# LEXICOGRAPHIC AND REVERSE LEXICOGRAPHIC QUADRATIC GRÖBNER BASES OF CUT IDEALS 

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#### Abstract

Hibi conjectured that if a toric ideal has a quadratic Gröbner basis, then the toric ideal has either a lexicographic or a reverse lexicographic quadratic Gröbner basis. In this paper, we present a cut ideal of a graph that serves as a counterexample to this conjecture. We also discuss the existence of a quadratic Gröbner basis of a cut ideal of a cycle. Nagel and Petrović claimed that a cut ideal of a cycle has a lexicographic quadratic Gröbner basis using the results of Chifman and Petrović. However, we point out that the results of Chifman and Petrović used by Nagel and Petrović are incorrect for cycles of length greater than or equal to 6 . Hence the existence of a quadratic Gröbner basis for the cut ideal of a cycle (a ring graph) is an open question. We also provide a lexicographic quadratic Gröbner basis of a cut ideal of a cycle of length less than or equal to 7 .


## Introduction

A $d \times n$ integer matrix $A=\left(\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \ldots, \boldsymbol{a}_{n}\right)$ is called a configuration if there exists a vector $\boldsymbol{c} \in \mathbb{R}^{d}$ such that for all $1 \leq i \leq n$, the inner product $\boldsymbol{a}_{i} \cdot \boldsymbol{c}$ is equal to 1 . Let $K$ be a field and let $K[\boldsymbol{x}]=K\left[x_{1}, x_{2}, \ldots, x_{n}\right]$ be a polynomial ring in $n$ variables. For an integer vector $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{d}\right) \in \mathbb{Z}^{d}$, we define the Laurent monomial $\boldsymbol{t}^{\alpha}=t_{1}^{\alpha_{1}} t_{2}^{\alpha_{2}} \ldots t_{d}^{\alpha_{d}} \in K\left[t_{1}^{ \pm 1}, t_{2}^{ \pm 1}, \ldots, t_{d}^{ \pm 1}\right]$ and $K[A]=K\left[\boldsymbol{t}^{a_{1}}, \boldsymbol{t}^{a_{2}}, \ldots, \boldsymbol{t}^{a_{n}}\right]$. Let $\pi$ be a homomorphism $\pi: K[\boldsymbol{x}] \rightarrow K[A]$, where $\pi\left(x_{i}\right)=\boldsymbol{t}^{\boldsymbol{a}_{i}}$. The kernel of $\pi$ is called the toric ideal of $A$ and is denoted by $I_{A}$. It is known [13, 21] that $I_{A}$ is generated by homogeneous binomials associated to the kernel of $A$. For a configuration $A$, let $\operatorname{Ker}_{\mathbb{Z}} A=\left\{\boldsymbol{b} \in \mathbb{Z}^{n} \mid A \boldsymbol{b}=\mathbf{0}\right\}$. For each $\boldsymbol{b}=\left(b_{1}, \ldots, b_{n}\right) \in \operatorname{Ker}_{\mathbb{Z}} A$, we define

$$
f_{\boldsymbol{b}}=\prod_{b_{i}>0} x_{i}{ }^{b_{i}}-\prod_{b_{j}<0} x_{j}^{-b_{j}} \in K[\boldsymbol{x}] .
$$

Then $I_{A}=\left\langle f_{\boldsymbol{b}} \mid \boldsymbol{b} \in \operatorname{Ker}_{\mathbb{Z}} A\right\rangle$. Commutative algebraists are interested in the following properties:
(1) The toric ideal $I_{A}$ is generated by quadratic binomials;
(2) The toric ring $K[A]$ is Koszul;
(3) There exists a monomial order satisfying that a Gröbner basis of $I_{A}$ consists of quadratic binomials.
The implication $(3) \Rightarrow(2) \Rightarrow(1)$ is true, but both $(1) \Rightarrow(2)$ and $(2) \Rightarrow(3)$ are false in general (for example, see [10, 14]). Several classes of toric ideals with lexicographic/reverse lexicographic quadratic Gröbner bases are known (for example, see [4, 6, 15, 17, 18, 20]). In contrast, in [2, 3, 19], sorting monomial orders (which are not necessarily lexicographic or reverse lexicographic) are used to construct a quadratic Gröbner basis. The monomial orders appearing in the theory of toric fiber products [23] constitute another example that is not necessarily lexicographic or reverse lexicographic. The following conjecture was presented by Hibi.

[^0]Conjecture 0.1. Suppose that the toric ideal $I_{A}$ has a quadratic Gröbner basis. Then $I_{A}$ has either a lexicographic or reverse lexicographic quadratic Gröbner basis.

In the present paper, we will present a cut ideal of a graph as a counterexample to this conjecture.

Now, we define the cut ideal of a graph. Let $G$ be a finite connected simple graph with the vertex set $V(G)=\{1,2, \ldots, m\}$ and the edge set $E(G)=\left\{e_{1}, e_{2}, \ldots, e_{r}\right\}$. Given a subset $C$ of $V(G)$, we define a vector $\delta_{C}=\left(d_{1}, d_{2}, \ldots, d_{r}\right) \in\{0,1\}^{r}$ by

$$
d_{i}= \begin{cases}1 & \left|C \cap e_{i}\right|=1\left(e_{i}=\{j, k\}\right) \\ 0 & \text { otherwise }\end{cases}
$$

We consider the configuration

$$
A_{G}=\left(\begin{array}{cccc}
\delta_{C_{1}} & \delta_{C_{2}} & \cdots & \delta_{C_{N}} \\
1 & 1 & \ldots & 1
\end{array}\right)
$$

where $\left\{\delta_{C} \mid C \subset V(G)\right\}=\left\{\delta_{C_{1}}, \delta_{C_{2}}, \ldots, \delta_{C_{N}}\right\}$ and $N=2^{m-1}$. The toric ideal of $A_{G}$ is called the cut ideal of $G$ and is denoted by $I_{G}$ (see [22] for details). This definition of the cut ideal is different from that in [22]. However, the two definitions are equivalent. In fact, in [22] they say that "Indeed, the convex hull of the exponent vectors $\phi_{G}$ is affinely isomorphic to $\operatorname{Cut}^{\square}(G)$." Here $\operatorname{Cut}^{\square}(G)$ is the convex hull of $\left\{\delta_{C} \mid C \subset V(G)\right\}$. We illustrate this equivalence by an example.
Example 0.2. Let $G$ be a cycle of length 4 with $V(G)=\{1,2,3,4\}, E(G)=\left\{e_{1}=\right.$ $\left.\{1,2\}, e_{2}=\{2,3\}, e_{3}=\{3,4\}, e_{4}=\{1,4\}\right\}$. Then $A_{G}$ is

$$
A_{G}=\left(\begin{array}{llllllll}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

Here, the $i$-th row of $A_{G}$ is indexed by the edge $e_{i}$ and the $j$-th column of $A_{G}$ is indexed by the subset $C_{j} \subset\{1,2,3,4\}$, where $C_{1}=\phi, C_{2}=\{2\}, C_{3}=\{2,3\}, C_{4}=$ $\{2,3,4\}, C_{5}=\{3\}, C_{6}=\{3,4\}, C_{7}=\{4\}, C_{8}=\{2,4\}$. On the other hand, in [22], the cut ideal of $G$ is defined as the kernel of homomorphism $\phi_{G}: K\left[q_{\mid 1234}, q_{2 \mid 134}, q_{23 \mid 14}\right.$, $\left.q_{234 \mid 1}, q_{3 \mid 124}, q_{34 \mid 12}, q_{4 \mid 123}, q_{24 \mid 13}\right] \rightarrow K\left[s_{12}, s_{23}, s_{34}, s_{14}, t_{12}, t_{23}, t_{34}, t_{14}\right]$ with

$$
\begin{aligned}
q_{| | 234} & \mapsto t_{12} t_{23} t_{34} t_{14} & q_{2 \mid 134} \mapsto s_{12} s_{23} t_{34} t_{14} \\
q_{23 \mid 14} & \mapsto s_{12} t_{23} s_{34} t_{14} & q_{234 \mid 1} \mapsto s_{12} t_{23} t_{34} s_{14} \\
q_{3 \mid 124} & \mapsto t_{12} s_{23} s_{34} t_{14} & q_{34 \mid 12} \mapsto t_{12} s_{23} t_{34} s_{14} \\
q_{4 \mid 123} & \mapsto t_{12} t_{23} s_{34} s_{14} & q_{24 \mid 13} \mapsto s_{12} s_{23} s_{34} s_{14} .
\end{aligned}
$$

So, the cut ideal defined in [22] is the toric ideal of the following configuration $A_{G}^{\prime}$ :

| $s_{12}$ |
| :--- |
| $s_{23}$ |
| $s_{34}$ |
| $s_{14}$ |
| $t_{12}$ |
| $t_{23}$ |
| $t_{34}$ |
| $t_{14}$ |\(\left(\begin{array}{llllllll}0 \& 1 \& 1 \& 1 \& 0 \& 0 \& 0 \& 1 <br>

0 \& 1 \& 0 \& 0 \& 1 \& 1 \& 0 \& 1 <br>
0 \& 0 \& 1 \& 0 \& 1 \& 0 \& 1 \& 1 <br>
0 \& 0 \& 0 \& 1 \& 0 \& 1 \& 1 \& 1 <br>
\hline 1 \& 0 \& 0 \& 0 \& 1 \& 1 \& 1 \& 0 <br>
1 \& 0 \& 1 \& 1 \& 0 \& 0 \& 1 \& 0 <br>
1 \& 1 \& 0 \& 1 \& 0 \& 1 \& 0 \& 0 <br>
1 \& 1 \& 0 \& 1 \& 0 \& 0 \& 0\end{array}\right)\),
where $j$-th column is indexed by $j$-th element of $\left(q_{\mid 1234}, q_{2 \mid 134}, q_{23 \mid 14}, q_{234 \mid 1}, q_{3 \mid 124}, q_{34 \mid 12}\right.$, $q_{4 \mid 123}, q_{24 \mid 13}$ ). We obtain the following matrix by elementary row operations from $A_{G}^{\prime}$ :

$$
\binom{A_{G}}{O}
$$

where $O$ is a $3 \times 8$ zero-matrix. Therefore, $\operatorname{Ker}_{\mathbb{Z}} A_{G}=\operatorname{Ker}_{\mathbb{Z}} A_{G}^{\prime}$.
We introduce important known results on the quadratic Gröbner bases of cut ideals. An edge contraction for a graph $G$ is an operation that merges two vertices joined by the edge $e$ after removing $e$ from $G$. A graph $H$ is called a minor of the graph $G$ if $H$ is obtained by deleting some edges and vertices and contracting some edges. In this paper, $K_{n}, K_{m, n}$, and $\mathcal{C}_{n}$ stand for the complete graph with $n$ vertices, the complete bipartite graph on the vertex set $\{1,2, \ldots, m\} \cup\{m+1, m+2, \ldots, m+n\}$ and the cycle of length $n$, respectively.

Proposition 0.3 ([8]). Let $G$ be a graph. Then $I_{G}$ is generated by quadratic binomials if and only if $G$ is free of $K_{4}$ minors.

Proposition $0.4([20])$. Let $G$ be a graph. Then $K\left[A_{G}\right]$ is strongly Koszul if and only if $G$ is free of $\left(K_{4}, \mathcal{C}_{5}\right)$ minors. In addition, if $K\left[A_{G}\right]$ is strongly Koszul, then $I_{G}$ has a quadratic Gröbner basis.

Nagel and Petrović [11, Proposition 3.2] claimed that if $G$ is a cycle, then $I_{G}$ has a (lexicographic) quadratic Gröbner basis. However, [5, Propositions 2 and 3], which are used in the proof of [11, Proposition 3.2], contain some errors. We will explain this in Section 2. In contrast, the following problem is open.

Problem 0.5. Classify the graphs whose cut ideals have a quadratic Gröbner basis.
This paper comprises Sections 1 and 2. In Section 1, we show some results concerning the existence of a lexicographic/reverse lexicographic quadratic Gröbner basis of cut ideals. Then, we give a graph whose cut ideal is a counterexample to Conjecture 0.1. In Section 2, we study the cut ideal of a cycle. First, we point out an error in the lexicographic quadratic Gröbner basis of cut ideals of cycles given in [5, Proposition 3] (and introduced in [11]). Finally, we construct a lexicographic quadratic Gröbner basis of the cut ideal of a cycle of length $\leq 7$.

## 1. Lexicographic and reverse lexicographic Gröbner bases

In this section, we present necessary conditions for cut ideals to have a lexicographic/reverse lexicographic quadratic Gröbner basis. Using these results, we present a graph whose cut ideal is a counterexample to Conjecture 0.1.

First, we study reverse lexicographic quadratic Gröbner bases of cut ideals. The following was proved in [22, Theorem 1.3].

Proposition 1.1. Let $G$ be a graph. Then the graph $G$ is free of $K_{5}$ minors and has no induced cycles of length $\geq 5$ if and only if there exists a reverse lexicographic order such that the initial ideal of $I_{G}$ is squarefree.

Using that fact that $A_{G}$ is a $(0,1)$ matrix and Proposition 1.1, we are able to prove the following.

Proposition 1.2. Suppose that a graph $G$ has an induced cycle of length $\geq 5$. Then $I_{G}$ has no reverse lexicographic quadratic Gröbner bases.

Proof. Suppose that $I_{G}$ has a reverse lexicographic quadratic reduced Gröbner basis $\mathcal{G}$. Any toric ideal is prime in general, and hence $\mathcal{G}$ consists of irreducible binomials. Since $A_{G}$ is a configuration, $\mathcal{G}$ consists of homogeneous binomials. Moreover, since $A_{G}$ is a $(0,1)$ matrix, there exist no nonzero binomials of the form $x_{i}^{2}-x_{j} x_{k}$ in $I_{G}$. In fact, if $x_{i}^{2}-x_{j} x_{k} \neq 0$ belongs to $I_{G}$, then $2 \delta_{C_{i}}=\delta_{C_{j}}+\delta_{C_{k}}$. However, this is impossible since $\delta_{C_{i}}, \delta_{C_{j}}, \delta_{C_{k}}$ are $(0,1)$-vectors. It therefore follows that the initial ideal is generated by squarefree monomials. By proposition 1.1, $G$ has no induced cycle of length $\geq 5$.

Second, we study the lexicographic quadratic Gröbner bases of cut ideals. Let $G$ be a complete bipartite graph $K_{2,3}$, as shown in Fig. 1. The configuration $A_{G}$ is


Figure 1. Complete bipartite graph $K_{2,3}$.

$$
A_{G}=\left(\begin{array}{llllllllllllllll}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\hline 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

Here, the $i$-th row of $A_{G}$ is indexed by the edge $e_{i}$ and the $j$-th column of $A_{G}$ is indexed by the subset $C_{j} \subset\{1,2,3,4,5\}$, where $C_{1}=\emptyset, C_{2}=\{5\}, C_{3}=\{4\}, C_{4}=$ $\{4,5\}, C_{5}=\{2,3,4,5\}, C_{6}=\{2,3,4\}, C_{7}=\{2,3,5\}, C_{8}=\{2,3\}, C_{9}=\{2,4,5\}, C_{10}=$ $\{2,4\}, C_{11}=\{2,5\}, C_{12}=\{2\}, C_{13}=\{3\}, C_{14}=\{3,5\}, C_{15}=\{3,4\}, C_{16}=\{3,4,5\}$.
The configuration $A_{G}$ has a symmetry group, called switching in [7, as follows.
Given subsets $A, B \subset\{1,2,3,4,5\}$, let $A \triangle B$ denote the symmetric difference $(A \cup$ $B) \backslash(A \cap B)$ of them. From the general theory of cuts, for any $C, C^{\prime} \subset\{1,2,3,4,5\}, \delta_{C}+$ $\delta_{C^{\prime}}=\delta_{C \Delta C^{\prime}}$ in $\mathbb{F}_{2}^{6}$. Hence each $C \subset\{1,2,3,4,5\}$ gives a permutation $\psi_{C}$ on $\left(\delta_{C_{1}}, \cdots, \delta_{C_{16}}\right)$ defined by

$$
\psi_{C}\left(\delta_{C_{1}}, \cdots, \delta_{C_{16}}\right)=\left(\delta_{C_{i_{1}}}, \cdots, \delta_{C_{i_{16}}}\right)
$$

where $\delta_{C_{k}}+\delta_{C}=\delta_{C_{i_{k}}}$ in $\mathbb{F}_{2}^{6}$. The permutation $\psi_{C}$ naturally induces an action on $K[\boldsymbol{x}]$ by $\psi_{C}\left(x_{k}\right)=x_{i_{k}}$. Since

$$
\left(\begin{array}{ccc}
\delta_{C_{1}}+\delta_{C} & \cdots & \delta_{C_{16}}+\delta_{C} \\
1 & \cdots & 1
\end{array}\right)
$$

is obtained by elementary row operations from

$$
\left(\begin{array}{ccc}
\delta_{C_{1}} & \cdots & \delta_{C_{16}} \\
1 & \cdots & 1
\end{array}\right)
$$

their kernels are the same. Hence we have $\psi_{C}\left(I_{G}\right)=I_{G}$. We show that $I_{G}$ has no lexicographic quadratic Gröbner bases by using these symmetries.

Proposition 1.3. The cut ideal of the complete bipartite graph $K_{2,3}$ is generated by quadratic binomials and has no lexicographic quadratic Gröbner bases.

Proof. Since $K_{2,3}$ is free of $K_{4}$ minors, $I_{K_{2,3}}$ is generated by quadratic binomials according to Proposition 0.3. Let $<$ be a lexicographic order on $K[\mathbf{x}]$. Suppose that the initial ideal of $I_{K_{2,3}}$ with respect to $<$ is quadratic. Let $\mathcal{M}$ be the set of all monomials in $K[\mathbf{x}]$ and let

$$
S=\left\{u \in \mathcal{M} \mid \pi(u)=t_{1} t_{2} t_{3} t_{4} t_{5} t_{6} t_{7}^{2}\right\}
$$

Then we have

$$
S=\left\{x_{1} x_{16}, x_{2} x_{15}, x_{3} x_{14}, x_{4} x_{13}, x_{5} x_{12}, x_{6} x_{11}, x_{7} x_{10}, x_{8} x_{9}\right\}
$$

For each element $x_{i} x_{17-i} \in S, \psi_{C_{i}}\left(x_{i} x_{17-i}\right)=x_{1} x_{16}$ for $i=2, \ldots, 8$. (For example, $\psi_{C_{2}}\left(x_{2} x_{15}\right)=x_{1} x_{16}$ for $C_{2}=\{5\}$ since $\delta_{C_{2}}+\delta_{C_{2}}=\delta_{C_{1}}$ and $\delta_{C_{15}}+\delta_{C_{2}}=\delta_{C_{16}}$ in $\mathbb{F}_{2}^{6}$.) Hence we may assume that $x_{1} x_{16}$ is the smallest monomial in $S$ with respect to $<$. It then follows that $x_{1} x_{16} \notin \mathrm{in}_{<}\left(I_{K_{2,3}}\right)$. We now consider the following 8 cubic binomials of $I_{K_{2,3}}$ :

$$
\begin{aligned}
& f_{1}=x_{6} x_{7} x_{9}-x_{1} x_{5} x_{16} \\
& f_{2}=x_{5} x_{8} x_{10}-x_{1} x_{6} x_{16} \\
& f_{3}=x_{5} x_{8} x_{11}-x_{1} x_{7} x_{16} \\
& f_{4}=x_{6} x_{7} x_{12}-x_{1} x_{8} x_{16} \\
& f_{5}=x_{5} x_{0} x_{11}-x_{1} x_{9} x_{16} \\
& f_{6}=x_{6} x_{9} x_{12}-x_{1} x_{10} x_{16} \\
& f_{7}=x_{7} x_{9} x_{12}-x_{1} x_{11} x_{16} \\
& f_{8}=x_{8} x_{10} x_{11}-x_{1} x_{12} x_{16}
\end{aligned}
$$

Suppose that there exists a nonzero binomial $x_{1} x_{i}-x_{j} x_{k} \in I_{K_{2,3}}$ with $i \in\{5, \ldots, 12\}$. Then we have $\delta_{C_{i}}=\delta_{C_{j}}+\delta_{C_{k}}$. Since $\delta_{C_{i}}$ contains exactly 3 ones, so does $\delta_{C_{j}}+\delta_{C_{k}}$. It then follows that one of $C_{j}$ and $C_{k}$ is $C_{1}$ and hence $x_{1} x_{i}-x_{j} x_{k}=0$. Similarly, suppose that there exists a nonzero binomial $x_{i} x_{16}-x_{j} x_{k} \in I_{K_{2,3}}$ with $i \in\{5, \ldots, 12\}$. Then we have $\delta_{C_{i}}+\delta_{C_{16}}=\delta_{C_{j}}+\delta_{C_{k}}$. Since the sum of the components of $\delta_{C_{i}}+$ $\delta_{C_{16}}$ is 9 , it follows that one of $C_{j}$ and $C_{k}$ is $C_{16}$ and hence $x_{i} x_{16}-x_{j} x_{k}=0$. Thus $x_{1} x_{16}, x_{1} x_{i}, x_{i} x_{16} \notin \mathrm{in}_{<}\left(I_{K_{2,3}}\right)$ for each $i \in\{5, \ldots, 12\}$. If $x_{1} x_{i} x_{16}$ belongs to $\mathrm{in}_{<}\left(I_{K_{2,3}}\right)$ for some $i \in\{5, \ldots, 12\}$, then the cubic monomial $x_{1} x_{i} x_{16}$ belongs to the minimal set of monomial generators of $\mathrm{in}_{<}\left(I_{K_{2,3}}\right)$. This contradicts the hypothesis that $\operatorname{in}_{<}\left(I_{K_{2,3}}\right)$ is generated by quadratic monomials. Hence each $x_{1} x_{i} x_{16}$ does not belong to in $\mathrm{C}_{<}\left(I_{K_{2,3}}\right)$. Thus the initial monomial of each cubic binomial $f_{i}(1 \leq i \leq 8)$ above is the first monomial. Let $R=K\left[x_{1}, x_{5}, x_{6}, \ldots, x_{12}, x_{16}\right]$. Note that each $f_{i}$ belongs to $R$. Let $x_{k}(k \in\{1,5,6, \ldots, 12,16\})$ be the greatest variable in $R$ with respect to the lexicographic order. Then $x_{k}$ appears in the second monomial of $f_{j}$ for some $j$. Since < is a lexicographic order, the initial monomial of $f_{j}$ is the second monomial, a contradiction.

Remark 1.4. Shibata [20] showed that the cut ideal of the complete bipartite graph $K_{2, m}$ has a quadratic Gröbner basis with respect to a reverse lexicographic order.

Let $A=\left(\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \ldots, \boldsymbol{a}_{n}\right)$ be a $d \times n$ configuration and let $B=\left(\boldsymbol{a}_{i_{1}}, \boldsymbol{a}_{i_{2}}, \ldots, \boldsymbol{a}_{i_{m}}\right)$ be a submatrix of $A$. Then $K[B]$ is called a combinatorial pure subring of $K[A]$ if
there exists a vector $\boldsymbol{c} \in \mathbb{R}^{d}$ such that

$$
\boldsymbol{a}_{i} \cdot \boldsymbol{c}\left\{\begin{array}{cc}
=1 & i \in\left\{i_{1}, i_{2}, \ldots, i_{m}\right\}, \\
<1 & \text { otherwise }
\end{array}\right.
$$

That is, $K[B]$ is a combinatorial pure subring of $K[A]$ if and only if there exists a face $F$ of the convex hull of $A$ such that $\left\{\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \ldots, \boldsymbol{a}_{n}\right\} \cap F=\left\{\boldsymbol{a}_{i_{1}}, \boldsymbol{a}_{i_{2}}, \ldots, \boldsymbol{a}_{i_{m}}\right\}$. It is known that a combinatorial pure subring $K[B]$ inherits numerous properties of $K[A]$ (see [13]). In particular, we have the following:
Proposition 1.5. Suppose that $K[B]$ is a combinatorial pure subring of $K[A]$. If $I_{A}$ has a lexicographic quadratic Gröbner basis, then so does $I_{B}$.

Suppose that a graph $H$ is obtained by an edge contraction from a graph $G$; then it is known from [22, Lemma $3.2(2)]$ that $K\left[A_{H}\right]$ is a combinatorial pure subring of $K\left[A_{G}\right]$. Thus we have the following from Propositions 1.3 and 1.5 .

Proposition 1.6. Let $G$ be a graph. Suppose that $K_{2,3}$ is obtained by a sequence of contractions from $G$. Then $I_{G}$ has no lexicographic quadratic Gröbner bases.

Let $G$ be a graph with 6 vertices and 7 edges, as shown in Fig. 2. Then the


Figure 2. A counterexample to Conjecture 0.1.
configuration $A_{G}$ is

$$
\left(\begin{array}{llllllllllllllllllllllllllllllll}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
$$

Here, the $i$-th row of $A_{G}$ is indexed by the edge $e_{i}$ and the $j$-th column of $A_{G}$ is indexed by the subset $C_{j} \subset\{1,2,3,4,5,6\}$, where $C_{1}=\emptyset, C_{2}=\{6\}, C_{3}=\{5\}, C_{4}=$ $\{5,6\}, C_{5}=\{4\}, C_{6}=\{4,6\}, C_{7}=\{4,5\}, C_{8}=\{4,5,6\}, C_{9}=\{2,4,5\}, C_{10}=$ $\{2,4,5,6\}, C_{11}=\{2,4\}, C_{12}=\{2,4,6\}, C_{13}=\{2,5\}, C_{14}=\{2,5,6\}, C_{15}=\{2\}, C_{16}=$ $\{2,6\}, C_{17}=\{3\}, C_{18}=\{3,6\}, C_{19}=\{3,5\}, C_{20}=\{3,5,6\}, C_{21}=\{2,3,4,5\}, C_{22}=$ $\{2,3,4,5,6\}, C_{23}=\{2,3,4\}, C_{24}=\{2,3,4,6\}, C_{25}=\{2,3,5\}, C_{26}=\{2,3,5,6\}, C_{27}=$ $\{2,3\}, C_{28}=\{2,3,6\}, C_{29}=\{3,4\}, C_{30}=\{3,4,6\}, C_{31}=\{3,4,5\}, C_{32}=\{3,4,5,6\}$. The configuration $A_{G}$ contains six combinatorial pure subrings which are isomorphic to $A_{K_{2,3}}$. By considering weight vectors such that the reduced Gröbner basis of $I_{K_{2,3}}$ is quadratic, we found a weight vector $\boldsymbol{w} \in \mathbb{R}^{32}$ such that the reduced Gröbner basis of $I_{G}$ is also quadratic. Let $\boldsymbol{w}=(25,24,24,45,46,44,37,37,47,47,63,107,47,25,24$, $46,36,33,20,26,102,87,80,103,92,35,25,26,53,37,22,27)$. The following Gröbner basis of $I_{G}$ with respect to $\boldsymbol{w}$ is quadratic:
$\left\{-x_{20} x_{31}+x_{19} x_{32},-x_{15} x_{3}+x_{14} x_{2}, x_{28} x_{20}-x_{27} x_{19},-x_{27} x_{31}+x_{28} x_{32}, x_{18} x_{31}-x_{30} x_{19}\right.$,
$x_{3} x_{32}-x_{8} x_{19}, x_{3} x_{31}-x_{7} x_{19}, x_{2} x_{19}-x_{18} x_{3},-x_{15} x_{19}+x_{18} x_{14},-x_{26} x_{15}+x_{27} x_{14}$,
$x_{27} x_{3}-x_{26} x_{2}, x_{1} x_{19}-x_{17} x_{3},-x_{17} x_{2}+x_{1} x_{18}, x_{2} x_{31}-x_{30} x_{3},-x_{15} x_{31}+x_{30} x_{14}$,
$-x_{30} x_{20}+x_{18} x_{32}, x_{7} x_{27}-x_{28} x_{8}, x_{7} x_{20}-x_{3} x_{32},-x_{8} x_{31}+x_{7} x_{32}, x_{2} x_{31}-x_{6} x_{19}$, $x_{3} x_{20}-x_{4} x_{19},-x_{1} x_{31}+x_{5} x_{19}, x_{4} x_{31}-x_{3} x_{32}, x_{27} x_{19}-x_{18} x_{26},-x_{6} x_{3}+x_{7} x_{2}$, $-x_{7} x_{15}+x_{6} x_{14},-x_{6} x_{20}+x_{2} x_{32}, x_{2} x_{32}-x_{8} x_{18}, x_{2} x_{31}-x_{7} x_{18},-x_{5} x_{3}+x_{1} x_{7}$,
$-x_{5} x_{2}+x_{1} x_{6}, x_{27} x_{3}-x_{4} x_{28}, x_{28} x_{15}-x_{16} x_{27},-x_{8} x_{20}+x_{4} x_{32}, x_{27} x_{31}-x_{30} x_{26}$,
$x_{1} x_{32}-x_{10} x_{27}, x_{2} x_{32}-x_{9} x_{27}, x_{16} x_{20}-x_{15} x_{19}, x_{5} x_{20}-x_{1} x_{32},-x_{15} x_{3}+x_{13} x_{1}$,
$-x_{10} x_{2}+x_{9} x_{1},-x_{15} x_{31}+x_{16} x_{32}, x_{17} x_{31}-x_{29} x_{19}, x_{1} x_{31}-x_{10} x_{28}, x_{2} x_{31}-x_{9} x_{28}$,
$-x_{1} x_{32}+x_{17} x_{8}, x_{1} x_{31}-x_{17} x_{7}, x_{6} x_{32}-x_{8} x_{30}, x_{6} x_{31}-x_{7} x_{30}, x_{1} x_{31}-x_{29} x_{3}$,
$-x_{29} x_{2}+x_{1} x_{30}, x_{30} x_{2}-x_{6} x_{18}, x_{2} x_{20}-x_{4} x_{18},-x_{29} x_{20}+x_{17} x_{32},-x_{6} x_{26}+x_{7} x_{27}$,
$x_{5} x_{18}-x_{1} x_{30},-x_{1} x_{30}+x_{17} x_{6}, x_{28} x_{14}-x_{16} x_{26}, x_{1} x_{20}-x_{17} x_{4}, x_{2} x_{32}-x_{4} x_{30}$,
$x_{3} x_{32}-x_{9} x_{26}, x_{29} x_{1}-x_{17} x_{5}, x_{8} x_{3}-x_{4} x_{7},-x_{11} x_{19}+x_{10} x_{18}, x_{15} x_{19}-x_{13} x_{17}$,
$x_{10} x_{18}-x_{9} x_{17},-x_{7} x_{15}+x_{16} x_{8},-x_{11} x_{31}+x_{10} x_{30},-x_{29} x_{18}+x_{17} x_{30},-x_{11} x_{3}+x_{10} x_{2}$,
$-x_{10} x_{15}+x_{11} x_{14},-x_{1} x_{30}+x_{11} x_{28}, x_{8} x_{2}-x_{4} x_{6}, x_{5} x_{32}-x_{29} x_{8}, x_{5} x_{31}-x_{29} x_{7}$,
$x_{15} x_{3}-x_{16} x_{4}, x_{1} x_{8}-x_{5} x_{4}, x_{7} x_{15}-x_{13} x_{5},-x_{10} x_{6}+x_{9} x_{5}, x_{9} x_{14}-x_{13} x_{10}$,
$x_{5} x_{30}-x_{29} x_{6},-x_{1} x_{32}+x_{29} x_{4},-x_{1} x_{32}+x_{11} x_{26}, x_{15} x_{31}-x_{13} x_{29},-x_{10} x_{30}+x_{9} x_{29}$,
$-x_{11} x_{7}+x_{10} x_{6},-x_{23} x_{15}+x_{11} x_{27},-x_{1} x_{32}+x_{23} x_{14}, x_{9} x_{15}-x_{11} x_{13},-x_{1} x_{32}+x_{22} x_{15}$,
$x_{23} x_{3}-x_{22} x_{2}, x_{22} x_{14}-x_{10} x_{26},-x_{23} x_{26}+x_{22} x_{27},-x_{25} x_{15}+x_{13} x_{27},-x_{25} x_{14}+x_{13} x_{26}$,
$-x_{27} x_{3}+x_{25} x_{1}, x_{23} x_{19}-x_{22} x_{18},-x_{23} x_{31}+x_{22} x_{30},-x_{1} x_{30}+x_{23} x_{16}, x_{2} x_{32}-x_{21} x_{15}$,
$x_{24} x_{15}-x_{1} x_{30},-x_{2} x_{32}+x_{23} x_{13},-x_{3} x_{32}+x_{21} x_{14}, x_{23} x_{3}-x_{21} x_{1},-x_{24} x_{27}+x_{23} x_{28}$
$, x_{27} x_{19}-x_{25} x_{17}, x_{1} x_{31}-x_{24} x_{14}, x_{24} x_{20}-x_{23} x_{19}, x_{23} x_{31}-x_{24} x_{32},-x_{23} x_{7}+x_{22} x_{6}$,
$-x_{12} x_{15}+x_{11} x_{16},-x_{12} x_{27}+x_{1} x_{30},-x_{12} x_{14}+x_{10} x_{16}, x_{1} x_{31}-x_{22} x_{16}, x_{12} x_{20}-x_{10} x_{18}$,
$-x_{12} x_{32}+x_{10} x_{30},-x_{3} x_{32}+x_{22} x_{13},-x_{24} x_{26}+x_{22} x_{28}, x_{13} x_{28}-x_{25} x_{16},-x_{7} x_{27}+x_{25} x_{5}$,
$x_{23} x_{19}-x_{21} x_{17}, x_{3} x_{32}-x_{25} x_{10},-x_{24} x_{8}+x_{23} x_{7}, x_{1} x_{31}-x_{12} x_{26},-x_{12} x_{8}+x_{10} x_{6}$,
$x_{27} x_{31}-x_{25} x_{29},-x_{23} x_{3}+x_{24} x_{4}, x_{2} x_{31}-x_{21} x_{16},-x_{23} x_{7}+x_{21} x_{5},-x_{12} x_{28}+x_{24} x_{16}$,
$-x_{25} x_{9}+x_{21} x_{13},-x_{21} x_{10}+x_{22} x_{9}, x_{2} x_{31}-x_{24} x_{13},-x_{23} x_{10}+x_{22} x_{11}, x_{10} x_{2}-x_{12} x_{4}$,
$x_{9} x_{16}-x_{12} x_{13},-x_{23} x_{31}+x_{21} x_{29}, x_{2} x_{32}-x_{25} x_{11}, x_{23} x_{9}-x_{21} x_{11}, x_{21} x_{27}-x_{25} x_{23}$,
$x_{21} x_{26}-x_{25} x_{22}, x_{24} x_{11}-x_{12} x_{23}, x_{24} x_{10}-x_{12} x_{22}, x_{21} x_{28}-x_{24} x_{25}, x_{2} x_{31}-x_{12} x_{25}$,
$\left.x_{24} x_{9}-x_{12} x_{21}\right\}$.
For the sake of reliability, we computed this using several different software packages (CoCoA [1], Risa/Asir [12], and so on). The code for the computation is available in
https://sci-tech.ksc.kwansei.ac.jp/~hohsugi/R_Sakamoto/code_cutideal
For example, if we input

```
M:=MakeTermOrd(mat ([[25, 24, 24, 45, 46, 44, 37, 37, 47, 47, 63, 107, 47, 25, 24,
46, 36, 33, 20, 26, 102, 87, 80, 103, 92, 35, 25, 26, 53, 37, 22, 27]]));
R:= NewPolyRing(QQ, SymbolRange("x",1,32 ), M, 1);
use R;
A:=mat([
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1],
[0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,1,1,1,1],
[0,0,0,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1],
[0,0,0,0,1,1,1,1,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1],
[0,0,1,1,0,0,1,1,1,1,0,0,1,1,0,0,0,0,1,1,1,1,0,0,1,1,0,0,0,0,1,1],
[0,1,1,0,0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0],
[0,1,0,1,0,1,0,1,1,0,1,0,1,0,1,0,0,1,0,1,1,0,1,0,1,0,1,0,0,1,0,1],
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]
]);
ReducedGBasis( toric(A) );
```

to CoCoA, then we can obtain the reduced Gröbner basis in several seconds. The
monomial order $\boldsymbol{w}$ is neither lexicographic nor reverse lexicographic. In fact, all monomial orders for which the reduced Gröbner bases of $I_{G}$ consist of quadratic binomials are neither lexicographic nor reverse lexicographic.
Theorem 1.7. Let $G$ be the graph of Fig. 2. Then $I_{G}$ has quadratic Gröbner bases, none of which are either lexicographic or reverse lexicographic. In particular, $I_{G}$ is a counterexample to Conjecture 0.1.
Proof. Since $G$ has an induced cycle of length $5, I_{G}$ has no reverse lexicographic quadratic Gröbner bases by Proposition 1.2 . Moreover, since $K_{2,3}$ is obtained by contraction of an edge of $G, I_{G}$ has no lexicographic quadratic Gröbner bases by Proposition 1.6 .

## 2. Squarefree Veronese subrings and cut ideals of cycles

If a graph $G$ is a cycle, then the cut ideal $I_{G}$ is generated by quadratic binomials by Proposition 0.3. Nagel-Petrović [11, Proposition 3.2] claimed that the cut ideal of a cycle has a quadratic Gröbner basis with respect to a lexicographic order. This claim relies on the following claims in Chifman-Petrović [5]:

Claim 1 (5, Proposition 2]) Let $I_{m}$ be the toric ideal of phylogenetic invariants for the general group-based model on the claw tree $K_{1, m}$ (defined later) which coincides with the cut ideals of the cycle of length $m+1$. Then $I_{m}$ is generated by $Q_{m}$ (defined later) which consists of quadratic binomials.

Claim 2 ([5, Proposition 3]) The set $Q_{m}$ is a lexicographic Gröbner basis of $I_{m}$ for any $m \geq 4$.

However, Claim 1 is not true for any $m \geq 5$. Therefore, Claim 2 is not true for any $m \geq 5$. Moreover, with respect to a lexicographic order given in [5], the reduced Gröbner basis of $I_{m}$ is not quadratic for any $m \geq 5$. In this section, we point out an error in the proof of [5, Propositions 2 and 3] for the cut ideal of the cycle and present a lexicographic order for which the reduced Gröbner basis of the cut ideal of the cycle of length 7 consists of quadratic binomials.

First, we explain an error in the proof of [5, Propositions 2 and 3]. For each $m$-dimensional $(0,1)$ vector $\left(i_{1}, i_{2}, \ldots, i_{m}\right)$, we associate a variable $q_{i_{1} i_{2} \cdots i_{m}}$. Let $K\left[q_{i_{1} i_{2} \ldots i_{m}} \mid i_{1}, i_{2}, \ldots, i_{m} \in\{0,1\}\right]$ and $K\left[a_{i_{j}}^{(j)} \mid i_{j} \in\{0,1\}, j=1, \ldots, m+1\right]$ be polynomial rings over $K$. Let

$$
\varphi_{m}: K\left[q_{i_{1} i_{2} \ldots i_{m}} \mid i_{1}, i_{2}, \ldots, i_{m} \in\{0,1\}\right] \rightarrow K\left[a_{i_{j}}^{(j)} \mid i_{j} \in\{0,1\}, j=1, \ldots, m+1\right]
$$

be a homomorphism such that $\varphi_{m}\left(q_{i_{1} i_{2} \ldots i_{m}}\right)=a_{i_{1}}^{(1)} a_{i_{2}}^{(2)} \ldots a_{i_{m}}^{(m)} a_{i_{1}+i_{2}+\cdots+i_{m}(\bmod 2)}^{(m+1)}$ and let $I_{m}$ be the kernel of $\varphi_{m}$. According to [11], the ideal $I_{m}$ is the cut ideal of the cycle of length $m+1$. Let $Q_{m}$ be a set of all quadratic binomials

$$
q_{i_{1} i_{2} \cdots i_{m}} q_{j_{1} j_{2} \cdots j_{m}}-q_{k_{1} k_{2} \cdots k_{m}} q_{l_{1} l_{2} \cdots l_{m}} \in I_{m}
$$

satisfying one of the following properties:
(1) For some $1 \leq a \leq m$ and $j \in\{0,1\}$,

$$
i_{a}=j_{a}=j=k_{a}=l_{a}
$$

and the binomial

$$
q_{i_{1} \ldots i_{a-1} i_{a+1} \ldots i_{m}} q_{j_{1} \ldots j_{a-1} j_{a+1} \ldots j_{m}}-q_{k_{1} \ldots k_{a-1} k_{a+1} \ldots k_{m}} q_{l_{1} \ldots l_{a-1} l_{a+1} \ldots l_{m}}
$$

belongs to $I_{m-1}$;
(2) For each $1 \leq b \leq m$,

$$
i_{b}+j_{b}=1=k_{b}+l_{b}
$$

and the binomial

$$
q_{i_{1} \ldots i_{b-1} i_{b+1} \ldots i_{m}} q_{j_{1} \ldots j_{b-1} j_{b+1} \ldots j_{m}}-q_{k_{1} \ldots k_{b-1} k_{b+1} \ldots k_{m}} q_{l_{1} \ldots l_{b-1} l_{b+1} \ldots l_{m}}
$$

belongs to $I_{m-1}$.
In [5, Proposition 2], $I_{m}$ is claimed to be generated by $Q_{m}$ for any $m \geq 4$. However, this is incorrect for $m \geq 5$. Now, we consider the quadratic binomial

$$
q=q_{10101} q_{01010}-q_{11111} q_{00000}
$$

and the binomial $q^{\prime}=q_{1101} q_{0010}-q_{1111} q_{0000}$. Since

$$
\varphi_{5}\left(q_{10101} q_{01010}\right)=\varphi_{5}\left(q_{11111} q_{00000}\right)=a_{0}^{(1)} a_{0}^{(2)} a_{0}^{(3)} a_{0}^{(4)} a_{0}^{(5)} a_{0}^{(6)} a_{1}^{(1)} a_{1}^{(2)} a_{1}^{(3)} a_{1}^{(4)} a_{1}^{(5)} a_{1}^{(6)}
$$

$q$ belongs to $I_{5}$. On the other hand, since

$$
\begin{aligned}
& \varphi_{4}\left(q_{1101} q_{0010}\right)=a_{0}^{(1)} a_{0}^{(2)} a_{0}^{(3)} a_{0}^{(4)} a_{1}^{(1)} a_{1}^{(2)} a_{1}^{(3)} a_{1}^{(4)}\left(a_{1}^{(5)}\right)^{2}, \\
& \varphi_{4}\left(q_{1111} q_{0000}\right)=a_{0}^{(1)} a_{0}^{(2)} a_{0}^{(3)} a_{0}^{(4)} a_{1}^{(1)} a_{1}^{(2)} a_{1}^{(3)} a_{1}^{(4)}\left(a_{0}^{(5)}\right)^{2}
\end{aligned}
$$

$q^{\prime}$ does not belong to $I_{4}$. Hence $q$ does not belong to $Q_{5}$. The following proposition shows that $q$ is not generated by $Q_{5}$.

Proposition 2.1. Let

$$
P=\left\{q_{i_{1} i_{2} i_{3} i_{4} i_{5}} q_{j_{1} j_{2} j_{3} j_{4} j_{5}} \mid i_{k}+j_{k}=1, i_{k}, j_{k} \in\{0,1\} \text { for } 1 \leq k \leq 5\right\}
$$

Then any nonzero binomial $q=u-v$ where $u, v \in P$ does not belong to $Q_{5}$.
Proof. Let

$$
q=q_{i_{1} i_{2} i_{3} i_{4} i_{5}{ }_{j} q_{j_{1} j_{2} j_{3} j_{4} j_{5}}-q_{i_{1}^{\prime} i_{2}^{\prime} i_{3}^{\prime} i_{4}^{\prime} i_{5}^{\prime}}^{q_{1}^{\prime} j_{2}^{\prime} j_{3}^{\prime} j_{4}^{\prime} j_{5}^{\prime}}, ~}^{\text {and }}
$$

be a nonzero binomial where $i_{k}+j_{k}=i_{k}^{\prime}+j_{k}^{\prime}=1$ and $i_{k}, j_{k}, i_{k}^{\prime}, j_{k}^{\prime} \in\{0,1\}$ for $1 \leq k \leq 5$. It is trivial that $q$ does not satisfy property (1). Since $i_{k}+j_{k}=i_{k}^{\prime}+j_{k}^{\prime}=1$ for $1 \leq k \leq 5$, we have

$$
\sum_{k=1}^{5} i_{k}+\sum_{k=1}^{5} j_{k}=\sum_{k=1}^{5} i_{k}^{\prime}+\sum_{k=1}^{5} j_{k}^{\prime}=5
$$

Hence we may assume that $\sum_{k=1}^{5} i_{k} \equiv \sum_{k=1}^{5} i_{k}^{\prime} \equiv 1$ and $\sum_{k=1}^{5} j_{k} \equiv \sum_{k=1}^{5} j_{k}^{\prime} \equiv 0$ modulo 2. Since $q$ is not zero, $q_{i_{1} i_{2} i_{3} i_{4} i_{5}} \neq q_{i_{1}^{\prime} i_{2}^{\prime} i_{3}^{\prime} i_{4}^{\prime} i_{5}^{\prime}}$. Thus we may assume that $i_{k}=1$ and $i_{k}^{\prime}=0$ for some $1 \leq k \leq 5$ (by exchanging $q_{i_{1} i_{2} i_{3} i_{4} i_{5}}$ and $q_{i_{1}^{\prime} i_{2}^{\prime} i_{3}^{\prime} i_{4}^{\prime} i_{5}^{\prime}}$ if we need). Then $j_{k}=0$ and $j_{k}^{\prime}=1$. For example, if $k=1$, then

$$
q^{\prime}=q_{i_{2} i_{3} i_{4} i_{5}} q_{j_{2} j_{3} j_{4} j_{5}}-q_{i_{2}^{\prime} i_{3}^{\prime} i_{4}^{\prime} i_{5}^{\prime}}{q j_{2}^{\prime} j_{3}^{\prime} j_{4}^{\prime} j_{5}^{\prime}}^{\prime}
$$

does not belong to $I_{4}$ since $i_{2}+i_{3}+i_{4}+i_{5} \equiv j_{2}+j_{3}+j_{4}+j_{5} \equiv 0$ and $i_{2}^{\prime}+i_{3}^{\prime}+i_{4}^{\prime}+i_{5}^{\prime} \equiv$ $j_{2}^{\prime}+j_{3}^{\prime}+j_{4}^{\prime}+j_{5}^{\prime} \equiv 1$. Thus $q$ does not satisfy property (2).

Thus, $I_{5}$ is not generated by $Q_{5}$. This is the error in the proof of [5, Proposition 2]. By this error, instead of $I_{m}$, an ideal that is strictly smaller than $I_{m}$ is considered in the proof of [5, Proposition 3]. Unfortunately, with respect to a lexicographic order considered in [5, Proposition 3], the reduced Gröbner basis of $I_{m}$ is not quadratic for $m \geq 5$. The computation for $m=5$ is given in

| degree | the number of binomials |
| :---: | :---: |
| 2 | 195 |
| 3 | 10 |
| 4 | 2 |

Table 1. The number of binomials in the reduced Gröbner basis of $I_{5}$.

We describe the number of binomials in the reduced Gröbner basis with respect to a lexicographic order in 5 for $m=5$ in Table 1.

Thus the existence of a quadratic Gröbner basis of the cut ideal of a cycle is now an open problem. However, we will show that there exists a lexicographic order such that the reduced Gröbner basis of the cut ideal of a cycle of length 7 consists of quadratic binomials. In general, if $G$ is a cycle of length $m$, then it is known that $\left\{\delta_{C} \mid C \subset V(G)\right\}=\left\{\left(d_{1}, \ldots, d_{m}\right) \in\{0,1\}^{m} \mid d_{1}+\cdots+d_{m}\right.$ is even $\}$. Let $G$ be the cycle of length 7 . Then we have

$$
A_{G}=\left(\begin{array}{cccccccccc}
\mathbf{0} & & A & & & B & & & C & \\
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1
\end{array}\right)
$$

where

$$
\begin{aligned}
& A=\left(\begin{array}{lllllllllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}\right),
\end{aligned}
$$

$$
\begin{aligned}
& C=\left(\begin{array}{lllllll}
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right) .
\end{aligned}
$$

In general, the configuration of the $(m, r)$-squarefree Veronese subring is the configuration whose columns are

$$
\left\{\left(d_{1}, \ldots, d_{m}\right) \in\{0,1\}^{m} \mid d_{1}+\cdots+d_{m}=r\right\}
$$

The matrix $A$ is a configuration of the $(7,2)$-squarefree Veronese subring, and $B$ is a configuration of the $(7,4)$-squarefree Veronese subring. According to [16, Theorem 1.4], there is a lexicographic order such that the reduced Gröbner basis of the toric ideal of the $(m, 2)$-squarefree Veronese subring consists of quadratic binomials for any integer $m \geq 2$. However, it is not known whether there is a lexicographic order such that the reduced Gröbner basis of $I_{B}$ consists of quadratic binomials. Now, we consider the following question:

Question 2.2. If we use lexicographic orders such that the reduced Gröbner bases of $I_{A}$ and $I_{B}$ consist of quadratic binomials, do we obtain a lexicographic order such that the reduced Gröbner basis of $I_{A_{G}}$ consists of quadratic binomials?

To answer this question, we look for a lexicographic order $>_{1}$ such that the reduced Gröbner basis of $I_{B} \subset K\left[y_{1}, y_{2}, \ldots, y_{35}\right]$ consists of quadratic binomials. For $i=1,2, \ldots, 7$, we consider the subconfiguration $B_{i}$ of $B$ with column vectors consisting of all column vectors of $B$ whose $i$-th component is one. We consider combining lexicographic orders such that the reduced Gröbner bases of $I_{B_{i}}$ consist of quadratic binomials. We write down the lexicographic order $>_{1}$ :
$y_{1}>y_{2}>y_{4}>y_{3}>y_{5}>y_{7}>y_{6}>y_{10}>y_{9}>y_{8}>y_{11}>y_{13}>y_{12}>y_{16}>y_{15}>$ $y_{14}>y_{20}>y_{19}>y_{18}>y_{17}>y_{21}>y_{23}>y_{22}>y_{26}>y_{25}>y_{24}>y_{30}>y_{29}>y_{28}>$ $y_{27}>y_{35}>y_{34}>y_{33}>y_{32}>y_{31}$.
Next, we consider combining two lexicographic orders such that the reduced Gröbner bases of $I_{A}$ and $I_{B}$ consist of quadratic binomials. We fix the order
$x_{23}>x_{24}>x_{26}>x_{25}>x_{27}>x_{29}>x_{28}>x_{32}>x_{31}>x_{30}>x_{33}>x_{35}>x_{34}>$ $x_{38}>x_{37}>x_{36}>x_{42}>x_{41}>x_{40}>x_{39}>x_{43}>x_{45}>x_{44}>x_{48}>x_{47}>x_{46}>$ $x_{52}>x_{51}>x_{50}>x_{49}>x_{57}>x_{56}>x_{55}>x_{54}>x_{53}$
which corresponds to the lexicographic order $>_{1}$ and look for the order such that the reduced Gröbner basis of $I_{A_{G}}$ consists of quadratic binomials by modifying the order for $I_{A}$ using computational experiments. A desired lexicographic order is
$x_{1}>x_{17}>x_{18}>x_{19}>x_{22}>x_{20}>x_{21}>x_{13}>x_{14}>x_{15}>x_{16}>x_{2}>x_{3}>x_{4}>$ $x_{5}>x_{6}>x_{7}>x_{8}>x_{9}>x_{10}>x_{11}>x_{12}>x_{23}>x_{24}>x_{26}>x_{25}>x_{27}>x_{29}>$ $x_{28}>x_{32}>x_{31}>x_{30}>x_{33}>x_{35}>x_{34}>x_{38}>x_{37}>x_{36}>x_{42}>x_{41}>x_{40}>$ $x_{39}>x_{43}>x_{45}>x_{44}>x_{48}>x_{47}>x_{46}>x_{52}>x_{51}>x_{50}>x_{49}>x_{57}>x_{56}>$ $x_{55}>x_{54}>x_{53}>x_{58}>x_{59}>x_{60}>x_{61}>x_{62}>x_{63}>x_{64}$.
The reduced Gröbner basis of $I_{G}$ consists of 1050 quadratic binomials. The computation is given in
https://sci-tech.ksc.kwansei.ac.jp/~hohsugi/R_Sakamoto/code_cutideal
Note that any cycle of length $\leq 6$ is obtained by the sequence of contractions from $G$. Thus, we have the following.

Theorem 2.3. Let $G$ be a cycle of length $\leq 7$. Then $I_{G}$ has a lexicographic quadratic Gröbner basis.
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