



The University of Manchester Research

Modeling complex linguistic expressions in qualitative decision making

DOI: 10.1016/j.knosys.2017.12.030

Document Version

Accepted author manuscript

Link to publication record in Manchester Research Explorer

Citation for published version (APA):

Wang, H., Xu, Z., & Zeng, X. J. (2018). Modeling complex linguistic expressions in qualitative decision making: An overview. *Knowledge-Based Systems*, *144*, 174-187. https://doi.org/10.1016/j.knosys.2017.12.030

Published in: Knowledge-Based Systems

Citing this paper

Please note that where the full-text provided on Manchester Research Explorer is the Author Accepted Manuscript or Proof version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version.

General rights

Copyright and moral rights for the publications made accessible in the Research Explorer are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Takedown policy

If you believe that this document breaches copyright please refer to the University of Manchester's Takedown Procedures [http://man.ac.uk/04Y6Bo] or contact uml.scholarlycommunications@manchester.ac.uk providing relevant details, so we can investigate your claim.



Modeling complex linguistic expressions in qualitative decision making: An overview

Hai Wang^{a,b}, Zeshui Xu^{c,d,*}, Xiao-Jun Zeng^b

 ^a School of Government Audit, Nanjing Audit University, Nanjing, Jiangsu 211815, China
 ^b School of Computer Science, University of Manchester, Manchester, M13 9PL, United Kingdom
 ^c Business School, Sichuan University, Chengdu, Sichuan, 610064, China
 ^d School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing 210044, China

Abstract

The increasing complexity of real-world problems drives the experts to consider complex linguistic expressions instead of single linguistic terms to represent their linguistic opinions under uncertainties. Based on some classical linguistic representational models, a number of techniques of modeling complex linguistic expressions have been proposed. The main purpose of this paper is to present a systematical overview on these techniques, especially their focused linguistic expressions and the associated computational essentials. According to the features of the underlying linguistic expressions, the existing techniques are classified into two categories: the models of natural linguistic expressions, such as uncertain linguistic terms and hesitant fuzzy linguistic term sets, which focus on frequently used expressions in natural languages, and the models of artificial linguistic expressions, such as discrete fuzzy numbers and probabilistic linguistic term sets, which consider special types of expressions artificially constructed by linguistic terms and additional information. After the presentation of comparative analyses

^{*}Corresponding author.

E-mail addresses: wanghai17@sina.com (H. Wang), xuzeshui@263.com (Z.S. Xu), x.zeng@manchester.ac.uk (X. Zeng).

on the existing techniques, we figure out some current challenges and provide some possible directions for further developments.

Keywords: Complex linguistic expressions; Fuzzy linguistic approach; Linguistic term sets; Hesitant fuzzy linguistic term set; Uncertain linguistic term.

1. Introduction

The increasing complexity and uncertainties of real-world problems drive agents to manipulate qualitative information for decision making. Qualitative decision making (QDM) is a kind of commonly considered process to operate qualitative information. To model the uncertainties involved in the decision information, fuzzy logic based on fuzzy set theory [1] has been widely acknowledged as a solution to develop QDM processes, such as linguistic decision making processes. Basically, fuzzy logic is not fuzzy but a precise logic of imprecision and approximate reasoning. It could be regarded as an attempt at formalization and mechanization of the remarkable human capability to converse, reason and make rational decisions in an imprecise, uncertain, incomplete and even conflicting environment [2]. Standing on fuzzy logic, the fuzzy linguistic approach [3], especially techniques for computing with words (CWW) [4-6], is exceedingly suitable for the QDM problems under various kinds of uncertainties. One noticeable feature of the fuzzy linguistic approach is the use of linguistic variables to collect information represented by natural or artificial languages. Thus, the developments of the fuzzy linguistic approach usually focus on: (1) the range of possible values which can be assigned to a linguistic variable; and (2) the techniques for operating and reasoning with these values.

Classical linguistic models assume that the value of a given linguistic variable should be a single

linguistic term selected from a predefined, maybe discrete, linguistic term set (LTS) which is a granular partition of a domain [7, 8]. In the semantic model [9], linguistic terms are represented and computed by their semantics taking the form of fuzzy numbers. In the ordered structure model [10], a total order on the LTSs and some basic operations are predefined. Both models incorporate an approximation step, which may lead to the loss of information, to ensure the interpretability of the computational results. Two accurate linguistic models overcome this issue by extending the discrete LTS to a continuous version. The linguistic 2-tuple model [11] converts a LTS with g+1 terms into a numerical interval [0, g] by a symbolic translation. The virtual linguistic model [12, 13] introduces semantics of virtual linguistic terms by means of linguistic modifiers. The two models frequently serve as the basic model to develop sophisticated linguistic decision making processes.

Although linguistic terms are the appropriate tools for describing vague concepts in natural language, the employment of individual linguistic terms might be very hard to express the experts' opinions exactly due to the complex decision making situations and the experts' granules of knowledge [14]. In fact, using a predefined linguistic term would restrict to present preferences freely, because: (1) the linguistic term selected from the LTS may not coincide with the expert's preference; and thus (2) the expert has to balance among several linguistic terms [15]. To decrease such factitious uncertainties of selecting and balancing linguistic terms, it is rational to allow the experts to use more than one linguistic term. This results in the consideration of complex linguistic expressions in QDM. Roughly, complex linguistic expressions, such as comparative linguistic expressions, refer to the linguistic information involving more than one linguistic term, expressed by either natural or artificial languages by means of linguistic terms, connectives and linguistic hedges. The extension from a single linguistic term to a complex linguistic expression facilitates the elicitation and representation of the experts' preferences in an elaborated manner.

Ideally, the use of complex linguistic expressions enables the experts to express their opinions naturally and freely. But handling such information is definitely much more complicated than computing with single terms. Till now, some researches have devoted several attempts to modeling and computing with some specific types of complex linguistic expressions, such as uncertain linguistic terms (ULTs) [16], hesitant fuzzy linguistic term sets (HFLTSs) [17], extended hesitant fuzzy linguistic term sets (EHFLTSs) [18], proportional terms [19], distribution assessments [20], discrete fuzzy numbers (DFNs) [21, 22], probabilistic linguistic term sets (PLTSs) [23], linguistic hesitant fuzzy sets (LHFSs) [24], 2-dimension linguistic terms (2DLTs) [25], the evidential reasoning (ER) algorithm [26], the label semantics model [27] and so on. These models are summarized in Table 1.

Model	Reference	Characteristics
ULTs	[16]	A linguistic interval formed by two terms
HFLTSs	[17]	A subset of ordered finite consecutive terms
EHFLTSs	[18]	A subset of ordered finite terms
Label semantic model	[27]	Expressions formed by several terms and logical connectives
ER algorithm	[26]	A basic probability assignment on the several terms
Proportional terms	[19]	A probability distribution of two consecutive terms
Distribution assessments	[20]	A probability distribution of several terms
DFNs	[21, 22]	Several numerical scales associated with membership degrees
PLTSs	[23]	An incomplete probability distribution of several terms
LHFSs	[24]	Several terms with hesitant fuzzy membership degrees
2DLTs	[25]	A term associated with another term to indicate confidence

Table 1. A summary of techniques for modeling complex linguistic expressions.

The consideration of multiple linguistic terms is a bit similar to the motivation of developing multi-granularity linguistic decision making techniques [28-30]. In reality, when the knowledge granule of an expert is coarser than that of a given LTS, he/she may consider expressing the opinion by complex linguistic expressions, or alternatively, he/she may seek for another LTS with a relatively coarse granule.

However, the techniques of modeling complex linguistic expressions will not totally substitute the multi-granularity techniques, and vice versa. For one thing, the multi-granularity techniques are frequently considered because the different granules come from distinct information sources. Taking the evaluation of students for instance, we may use a LTS to evaluate the grades of a lesson, and use another LTS to evaluate the research potential. For another, considering complex linguistic expressions is a much more natural pattern to express opinions than considering another LTS with coarser granularity. This is because the experts do not have to turn their attention to another LTS. But if multi-granularity linguistic information can be collected, computing with this information is generally easier than computing with information elicited by complex linguistic expressions.

This paper is devoted to providing a systematic overview on the current techniques of modeling complex linguistic expressions. Especially, we shall focus on the types of linguistic expressions which can be represented and elicited. From the perspective of CWW, the current techniques are classified into the models of natural linguistic expressions (which frequently appear in natural languages) and the models of artificial linguistic expressions (which require some specific information, such as possibilities, that does not directly appear in natural languages). After exploring the main idea of current techniques, we will present a brief discussion on each technique and highlight some current issues and challenges of modeling complex linguistic expressions. In order that, the rest part is organized as follows: Section 2 recalls the framework of linguistic decision making as well as some classical linguistic expressions respectively. We mainly pay our attention on the focused linguistic expressions and computational essentials. Section 5 presents a comparative discussion on the reviewed models. Some challenges and possible directions are figured out in Section 6. Finally, Section 7 concludes the paper.

2. The framework of linguistic decision making

We will recall the framework of the fuzzy linguistic approach, the concept of linguistic variables, the developments of values of linguistic variables and several classical linguistic models in this section.

2.1. Fuzzy linguistic approach and complex linguistic expressions

The fuzzy linguistic approach based on fuzzy logic is a common scheme for solving complex realworld problems which are usually ill-defined because of the involved incomplete, vague and uncertain information. The key idea is the use of linguistic variables to manage and model the inherent vagueness and uncertainty of the linguistic descriptors. Basically, the values of a linguistic variable are not numbers but words or sentences in a natural or artificial language so that qualitative information can be modeled by simulating the human cognitive processes. The concept of a linguistic variable is defined by a 4-tuple as follows:

Definition 1 [31]. A linguistic variable is characterized by a quintuple (X, S(X), U, G, M), where X is the name of the variable; S(X) (or simply S) denotes the term set of X with each term being a fuzzy variable denoted generically by s and ranging over the domain U which is associated with the base variable u; G is a syntactic rule for generating the names, s, of values of X; and M is a semantic rule for associating with each s its meaning, M(s), which is a fuzzy set of U.

The three denotations, i.e., the name s, its semantics M(s) and its restriction R(s) can be used interchangeably [31]. Given a domain U, the set S is a fuzzy partition of the domain which also implies the granularity of uncertainty. Generally, a set of g+1 linguistic terms, associated with their semantics, are denoted as:

$$S = \{s_0, s_1, \dots, s_g\}$$
(1)

6

Generally, the semantics of linguistic terms is specified due to the problem in hand. Thus, the linguistic terms may be not uniformly distributed or balanced in the LTS. For example, some unbalanced LTSs can be found in Refs. [32, 33].

To highlight complex linguistic expressions, it is usually assumed that the linguistic terms in S are words or phrases in a natural language. Complex linguistic expressions can be generated on the basis of the terms in S. According to the traditional and customary use of languages, we classify complex linguistic expressions by the following two classes:

- (1) Natural linguistic expressions: complex linguistic expressions which frequently appears in a natural language, like comparative linguistic expressions. For example, some natural linguistic expressions could be "between low and good", "low or medium or good", "at least very good", "more or less medium", "not good".
- (2) Artificial linguistic expressions: complex linguistic expressions that are generated in an artificial manner that is not as close to natural languages as natural linguistic expressions. This is usually caused by the case where some additional information (maybe taking the form of numerical values) are required to mine individual opinions as precisely as possible. For instance, an artificial linguistic expression might be "*low or medium or good, furthermore, the possibility of low is 0.2, the possibility of medium is 0.5 and the possibility of good is 0.3*".

To make decisions with linguistic information, the following decision making scheme is frequently considered [34]: (1) the selection of LTSs with semantics; (2) the selection of aggregation operators for linguistic information; (3) aggregation; and (4) exploitation. The linguistic expressions cannot be operated directly in decision making processes. Usually, an elicitation process is required to transform the complex linguistic expressions into their symbolic or mathematical representation so that linguistic computational

models can work.

2.2. Classical linguistic representational and computational models

For a better understanding of the overview in the next sections, we recall some linguistic models that usually serve as the foundation of modeling complex linguistic expressions.

The semantic based model [9] operates linguistic terms by computing with their semantics directly. Given n linguistic terms associated with semantics, the computational process is denoted as:

$$S^{n} \xrightarrow{\bar{F}} F(\mathbf{R}) \xrightarrow{app_{1}} S$$
⁽²⁾

where *S* is the LTS, \tilde{F} is an aggregation operator based on the extension principle [35] and $F(\mathbf{R})$ is the set of fuzzy sets over the set of real numbers \mathbf{R} . As semantics are expressed by fuzzy sets defined on the reference domain, the aggregation results are generally fuzzy sets which might not match the semantics of any original linguistic terms. Thus, an approximation procedure is included to ensure the interpretability of the aggregation results.

The ordered structure based model [10, 36, 37] does not operate semantics directly. Instead, it assumes that the following conditions are satisfied by a LTS S:

- (1) Linear order: if $\alpha > \beta$, then $s_{\alpha} > s_{\beta}$;
- (2) Max and min operators: if $s_{\alpha} \ge s_{\beta}$, then $\max(s_{\alpha}, s_{\beta}) = s_{\alpha}$, $\min(s_{\alpha}, s_{\beta}) = s_{\beta}$;
- (3) Negation operator: $neg(s_{\alpha}) = s_{\tau-\alpha}$.

The framework of decision making within the ordered structure model can be depicted as:

$$S^{n} \xrightarrow{A} [0, \tau] \xrightarrow{app_{2}} \{0, 1, \dots, g\}$$

$$(3)$$

where A is a linguistic aggregation operator, the approximation function app_2 is employed to provide interpretable computational results. Contrast to Eq. (2), it is obvious that the process depicted in Eq. (3) extends the use of indices of linguistic terms.

Notice that the above mentioned two models can serve as CWW techniques as their outputs are interpretable. But, the necessity of approximation leads to the loss of information. Thereafter, two accurate linguistic models have been proposed, which are the linguistic 2-tuple model [11] and the virtual linguistic model [12, 13].

The linguistic domain of the linguistic 2-tuple model can be treated as continuous. Based on the ordered structure model, linguistic information is presented by a pair of values (s_i, α) , where s_i is a linguistic term in *S* and $\alpha \in [-0.5, 0.5)$ is a number to represent the symbolic translation. Specifically, a numerical value $\beta \in [0, g]$ is included to represent the equivalent information of a linguistic 2-tuple (s_i, α) . The linguistic 2-tuple that expresses the equivalent information to β is obtained by:

$$\Delta:[0,g] \to S \times [-0.5, 0.5)$$

$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i & \alpha \in [-0.5, 0.5) \end{cases}$$
(4)

The inverse function of Δ , denoted by Δ^{-1} , is also defined such that $\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$. Based on the pair of functions Δ and Δ^{-1} , the computational foundation can be defined:

- (1) Linear order: $(s_i, \alpha_1) \leq (s_j, \alpha_2)$, if $\Delta^{-1}(s_i, \alpha_1) \leq \Delta^{-1}(s_j, \alpha_2)$;
- (2) Basic operation: $\lambda_1(s_i, \alpha_1) \oplus \lambda_2(s_j, \alpha_2) = \Delta(\lambda_1 \Delta^{-1}(s_i, \alpha_1) + \lambda_2 \Delta^{-1}(s_j, \alpha_2))$, where $\lambda_1, \lambda_2 \in [0, 1]$;
- (3) Negation operator: $neg((s_i, \alpha_1)) = \Delta(g \Delta^{-1}(s_i, \alpha_1)).$

where (s_i, α_1) and (s_i, α_2) are two linguistic 2-tuples.

As another accurate linguistic model, the virtual linguistic model provides an approach to operate the indices of terms directly. Given a LTS S with semantics, as depicted in Eq. (1), the virtual linguistic model extends S to a continuous form:

$$\overline{S} = \{s_{\alpha} | \alpha \in [0, g]\}$$
(5)

9

Given a linguistic term $s_{\alpha} \in \overline{S}$, if $s_{\alpha} \in S$, then it is called an original term or atomic term; otherwise, it is called a virtual term. From the perspective of CWW, a virtual term can be generated by an atomic term and a real number. Moreover, based on the semantics of atomic terms, the semantics of a virtual term are defined on the basis of the semantics of its nearest atomic term and a linguistic modifier [13]. In some special situations, the following symmetric version of LTS would be more convenient [38]:

$$S = \{s_t | t = -\gamma, \dots, -1, 0, 1, \dots, \gamma\}$$
(6)

Accordingly, Eq. (5) can be rewritten as $\overline{S} = \{s_{\alpha} | \alpha \in [-\gamma, \gamma]\}$. Given two virtual terms $s_{\alpha}, s_{\beta} \in \overline{S}$, the computational model is characterized by the following aspects:

- (1) Linear order: $s_{\alpha} \leq s_{\beta} \Leftrightarrow \alpha \leq \beta$;
- (2) Basic operations: $\lambda_1 s_{\alpha} \oplus \lambda_2 s_{\beta} = s_{\lambda_1 \alpha + \lambda_2 \beta}$ and $(s_{\alpha})^{\lambda_1} \otimes (s_{\beta})^{\lambda_2} = s_{\alpha^{\lambda_1} \beta^{\lambda_2}}$, where $\lambda_1, \lambda_2 \in [0, 1]$;
- (3) Negation operator: $neg(s_{\alpha}) = s_{\tau-\alpha}$. If the LTS is denoted by Eq. (6), then $neg(s_{\alpha}) = s_{-\alpha}$.

From a mathematical point of view, the computational process and results of this model are equivalent to those of the linguistic 2-tuple model [39]. The virtual linguistic model is easier than the linguistic 2-tuple model because it computes the indices of term directly. However, it is not so straightforward to interpret the semantics of virtual terms.

3. Techniques for modeling natural linguistic expressions

This section focuses on four techniques, i.e., ULTs, HFLTSs, EHFLTSs and linguistic expressions based on the label semantics (LEoLS), which deal with natural linguistic expressions. Especially, the former three can be elicited by natural linguistic expressions directly.

3.1. Uncertain linguistic terms

3.1.1. Elicitation and representation of ULTs

Given a predefined LTS, ULTs may be considered if the individual granule of knowledge does not meet the granule defined by the LTS due to time pressure, lack of knowledge, and his/her limited expertise. An ULT is formally defined as:

Definition 2 [16]. Given a continuous LTS \overline{S} , an uncertain linguistic term, denoted as \tilde{s} , is a linguistic interval with its lower and upper limits being linguistic terms. That is, $\tilde{s} = [s_{\alpha}, s_{\beta}]$, where $s_{\alpha}, s_{\beta} \in \overline{S}$.

It is obvious that this model is based on the virtual linguistic model. An ULT is an ordered infinite consecutive subset of \overline{S} and can be elicited by the following natural linguistic expression:

Between
$$s_{\alpha}$$
 and s_{β}

An example of ULTs can be found in Fig. 1. As suggested by Rodríguez et al. [40], this form of expressions is named as comparative linguistic expressions. Specifically, the following transformation function can be employed to elicit ULTs by the above form of natural linguistic expressions:



Fig. 1. An example of a ULT

Definition 3. The transformation function, denoted by TF_U , to obtain ULTs is defined as:

 $TF_U(between \ s_{\alpha} \ and \ s_{\beta}) = [s_{\alpha}, \ s_{\beta}].$

The set of ULTs based on S is denoted by Y. A relatively new version of ULTs was proposed by Zhang [41] recently, based on the linguistic 2-tuple model. Notice that the new version maintains the main idea of Xu [16]. The basic computational model is also equivalent to the Xu [16]'s version.

3.1.2. Computational essential of ULTs

The basic computational model of ULTs can be found in Xu [16]. Based on the virtual linguistic model, some basic operations were defined:

Definition 4 [16]. Let $[s_{\alpha_1}, s_{\beta_1}]$ and $[s_{\alpha_2}, s_{\beta_2}]$ be two ULTs, $\lambda_1, \lambda_2 \in [0,1]$, then

$$\lambda_1[s_{\alpha_1},s_{\beta_1}] \oplus \lambda_2[s_{\alpha_2},s_{\beta_2}] = [s_{\lambda_1\alpha_1+\lambda_2\alpha_2},s_{\lambda_1\beta_1+\lambda_2\beta_2}]$$

According to the degree of possibility of one interval being greater than another, a partial order on the set of ULTs was proposed by Xu [16] as follows:

Definition 5 [16]. Let $[s_{\alpha_1}, s_{\beta_1}]$ and $[s_{\alpha_2}, s_{\beta_2}]$ be two ULTs, then an order relation on Y is defined by:

$$[s_{\alpha_1}, s_{\beta_1}] < [s_{\alpha_2}, s_{\beta_2}] \Leftrightarrow p([s_{\alpha_1}, s_{\beta_1}] \ge [s_{\alpha_2}, s_{\beta_2}]) > 0.5$$

where $p([s_{\alpha_1}, s_{\beta_1}] \ge [s_{\alpha_2}, s_{\beta_2}]) = \min\{\max\{(\beta_1 - \alpha_2)/(\beta_1 - \alpha_1 + \beta_2 - \alpha_2), 0\}, 1\}.$

A simple total order on Y can be found in Falco et al. [42]. A large number of aggregation operators, based on arithmetical mean [16], geometric mean [43], harmonic mean [44], Bonferroni mean [45], Choquet integral [46] and so on, have been defined for fusing a collection of ULTs.

3.2. Hesitant fuzzy linguistic term sets

3.2.1. Elicitation and representation of HFLTSs

In some complex decision making situations, a single linguistic term might be not accurate enough to express linguistic opinions if uncertainty exists. In the case where the linguistic information could be expressed by comparative linguistic expressions, the concept of HFLTSs is an outstanding tool to model the hesitation among several possible terms. Based on the ordered structure model, the concept is defined below:

Definition 6 [17]. Given LTS S, a HFLTS, denoted by h_s , is an ordered finite subset of the consecutive LTS S.

An example of a HFLTS is shown in Fig. 2. Formally, a HFLTS can be denoted by

$$h_{s} = \{s_{i}, s_{i+1}, \dots, s_{i}\}$$
(7)

where $s_i, s_{i+1}, \dots, s_j \in S$. Generally, comparative linguistic expressions could take several forms. To facilitate eliciting the most frequently used forms of comparative linguistic expressions, a more complicated transformation function was defined:



Fig. 2. An example of a HFLTS (colored in red) and its fuzzy envelope (colored in blue)

Definition 7 [17]. The transformation function, denoted by TF_H , to transform comparative linguistic expressions into HFLTSs is defined as:

 $TF_{H}(s_{i}) = \{s_{i}\},$ $TF_{H}(at most s_{i}) = \{s_{j} | s_{j} \in S, s_{j} \leq s_{i}\},$ $TF_{H}(lower than s_{i}) = \{s_{j} | s_{j} \in S, s_{j} < s_{i}\},$

$$TF_{H}(at \ least \ s_{i}) = \{s_{j} \mid s_{j} \in S, s_{j} \ge s_{i}\},$$

$$TF_{H}(greater \ than \ s_{i}) = \{s_{j} \mid s_{j} \in S, s_{j} > s_{i}\},$$

$$TF_{H}(between \ s_{i} \ and \ s_{j}) = \{s_{k} \mid s_{k} \in S, s_{i} \le s_{k} \le s_{j}\},$$

where $s_i \in S$ and S is a LTS.

3.2.2. Computational essential of HFLTSs

QDM with HFLTSs is being a hot topic in recent years. Basically, the computational strategies of HFLTSs are bipartite.

The first strategy treats a HFLTS as an indivisible entity and transforms HFLTSs into their envelopes (or fuzzy envelopes). The concept of envelopes was defined as follows:

Definition 8 [17]. Given a HFLTS h_s as in Eq. (7), its envelope, denoted by $env(h_s)$, is defined by an ULT whose limits are the upper and lower bounds of h_s , i.e.,

$$env(h_s) = [s_i, s_j] \tag{8}$$

The concept of envelopes implies that the terms in a HFLTS are equally important. The fuzzy envelope of a HFLTS was initially defined to present a fuzzy representation of the HFLTS.

Definition 9 [47]. *Given a HFLTS* h_s *as in Eq. (7), its fuzzy envelope, denoted by* $env_F(h_s)$, *is defined as:*

$$env_F(h_S) = T(a,b,c,d) \tag{9}$$

where $T(\cdot)$ is the trapezoidal fuzzy membership function.

The use of envelopes makes computing with HFLTSs rely on the techniques of computing with ULTs. For instance, an order relation on the set of HFLTSs was defined by the preference degree of two intervals [17], which is similar to the order defined in Definition 5. The aggregation of a collection of

HFLTSs can be derived by aggregating the boundary terms which form the envelopes [40]. The use of fuzzy envelopes is somewhat similar to the framework of semantic decision making because HFLTSs are transformed into trapezoidal fuzzy numbers (see Refs. [48, 49] for examples).

Bearing in the key idea of hesitant fuzzy sets [50], the second strategy tries to compute with all possible linguistic terms involved in a HFLTS. In order to deal with these possible terms at the same time, the representational form of HFLTSs is often rewritten as follows:

Definition 10 [51]. Let $X = \{x_i | i = 1, 2, ..., N\}$ be a reference set and $S = \{s_0, s_1, ..., s_\tau\}$ be a LTS. A *HFLTS on* X, denoted by H_s , is in mathematical terms of

$$H_s = \{ \langle x_i, h_s(x_i) \rangle \mid x_i \in X \}$$

$$(10)$$

where the hesitant fuzzy linguistic element $h_s(x_i)$ is a set of some values in S and can be expressed as:

$$h_{S}(x_{i}) = \{s_{\phi_{i}}(x_{i}) \mid s_{\phi_{i}}(x_{i}) \in S, l = 1, 2, \dots, L\}$$
(11)

with L being the number of linguistic terms in $h_s(x_i)$.

Here, the concept of hesitant fuzzy linguistic element is equivalent to the concept of HFLTS in Definition 6. Eq. (11) highlights each possible term and thus it enables us to compute with each term. More than a half of the current developments are based on this strategy.

Some new operations have been defined for the convenience of computing with possible terms. For instance, the operations defined by Wei et al. [52] possess better mathematical properties than that of Rodríguez et al. [17]; the operations in Gou et al. [53] are based on the idea of operating hesitant fuzzy elements [54]. As can be seen in Liao et al. [55] and Wei et al. [52], several different versions of order relations on the set of all HFLTSs are based on the relations among each pair of possible terms. Based on the basic operations and the order relations, some aggregation methods, such as the ordered weighted averaging [52, 56] and the Bonferroni means [57], have been proposed.

3.3. Extended hesitant fuzzy linguistic term sets

3.3.1. Elicitation and representation of EHFLTSs

Initially, EFHLTSs were defined to handle more types of uncertainties in group decision making (GDM) [18]. Indeed, this concept extends the concept of HFLTS by canceling the limitation of consecutiveness.

Definition 11 [18]. Given a LTS S, an EHFLTS, denoted by \dot{h}_s , is an ordered finite subset of the LTS S. Formally $\dot{h}_s = \{s_\alpha | s_\alpha \in S\}.$

It is obvious that the EHFLTS is equivalent to the hesitant fuzzy linguistic element defined in Definition 10 from a mathematical point of view. Therefore, the abovementioned second computational strategy of HFLTSs is, actually, computing with EHFLTSs. The hesitant fuzzy linguistic sets defined in Zhang and Wu [58] are also EHFLTSs. Moreover, Ma et al. [15] suggested using a g+1 dimensional vector $v = (v_0, v_1, ..., v_g)^T$ to represent an EHFLTSs \dot{h}_s based on the LTS in Eq. (1), where $v_k = 1$ means the linguistic term s_k is selected by the expert and 0 in otherwise, k = 0, 1, ..., g. The derivation of EHFLTSs can be implemented by two distinct manners. Firstly, an EHFLTS $\dot{h}_s = \{s_{\alpha_1}, s_{\alpha_2}, ..., s_{\alpha_n}\}$ can be elicited by the following form of linguistic expressions:

$$s_{\alpha_1}$$
 or s_{α_2} or \cdots or s_{α_n}

Therefore, it can be obtained by the following transformation function:

Definition 12. The transformation function, denoted by TF_E , to obtain EHFLTSs is defined as:

$$TF_U(s_{\alpha_1} \text{ or } s_{\alpha_2} \text{ or } \cdots \text{ or } s_{\alpha_n}) = \{s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}\}.$$

The set of all EHFLTSs based on S is denoted as EH. Secondly, according to the relation between Definition 6 and Definition 11, EHFLSs can be generated by combining a group's opinion directly:

Theorem 1 (Construction axiom) [59]. The union of HFLTSs results in EHFLTSs.

3.3.2. Computational essential of EHFLTSs

To begin with, one may realize that some techniques of handling EHFLTSs have been developed in the literature reviewed in Section 3.2.2, where they are considered to operate each possible term of HFLTSs. Besides, some basic operations were proposed by Wang [18] and Zhang and Wu [58] based on the virtual linguistic model.

Definition 13 [18]. Given two EHFLTSs \dot{h}_{s}^{1} , $\dot{h}_{s}^{2} \in \text{EH}$, $\lambda \in [0,1]$, then:

- (1) $\dot{h}_{s}^{1} \oplus \dot{h}_{s}^{2} = \bigcup_{s_{i} \in \dot{h}_{s}^{1}, s_{i} \in \dot{h}_{s}^{2}} \{s_{i+j}\};$
- (2) $\lambda \dot{h}_{S}^{1} = \bigcup_{s \in \dot{h}_{S}^{1}} \{s_{\lambda i}\}.$

After normalizing the numbers of linguistic terms in EHFLTSs, Wang and Xu [60] defined the following total orders on EH:

Definition 14 [60]. The order \leq is called a total order on EH if

- (1) \leq is a linear order on EH;
- (2) Given $\dot{h}_{s}^{1} = \{s_{\alpha_{1}}^{1}, s_{\alpha_{2}}^{1}, \dots, s_{\alpha_{n}}^{1}\}$, $\dot{h}_{s}^{2} = \{s_{\beta_{1}}^{2}, s_{\beta_{2}}^{2}, \dots, s_{\beta_{n}}^{2}\}$, $\dot{h}_{s}^{1} \leq \dot{h}_{s}^{2}$ whenever $s_{\alpha_{i}}^{1} \leq s_{\beta_{i}}^{2}$ for any $i \in \{1, 2, \dots, n\}$.

Based on Definition 14, total orders on EH can be specified by a set of n aggregation functions defined on EH. Obviously, this definition can serve as the total order of the set of HFLTSs as well. The following extension principle was provided to borrow the existing operators of aggregating virtual linguistic terms for synthesizing EHFLTSs.

Definition 15 [18]. Let Θ be a function $\Theta: \overline{S}^n \to \overline{S}$, $EH = \{\dot{h}_S^1, \dot{h}_S^2, \dots, \dot{h}_S^n\}$. Then the extension of Θ on EH is defined by:

$$\Theta_{EH}(\dot{h}_{S}^{1},\dot{h}_{S}^{2},\ldots,\dot{h}_{S}^{n}) = \bigcup_{(s_{\alpha_{1}},s_{\alpha_{2}},\ldots,s_{\alpha_{n}})\in\dot{h}_{S}^{1}\times\dot{h}_{S}^{2}\times\cdots\times\dot{h}_{S}^{n}} \{\Theta(s_{\alpha_{1}},s_{\alpha_{2}},\ldots,s_{\alpha_{n}})\}$$

Different from the above results based on accurate linguistic models, the computational model proposed by Ma et al. [15] utilizes the semantics of possible linguistic terms to measure the understandable degree and the consistent degree of linguistic expressions. The following definition of determinacy was proposed to measure the understandable degree which the expert has on an EHFLTS:

Definition 16 [15]. *The determinacy of an EHFLTS presented by an expert, denoted by* $Det(\dot{h}_s)$, *is:*

$$Det(\dot{h}_{s}) = 1 - \left(\int_{U} \mathbf{F}_{\dot{h}_{s}} dU\right) / \int_{U} dU$$
(12)

where $F_{\dot{h}_s}$ is the membership function of the EHFLTS \dot{h}_s and U is the domain.

This concept is to indicate the degree of uncertainty in the linguistic expression. Furthermore, the consistency was defined to imply the rationality of the provided linguistic expression as follows:

Definition 17 [15]. Let \dot{h}_s be an EHFLTS and F_i be the membership function corresponds to the linguistic term s_{α_i} , where $s_{\alpha_i} \in \dot{h}_s$. Then the consistency of \dot{h}_s is:

$$Con(h_s) = \bigcup \{ \alpha : \bigcap_{s_{\alpha} \in \dot{h}_s} (\mathbf{F}_i)_{\alpha} \neq 0 \}$$
(13)

where $(F_i)_{\alpha}$ is the α -cut of F_i .

Then an EHFLTS can be synthesized by assigning each possible linguistic term s_{α_i} a number derived by $Det(s_{\alpha_i}) \cdot Det(\dot{h}_s) \cdot Con(\dot{h}_s)$.

3.4. Linguistic expressions based on label semantics model

3.4.1. Representation of LEoLSs

The label semantics model, developed by Tang and Zheng [27], is another methodology to represent natural linguistic expressions in term of label descriptions, appropriateness measures and mass

assignments. Different from the perspective of Zadeh's CWW framework, this model does not utilize fuzzy sets to represent the semantics of linguistic terms. Based on a predefined LTS and logical connectives, the LEoLSs can be defined below:

Definition 18 [27]. Let S be a LTS. The set of LEoLSs, denoted by LE_{LS} , is defined recursively as follows:

- (1) if $s_{\alpha} \in S$ then $s_{\alpha} \in LE_{LS}$;
- (2) if $\theta, \varphi \in LE_{LS}$ then $\neg \theta, \theta \lor \varphi, \theta \land \varphi, \theta \to \varphi \in LE_{LS}$.

It is explicit that, although the linguistic expressions in LE_{LS} do not begin with natural languages, the above definition provides a wide range of natural linguistic expressions. In fact, the recursively use of $\theta \lor \varphi$ includes the set of all EHFLTSs. The expression $\neg \theta$ means *not* θ . The expression $\theta \land \varphi$ often appears in natural language as well. For instance, given a color described by RGB mode, one may say that it is "both red and pink". The expression $\theta \rightarrow \varphi$ includes the assertions like "*if someone is very tall then he is tall*".

3.4.2. Computational essential of LEoLSs

In the label semantics model, the collective linguistic opinion of an alternative can be also represented by a LEoLS according to Definition 18. Thus, the essential of decision making with LEoLSs is to understand the semantics of linguistic expressions in LE_{LS} . This was achieved by defining fuzzy relation among linguistic expressions [27]. Firstly, the set of all appropriate expressions to describe an object was defined by a λ -mapping:

Definition 19 [27]. Every linguistic expression $\theta \in LE_{LS}$ is associated with a set of subsets of S, denoted by $\lambda(\theta)$ and defined recursively as:

- (1) $\lambda(s_{\alpha}) = \{\dot{h}_{s} \subseteq S \mid s_{\alpha} \in \dot{h}_{s}\} \text{ for any } s_{\alpha} \in S;$
- (2) $\lambda(\theta \wedge \varphi) = \lambda(\theta) \cap \lambda(\varphi);$
- (3) $\lambda(\theta \lor \varphi) = \lambda(\theta) \bigcup \lambda(\varphi);$
- (4) $\lambda(\theta \to \varphi) = (\lambda(\theta))^c \bigcup \lambda(\varphi);$

(5)
$$\lambda(\neg \theta) = (\lambda(\theta))^c$$
.

A specific appropriateness measure was also presented in Lawry [61]. Then a mass assignment on the subset of *S* can be defined and the membership function of each linguistic term $s_{\alpha} \in S$ could be derived. Accordingly, a fuzzy relation which reflects the semantic similarities among linguistic terms can be determined by the derived membership functions. The synthesized opinion can be approximated based on the similarities between a linguistic expression and the linguistic terms [27].

3.5. A comparative analysis

From the perspective of modeling linguistic expressions, ULTs, HFLTSs and EHFLTSs start with specific types of natural linguistic expressions and thus can be employed straightforward in QDM. LEoLSs tend to model more expressions. Because of the use of logical operations, they are not direct enough for the decision makers. The effectiveness of LEoLSs could be exploited if they are considered in an expert system. Fig. 3 illustrates the ranges of values of HFLTS, EHFLTSs and LEoLSs. ULTs are incomparable because virtual linguistic terms are involved.

The relationships among ULTs, HFLTSs and EHFLTSs are very interesting. The focused natural linguistic expressions are very similar to each other. HFLTSs can model more types of expressions than others. However, the computation of HFLTSs highly depends on ULTs and EHFLTSs. If HFLTSs are computed based on their envelopes, then ULTs play the central role; if they are computed based on

possible terms, then the computational items are actually EHFLTSs. In summary, HFLTSs are good at modeling comparative linguistic expressions, and ULTs and EHFLTSs are essential in the computational processes.



Fig. 3. Graphical interpretation of the ranges of values of HFLTSs, EHFLTSs and LEoLSs

Most of the four models treat their involved linguistic terms equally, except for HFLTSs. In fact, if HFLTSs are computed with fuzzy envelopes, some possible terms are treated more important than others. Taking the expression "greater than s_i " for example, the terms which are closer to s_g are more important than others.

The lack of consecutiveness may cause some limitations of applications. EHFLTSs are not reasonable for representing individual opinions if the involved linguistic terms are not consecutive in S. Although LEoLSs include more types of natural linguistic expressions, some of them may be useless in QDM. For instance, the expression "*not medium or good*" (based on $S = \{very \ low, \ low, \ medium, \ good$, *very good*}) is a LEoLS. But it is difficult to make a decision based on the terms $\{very \ low, \ low, \ very \ good\}$. This is a major reason why an appropriateness measure was defined [61]. Some other features of the four models are summarized in Table 1.

	ULT	HFLTS	EHFLTS	LEoLSs
Base model	Virtual linguistic model	Ordered structure model	Virtual linguistic model	Not fuzzy set based model
Number of terms	Infinite	Finite	Finite	Finite
Consecutiveness	Consecutive on \overline{S}	Consecutive on S	Not consecutive	Not consecutive
Importance of possible terms	Indifferent	Not clear	Indifferent	Indifferent
Available setting	IDM and GDM	IDM and GDM	GDM	IDM and GDM
Computational strategy	Compute with boundaries	Compute with envelopes; compute with possible terms	Compute with possible terms	Compute with similarity relation

Table 1. A summary on four techniques which model natural linguistic expressions

4. Techniques for modeling artificial linguistic expressions

This section will focus on seven techniques which model specific types of artificial linguistic expressions, such as the ER algorithm, proportional terms, distribution assessments, DFNs, PLTs, LHFSs and 2DLTs.

4.1. Linguistic expressions based on the ER framework

4.1.1. Focused linguistic expressions and representation

In some subjective evaluations under uncertainty, such as evaluating the quietness of an engine, an expert may state that he is "50% sure the engine is good and 30% sure it is excellent". A belief structure is employed to describe this kind of information under the framework of multi-attribute decision making (MADM) with the ER algorithm [26]. For convenience, given a LTS $S = \{s_n | n = 1, 2, ..., N\}$ and a set of L attributes $E = \{e_i | i = 1, 2, ..., L\}$ associated with weights $w = (w_1, w_2, ..., w_L)^T$ such that $0 \le w_i \le 1$ and $\sum_{i=1}^{L} w_i = 1$, the belief structure can be modeled by using the following expectations on the attribute e_i (i = 1, 2, ..., L) [26]:

$$B(e_i) = \{(s_n, \beta_{n,i}) \mid n = 1, 2, \dots, N\}, \quad i = 1, 2, \dots, L.$$
(14)

where $\beta_{n,i} \ge 0$ and $\sum_{n=1}^{N} \beta_{n,i} \le 1$. $\beta_{n,i}$ is called a degree of belief. If $\sum_{n=1}^{N} \beta_{n,i} = 1$, then the assessment is complete; and if $\sum_{n=1}^{N} \beta_{n,i} < 1$, then it is incomplete and $1 - \sum_{n=1}^{N} \beta_{n,i}$ is called the degree of ignorance.

4.1.2. Computational essential of the ER algorithm

The following ER algorithm [26] is widely acknowledged to aggregate the assessments represented in Eq. (14):

$$\begin{split} m_{n,i} &= w_i \beta_{n,i}, \quad n = 1, 2, \dots, N, \quad i = 1, 2, \dots, L, \\ m_{S,i} &= 1 - \sum_{n=1}^{N} m_{n,i} = 1 - w_i \sum_{n=1}^{N} \beta_{n,i}, \quad i = 1, 2, \dots, L, \\ \{s_n\} &: m_{n,I(i+1)} = K_{I(i+1)} (m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{S,i+1} + m_{S,I(i)} m_{n,i+1}), \quad n = 1, 2, \dots, N \\ \{S\} &: m_{S,I(i+1)} = K_{I(i+1)} m_{S,I(i)} m_{S,i+1}, \\ K_{I(i+1)} &= [1 - \sum_{t=1}^{N} \sum_{j=1, j \neq t}^{N} m_{t,I(i)} m_{j,i+1}]^{-1}, \quad i = 1, 2, \dots, L - 1. \end{split}$$

where $m_{n,I(i)}$ is the probability mass defined as the degree to which all the first *i* attributes support the hypothesis that the object satisfies the linguistic term s_n . The combined degree of belief is then given directly by:

$$\beta_n = m_{n,I(L)}, \quad n = 1, 2, ..., N,$$

 $\beta_s = m_{s,I(L)} = 1 - \sum_{n=1}^N \beta_n.$

To enhance the capability of aggregating information under uncertainty, utility intervals, whose lower bound is β_n and upper bound is $\beta_n + \beta_s$, were generated to model the impact of ignorance [26, 62].

4.2. Proportional terms

4.2.1. Focused linguistic expressions and representation

Wang and Hao [19] presented a novel version of the linguistic 2-tuple model for the case when the opinion is distributed between two consecutive linguistic terms. For instance, the evaluation grades could form the artificial linguistic expressions like "20% A and 80% B". Based on the ordered structure model, the concept of proportional terms is defined as follows:

Definition 20 [19]. Let *S* be an ordered LTS, I = [0,1] and

$$IS = I \times S = \{(\alpha, s_i) \mid \alpha \in [0, 1], s_i \in S\}$$
(15)

Given a pair of two successive terms (s_i, s_{i+1}) , two elements (α, s_i) , $(1-\alpha, s_{i+1})$ of IS is called a symbolic proportion pair, where $s_i, s_{i+1} \in S$.

A symbolic proportion pair (α, s_i) , $(1-\alpha, s_{i+1})$ can be denoted by $(\alpha s_i, (1-\alpha)s_{i+1})$. Let P_s be the set of all symbolic proportion pairs generated by *S*. Wang and Hao [19] assumed that if the opinion is a proportion of two linguistic terms, then the underlain real value should be another term located between the two, taking the form of a 2-tuple term. From this view, the proportional term is actually an alternative model of the linguistic 2-tuple model. Their relationship can be described by a one-to-one mapping:

Definition 21 [19]. Let S, \tilde{S} and P_s be an ordered LTS, the set of linguistic 2-tuples and the set of all symbolic proportion pairs generated by S, respectively. The function $h: P_s \to \tilde{S}$ is defined by:

$$h((\alpha s_i, (1-\alpha)s_{i+1})) = \begin{cases} (s_{i+1}, -\alpha), & 0 \le \alpha \le 0.5\\ (s_i, 1-\alpha), & 0.5 < \alpha \le 1 \end{cases}$$
(16)

4.2.2. Computational essential of proportional terms

Innovated by the linguistic 2-tuple model, a basic operation is defined below:

Definition 22 [19]. Let *S* and P_s be an ordered LTS and the set of all symbolic proportion pairs generated by *S*, respectively. The position index function of ordinal 2-tuples $\pi: P_s \to [0,g]$ is defined by:

$$\pi((\alpha s_i, (1-\alpha)s_{i+1})) = i + (1-\alpha) \tag{16}$$

where $s_i, s_{i+1} \in S$ and $\alpha \in [0,1]$.

The position index function is bijective. Moreover, $\pi = \Delta^{-1} \circ h$. A linguistic computational model for proportional terms was developed based on the function [19].

Firstly, the order relation on P_s was defined. According to Eq. (16), for any $(\alpha s_i, (1-\alpha)s_{i+1})$, $(\beta s_j, (1-\beta)s_{j+1}) \in P_s$, $(\alpha s_i, (1-\alpha)s_{i+1}) < (\beta s_j, (1-\beta)s_{j+1}) \Leftrightarrow i + (1-\alpha) < j + (1-\beta)$. Motivated by the traditional lexicographical order on the set of 2-dimentional vector (i, α) , the following order was defined:

(1) If i < j, then if i = j - 1 and $\alpha = 0$, $\beta = 1$, $(\alpha s_i, (1 - \alpha) s_{i+1}) = (\beta s_j, (1 - \beta) s_{j+1})$; else $(\alpha s_i, (1 - \alpha) s_{i+1}) < (\beta s_j, (1 - \beta) s_{j+1})$;

(2) If i = j, then: (a) if $\alpha = \beta$, then $(\alpha s_i, (1 - \alpha) s_{i+1}) = (\beta s_j, (1 - \beta) s_{j+1})$; (b) if $\alpha < \beta$, then $(\alpha s_i, (1 - \alpha) s_{i+1}) < (\beta s_j, (1 - \beta) s_{j+1})$;

The negation operator was defined by extending the traditional negation in the ordered structure model: $Neg((\alpha s_i, (1-\alpha)s_{i+1})) = ((1-\alpha)Neg(s_{i+1}), \alpha Neg(s_i)) = ((1-\alpha)s_{g-i-1}, \alpha s_{g-i}).$

Some aggregation operators based on the idea of ordered weighted averaging and quasi-arithmetic averaging were also proposed in Ref. [19]. In fact, according to Definition 21, any aggregation operators defined for linguistic 2-tuple can be considered as well.

4.3. Distribution assessments

4.3.1. Focused linguistic expressions and representation

This model is devoted to generalizing proportional terms. In distribution assessments, symbolic proportions are assigned to all linguistic terms of a given LTS [20]. The focused form of linguistic expressions can be illustrated by an example of evaluating football players. Suppose that the coach uses the LTS $S = \{poor, medium, good\}$ to evaluate the team of players. After a season of matches, the evaluation of one player might be: the frequencies of "*poor*", "*medium*" and "*good*" performances are 0.2, 0.3 and 0.5 respectively. This information could be collected by a distribution assessment: {(*poor*, 0.2), (*medium*, 0.3), (*good*, 0.5)}. Formally:

Definition 23 [20]. Let *S* be a LTS. A distribution assessment is defined by:

$$m = \{ (s_i, \beta_i) \mid s_i \in S, \sum_{s_i \in S} \beta_i = 1, \beta_i \ge 0 \}$$
(17)

where β_i is called the symbolic proportion of s_i .

4.3.2. Computational essential of distribution assessments

The computational model proposed in Zhang et al. [20] could serve as the basis of QDM with distribution assessments. Following the idea of expected value in probability theory, the expectation of a distribution assessment is defined as follows:

Definition 24 [20]. Let *m* be a distribution assessment defined above. The expectation of *m* is:

$$E(m) = \sum_{s, \in S} \beta_i s_i \tag{18}$$

The operations included in Eq. (18) are the ones in the virtual linguistic model, which have been recalled in Section 2. Based on the expectation, a partial order on the set of distribution assessments is

defined by [20]:

$$m_1 < m_2 \Leftrightarrow E(m_1) < E(m_2)$$

if $E(m_1) = E(m_2)$, then the two distribution assessments are said to be indifferent.

The negation was defined by fetching the probability of each linguistic term from its negation [20]:

$$Neg(\{(s_i, \beta_i) | s_i \in S\}) = \{(s_i, \beta_i) | s_i \in S, j = Ind(Neg(s_i))\}$$

where the function Ind returns the index of the input linguistic term.

Two aggregation operators based on the weighted aggregating and ordered weighted aggregating were defined as well [20]. Thanks to the use of probability distributions, the operators possess some wonderful properties, such as boundedness and monotonicity.

4.4. Subjective evaluations based on DFNs

4.4.1. Focused linguistic expressions and representation

Riera et al. [22] demonstrated that the use of DFNs can offer a great flexibility for representing the experts' subjective evaluations. Specifically, in the sense of HFLTSs, we are sure about which terms are involved. Besides, if the experts do not reject the possibility of other terms, then the other terms associated with the degrees of possibility should be modeled as well. For example, if an expert considers linguistic expression "*between good and very good*" to represent an alternative, and he/she cannot completely deny the possibility of the alternative being "*medium*" or "*perfect*", then the opinion might be represented as: {0.5/medium, 1/good, 1/very good, 0.6/perfect}.

Let $A: R \to [0,1]$ be a fuzzy subset of R, $A^{\alpha} = \{x \in R \mid A(x) \ge \alpha\}$ be the α -cut of A (for any $\alpha \in (0,1]$), A^1 be the core of A, and $supp(A) = \{x \in R \mid A(x) > 0\}$ be the support of A. In the framework of DFNs, an ordered infinite LTS with g+1 terms, as in Eq. (1), is usually mapped into a finite

chain

$$L_{g} = \{0, 1, \dots, g\}$$
(19)

DFNs are defined as follows:

Definition 25 [21]. A fuzzy subset of R with the membership function $A: R \rightarrow [0,1]$ is called a DFN if there exist $x_1, x_2, ..., x_n \in R$ with $x_1 < x_2 < ... < x_n$ m such that $supp(A) = \{x_1, x_2, ..., x_n\}$, and there are two natural numbers s, t with $1 \le s \le t \le n$ such that:

- (1) For any natural number $i \in [s, t]$, $A(x_i) = 1$;
- (2) For each natural number i, j with $1 \le i \le j \le s$, $A(x_i) \le A(x_j)$;
- (3) For each natural number i, j with $t \le i \le j \le n$, $A(x_i) \ge A(x_j)$.

The set of all DFNs, whose support is a sub-interval of L_g , is denoted by A^{L_g} . Based on the partial order on A^{L_g} defined in Riera and Torrens [63], a subjective evaluation could be interpreted as a normal convex fuzzy subset defined on an ordered chain:

Definition 26 [22]. Let $L_g = \{0,1,...,g\}$ be a finite ordered chain. We call a subjective evaluation to each DFN belonging to the partially ordered set A^{L_g} .

Comparative linguistic expressions in Definition 7 can be interpreted as the following subjective evaluations:

Between s_i and s_j : $\{A \in A^{L_g} | core(A) = [s_i, s_j]\};$ Worse than s_i : $\{A \in A^{L_g} | core(A) = [s_0, s_{i-1}]\};$ At most s_i : $\{A \in A^{L_g} | core(A) = [s_0, s_i]\};$ Better than s_i : $\{A \in A^{L_g} | core(A) = [s_{i+1}, s_n]\};$ At least s_i : $\{A \in A^{L_g} | core(A) = [s_i, s_n]\}.$

Fig. 4 illustrates that two DFNs could interpret the comparative linguistic expressions "between bad

and good". For one thing, it could enhance the capability of incorporating more accurate information by assigning different membership degrees to the linguistic terms. For another, the necessity of membership degrees makes the model not be a representational tool for natural linguistic expressions like HFLTSs.



(b) DFN whose support is {*VB*, *B*, *F*, *G*, *VG*}

Fig. 4. Graphical representations of DFNs which can interpret "between bad and good"

4.4.2. Computational essential of DFN-based subjective assessments

Let *A* and *B* be two DFNs and $A^{\alpha} = \{x_1^{\alpha}, x_2^{\alpha}, x_p^{\alpha}\}, B^{\alpha} = \{y_1^{\alpha}, y_2^{\alpha}, y_k^{\alpha}\}$ be the corresponding α -cuts. A partial order on A^{L_g} can be defined by [63]:

$$A \le B \Leftrightarrow MIN(A, B) = A$$
 (or equivalently $MAX(A, B) = B$) (20)

where MIN(A, B) and MAX(A, B) are the DFNs whose α -cuts are $min(A, B)^{\alpha}$ and $max(A, B)^{\alpha}$,

respectively, and

$$\min(A, B)^{\alpha} = \{ z \in supp(A) \land supp(B) \mid \min(x_1^{\alpha}, y_1^{\alpha}) \le z \le \min(x_p^{\alpha}, y_k^{\alpha}) \}$$
$$\max(A, B)^{\alpha} = \{ z \in supp(A) \lor supp(B) \mid \max(x_1^{\alpha}, y_1^{\alpha}) \le z \le \max(x_p^{\alpha}, y_k^{\alpha}) \}$$

for each $\alpha \in [0,1]$. The triplet (A^{*L_g*}, *MIN*, *MAX*) is a bounded distributive lattice [64].

For aggregating a collection of DFNs, Riera and Torrens [63] suggested to extend the existing discrete aggregation functions on the chain L_g to the aggregation function on the bounded distributive lattice. The following definition was proposed to define a binary aggregation function on A^{L_g} :

Definition 27 [63]. Given a binary aggregation function F on the finite chain L_g , the binary operation on A^{L_g} defined as:

$$F: A^{L_g} \times A^{L_g} \to A^{L_g}$$
$$(A, B) \to F (A, B)$$

is called the extension of F to A^{L_g} , where F(A,B) is the DFN whose α -cuts are (for each $\alpha \in [0,1]$):

$$\{z \in L_{\varphi} \mid \min F(A^{\alpha}, B^{\alpha}) \le z \le \max F(A^{\alpha}, B^{\alpha})\}$$

The defined function F is a binary aggregation function on A^{L_s} . Based on which, other aggregation functions can be extended to suit the case where the number of inputs is greater than 2.

4.5. Probabilistic linguistic term sets

4.5.1. Focused linguistic expressions and representation

PLTSs are proposed to capture the relative importance of linguistic terms included in a HFLTS [23]. For instance, according to an online survey of the comfortable degree of a vehicle, 20 respondents (out of 100) stated it is "*very high*", 65 respondents stated it is "*high*", ten believed it is "*slightly high*" and others did not say anything. The collective information can be denoted as: {(*very high*, 0.2), (*high*, 0.65), (*slightly high*, 0.1)}. Obviously, the focused linguistic expressions are similar to those of the ER framework. But there are some differences. The ER based model treats the numerical information as the degrees of belief, whereas this model considers it as probabilities. In order to propose a new CWW model, PLTSs were defined as follows:

Definition 28 [23]. Given a LTS S, a PLTS is defined as:

$$L(p) = \{s^{(k)}(p^{(k)}) \mid s^{(k)} \in S, p^{(k)} \ge 0, k = 1, 2, \dots, \#L(p), \sum_{k=1}^{\#L(p)} p^{(k)} \le 1\}$$

where $s^{(k)}(p^{(k)})$ is the linguistic term $s^{(k)}$ associated with its probability $p^{(k)}$, and #L(p) is the number of linguistic terms in L(p).

A PLTS permits that its probabilistic information is partial unknown, which means that a certain degree of ignorance is allowed. To facilitate the computational process, PLTSs can be normalized by two steps [23]:

(1) Normalize the probabilistic distribution: replace $p^{(k)}$ by $p^{(k)} / \sum_{k=1}^{\#L(p)} p^{(k)}$, k = 1, 2, ..., #L(p). (2) Normalize the lengths of PLTSs: for two PLTSs $L_1(p)$ and $L_2(p)$ with $\#L_1(p) > \#L_2(p)$,

add $s^{(k_{-}\min)}(0)$ to $L_{2}(p)$ (repeat $#L_{1}(p)-#L_{2}(p)$ times).

For a given PLTS L(p), the normalized PLTS is denoted by $L^{N}(p)$. The set of all PLTSs based on *S* is denoted by $\Lambda(p)$.

4.5.2. Computational essential of PLTSs

A partial order on $\Lambda(p)$ is defined as follows:

Definition 29 [23]. Given a PLTS L(p), let $E(L(p)) = \sum_{k=1}^{\#L(p)} Ind(s^{(k)}) p^{(k)} / \sum_{k=1}^{\#L(p)} p^{(k)}$ and $\sigma(L(p)) = (\sum_{k=1}^{\#L(p)} (p^{(k)}(Ind(s^{(k)}) - E(L(p))))^2)^{1/2} / \sum_{k=1}^{\#L(p)} p^{(k)}$. Then for any two PLTSs $L_1(p)$,

 $L_{\!\!2}(p)\!\in\Lambda(p)\,,$

$$L_{1}(p) < L_{2}(p) \Leftrightarrow (E(L_{1}(p)) < E(L_{1}(p))) \lor ((E(L_{1}(p)) = E(L_{1}(p))) \land (\sigma(L_{1}(p)) > \sigma(L_{1}(p))))$$

Some basic operations were defined in Ref. [23] based on the virtual linguistic model. But the computational results are generally EHFLTSs rather than PLTSs. The improved version of basic operations defined by Zhang et al. [65] are as follows:

Definition 30 [65]. *Given two normalized PLTSs* $L_1^N(p), L_2^N(p) \in \Lambda(p)$, then

(1)
$$L_1^N(p) \oplus L_2^N(p) \in_{k_1 \in \{1, 2, \dots, \#L_1^N(p)\}, k_2 \in \{1, 2, \dots, \#L_2^N(p)\}} \bigcup \{s_1^{(k_1)} \oplus s_2^{(k_2)}(p_1^{(k_1)} \oplus p_2^{(k_2)})\};$$

(2)
$$\lambda L_1^N(p) \in_{k_1 \in \{1, 2, \dots, \#L_1^N(p)\}} \bigcup \{\lambda s_1^{(k_1)}(p_1^{(k_1)})\}, where \ \lambda \in [0, 1].$$

Some other basic operations and fuzzy measures can be found in Ref. [53, 66, 67]. For fusing PLTSs, two aggregation functions based on weighted arithmetical averaging and weighted geometric averaging were defined in Ref. [23].

4.6. Linguistic hesitant fuzzy sets

4.6.1. Focused linguistic expressions and representation

LHFSs were proposed to model a class of more complicated case where the degree of a linguistic term being the real value of a linguistic variable is represented by hesitant fuzzy elements. Taking the evaluation of the quietness of a refrigerator for example [24], an expert may hesitate to give the values 0.1 or 0.2 for "*slightly good*", the values 0.4 or 0.5 for "*good*" and 0.1, 0.2 or 0.25 for "*very good*". This could be collected as {(*slightly good*, 0,1, 0,2), (*good*, 0.4, 0.5), (*very good*, 0.1, 0.2, 0.25)}. In each 2-tuple, the numerical values imply the possible membership degrees caused by the expert's hesitancy and uncertainty.

Definition 31 [24]. *Given a LTS S*, *a LHFS in S*, *denoted by LH*, *is a set that when applied to the* 32

linguistic terms of S it returns a subset of S associated with several values in [0,1]. Formally,

$$LH = \{(s^{(k)}, lh^{(k)}) | s^{(k)} \in S, lh^{(k)} = \{r_1, r_2, \dots, r_{m_k}\}\}$$

The set of all LHFSs based on S is denoted by AH. Recently, several extensions of LHFSs have been proposed from the perspective of generalizing the form of membership degrees, such as the linguistic interval-valued hesitant fuzzy sets [68].

4.6.2. Computational essential of LHFSs

Similar to the idea of Definition 29, a partial order on AH was defined as follows:

Definition 32 [24]. Given a LHFS $LH \in \Lambda H$, denote $E(LH) = s_{e(LH)}$ and $D(LH) = s_{v(LH)}$, where

$$e(LH) = \left(\sum_{k \in ind(LH)} \left(k \sum_{r \in lh^{(k)}} r / |lh^{(k)}|\right)\right) / |ind(LH)|,$$

$$v(LH) = \left(\sum_{k \in ind(LH)} \left(\left(k \sum_{r \in lh^{(k)}} r / |lh^{(k)}|\right) - e(LH)\right)^{2}\right) / |ind(LH)|,$$

and ind(LH) returns the indices of linguistic terms included in LH and the operation $|\cdot|$ returns the cardinality of a set. Then given two LHFSs $LH_1, LH_2 \in \Lambda H$,

$$LH_1 < LH_2 \Leftrightarrow (E(LH_1) < E(LH_2)) \lor ((E(LH_1) = E(LH_2)) \land (D(LH_1) > D(LH_2)))$$

Some basic operations were developed as well:

Definition 33 [24]. *Given two* LHFSs $LH_1, LH_2 \in \Lambda H$,

(1)
$$LH_1 \oplus LH_2 = \bigcup_{(s^{(k)}, |h^{(k)}) \in LH_{\nu}(s^{(i)}, |h^{(i)}) \in LH_2} (s^{(k)} \oplus s^{(i)}, \bigcup_{r \in |h^{(k)}, r, \in |h^{(i)}} \{1 - (1 - r_i)(1 - r_j)\})\};$$

(2) $\lambda LH_1 = \bigcup_{(s^{(k)}, |h^{(k)}) \in LH_1} (\lambda s^{(k)}, \bigcup_{r_i \in |h^{(k)}} \{1 - (1 - r_i)^{\lambda}\})\}, where \ \lambda \in [0, 1].$

Based on the basic operations, two generalized hybrid aggregation operators and two Shapley weighted averaging operators were developed [24].

4.7. 2-dimension linguistic terms

4.7.1. Focused linguistic expressions and representation

Instead of considering multiple linguistic terms under uncertainty, 2DLTs utilize a special linguistic term to express the confidence level of the provided linguistic term [25]. A normal instance can be found in the online blind review system of journal papers. Referees are required not only to express their opinions about the manuscript, but also to state the degree of confidence of the opinions. This kind of information is naturally collected by a 2-tuple, such as (*familiar*, *very good*), where the first component means that the expert is "*familiar*" with the field of the manuscript and the second component implies that he/she thinks the quality of the manuscript is "*very good*". The definition is as follows:

Definition 34 [25]. Given two LTSs $S = \{s_0, s_1, ..., s_g\}$ and $H = \{h_0, h_1, ..., h_\tau\}$, a 2-tuple $\hat{r} = (s_i, h_j)$ is called a 2DLT, in which $h_j \in H$ is the assessment information about the alternative and $s_j \in S$ represents the self-assessment of the expert regarding the assessment.

4.7.2. Computational essential of 2DLTs

The computational model proposed in Ref. [25] is based on the direct product and the lattice implication algebra defined on two Lukasiewicz implication algebras [69]:

Definition 35 [25]. Given two LTSs $S = \{s_0, s_1, \dots, s_g\}$, $H = \{h_0, h_1, \dots, h_r\}$, and a lattice implication algebra $LIA_{(g+1)\times(\tau+1)} = (S \times H, \lor, \land, \rightarrow, \neg)$. Let a mapping $f : S \times H \rightarrow LIA_{(g+1)\times(\tau+1)}$ be defined such that $f((s_i, h_j)) = (\dot{s}_i, \dot{h}_j)$, where $\dot{s}_i \in S$, $\dot{h}_j \in H$. Then if f is bijective, for any $(s_i, h_j), (s_k, h_l) \in S \times H$, and (1) $(s_i, h_j) \lor (s_k, h_l) = f^{-1}(f((s_i, h_j)) \lor f((s_k, h_l)))$, (2) $(s_i, h_i) \land (s_k, h_l) = f^{-1}(f((s_i, h_i)) \land f((s_k, h_l)))$,

(3)
$$(s_i, h_j) \to (s_k, h_l) = f^{-1}(f((s_i, h_j)) \to f((s_k, h_l))),$$

(4)
$$\neg (s_i, h_j) \land (s_k, h_l) = f^{-1}(\neg f((s_i, h_j))),$$

then the lattice implication algebra $2DLLIA = (S \times H, \lor, \land, \rightarrow, \neg, (s_0, h_0), (s_g, h_\tau))$ is called a 2-dimension linguistic lattice implication algebra.

The Hasse Diagram of 2-dimention linguistic lattice implication algebra is shown in Fig. 5. Associated with the risk attitude of the decision maker, represented by a parameter δ , partial orders were defined as follows:



Fig. 5. The Hasse Diagram of 2-dimention linguistic lattice implication algebra

Definition 36 [25]. Let (s_i, h_j) and (s_k, h_t) be two 2DLTs of 2DLLIA defined in Definition 35, δ be a positive real number. Then two weak partial orders are defined by:

- (1) less than: $(s_i, h_j) \leq (s_k, h_t) \Leftrightarrow ((i < k) \land (j \leq t)) \lor ((i \leq k) \land (j < t));$
- (2) weakly less than: $(s_i, h_j) \leq (s_k, h_t) \Leftrightarrow (j \leq t) \land 0 < (i-k) < \delta$.

Inspired by the linguistic 2-tuple model, a 2DLT can be represented by two linguistic 2-tuples.

Accordingly, Zhu et al. [25] developed two aggregation operators for 2DLTs. Moreover, a number of aggregation operators were defined in Ref. [70] based on a generalized triangular fuzzy representation of 2DLTs.

4.8. A comparative analysis

These models require not only linguistic terms but also additional numerical or linguistic information taking the form of probabilistic distributions, mass distributions, membership degrees and linguistic terms. The motivation of collecting the additional information is to represent the experts' opinions as accurate as possible. On the surface, it seems that these models are more sophisticated than those in Section 3.

The additional information is definitely helpful to distinguish and weight the possible terms. From a mathematical point of view, several types of numerical values are interpreted and modeled by probability, the degree of belief and the degree of membership. If probability and the degree of belief are considered, then the values should be normalized. Especially, as can be seen in ER-based model, proportional terms, distribution assessments, and PLTSs, frequencies are usually natural to be considered as the evidences of both probability and degree of belief. However, in the case of DFNs and LHFSs, there is not any clear statement about how the membership degrees can be derived. This fact might increase the potential difficulty of their applications.

From the perspective of CWW, some sophisticated models do not compute with linguistic terms at all. As can be seen in the ER-based model and the distribution assessments, the computational essence is to compute with only the additional information, i.e., (g+1)-dimensional vectors (given a LTS with g+1 linguistic terms). The linguistic terms in these cases only serve as a set of evaluation scales. Although this

is not enough to say the models suffer from drawbacks, the lack of computing with linguistic terms may lead to not sufficiently using of available information.

Some of the features of these models are summarized in Table 2.

	Base model	Additional information	Ignorance	Computational strategy
ER-based linguistic expressions	Mass assignment	Degree of belief	Enable	Compute with degrees of believes
Proportional terms	Linguistic 2-tuple model	Probabilistic distribution	Disenable	Compute with terms and probabilities
Distribution assessments Subjective evaluations	Linguistic 2-tuple model	Probabilistic distribution	Disenable	Compute with probabilities
	Ordered structure model	Membership degree	N/A	Compute with membership degrees
PLTSs	Virtual linguistic model	Probabilistic distribution	Enable	Compute with terms and probabilities
LHFSs	Virtual linguistic model	Membership degree	N/A	Compute with terms and membership degrees
2DLTs	Accurate linguistic model	Condifence level	N/A	Compute with terms

Table 2. A summary on seven techniques which model artificial linguistic expressions

5. Further discussions

We have reviewed the main ideas of models for both natural and artificial linguistic expressions. Techniques for natural linguistic expressions collect and model all the involved linguistic terms. Tools for artificial linguistic expressions require additional information which is associated with the linguistic terms. The former aims at handling natural languages in QDM. And the latter focuses on more elaborate presentation of human opinions. Fig. 6 lists the reviewed techniques graphically according to the focused linguistic expressions and the complexity degree of collecting required information. ULTs, HFLTSs and EHFLTSs model specific natural linguistic expressions which are frequently emerged in human thinking. Thus, it is very easy to obtain these kinds of information, i.e., some linguistic terms. LHFSs model artificial linguistic expressions where each possible term should be associated with a hesitant fuzzy element. Therefore, we think that its required information is more complex than others.



Fig.6. A graphical summary of the techniques of modeling complex linguistic expressions

Although the techniques are classified into two categories, it is irrational to say which class of techniques is generally better. The first class is devoted to modeling natural languages in which additional information such as probabilities and membership degrees is not necessary. The motivation of the other class is not to improve the first class. The models in the second class are proposed to model specific scenarios where the required types of additional information are available. The models are also valuable for some specific cases though the necessity of addition information leads to be less natural than the linguistic convention.

However, it may be not so interesting to extend the forms of the additional information involved in some models for artificial linguistic expressions. Because the collection of required information in these models is more or less complicated, a principle of such extensions should be that they could facilitate the

38

representation of uncertainty and the collection of required information. For instance, in the ER framework, if the distribution of degrees of belief is hard to obtain, then an interval-valued degree of belief can be considered [71]. Analogously, LHFSs have been extended by generalizing each hesitant fuzzy element to an interval-valued hesitant fuzzy element [72] or an intuitionistic hesitant fuzzy element [73]. See Refs. [68, 74] for details. However, the use of hesitant fuzzy elements in LHFSs results in the complexity of both information collection and computing. Although the extension may alleviate the complexity of information collection to a certain degree, it would increase the computational complexity, and the effectiveness of the extension is miniature.

When selecting certain models for applications, two criteria could be considered. (1) Simplicity. In order to represent the collected linguistic expressions, the simple model is the best. It is sufficient if the selected model could include all the available information in the linguistic expressions. For instance, HFLTSs can be regarded as special cases of PLTSs mathematically. Thus, it is generally better to represent comparative linguistic expressions by means of HFLTSs than PLTSs. (2) Accuracy. When facing uncertainties, any available information could be valuable for final decisions. The selected model should be capable to represent the information in a correct and accurate manner. For example, if the additional information associated with each possible term takes the form of probabilities, rather than membership degrees, then PLTSs would be more accurate than DFNs.

6. Current challenges and possible directions

Based on the above analysis and some recent developments of the reviewed techniques, we shall address some current challenges of both modeling complex linguistic expressions and using the existing models in QDM to point out some possible directions for further investigations. Generally, fuzzy logic is useful and effective for modeling qualitative assessments, including single linguistic terms and complex linguistic expressions. To facilitate QDM by using the existing models, the following aspects could be considered:

(1) Theories and processes should be developed strictly on the basis of real-world applications. Currently, some models have been proposed to extend and generalize the existing ones just from the theoretical perspective. This is not enough. The resultant models may be not applicable or practicable in real applications. Some QDM processes have also been developed by simply extending the idea of famous decision making processes. Especially, a large number of aggregation operators have been extended to several models reviewed in this paper. As advocated in Rodríguez et al. [75], simple extensions without sound theoretical or practical justification make no sense. New aggregation functions are welcome if they are driven by real world applications and/or if they fuse information in a novel manner. Simultaneously, it is not a good idea to extend the existing decision making processes arbitrarily. The extension is interesting only if it could solve at least one new problem.

(2) Although there are many models, the corresponding decision making processes are quite limited. As stated in (1), many processes have been presented by extending some famous processes which are popular in uncertain decision making. But very little new idea has been introduced. For instance, the concept of aspiration levels rests at the central role of bounded rationality [76]. In practice, decisions could plausibly be made by accepting the first solution which meets a sufficient good aspiration level rather than seeking for the one with the highest performances [77]. This idea has been considered to deal with ULTs in a real case study [78]. However, it has not been considered in any other models reviewed in this paper. Other outstanding decision making patterns could also be employed if they could introduce new solutions for some applications.

(3) It is interesting if a new decision making process could handle multiple types of linguistic information. As have been discussed, the models for natural linguistic expressions focus on specific types of linguistic expressions and other models for artificial linguistic expressions pay their attention to linguistic information within specific scenarios. The definition of each model presents inherent limitation in applications. However, the real-world problems maybe not match the ideal situations defined in any models. Thus, it would be powerful if more than one model can be combined together to solve some practical problems. This is somewhat like the idea of decision making with heterogeneous information.

To model complex linguistic expressions, the following aspects are interesting:

(1) More types of complex linguistic expressions should be modeled. The existing models of natural linguistic expressions focus mainly on the comparative linguistic expressions. The other models do not essentially increase the ranges of linguistic phrases or expressions of a linguistic variable. Linguistic expressions which can be modeled currently are still limited. A typical instance is the linguistic hedges. The linguistic hedges, especially the weakened hedges such as "more or less" and "roughly", are an exceedingly frequent manner to express the uncertainty of using linguistic terms. For example, a linguistic expression might be "more or less good". It is not a linguistic term neighboring to "good". Instead, it expresses the uncertainty of using the single term "good". It means that "good" might be the real value of the linguistic variable. However, the terms neighboring to "good" might be possible as well. The possibility of other terms neighboring to "good" being the real value is less than that of "good". If a similarity measure is defined appropriately, then it is intuitive that someone is "more or less good" if and only if it is similar to some others which are "good" [79]. Till now, linguistic hedges have not been systematically investigated in QDM.

(2) It is excellent if a novel model could combine the strengths of the both classes of the existing

models [14]. The models of natural linguistic expressions start with natural languages and can be formalized by certain grammars. Mathematically, they can be regarded as the special cases of some models of artificial linguistic expressions. The latter class of models includes additional information and thus provides more flexibility of representing and handling information. It would be very interesting if a linguistic representational model or a decision making model could enhance the flexibility of the first class of models as well as keeping their features.

(3) The underlying LTS in complex linguistic expressions should be paid more attention. Complex linguistic expressions are always generated based on the semantics of linguistic terms in the LTS. For example, the semantics of the expression "*between good and very good*" highly depend on the semantics of "*good*" and "*very good*". Roughly, linguistic terms could be uniformly distributed or non-uniformly distributed, balanced or unbalanced, in the LTS. The models for modeling complex linguistic expressions should be constructed based on the consideration of the underlying LTS.

Moreover, the techniques for modeling and operating linguistic expressions enable us to process natural language, especially in intelligent decision support system. For one thing, these techniques could help to understand the meaning, such as sentimental orientations, of customers. The opinions and feelings included in texts and videos are frequently expressed by means of linguistic expressions, such as single terms, comparative linguistic expressions, and linguistic hedges. Based on specific models, the meaning of these expressions could be understood exactly. And then the overall opinions can be computed by the techniques of information fusion. For another, when dealing with massive data, linguistic expressions are the natural way to present an understandable view for users. Complex linguistic expressions enable a flexible manner to exhibit different views with different granules. This is essential for human-computer interaction in the big data era.

7. Conclusions

Because of the sharply increasing complexity of real-world problems, the use of single linguistic terms is often not enough to cope with uncertainties which come from multiple information sources. Thereby, the experts have to consider complex linguistic expressions from a natural or artificial way. Till now, there are a number of models to represent and operate several types of complex linguistic expressions, such as ULTs, HFLTS, EHFLTS, LEoLSs, ER-based model, proportional terms, distribution assessments, DFN-based subjective evaluations, PLTSs, LHFSs and 2DLTs. The models provide the excited tools to enhance the capability of QDM.

This survey has been devoted to presenting a systematical review on the existing models, especially on the focused linguistic expressions and the computational essentials. The main contribution can be concluded as follows:

(1) The existing models are reviewed based on a novel taxonomy which classifies the models by the focused natural or artificial linguistic expressions. Thereafter, the characteristics of each class of models can be exploited and compared.

(2) The major limitations of the existing models are addressed. Based on which, we outline the principles and the directions of further developments of the models. Especially, we argue that simple extensions of the existing ideas of decision making processes should be eliminated. Superior and fresh decision making processes are welcome for the purpose of solving real-world decision making applications.

(3) Some possible directions of developing new tools are figured out to model more types of complex linguistic expressions. Especially, we highlight the necessity and a possible solution of modeling the weakened linguistic hedges in QDM.

Acknowledgments

This work was funded by the National Natural Science Foundation of China (Nos. 71571123, 71771155, 71601092, 61640220), the Key University Science Research Project of Jiangsu Province (No. 16KJA520002) and the China Scholarship Council.

References

- [1] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338-353.
- [2] L.A. Zadeh, Is there a need for fuzzy logic?, Inf. Sci. 178 (2008) 2751-2779.
- [3] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inf. Sci. 8 (1975) 199-249.
- [4] L.A. Zadeh, Fuzzy logic equals Computing with words, IEEE Trans. Fuzzy Syst. 4 (1996) 103-111.
- [5] F. Herrera, S. Alonso, F. Chiclana, E. Herrera-Viedma, Computing with words in decision making: Foundations, trends and prospects, Fuzzy Optim. Decis. Ma. 8 (2009) 337-364.
- [6] Z.S. Xu, Linguistic Decision Making: Theory and Methods, in, Springer-Verlag, Berlin, Heidelberg, 2012.
- [7] F.J. Cabrerizo, E. Herrera-Viedma, W. Pedrycz, A method based on PSO and granular computing of linguistic information to solve group decision making problems defined in heterogeneous contexts, Eur. J. Oper. Res. 230 (2013) 624-633.
- [8] W. Pedrycz, M.L. Song, A granulation of linguistic information in AHP decision-making problems, Inf. Fusion 17 (2014) 93-101.
- [9] R. Degani, G. Bortolan, The problem of linguistic approximation in clinical decision making, Int. J. Approx. Reason. 2 (1988) 143-162.
- [10] R.R. Yager, A new methodology for ordinal multiobjective decisions based on fuzzy sets, Decis. Sci. 12 (1981) 589-600.
- [11] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (2000) 746-752.
- [12] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004) 19-30.

- [13] Z.S. Xu, H. Wang, On the syntax and semantics of virtual linguistic terms for information fusion in decision making, Inf. Fusion 34 (2017) 43-48.
- [14] R.M. Rodríguez, A. Labella, L. Martínez, An overview on fuzzy modelling of complex linguistic preferences in decision making, Int. J. Comput. Int. Sys. 9 (2016) 81-94.
- [15] J. Ma, D. Ruan, Y. Xu, G. Zhang, A fuzzy-set approach to treat determinacy and consistency of linguistic terms in multi-criteria decision making, Int. J. Approx. Reason. 44 (2007) 165-181.
- [16] Z.S. Xu, Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment, Inf. Sci. 168 (2004) 171-184.
- [17] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2012) 109-119.
- [18] H. Wang, Extended hesitant fuzzy linguistic term sets and their aggregation in group decision making, Int. J. Comput. Int. Sys. 8 (2015) 14-33.
- [19] J.H. Wang, J.Y. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 14 (2006) 435-445.
- [20] G.Q. Zhang, Y.C. Dong, Y.F. Xu, Consistency and consensus measures for linguistic preference relations based on distribution assessments, Inf. Fusion 17 (2014) 46-55.
- [21] W. Voxman, Canonical representations of discrete fuzzy numbers, Fuzzy Sets Syst. 118 (2001) 457-466.
- [22] J.V. Riera, S. Massanet, E. Herrera-Viedma, J. Torrens, Some interesting properties of the fuzzy linguistic model based on discrete fuzzy numbers to manage hesitant fuzzy linguistic information, Appl. Soft Comput. 36 (2015) 383-391.
- [23] Q. Pang, H. Wang, Z.S. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, Inf. Sci. 369 (2016) 128-143.
- [24] F.Y. Meng, X.H. Chen, Q. Zhang, Multi-attribute decision analysis under a linguistic hesitant fuzzy environment, Inf. Sci. 267 (2014) 287-305.
- [25] H. Zhu, J. Zhao, Y. Xu, 2-dimension linguistic computational model with 2-tuples for multi-attribute group decision making, Knowl. Based Syst. 103 (2016) 132-142.
- [26] J.B. Yang, D.L. Xu, On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty, IEEE Trans. Syst. Man Cy. A 32 (2002) 289-304.
- [27] Y.C. Tang, J.C. Zheng, Linguistic modelling based on semantic similarity relation among linguistic

labels, Fuzzy Sets Syst 157 (2006) 1662-1673.

- [28] J.A. Morente-Molinera, I.J. Pérez, M.R. Ureña, E. Herrera-Viedma, On multi-granular fuzzy linguistic modeling in group decision making problems: a systematic review and future trends, Knowl. Based Syst. 74 (2015) 49-60.
- [29] F.J. Cabrerizo, S. Alonso, E. Herrera-Viedma, A consensus model for group decision making problems with unbalanced fuzzy linguistic information, Int. J. Inf. Tech. Decis. 8 (2009) 109-131.
- [30] Z.S. Xu, H. Wang, Managing multi-granularity linguistic information in qualitative group decision making: an overview, Granul. Comput. 1 (2016) 21-35.
- [31] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-II, Inf. Sci. 8 (1975) 301-357.
- [32] F.J. Cabrerizo, R. Al-Hmouz, A. Morfeq, A.S. Balamash, M.A. Martínez, E. Herrera-Viedma, Soft consensus measures in group decision making using unbalanced fuzzy linguistic information, Soft comp. 21 (2017) 3037–3050.
- [33] Y. Dong, Y. Wu, H. Zhang, G. Zhang, Multi-granular unbalanced linguistic distribution assessments with interval symbolic proportions, Knowl. Based Syst. 82 (2015) 139-151.
- [34] F. Herrera, E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, Fuzzy Sets Syst, 115 (2000) 67-82.
- [35] D. Dubois, H. Prade, Fuzzy sets and systems: Theory and applications, Academic press, New York, 1980.
- [36] M. Delgado, J.L. Verdegay, M.A. Vila, On aggregation operations of linguistic labels, Int. J. Intell. Syst. 8 (1993) 351-370.
- [37] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, Direct approach processes in group decision making using linguistic OWA operators, Fuzzy Sets Syst. 79 (1996) 175-190.
- [38] Z.S. Xu, Deviation measures of linguistic preference relations in group decision making, Omega 33 (2005) 249-254.
- [39] Y. Dong, Y. Xu, S. Yu, Linguistic multiperson decision making based on the use of multiple preference relations, Fuzzy Sets Syst. 160 (2009) 603-623.
- [40] R.M. Rodríguez, L. Martínez, F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, Inf. Sci. 241 (2013) 28-42.
- [41] H. Zhang, The multiattribute group decision making method based on aggregation operators with

interval-valued 2-tuple linguistic information, Math. Comput. Model. 56 (2012) 27-35.

- [42] E. Falco, J.L. Garcia-Lapresta, L. Rosello, Allowing agents to be imprecise: A proposal using multiple linguistic terms, Inf. Sci. 258 (2014) 249-265.
- [43] H. Zhang, Some interval-valued 2-tuple linguistic aggregation operators and application in multiattribute group decision making, Appl. Math. Model. 37 (2013) 4269-4282.
- [44] B. Peng, C.M. Ye, S.Z. Zeng, Uncertain pure linguistic hybrid harmonic averaging operator and generalized interval aggregation operator based approach to group decision making, Knowl. Based Syst. 36 (2012) 175-181.
- [45] X. Liu, Z. Tao, H. Chen, L. Zhou, A new interval-valued 2-tuple linguistic bonferroni mean operator and its application to multiattribute group decision making, Int. J. Fuzzy Syst., (2016) 1-23.
- [46] J.Q. Wang, D.D. Wang, H.Y. Zhang, X.H. Chen, Multi-criteria group decision making method based on interval 2-tuple linguistic information and Choquet integral aggregation operators, Soft Comput. 19 (2015) 389-405.
- [47] H.B. Liu, R.M. Rodríguez, A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making, Inf. Sci. 258 (2014) 220-238.
- [48] H.B. Liu, J.F. Cai, L. Jiang, On improving the additive consistency of the fuzzy preference relations based on comparative linguistic expressions, Int. J. Intell. Syst. 29 (2014) 544-559.
- [49] X.L. Zhang, Z.S. Xu, H. Wang, Heterogeneous multiple criteria group decision making with incomplete weight information: A deviation modeling approach, Inf. Fusion 25 (2015) 49-62.
- [50] V. Torra, Hesitant Fuzzy Sets, Int. J. Intell. Syst. 25 (2010) 529-539.
- [51] H.C. Liao, Z.S. Xu, X.J. Zeng, J.M. Merigo, Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets, Knowl. Based Syst. 76 (2015) 127-138.
- [52] C.P. Wei, N. Zhao, X.J. Tang, Operators and comparisons of hesitant fuzzy linguistic term sets, IEEE Trans. Fuzzy Syst. 22 (2014) 575-585.
- [53] X.J. Gou, Z.S. Xu, Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets, Inf. Sci. 372 (2016) 407-427.
- [54] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, Int. J. Approx. Reason. 52 (2011) 395-407.
- [55] H.C. Liao, Z.S. Xu, X.J. Zeng, Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making, IEEE Trans. Fuzzy Syst. 23 (2015) 1343-1355.

- [56] Z.B. Wu, J.P. Xu, Possibility distribution-based approach for MAGDM With hesitant fuzzy linguistic information, IEEE Trans. Cybernet. 46 (2016) 694-705.
- [57] X.J. Gou, Z.S. Xu, H.C. Liao, Multiple criteria decision making based on Bonferroni means with hesitant fuzzy linguistic information, Soft Comput. (2016) 1-15.
- [58] Z.M. Zhang, C. Wu, Hesitant fuzzy linguistic aggregation operators and their applications to multiple attribute group decision making, J. Intell. Fuzzy Syst. 26 (2014) 2185-2202.
- [59] H. Wang, Z.S. Xu, Some consistency measures of extended hesitant fuzzy linguistic preference relations, Inf. Sci. 297 (2015) 316-331.
- [60] H. Wang, Z.S. Xu, Total orders of extended hesitant fuzzy linguistic term sets: Definitions, generations and applications, Knowl. Based Syst. 107 (2016) 142-154.
- [61] J. Lawry, A framework for linguistic modelling, Artif. Intell. 155 (2004) 1-39.
- [62] J.B. Yang, Rule and utility based evidential reasoning approach for multiattribute decision analysis under uncertainties, Eur. J. Oper. Res. 131 (2001) 31-61.
- [63] J.V. Riera, J. Torrens, Aggregation of subjective evaluations based on discrete fuzzy numbers, Fuzzy Sets Syst. 191 (2012) 21-40.
- [64] J. Casasnovas, J.V. Riera, Extension of discrete t-norms and t-conorms to discrete fuzzy numbers, Fuzzy Sets Syst. 167 (2011) 65-81.
- [65] Y.X. Zhang, Z.S. Xu, H. Wang, H.C. Liao, Consistency-based risk assessment with probabilistic linguistic preference relation, Appl. Soft Comput. 49 (2016) 817-833.
- [66] Y.L. Zhai, Z.S. Xu, H.C. Liao, Probabilistic linguistic vector-term set and its application in group decision making with multi-granular linguistic information, Appl. Soft Comput. 49(2016) 801-816.
- [67] Y. Zhang, Z. Xu, H.C. Liao, A consensus process for group decision making with probabilistic linguistic preference relations, Inf. Sci. 414 (2017) 260-275.
- [68] J.Q. Wang, J.T. Wu, J. Wang, H.Y. Zhang, X.H. Chen, Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems, Inf. Sci. 288 (2014) 55-72.
- [69] S.W. Chen, J. Liu, H. Wang, Y. Xu, J.C. Augusto, A linguistic multi-criteria decision making approach based on logical reasoning, Inf. Sci. 258 (2014) 266-276.
- [70] X.H. Yu, Z.S. Xu, S.S. Liu, Q. Chen, Multicriteria decision making with 2-dimension linguistic aggregation techniques, Int. J. Intell. Syst. 27 (2012) 539-562.
- [71] D.L. Xu, J.B. Yang, Y.M. Wang, The evidential reasoning approach for multi-attribute decision

analysis under interval uncertainty, Eur. J. Oper. Res. 174 (2006) 1914-1943.

- [72] N. Chen, Z.S. Xu, M.M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, Knowl. Based Syst. 37 (2013) 528-540.
- [73] H. Wang, Z.S. Xu, Multi-groups decision making using intuitionistic-valued hesitant fuzzy information, Int. J. Comput. Int. Sys. 9 (2016) 468-482.
- [74] X.Y. Liu, Y.B. Ju, S.H. Yang, Hesitant intuitionistic fuzzy linguistic aggregation operators and their applications to multiple attribute decision making, J. Intell. Fuzzy Syst. 27 (2014) 1187-1201.
- [75] R.M. Rodríguez, B. Bedregal, H. Bustince, Y.C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y.J. Xu, Z.S. Xu, F. Herrera, A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress, Inf. Fusion 29 (2016) 89-97.
- [76] H.A. Simon, Rational choice and the structure of the environment, Psychol. Rev. 63 (1956) 129.
- [77] D.B. Brown, E.D. Giorgi, M. Sim, Aspirational preferences and their representation by risk measures, Manage. Sci. 58 (2012) 2095-2113.
- [78] B. Feng, F. Lai, Multi-attribute group decision making with aspirations: A case study, Omega 44 (2014) 136-147.
- [79] H. Wang, Z.S. Xu, Enabling weakened hedges in linguistic multi-criteria decision making, J. Southeast University 32 (2016) 125-131.