefficient must be divisible by 2 .

Now consider the case of $s$ being even. If $j-\ell<s-1$, the corresponding summand is divisible by 4 . Otherwise $j-\ell=s$ or $j-\ell=s-1$. We have seen that if $j-\ell=s$, then the coefficient is divisible by 4 . If $j-\ell=s-1$. Then

$$
\binom{s}{j-\ell}=s \quad \text { and } \quad 2^{s-j+\ell}=2 .
$$

Thus the summand is also divisible by 4 .
Example 4.2 Consider the case of $m=11$. We have

$$
(4 k+1)^{11} t_{k}+10515(4 k+1) t_{k}=\Delta_{k}\left(32 k^{4} p(k) t_{k}\right)
$$

where
$p(k)=\frac{1}{5}(4 k-1)^{8}-\frac{249}{20}(4 k-1)^{6}+\frac{20207}{60}(4 k-1)^{4}-\frac{89909}{20}(4 k-1)^{2}+\frac{524029}{20}$.

As an application, we obtain the following congruences.

Theorem 4.5 Let $m$ be a positive odd integer and $\mu=(m-1) / 2$. Denote

$$
S_{m}=\sum_{k=0}^{\frac{p-1}{2}}(4 k+1)^{m}\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{4} .
$$

Then there exists an integer $a_{m}$ such that for each prime $p>\mu$,

$$
S_{m} \equiv \frac{a_{m}}{\mu!} p \quad\left(\bmod p^{4}\right)
$$

Proof. By Lemma 4.4, there exist an integer $a_{m}$ and a polynomial $q_{m}(k) \in$ $\mathbb{Z}[k]$ such that

$$
(4 k+1)^{m} t_{k}-\frac{a_{m}}{\mu!}(4 k+1) t_{k}=\frac{1}{\mu!} \Delta_{k}\left(32 k^{4} q_{m}(k) t_{k}\right),
$$

where $t_{k}=\left(\frac{(1 / 2)_{k}}{(1)_{k}}\right)^{4}$. Summing over $k$ from 0 to $(p-1) / 2$, we obtain

$$
S_{m}-\frac{a_{m}}{\mu!} S_{1}=32 \omega^{4} \frac{q_{m}(4 \omega)}{\mu!}\left(\frac{(1 / 2)_{\omega}}{(1)_{\omega}}\right)^{4}
$$

where $\omega=(p+1) / 2$. When $p>\mu, 1 / \mu$ ! is a $p$-adic integer and

$$
\left(\frac{(1 / 2)_{\omega}}{(1)_{\omega}}\right)^{4} \equiv 0 \quad\left(\bmod p^{4}\right)
$$

Therefore,

$$
S_{m} \equiv \frac{a_{m}}{\mu!} S_{1} \quad\left(\bmod p^{4}\right)
$$

It is shown by Long [12] that

$$
S_{1} \equiv p \quad\left(\bmod p^{4}\right)
$$

completing the proof.
The integer $a_{m}$ and the polynomial $q_{m}(k)$ can be computed by the extended Zeilberger's algorithm.

By checking the initial values, we propose the following conjecture.

Conjecture 4.6 For any positive odd integer $m$, the coefficient $a_{m} /\left(\frac{m-1}{2}\right)$ ! is an integer.

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## References

[1] S.A. Abramov, The rational component of the solution of a first-order linear recurrence relation with rational right-hand side, Comput. Math. Math. Phys. 15 (1975) 1035-1039.
[2] S.A. Abramov, Indefinite sums of rational functions, in: Proc. ISSAC95, ACM Press, New York, 1995, 303-308.
[3] S.A. Abramov and M. Petkovšek, Minimal decomposition of indefinite hypergeometric sums, in: Proc. ISSAC2001 ACM Press, New York, 2001, 7-14.
[4] S.A. Abramov and M. Petkovšek, Rational normal forms and minimal decompositions of hypergeometric terms, J. Symbolic Comput. 33 (2002) 521-543.
[5] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. reine angew. Math. 518 (2000) 187212
[6] S. Chen, H. Huang, M. Kauers, and Z. Li, A modified AbramovPetkovšek reduction and creative telescoping for hypergeometric terms, Proceedings of the 2015 ACM on International Symposium on Symbolic and Algebraic Computation. ACM, 2015, p. 117-124.
[7] W. Y. C. Chen, Q.-H. Hou, and Y.-P. Mu, The extended Zeilberger algorithm with parameters, J. Symbolic Comput. 47(6) (2012) 643654.
[8] V.J.W. Guo, Some generalizations of a supercongruence of van Hamme, Integral Transforms Spec. Funct. 28(12) (2017) 888-899.
[9] V.J.W. Guo, Common q-analogues of some different supercongruences, Results Math., in press; https://doi.org/10.1007/s00025-019-1056-1.
[10] A. Jana and G. Kalita, Supercongruences for sums involving rising factorial $\left(\frac{1}{\ell}\right)_{k}^{3}$, Integral Transforms Spec. Funct., in press; https://doi.org/10.1080/10652469.2019.1604700.
[11] J.-C. Liu, Semi-automated proof of supercongruences on partial sum of hypergeometric series, J. Symbolic Comput., to appear.
[12] L. Long, Hypergeometric evaluation identities and supercongruences, Pac. J. Math. 249 (2011) 405-418.
[13] E. Mortenson, A p-adic supercongruence conjecture of van Hamme. Proc. Am. Math. Soc. 136 (2008) 4321-4328.
[14] R. Osburn and C. Schneider, Gaussian hypergeometric series and supercongruences, Math. Comp. 78(265) (2009) 275-292.
[15] R. Pirastu and V. Strehl, Rational summation and Gosper-Petkovsek representation, J. Symbolic Comput. 20 (1995) 617-635.
[16] Z.-W. Sun, On sums related to central binomial and trinomial coefficients, in: M. B. Nathanson (ed.), Combinatorial and Additive Number Theory: CANT 2011 and 2012, Springer Proc. in Math. \& Stat., Vol. 101, Springer, New York, 2014, pp. 257-312.
[17] Z.-W. Sun, A refinement of a congruence result by van Hamme and Mortenson, Ill. J. Math. 56 (2012) 967-979.
[18] Z.-W. Sun, Supecongruences involving products of two binomial coefficients, Finite Fields Appl. 22 (2013) 24-44.
[19] L. van Hamme, Some conjectures concerning partial sums of generalized hypergeometric series, in: p-adic Functional Analysis (Nijmegen, 1996), pp. 223-236, Lecture Notes in Pure and Appl. Math., Vol., 192, Dekker, 1997.
[20] S.-D. Wang, Some supercongruences involving $\binom{2 k}{k}^{4}$, J. Differ. Equ. Appl., to appear.
[21] W. Zudilin, Ramanujan-type supercongruences. J. Number Theory 129 (2009) 1848-1857.

