**Towards achieving consistent opinion fusion in group decision making with complete distributed preference relations**

Mi Zhou1,2, Meng Hu1,2\*, Yu-Wang Chen3\*, Ba-Yi Cheng1,2, Jian Wu4, Enrique Herrera-Viedma5,6

1School of Management, Hefei University of Technology, Hefei, Anhui 230009, China

2Ministry of Education Engineering Research Centre for Intelligent Decision-Making & Information System Technologies, Hefei, Anhui 230009, China

3Alliance Manchester Business School, The University of Manchester, Manchester M15 6PB, UK

4School of Economics and Management, Shanghai Maritime University, Shanghai 201306, China

5Andalusian Research Institute in Data Science and Computational Intelligence, Dept. of Computer Science and AI, University of Granada, Granada 18071, Spain

6Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia

\*Corresponding authors: menghu@mail.hfut.edu.cn; yu-wang.chen@manchester.ac.uk

**Abstract**

Belief distribution (BD) is a scheme of representing qualitative information with subjective uncertainty and imprecision. Distributed preference relation (DPR) extends BDs to the form of pairwise comparison by expressing the preferred, non-preferred, indifferent, and uncertain degrees of one decision alternative over another. However, previous studies on DPR only require comparison of adjacent alternatives, and consensus reaching is not considered fully in the decision making process. To solve this problem, a complete DPR model is presented in this paper to support group decision making (GDM). First, a consistency index is defined to measure the consistency level of the complete DPR representing experts’ judgments. Second, an automatic adjustment algorithm is proposed to improve the consistency of DPRs with unacceptable consistency to an acceptable level. Third, the evidential reasoning (ER) algorithm is utilized to aggregate all the DPRs together, and an optimization model is further constructed to generate experts’ weights, which maximizes the degree of consensus among experts. A GDM example is provided to illustrate the applicability and validity of the proposed DPR model, and comparative analysis demonstrates the potential of the proposed method in supporting real-world GDM problems.

**Keywords:** distributed preference relation; group decision making; consistency; consensus; evidential reasoning

**1. Introduction**

The explosive growth of information makes the real-world decision-making problems more and more complex. It is often difficult for individuals to make a firm decision, which leads to many problems that require multiple people to make decisions jointly [1,2]. In this context, group decision making (GDM) method which is a process of evaluating a set of alternatives by multiple decision makers (DMs) or experts, is widely used in various fields, such as supply chain management [3], location selection [4–6], corporate financing decisions [7], contractor selection [8], environment management [9], investment decision [10,11], product selection [12,13], life cycle assessment [14,15] and so on.

Due to the complexity of decision-making problems and the subjectivity of human cognition, it is important to characterize uncertainty in the judgment of DMs using the schemes, such as belief distribution (BD) [12,16–19], fuzzy set [20–22], and linguistic term set [23–26] instead of precise numerical values. Besides, compared with the direct assessment on alternatives, the indirect method of pairwise comparison is easier to be implemented [27]. Typical representations of pairwise comparison include multiplicative preference relation (MPR) [28–30], fuzzy preference relation (FPR) [31–34], intuitionistic FPR (IFPR) [35–38], hesitant LPR (HLPR) [39], probabilistic LPR (PLPR) [27], distribution LPR (DLPR) [40], DLPR with incomplete symbolic proportions (DLPR-I) [41], and distributed preference relation (DPR) [42]. The concept of DPR is designed to represent a distribution of evaluating the preferred, non-preferred, indifferent, and uncertain degrees of one alternative over another using a set of linguistic grades like ‘worse’, ‘indifferent’, ‘better’ and so on. It can represent preference relations including both the extent of superiority/inferiority and incomplete information in a rational way. A series of theoretical methods have been developed to ensure consistency [42,43] and derive the ranking order of alternatives [44]. Practically, DPR has been used to solve multiple attribute decision making (MADM) problems and GDM problems [45,46]. However, previous studies only require the comparison of adjacent alternatives, which leads to the lack of adequate comparison between alternatives and make the decision making process prone to bias, although it can relieve the burden for DMs to provide complete information. Moreover, the consensus process among DMs and the fusion method of DMs’ opinions is also significant for the GDM, which is not fully discussed in the existing research.

When using preference relation to deal with decision making problems, consistency is required to guarantee the scientific nature and rationality of decision-making. Specifically, there are two kinds of consistency, namely multiplicative consistency [29,32,47–49] and additive consistency [31,50–54]. Due to the bounded rationality of human being, it is difficult to achieve absolute consistency in preference relations. As such, acceptable consistency is often used as a metric. For example, Saaty [55] proposed the concept of consistency ratio (CR) under MPR. As long as CR is less than a specified threshold, the MPR can be considered to be acceptably consistent. Due to the complexity of linguistic term-based preference representations, the consistency is often judged by expectations [56–58]. Fu et al. [42] discussed the consistency of DPR by the score matrix of DPR derived from the score values of grades, and defined the additive consistency of the score matrix as the basis for judging the consistency of DPR. Specifically, their method only requires DMs to give the preference distributions between adjacent alternatives which form incomplete DPRs. After converting the incomplete DPR into incomplete score matrix, the missing scores are generated by using the additive consistency and a consistent complete score matrix is got. This method avoids the inconsistency of DPR ingeniously. However, only evaluating the adjacent alternatives is not sufficient when the decision-making issues are of great significance, such as impactful and strategic decisions for enterprises and governments. Therefore, it is necessary for each DM to compare sufficient pairs of alternatives and give a relatively complete DPR. Undoubtedly, it involves time costs for comparing more pairs of alternatives to construct complete DPRs, but in the circumstances of making impactful and strategic decisions, the significance of achieving a convincing and reliable decision outcome often outweighs the cost of investing more time to information collection and preference modelling. In addition, in impactful and strategic decision making, the alternatives have gone through preliminary evaluation and screening analyses, which usually leads to a small and manageable number of alternatives for final decision making. For instance, in strategic product selection, the number of final alternative products that need to be assessed can be very limited because many lower-performing alternatives may have already been eliminated for further consideration in the process of preliminary screening. To make the final choice, each pair of alternative products ought to be compared because an inappropriate production decision will cause irretrievable losses. If there are 5 alternative products, only 10 pairs of comparisons should be done, which is manageable in constructing complete DPRs from DMs. In another case of postgraduate reexamination in China, the ratio of shortlisting candidates for reexamination is usually set as 120%. Suppose 15 candidates are included in a group for reexamination and interview, the best-performing candidates can be easily selected for admission. Often, only a small number (e.g., 4 to 6) of candidates, who are on the board line of admission should be evaluated against each other meticulously for admission and rejection. In this paper, the concept of complete DPR is proposed along with its consistency measurement method. In addition, an adjustment algorithm is introduced to update DPR to satisfy the acceptable consistency.

There are two key correlated issues that ought to be considered in GDM: (ⅰ) How to aggregate DMs’ judgments into a collective one, and (ⅱ) How to design the consensus model among DMs. Regarding the first issue, there is lots of research in the field of preference relations. The methods based on weighted averaging (WA) operators and ordered weighted averaging (OWA) operators are widely used not only in numerical preference relations [32,59,60] but also in linguistic term-based preference relations [40,58,61]. On the basis of the fruitful research outcomes, the evidential reasoning (ER) approach [16,17,62–64] has been adopted in many uncertain decision-making circumstances with a probabilistic reasoning process. As for the consensus measurement among DMs, the general methods used in various GDM models can be divided into two types: one is to calculate the distance between each pair of DMs [12,40], the other one is to compute the distance between each DM and the collective evaluation [65,66]. Therefore, the dissimilarity measure is the key to quantify consensus. It has been studied on consensus when BD is used in GDM as a direct preference representation [12,67]. Nevertheless, how to measure the dissimilarity is still an open issue when DPRs are provided by DMs. In this paper, we will propose a comprehensive group decision-making process which takes into consideration the consensus with complete DPRs.

In summary, the main contributions of this paper can be summarized as follows:

(ⅰ) The concept of complete DPR is presented, followed by the definition of consistency measurement to quantify the consistency level of complete DPR using its corresponding score matrix.

(ⅱ) For DPR with unacceptable consistency, we propose an algorithm that can automatically adjust the DPR until the consistency condition is satisfied. The algorithm is convergent and can reduce the consistency of DPR below the threshold with relatively fewer iterations.

(ⅲ) An optimization model is constructed to maximize the consensus degree among DMs through which the objective weights of DMs can be derived. And the decision results based on the optimization model can be accepted by as many DMs as possible.

The reminder of the paper is organized as follows: Section 2 introduces the modeling of GDM problems with complete DPRs. In Section 3, the proposed GDM process with complete DPRs is described first and then the specific methods contained in the process are elaborated including consistency measurement, adjustment and optimization model considering consensus. Section 4 illustrates the application of the proposed methods by solving a GDM problem of evaluating students who compete for scholarships, and comparative analysis is conducted on the proposed method against other four representative linguistic-based preference relation methods. This paper is concluded in Section 5 with suggestions for further research.

**2. Description of GDM problem with complete DPRs**

In this section, the GDM problem with complete DPRs is presented along with the issues that need to be considered.

*2.1. The modeling of GDM problem with complete DPRs*

Suppose there are *M* alternatives $A=\{A\_{1},A\_{2},\cdots ,A\_{M}\}$，*T* experts $E=\{E\_{1},E\_{2},\cdots ,E\_{T}\}$ and a facilitator involved in a GDM problem. The relative weights of *T* experts are signified by $w=(w\_{1},w\_{2},\cdots ,w\_{T})$ where $0\leq w\_{t}\leq 1$ $(t=1,2,\cdots ,T)$ and $\sum\_{t=1}^{T}w\_{t}=1$.

**Definition 1.** [42]Let $A=\{A\_{1},A\_{2},\cdots ,A\_{M}\}$ be a set of alternatives, and expert $E\_{t}(t=1,2,\cdots ,T)$ compares two alternatives $A\_{i}$ and $A\_{k}$ by a set of linguistic grades $ Ω=\left\{H\_{1},H\_{2},\cdots ,H\_{N}\right\}$ in which *N*$(N\geq 3)$ is an odd number. The middle grade $H\_{(N+1)/2}$ stands for indifference, $H\_{(N+3)/2},\cdots ,H\_{N}$ the preferred grade whose intensity is increasing with subscript, $H\_{1},\cdots ,H\_{(N-1)/2}$ the non-preferred grade whose intensity is decreasing with subscript. Then, expert $E\_{t}$ can express his/her preference of alternatives $A\_{i}$ over $A\_{k}$ using a distribution denoted as follows:

$$d\_{ik}^{t}=\left\{\left(H\_{n},d\_{ik}^{t}\left(H\_{n}\right)\right),n=1,\cdots ,N;\left(Ω,d\_{ik}^{t}\left(Ω\right)\right)\right\}$$

where $d\_{ik}^{t}\left(H\_{n}\right)$ indicates the belief degree to the relation of $A\_{i}$ over $A\_{k}$ assigned to grade $H\_{n}$ by the *t*th expert. Specifically, in an individual decision-making problem, $d\_{ik}^{t}$ is replaced by $d\_{ik}$.

Here, $0\leq d\_{ik}^{t}\left(H\_{n}\right)\leq 1 $,$ d\_{ik}^{t}\left(H\_{n}\right)=d\_{ki}^{t}\left(H\_{N-n+1}\right)$ and $\sum\_{n=1}^{N}d\_{ik}^{t}\left(H\_{n}\right)\leq 1$ are three basic conditions. $d\_{ik}^{t}\left(Ω\right)$ indicates the degree of global ignorance in the relation between $A\_{i}$ and $A\_{k}$such that $d\_{ik}^{t}\left(Ω\right)=d\_{ki}^{t}\left(Ω\right)=1-\sum\_{n=1}^{N}d\_{ik}^{t}\left(H\_{n}\right)$. By comparing the alternatives $\left\{A\_{1},A\_{2},\cdots ,A\_{M}\right\}$ in pairs with distributions, a distributed preference relation (DPR) denoted by $D^{t}$ can be obtained such that $D^{t}=(d\_{ik}^{t})\_{M×M}$. And the diagonal line of $D^{t}$ satisfies $d\_{ii}^{t}\left(H\_{\frac{N+1}{2}}\right)=1$.

For the purpose of reducing the burden of information providing by experts, the model proposed by Fu et al. [42] requires experts to give distributions between adjacent alternatives $d\_{i,i+1}(i=1,\cdots ,M-1)$ which will form an incomplete DPR. However, this method makes it impossible for experts to fully compare sufficient alternatives. In general, comparing only the adjacent alternatives will cause a large deviation due to the bounded rationality of human beings. When the number of alternatives is not too large, or when the decision-making problems are of great significance, it is necessary to compare more pairs of alternatives. Thus, in this paper, each expert is required to give distributions on all pairs of alternatives which will form a complete DPR shown in Table 1.

**Table 1** Complete DPR $D^{t}$ given by expert $E\_{t}$

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| DPR | *A*1 | *A*2 | … | *A*M |
| *A*1 | $$\{\left(H\_{\frac{N+1}{2}},1\right)\}$$ | $$d\_{12}^{t}$$ | … | $$d\_{1M}^{t}$$ |
| *A*2 | $$d\_{21}^{t}$$ | $$\{\left(H\_{\frac{N+1}{2}},1\right)\}$$ | … | $$d\_{2M}^{t}$$ |
| … | … | … | … | … |
| *A*M | $$d\_{M1}^{t}$$ | $$d\_{M2}^{t}$$ | … | $$\{\left(H\_{\frac{N+1}{2}},1\right)\}$$ |

As can be seen from Table 1, each expert only needs to give the distributions above the diagonal line in the matrix. Distributions below the diagonal line can be derived from distributions above the diagonal line by using equation $d\_{ki}^{t}=\{\left(H\_{n},d\_{ik}^{t}\left(H\_{N-n+1}\right)\right),n=1,\cdots N;\left(Ω,d\_{ik}^{t}\left(Ω\right)\right)\}$.

In GDM problems, consensus among experts needs to be considered. To increase the degree of consensus, it is necessary for experts to fully communicate with each other about their opinions on the issue. The method proposed in this paper will take the consensus into account in the decision-making process. That is to say, experts will fully communicate with each other before they give their judgments. Therefore, this method is suitable for such decision-making background:

(1) The number of alternatives is limited so that each pair of alternatives can be fully compared;

(2) Small-scale group decision making (compared with large-scale group decision making where the number of experts involved exceeds 20);

In this context, the facilitator will guide experts in the decision-making process, where the following assumptions should be satisfied:

**Assumption 1.** In a GDM problem organized by a facilitator, it is required that

(ⅰ) Before the experts give their judgments, the facilitator helps them to fully understand the problem. But the facilitator should not provide experts with suggestions on alternative selection.

(ⅱ) In order to deepen the understanding of the problem by experts and avoid prejudice or divergence of judgments, experts should discuss the problem freely and fully with each other before presenting their judgments. There is no limit to the form of communication. For instance, experts may communicate with each other at a fixed place or in different places through technology means, such as video conference.

(ⅲ) After fully understanding the problem and discussion, each expert presents his/her own judgment independently.

*2.2. Analyze the GDM problem with complete DPRs*

*T* DPRs are to be generated from *T* experts involved in a GDM problem. It is necessary to guarantee the consistency of each DPR for ensuring the rationality of the subsequent decision-making process. Meanwhile, we also need to aggregate the preferences of all experts for the purpose to achieving group consensus.

For the consistency issue, DPR matrix is to be transformed into its corresponding score matrix, which is then used to define the consistency of the original complete DPR. Some relevant preliminaries are given in the following.

**Definition 2.** [42] Suppose the score value of grade $H\_{n}(n=1,\cdots ,N)$ is signified by $s(H\_{n})$ such that $0=s\left(H\_{\frac{N+1}{2}}\right)<s\left(H\_{\frac{N+1}{2}+1}\right)<\cdots <s\left(H\_{N}\right)=1$ and $s\left(H\_{n}\right)=-s\left(H\_{N-n+1}\right) (n=1,\cdots ,\frac{N-1}{2})$. Since the degree of ignorance $d\_{ik}(Ω)$ can be assigned to any grades, the distribution $d\_{ik}$ can be transformed into an interval number $S\_{ik}=[S\_{ik}^{l},S\_{ik}^{u}]$, where

$$\begin{array}{c} S\_{ik}^{l}=\sum\_{n=1}^{N}d\_{ik}\left(H\_{n}\right)∙s\left(H\_{n}\right)+d\_{ik}\left(Ω\right)∙s\left(H\_{1}\right) \left(1\right)\end{array}$$

$$\begin{array}{c} S\_{ik}^{u}=\sum\_{n=1}^{N}d\_{ik}\left(H\_{n}\right)∙s\left(H\_{n}\right)+d\_{ik}\left(Ω\right)∙s\left(H\_{N}\right) \left(2\right)\end{array}$$

By transforming each distribution $d\_{ik}$ of DPR matrix *D* into a score interval $S\_{ik}$, the score matrix denoted by $S=(S\_{ik})\_{M×M}$ is generated such that $S\_{ik}^{l}+S\_{ki}^{u}=0,∀i,k\in \left\{1,\cdots ,M\right\}$ and $S\_{ik}^{u}+S\_{ki}^{l}=0,∀i,k\in \left\{1,\cdots ,M\right\}.$

**Definition 3.** [45] By using a function $f:\left[-1,1\right]×\left[-1,1\right]\rightarrow [-1,1]$, *S* is defined to be consistent if the following two conditions are satisfied:

$$\begin{array}{c} f\left(S\_{ij}^{l},S\_{jk}^{l}\right)+S\_{ki}^{u}=0,∀i,j,k\in \left\{1,\cdots ,M\right\} \left(3\right)\end{array}$$

$$\begin{array}{c} f\left(S\_{ij}^{u},S\_{jk}^{u}\right)+S\_{ki}^{l}=0,∀i,j,k\in \left\{1,\cdots ,M\right\}. \left(4\right)\end{array}$$

Given *y* and *z* such that $y\in [-1,1]$ and $z\in [-1,1]$, the function $f:\left[-1,1\right]×\left[-1,1\right]\rightarrow [-1,1]$ used to combine *y* with *z* satisfies the following properties [31,68,69]:

(1)$f\left(y,z\right)>max⁡\{y,z\}$ if $y\in (0,1)$ and $z\in (0,1)$,

(2)$ f\left(y,z\right)<min⁡\{y,z\}$ if $y\in (-1,0)$ and $z\in (-1,0)$,

(3)$min⁡\{y,z\}\leq f(y,z)\leq max⁡\{y,z\}$ if $y∙z\leq 0$,

(4)$f\left(1,z\right)=f\left(y,1\right)=1$ if $z\in (0,1]$ and $y\in (0,1]$,

(5)$ f\left(-1,z\right)=f\left(y,-1\right)=-1$ if $z\in [-1,0)$ and $y\in [-1,0)$,

(6)$f\left(y,z\_{1}\right)>f(y,z\_{2})$ if $-1<y<1$ and $z\_{1}>z\_{2}$, and

(7)$f\left(y\_{1},z\right)>f(y\_{2},z)$ if $-1<z<1$ and $y\_{1}>y\_{2}$.

Any function satisfying the above properties can be used to construct a consistent score matrix. The detailed consistency definition of DPR can be referred to the literature [42].

**3. GDM method with complete DPRs**

 In this section, a flowchart is firstly given to emphasize the key steps in the proposed method.

*3.1. Procedure of the proposed method*

The specific decision-making process for GDM problems with complete DPRs is depicted in Fig.1.

**Fig. 1.** Decision-making process of GDM with complete DPRs

**Step 1:** Determine the basic parameters of GDM problem with complete DPRs.

Specify a set of linguistic grades $Ω=\{H\_{1},H\_{2},\cdots ,H\_{N}\}$ and the corresponding score values $s\left(H\_{n}\right) (n=1,2,\cdots ,N)$. Set the consistency threshold *τ* a priori, and give the criteria system for reference if possible.

**Step 2:** Let the experts to provide DPRs with respect to alternatives.

According to the given criteria system, experts compare each pair of alternatives to form a comprehensive impression based on their knowledge backgrounds and expertise. Then, experts fully discuss and exchange their opinions about alternatives. After discussion, experts give their DPRs $\{D^{1},D^{2},\cdots ,D^{T}\}$ independently.

**Step 3:** Consistency checking. (The details are illustrated in Sections 3.2.1 and 3.2.2.)

*substep 1*: Calculate the original score matrices $S^{t}$(or $S^{E\_{t}}$) $\left(t=1,2,\cdots ,T\right)$ corresponding to DPRs $D^{t}$ (or $D^{E\_{t}}$) $\left(t=1,2,\cdots ,T\right)$ by Eqs.(1) and (2).

 *substep 2*: Generate consistent score matrices $\hat{S}^{t}$(or $\hat{S}^{E\_{t}}$) $(t=1,2,\cdots ,T)$ by Algorithm I.

 *substep 3*: Calculate the consistency index $CI(E\_{t})(t=1,2,\cdots ,T)$ of each expert’s judgment by Eq.(11).

 *substep 4*: Compare the consistency index of each expert with the threshold *τ*. If the consistency index of each expert is no more than the threshold, i.e. $CI\left(E\_{t}\right)\leq τ$ for all $t=1,2,\cdots ,T$, go to Step 5. Otherwise, if the consistency indexes of some experts are greater than the threshold, go to Step 4 to automatically adjust the experts’ DPRs until all of the experts’ consistency indices accord with the requirement.

**Step 4:** Consistency improving. (The details are illustrated in Section 3.2.3.)

Adjusting the DPRs that don’t satisfy the consistency condition automatically by Algorithm II until their consistency indices are less than the threshold value.

**Step 5:** Aggregate DPRs. (The details are illustrated in Section 3.3.)

Suppose the weights of experts are $w\_{t}(t=1,2,\cdots ,T)$. Solve Model Ⅱ to generate experts’ weights and the collective DPR $D^{C}$ under the maximum consensus among experts.

**Step 6:** Generate a solution of the GDM problem using the collective DPR.

By using the possibility degree [42], the ranking order of alternatives can be derived. The possibility degree is calculated by the following equations:

|  |  |
| --- | --- |
| (ⅰ) $p\left(S\_{ik}\geq S\_{ki}\right)=0.5$ if $S\_{ik}^{l}=S\_{ik}^{u}=0$  | (5) |
| (ⅱ) $p\left(S\_{ik}\geq S\_{ki}\right)=1$ if $0\leq S\_{ik}^{l}<S\_{ik}^{u}$ or $0<S\_{ik}^{l}\leq S\_{ik}^{u}$  | (6) |
| (ⅲ) $p\left(S\_{ik}\geq S\_{ki}\right)=0$ if $S\_{ik}^{l}<S\_{ik}^{u}\leq 0$ or $S\_{ik}^{l}\leq S\_{ik}^{u}<0$ | (7) |
| (ⅳ) $p\left(S\_{ik}\geq S\_{ki}\right)=S\_{ik}^{u}/(S\_{ik}^{u}-S\_{ik}^{l})$ if $S\_{ik}^{l}<0<S\_{ik}^{u}$ | (8) |

*3.2. Consistency of individual DPR*

Three parts are contained in this section. First, for measuring the consistency of expert’s individual DPR, the consistent score matrix is constructed. Second, the consistency index of the original score matrix is generated as the consistency measure of DPR. Finally, in order to ensure the consistency of DPR, an algorithm is designed to adjust inconsistent DPR automatically.

3.2.1. Consistent score matrix

First, the DPR is transformed into the score matrix by using Eqs. (1) and (2). We call the score matrix transformed from complete DPR as the original score matrix, which is represented by $S=(S\_{ik})\_{M×M}$. Then we use the original score matrix and the consistency definition to construct a new score matrix, which is called the consistent score matrix represented by $\hat{S}=(\hat{S}\_{ik})\_{M×M}$.

The following consistency definition is equivalent to Def. 3, but the representation is more intuitive. We will use it to construct consistent score matrix.

**Definition 4.** Let $D=(d\_{ik})\_{M×M}$ be a DPR, $S$ be its score matrix. $D$ is called consistent if $S\_{ik}^{l}=f(S\_{ij}^{l},S\_{jk}^{l})$ and $S\_{ik}^{u}=f(S\_{ij}^{u},S\_{jk}^{u})$ for all $i,j,k=1,2,\cdots ,M$, where the function $f\left(y,z\right)$ satisfies the properties in Def. 3.

As interpreted in Section 2.2, any function satisfies the properties in Def. 3 can be used to construct consistent score matrix. In this paper, for the convenience of calculation, the following function is used:

$$\begin{array}{c} f\left(y,z\right)=\left\{\begin{matrix}y+z-yz,&if y\in \left[0,1\right] and z\in \left[0,1\right]\\y+z+yz,&if y\in \left[-1,0\right] and z\in \left[-1,0\right]\\y+z,&otherwise\end{matrix}\right. \left(9\right)\end{array}$$

Eq.(9) is just the situation that $b=0$ in Fu et al. [42]. According to Def. 4, Algorithm I is designed to construct consistent score matrix.

**Algorithm I.**

**Input:** Original score matrix calculated from DPR

**Output:** Consistent score matrix

**Step 1:** For adjacent alternatives $A\_{i}$ and $A\_{k}$ $(i,k\in \left\{1,2,…,M\right\},k-i=1 or k-i=-1)$,

the consistent scores $\hat{S}\_{ik}=S\_{ik}$, i.e. $\left[\hat{S}\_{ik}^{l},\hat{S}\_{ik}^{u}\right]=[S\_{ik}^{l},S\_{ik}^{u}]$;

**Step 2:** For $k-i>1$, the consistent scores $\hat{S}\_{ik}=\left[\hat{S}\_{ik}^{l},\hat{S}\_{ik}^{u}\right]$ are calculated by

$\hat{S}\_{ik}^{l}=\frac{\sum\_{j=i+1}^{k-1}f(S\_{ij}^{l},S\_{jk}^{l})}{k-i-1}$, $\hat{S}\_{ik}^{u}=\frac{\sum\_{j=i+1}^{k-1}f(S\_{ij}^{u},S\_{jk}^{u})}{k-i-1}$;

**Step 3:** For $k-i<-1$, $\left[\hat{S}\_{ik}^{l},\hat{S}\_{ik}^{u}\right]=\left[-\hat{S}\_{ki}^{u},-\hat{S}\_{ki}^{l}\right]$.

It can be seen from Algorithm I that the adjacent elements $S\_{i,i+1}$ in the original score matrix are retained in the consistent score matrix, whereas the non-adjacent elements are constructed by the functions in Step 2. After the elements above the diagonal line are determined in the consistent score matrix, the elements below the diagonal line can be generated according to Def. 1. In particular, all transitive possibilities are considered when calculating the consistent scores for non-adjacent alternatives. For instance, there is only one transmission step of $S\_{12}$ to $S\_{23}$ when calculating $\hat{S}\_{13}$. Comparatively, there are two transmission processes of $S\_{12}$ to $S\_{24}$ and $S\_{13}$ to $S\_{34}$ when calculating $\hat{S}\_{14}$. As described in Step 2, when there are multiple ways of transmission, all the candidate transmission ways are jointly considered. So Algorithm I is different from the consistent score matrix calculated by Fu et al. [42] in which only the distributions of adjacent alternatives are used to generate $\hat{S}$, while we use every elements in the complete DPR matrixto generate consistent score matrix $\hat{S}$.

To facilitate understanding, the following example is presented.

**Example 1.** Assume that we use linguistic evaluation grades *Ω=*{*worse, indifferent, better*} represented by $Ω=\{H\_{1},H\_{2},H\_{3}\}$ to illustrate preference relations on three alternatives $A=\{A\_{1},A\_{2},A\_{3}\}$. An expert gives the DPR matrix $D^{Example}$ presented in Table 2 which is obviously inconsistent because *A*1 is better than *A*2, *A*2 is slightly better than *A*3, but *A*1 is somewhat worse than *A*3.

**Table 2** The original DPR matrix $D^{Example}$ on three alternatives

|  |  |  |  |
| --- | --- | --- | --- |
| DPR | *A*1 | *A*2 | *A*3 |
| *A*1 | $$\{\left(H\_{2},1\right)\}$$ | $$\left\{\begin{array}{c}(H\_{2},0.5)\\\left(H\_{3},0.4\right)\\(Ω,0.1)\end{array}\right\}$$ | $$\left\{\begin{array}{c}(H\_{1},0.4)\\\left(H\_{2},0.3\right)\\(H\_{3},0.1)\\(Ω,0.2)\end{array}\right\}$$ |
| *A*2 | - | $$\{\left(H\_{2},1\right)\}$$ | $$\left\{\begin{array}{c}(H\_{2},0.8)\\\left(H\_{3},0.1\right)\\(Ω,0.1)\end{array}\right\}$$ |
| *A*3 | - | - | $$\{\left(H\_{2},1\right)\}$$ |

The scores of evaluation grades are assumed to be $s\left(H\_{1}\right)=-1$, $s\left(H\_{2}\right)=0$, $s\left(H\_{3}\right)=1$. So the original score matrix of DPR can be calculated by Eqs.(1) and (2) shown in Table 3.

**Table 3** Original score matrix $S^{Example}$ derived from DPR

|  |  |  |  |
| --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 |
| *A*1 | [0, 0] | [0.3, 0.5] | [-0.5, -0.1] |
| *A*2 | - | [0, 0] | [0, 0.2] |
| *A*3 | - | - | [0, 0] |

Table 4 shows the consistent score matrix $\hat{S}^{Example}$ generated by using Algorithm I and Eq.(9).

**Table 4** The generated consistent score matrix $\hat{S}^{Example}$

|  |  |  |  |
| --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 |
| *A*1 | [0, 0] | [0.3, 0.5] | [0.3, 0.6] |
| *A*2 | - | [0, 0] | [0, 0.2] |
| *A*3 | - | - | [0, 0] |

3.2.2. Consistency measurement

When the consistent score matrix is generated from the original DPR matrix, the calculation of the consistency level for each DPR matrix should be conducted. First, a formula for measuring the distance of interval numbers is introduced.

**Definition 5.** [70] Let $a=[a^{l},a^{u}]$ and $b=[b^{l},b^{u}]$ be two interval numbers, and $ c=\left[c^{l},c^{u}\right]=a∩b$, then the distance between *a* and *b* is defined as

$$\begin{array}{c} dist\left(a,b\right)=\sqrt{\frac{(a^{l}+a^{u}-b^{l}-b^{u})}{4}^{2}+\frac{1}{12}\left((a^{u}-a^{l})^{2}+(b^{u}-b^{l})^{2}\right)-\frac{1}{6}\left(c^{u}-c^{l}\right)^{2}} \left(10\right)\end{array}$$

For quantifying the consistency level of the score matrix derived from the original DPR matrix, the distance between the original score matrix *S* and the consistent score matrix $\hat{S}$ is calculated as the consistency index.

**Definition 6.** Let $S^{t}=\left(S\_{ik}^{t}\right)\_{M×M}=\left(S\_{ik}^{E\_{t}}\right)\_{M×M}$ be the original score matrix calculated from DPR given by expert$E\_{t}$, $\hat{S}^{t}=\left(\hat{S}\_{ik}^{t}\right)\_{M×M}=\left(\hat{S}\_{ik}^{E\_{t}}\right)\_{M×M}$ be the corresponding consistent score matrix calculated by Algorithm I. The consistency index of judgment by expert $E\_{t}$ is defined as

$$\begin{array}{c} CI\left(E\_{t}\right)=CI\left(D^{t}\right)=CI\left(S^{t}\right)=\frac{\sum\_{i=1}^{M-2}\sum\_{k=i+2}^{M}dist\left(S\_{ik}^{t},\hat{S}\_{ik}^{t}\right)}{\left(M-2\right)\left(M-1\right)} \left(11\right)\end{array}$$

Where $dist(S\_{ik}^{t},\hat{S}\_{ik}^{t})$ is calculated using Eq.(10) such that $dist\left(S\_{ik}^{t},\hat{S}\_{ik}^{t}\right)\in [0,2]$. Because we need to calculate the distance of $(M-2)(M-1)/2$ pairs of interval numbers, the value of $CI\left(E\_{t}\right)$ will be in the range of $[0,1]$.n

Since the consistency index is measured by the distance between original and consistent score matrix, then the smaller the consistency index, the higher the consistency degree of judgment. Especially, the judgment is called absolutely consistent if and only if $CI\left(E\_{t}\right)=0$ although it rarely happens in real-world decision making problems.

**Example 2.** (Continued with Example 1)

The consistency index of $D^{Example}$ can be calculated by Eqs. (10) and (11) as follows:

$$CI\left(D^{Example}\right)=CI\left(S^{Example}\right)=\frac{dist(S\_{13}^{Example},\hat{S}\_{13}^{Example})}{1×2}=0.76376$$

In addition to measure the level of consistency of expert’s judgment, consistency threshold denoted by *τ* should be set as an indicator to judge whether the DPR matrix is of acceptable consistent.

**Definition 7.** A DPR matrix is regarded as acceptably consistent if its consistency index satisfies the condition that $CI\left(E\_{t}\right)\leq τ$. Otherwise, it is unacceptable consistent when $CI\left(E\_{t}\right)>τ$.

Since the setting of threshold is complex and subjective, it should be set according to the situation of specific problem. Considering the preference representation of DPR, two main factors affecting the consistency level are the number of alternatives and evaluation grades. Specifically, the greater the scale of alternatives, the more probable that experts may make inconsistent judgment when comparing different pairs of alternatives. Similarly, with the increasing number of evaluation grades, the more detailed discrimination should be made by experts, which decrease the extent of a consistent judgment. Therefore, the threshold value is positively correlated with both the number of alternatives *M* and evaluation grades *N*. We give the conceptual function between the threshold value and these two parameters as follows:

$$τ=f(M,N)$$

The properties of function *f* are described as follows:

**Property 1.** Let *M* and *N* be the number of alternatives and evaluation grades such that $M\geq 2$, $N\geq 3$, then function *f* satisfies

(1) Boundary: $0\leq f\left(M,N\right)\leq 1$;

(2) Minimum: $f\left(M,N\right)=0$ iff $M=2$;

(3) Monotonicity: $f(M\_{1},N)>f(M\_{2},N)$ if $M\_{1}>M\_{2}$;

(4) Monotonicity: $f(M,N\_{1})>f(M,N\_{2})$ if $N\_{1}>N\_{2}$.

There are already some studies conducted on the setting of threshold [71]. Saaty [55] defined Consistency Ratio (CR) as the criterion of judging acceptable consistency which is generated by Consistency Index (CI) and Random Index (RI). And RI is obtained as the mean value of CI derived from randomly generated *M*-dimensional matrices. Based on the consistency definition, Saaty [29] set the threshold of CR as 0.1 in the analytic hierarchy process (AHP). Recent research [36,72–75] have used the same threshold value of 0.1 to study the fuzzy preference relation. Because the threshold setting is not the focus of this paper, and the number of alternatives and grades in the GDM problem discussed in this paper are not extensively large, we will not specifically analyze the threshold setting approach.

3.2.3. Automatic adjustment algorithm

When the DPR matrix given by the *t*th expert does not reach the consistency threshold such that $CI\left(E\_{t}\right)>τ$, it is necessary to update the DPR matrix in order to ensure the rationality and rigorism of the decision.

If the expert improves his/her judgment by himself/herself, the facilitator and the expert need mutual feedback on the issue of consistency. First, the facilitator should return the inconsistent judgment back to the expert and explain the reason. Second, the notified expert will improve his/her perspective for a more rational logistics. Finally, the facilitator calculates the consistency of the new judgment. This process will go on until the acceptable consistency is satisfied such that $CI\left(E\_{t}\right)\leq τ$. It is clear that this feedback process often consumes a lot of time and cost. Especially when the consistency index of an expert is relatively high, the benefit of such feedback is limited. Therefore, it is necessary to design an adjustment method which can improve the consistency of experts’ judgments automatically without intervention of experts as far as possible.

The issues that ought to be considered when designing an automatic adjustment method include: (ⅰ) which elements should be adjusted, and (ⅱ) how to adjust the selected elements. On the first issue, the elements that can assist to reach the consistency threshold as soon as possible are probably selected. According to Eq.(11), it can be seen that the consistency measurement is obtained by calculating the distance between the corresponding values of the original and consistent score matrices, following by the addition and averaging process. Therefore, the element with the largest distance has priority to be adjusted over others. Regarding the second issue, two aspects are to be considered. On one hand, the calculation process related to the adjustment mechanism should not be complex. On the other hand, the original judgments of the notified experts should be preserved as far as possible. Based on these considerations, the following algorithm is designed to adjust inconsistent DPR matrices automatically.

**Algorithm** **II.**

**Input:** Original DPR matrix *D* (inconsistent), original score matrix *S*, consistent score matrix $\hat{S}$, consistency threshold τ

**Output:** Acceptable consistent DPR matrix $\hat{D}$

**Step 1:** Judge whether $CI(E\_{t})\leq τ$ holds when $S\_{1M}$ is replaced by $\hat{S}\_{1M}$. If the condition holds, replace $S\_{1M}$ with $\hat{S}\_{1M}$ to form a new original score matrix *S* and go to Step 3 to map changes to distribution $d\_{1M}$. Otherwise, go to Step 2.

**Step 2:** Calculate $∆S\_{ik}=dist(S\_{ik},\hat{S}\_{ik}),i=1,\cdots ,M-2;k=i+2,\cdots ,M$ and find the maximum one $max∆S\_{ik}$. Replace $S\_{ik}$ by $\hat{S}\_{ik}$ to form a new score matrix *S*. Update the consistent score matrix $\hat{S}$ corresponding to *S* by Algorithm I. Go to Step 3 to map changes to distribution.

**Step 3:** Let $\hat{S}\_{ik}$ be the changed element from $S\_{ik}$. Correspondingly, the distribution $d\_{ik}=\{(H\_{n},d\_{ik}\left(H\_{n}\right),n=1,2,\cdots ,N;(Ω,d\_{ik}\left(Ω\right)\}$ is converted to a new one denoted by $\hat{d}\_{ik}=\{(H\_{n},\hat{d}\_{ik}\left(H\_{n}\right),n=1,2,\cdots ,N;(Ω,\hat{d}\_{ik}\left(Ω\right)\}$. Two replacement rules are considered here. One is a simple and fast process, the other one is a procedure that minimizes the difference between the expert’s original and adjusted judgment.

(ⅰ) The first rule of replacement is as follows:

For $\frac{\hat{S}\_{ik}^{u}+\hat{S}\_{ik}^{l}}{2}>0$,$ \hat{d}\_{ik}=\{\left(H\_{(N+1)/2},1-\hat{S}\_{ik}^{u}\right),\left(H\_{N},\frac{\hat{S}\_{ik}^{u}+\hat{S}\_{ik}^{l}}{2}\right),\left(Ω,\frac{\hat{S}\_{ik}^{u}-\hat{S}\_{ik}^{l}}{2}\right)\}$;

For $\frac{\hat{S}\_{ik}^{u}+\hat{S}\_{ik}^{l}}{2}<0$,$ \hat{d}\_{ik}=\{(H\_{1},-\frac{\hat{S}\_{ik}^{u}+\hat{S}\_{ik}^{l}}{2}),\left(H\_{(N+1)/2},1+\hat{S}\_{ik}^{l}\right),\left(Ω,\frac{\hat{S}\_{ik}^{u}-\hat{S}\_{ik}^{l}}{2}\right)\}$.

(ⅱ) The second replacement rule uses an optimization model to get a new distribution. The specific model is as follows:

**Model I.**

$$\begin{matrix}MAX Sim(d\_{ik},\hat{d}\_{ik})\\\begin{matrix}s.t.&\left\{\begin{array}{c}\sum\_{n=1}^{N}\hat{d}\_{ik}\left(H\_{n}\right)∙s\left(H\_{n}\right)+\hat{d}\_{ik}\left(Ω\right)∙s\left(H\_{1}\right)=\hat{S}\_{ik}^{l}\\\sum\_{n=1}^{N}\hat{d}\_{ik}\left(H\_{n}\right)∙s\left(H\_{n}\right)+\hat{d}\_{ik}\left(Ω\right)∙s\left(H\_{N}\right)=\hat{S}\_{ik}^{u}\\\begin{matrix}\sum\_{n=1}^{N}\hat{d}\_{ik}(H\_{n})+\hat{d}\_{ik}\left(Ω\right)=1\\\hat{d}\_{ik}\left(H\_{n}\right)\geq 0,n=1,2,\cdots ,N;\hat{d}\_{ik}(Ω)\geq 0\end{matrix}\end{array}\right.\end{matrix}\end{matrix}$$

Here, $\hat{d}\_{ik}\left(H\_{n}\right) (n=1,2,\cdots ,N)$ and $\hat{d}\_{ik}(Ω)$ are the variables to be solved. $Sim(d\_{ik},\hat{d}\_{ik})$ can be calculated by the equations presented in Def.8.

**Step 4:** Judge whether $CI(E\_{t})\leq τ$ holds. If the inequality holds, the algorithm ends. Otherwise, go to Step 1.

Steps 1 and 2 illustrate the identification rules (IR). And adjustment rules (AR) are specified in Step 3, where two methods are provided to generate a new distribution. The first method uses a fixed formula to get a new distribution only assigned to two evaluation grades and global ignorance, which is simple and fast. The second replacement rule can preserve the original judgment of expert to the largest extent. To achieve this, we use an optimization model to compute the new distribution. The goal of ModelⅠis to maximize the similarity between the original distribution and the new one. And the constraint demonstrates that the score interval of the new distribution is equal to that of the consistent score interval $\hat{S}\_{ik}$. In practical applications, which method to be applied depends on the specific decision-making circumstance. When the original judgment of expert needs to be preserved to a large extent, the second method is undoubtedly more applicable. However, with the increasing number of evaluation grades *N*, the second method requires more computing time and cost. As such, the first method will be much more efficient in this situation.

**Example 3.** (Continued with Example 2)

Because the consistency index is greater than the threshold, i.e.$ CI\left(D^{Example}\right)>0.1$, we adopt Algorithm II to adjust DPR $D^{Example}$. For there are only three alternatives in the example, $d\_{13}^{Example}$ becomes the only element that can be chosen. When we replace $S\_{13}^{Example}$ with $\hat{S}\_{13}^{Example}$, i.e. replace [-0.5, -0.1] with [0.3,0.6], the consistency index of the new score matrix will be 0 which is smaller than the threshold. Then we need to map [0.3,0.6] to DPR. Here we use the two methods presented in Step 3 of Algorithm II to calculate in turn.

(ⅰ) When the first replacement method is employed, the new distribution is $d\_{13}^{Example}=\{\left(H\_{2},0.4 \right),\left(H\_{3},0.45 \right),\left(Ω,0.15 \right)\}$.

(ⅱ) When the second replacement method is utilized, Model I will be constructed as follows:

$$\begin{matrix}MAX Sim\left(d\_{13},\hat{d}\_{13}\right)\\\begin{matrix}s.t.&\left\{\begin{array}{c}\sum\_{n=1}^{3}\hat{d}\_{13}\left(H\_{n}\right)∙s\left(H\_{n}\right)+\hat{d}\_{13}\left(Ω\right)∙s\left(H\_{1}\right)=\hat{S}\_{13}^{l}=0.3\\\sum\_{n=1}^{3}\hat{d}\_{13}\left(H\_{n}\right)∙s\left(H\_{n}\right)+\hat{d}\_{13}\left(Ω\right)∙s\left(H\_{N}\right)=\hat{S}\_{13}^{u}=0.6\\\begin{matrix}\sum\_{n=1}^{3}\hat{d}\_{13}(H\_{n})+\hat{d}\_{13}\left(Ω\right)=1\\\hat{d}\_{13}\left(H\_{n}\right)\geq 0,n=1,2,\cdots ,N;\hat{d}\_{13}(Ω)\geq 0\end{matrix}\end{array}\right.\end{matrix}\end{matrix}$$

By solving the optimization model, we get $\hat{d}\_{13}\left(H\_{1}\right)=0.03693$, $\hat{d}\_{13}\left(H\_{2}\right)=0.32614$, $\hat{d}\_{13}\left(H\_{3}\right)=0.48693$, $\hat{d}\_{13}\left(Ω\right)=0.15$, and the maximum similarity between the original distribution $d\_{13}$ and the new distribution $\hat{d}\_{13}$ is 0.70421. So the new distribution is $d\_{13}^{Example}=\{\left(H\_{1},0.03693\right),\left(H\_{2},0.32614 \right),\left(H\_{3},0.48693 \right),\left(Ω,0.15\right)\}$, which preserves the original judgment of expert to the largest extent.

The identification rules have not been used here because the number of alternatives is only three. In order to illustrate the effectiveness of identification rules, a more detailed analysis is presented in the illustrative example of Section 4. The proposed adjustment mechanism Algorithm II has some desirable properties.

**Property 2.** (*Convergence*) Let $(D)^{θ}$ be the DPR obtained after *θ* iterations by Algorithm II. The score matrix corresponding to $(D)^{θ}$ is $(S)^{θ}$, while the consistent score matrix corresponding to $(S)^{θ}$ is $(\hat{S})^{θ}$. Then

$$\lim\_{θ\to \infty }CI\left((S)^{θ}\right)=0<τ$$

**Proof.** When $θ\rightarrow \infty $, there will be $(S)^{θ}=(\hat{S})^{θ}$. Then $CI\left((S)^{θ}\right)=0$. Since $0<τ<1$, there will be $\lim\_{θ\to \infty }CI\left((S)^{θ}\right)=0<τ$.

**Property 3.** The second method to generate an adjusted new DPR in Step 3 of Algorithm II can preserve the original judgment of expert to the largest extent.

*3.3. Fusion of DPRs*

When the DPR matrix of each expert has been adjusted to be acceptably consistent, the opinions of different experts are to be fused to generate a collective judgment. In this section, an optimization model which considers the consensus among experts is proposed to combine the DPRs to a collective one.

3.3.1. Similarity measure of DPRs

First, an approach for measuring the distance between two distributions is introduced.

**Definition 8.** [76] Given two distributions such that $d\_{ik}^{r}=\left\{\left(H\_{n},d\_{ik}^{r}\left(H\_{n}\right)\right),n=1,\cdots ,N;\left(Ω,d\_{ik}^{r}\left(Ω\right)\right)\right\}(r=t,t^{'})$. The scores of grades $H\_{n}(n=1,\cdots N)$ are in the range of [0,1], then the dissimilarity measure between $d\_{ik}^{t}$and $d\_{ik}^{t^{'}}$ is constructed as

$$\begin{array}{c}Dissim\left(d\_{ik}^{t},d\_{ik}^{t^{'}}\right)\\=\sum\_{n=1}^{N-1}\sum\_{m=n+1}^{N}dist\left(d\_{ik}^{t△}\left(H\_{n}\right),d\_{ik}^{t^{'}△}\left(H\_{n}\right)\right)∙dist\left(d\_{ik}^{t△}\left(H\_{m}\right),d\_{ik}^{t^{'}△}\left(H\_{m}\right)\right)∙\left(s\left(H\_{m}\right)-s\left(H\_{n}\right)\right)\left(12\right)\end{array}$$

where $d\_{ik}^{r△}\left(H\_{n}\right)=[d\_{ik}^{r}\left(H\_{n}\right),d\_{ik}^{r}\left(H\_{n}\right)+d\_{ik}^{r}\left(Ω\right)] (r=t,t^{'})$. And $dist\left(d\_{ik}^{t△}\left(H\_{n}\right),d\_{ik}^{t^{'}△}\left(H\_{n}\right)\right)$ stands for the distance between interval numbers $d\_{ik}^{t△}\left(H\_{n}\right)$ and $d\_{ik}^{t^{'}△}\left(H\_{n}\right)$ which can be generated by Eq.(10).

Since the scores are bounded in the range of [-1,1], the value of dissimilarity measure will be in the range of [0,2]. Thus, the normalized similarity between $d\_{ik}^{t}$ and $d\_{ik}^{t^{'}}$ is generated as follows:

$$\begin{array}{c} Sim\left(d\_{ik}^{t},d\_{ik}^{t^{'}}\right)=1-\frac{Dissim\left(d\_{ik}^{t},d\_{ik}^{t^{'}}\right)}{2} \left(13\right)\end{array}$$

It is clear that Eq.(13) is in the range of [0,1]. In order to take into account the group consensus, a comprehensive similarity measure of experts’ judgments need to be defined first.

**Definition 9.** Given two complete DPRs $D^{t}$ and $D^{t^{'}}$ constructed by expert $E\_{t}$ and $E\_{t^{'}}$, the similarity measure between their judgments is defined as

$$\begin{array}{c} Sim(D^{t},D^{t^{'}})=\frac{2\sum\_{i=1}^{M-1}\sum\_{k=i+1}^{M}Sim\left(d\_{ik}^{t},d\_{ik}^{t^{'}}\right)}{M\left(M-1\right)} \left(14\right)\end{array}$$

**Property 4.** The similarity measure $Sim(D^{t},D^{t^{'}})$ between DPRs $D^{t}$ and $D^{t^{'}}$ shown in Def. 9 has the following properties:

(ⅰ) Boundary: $0\leq Sim(D^{t},D^{t^{'}})\leq 1$

(ⅱ) Maximum: $Sim(D^{t},D^{t^{'}})=1$ iff $D^{t}=D^{t^{'}}$

(ⅲ) Minimum: $Sim(D^{t},D^{t^{'}})=0$ when $d\_{ik}^{t}=\left\{\left(H\_{1},1\right)\right\}$ and $d\_{ik}^{t^{'}}=\left\{\left(H\_{N},1\right)\right\}$, or $d\_{ik}^{t}=\left\{\left(H\_{N},1\right)\right\}$ and $d\_{ik}^{t^{'}}=\left\{\left(H\_{1},1\right)\right\}$ $(∀i,k=1,\cdots ,M;i\ne k)$

(ⅳ) Symmetry: $Sim(D^{t},D^{t^{'}})=Sim(D^{t^{'}},D^{t})$.

3.3.2. Fusion of DPRs considering consensus

Based on the similarity measure defined in Eq.(14), the DPRs derived from *T* experts can be aggregated into a collective one represented by $D^{C}=(d\_{ik}^{c})\_{M×M}$, where $d\_{ik}^{c}=\{\left(H\_{n},d\_{ik}^{c}\left(H\_{n}\right)\right),n=1,\cdots ,N;\left(Ω,d\_{ik}^{c}\left(Ω\right)\right)\}$. Here, the ER approach is utilized to aggregate the distributions of different experts. Three representative combination rules are recursive ER algorithm [16], analytical ER algorithm [62], and the ER rule [17] considering both the weight and reliability of evidence. They have been applied in lots of domains such as data fusion [77], data classification [78], belief rule-based inference [79], MADM problems [18,19,64] and so on. In this paper, the analytical ER algorithm is applied to combine $D^{t}(t=1,\cdots ,T)$.

In Section 2, Assumption 1 reveals that experts have fully discussed the decision-making issue before giving their final judgments. To a certain extent, the consensus reaching may be faster compared with the procedure where the discussion is arranged after the judgments have been made. Nevertheless, it does not guarantee absolute consensus. In order to further improve the consensus, an appropriate set of expert weights is to be found before aggregating DPRs.

Based on the above illustration, an optimization model is constructed to generate expert weights and the collective DPR. Let $w\_{t}(t=1,\cdots ,T)$ be the weight of the *t*th expert. Then the model is constructed as follows:

**Model Ⅱ.**

$\begin{matrix}MAX \frac{\sum\_{t=1}^{T}Sim(D^{t},D^{C})}{T}\\\begin{matrix}s.t.&\left\{\begin{array}{c}Eqs.(12)-(14)\\0\leq w\_{t}\leq 1, t=1,\cdots ,T\\\sum\_{t=1}^{T}w\_{t}=1\end{array}\right.\end{matrix}\end{matrix}$

where $D^{C}=(d\_{ik}^{c})\_{M×M}$ is formed using the following ER algorithm [62]:

|  |  |
| --- | --- |
| $$m\_{n,t}=w\_{t}∙d\_{ik}^{t}\left(H\_{n}\right),n=1,\cdots ,N;t=1,\cdots ,T$$ | (15) |
| $m\_{Ω,t}=1-\sum\_{n=1}^{N}m\_{n,t}=1-w\_{t}∙\sum\_{n=1}^{N}d\_{ik}^{t}\left(H\_{n}\right),t=1,\cdots ,T$  | (16) |
| $$\overbar{m}\_{Ω,t}=1-w\_{t},t=1,\cdots ,T$$ | (17) |
| $\tilde{m}\_{Ω,t}=w\_{t}∙d\_{ik}^{t}\left(Ω\right)=w\_{t}∙\left(1-\sum\_{n=1}^{N}d\_{ik}^{t}\left(H\_{n}\right)\right),t=1,\cdots ,T$  | (18) |
| with $m\_{Ω,t}=\overbar{m}\_{Ω,t}+\tilde{m}\_{Ω,t}$ | (19) |
| $\left\{H\_{n}\right\}:m\_{n}=K∙\left[\prod\_{t=1}^{T}\left(m\_{n,t}+m\_{Ω,t}\right)-\prod\_{t=1}^{T}m\_{Ω,t}\right],n=1,\cdots ,N$  | (20) |
| $\left\{Ω\right\}:\tilde{m}\_{Ω}=K∙\left[\prod\_{t=1}^{T}m\_{Ω,t}-\prod\_{t=1}^{T}\overbar{m}\_{Ω,t}\right]$  | (21) |
| $\left\{Ω\right\}:\overbar{m}\_{Ω}=K∙\left[\prod\_{t=1}^{T}\overbar{m}\_{Ω,t}\right]$  | (22) |
| $K=\left[\sum\_{n=1}^{N}\prod\_{t=1}^{T}\left(m\_{n,t}+m\_{Ω,t}\right)-\left(N-1\right)∙\prod\_{t=1}^{T}m\_{Ω,t}\right]^{-1}$  | (23) |

Then the collective DPR is generated as:

|  |  |
| --- | --- |
| $$\left\{H\_{n}\right\}:d\_{ik}^{c}\left(H\_{n}\right)=\frac{m\_{n}}{1-\overbar{m}\_{Ω}},n=1,\cdots ,N$$ | (24) |
| $$\left\{Ω\right\}:d\_{ik}^{c}\left(Ω\right)=\frac{\tilde{m}\_{Ω}}{1-\overbar{m}\_{Ω}}$$ | (25) |

In Model II, $Sim\left(D^{t},D^{C}\right)(t=1,\cdots ,T)$ can be calculated by Eqs.(12)-(14) where $D^{t^{'}}$ is replaced by $D^{C}$. The objective function represents the average similarity between the DPR matrix of each expert and the aggregated opinion. $w\_{t}(t=1,\cdots ,T)$ are the *T* variables to be optimized for the attainment of a maximum consensus status based on the adjusted consistency DPR matrices.

**4. Illustrative example**

The evaluation of scholarship for graduate students in China is a typical GDM problem, in which multiple experts comprehensively evaluate a group of students from scientific research achievements, competition awards, and other aspects in the previous academic year, thus to select those who will be awarded the scholarship. In this section, a GDM problem of evaluating scholarship is investigated to demonstrate the application of the proposed method which considers consistency of DPR matrix and consensus among experts simultaneously.

*4.1. Description of the problem of evaluating scholarship*

In the case study, four teachers from graduate school are invited as experts for the evaluation of scholarship, denoted by $E=\{E\_{1},E\_{2},E\_{3},E\_{4}\}$. With the help of scholarship defense assistant who takes the role of a facilitator, experts need to compare five students $A=\{A\_{1},A\_{2},A\_{3},A\_{4},A\_{5}\}$ who compete for scholarships.

Table 5 shows the scholarship evaluation criteria system derived from the graduate school scholarship evaluation announcement, which is provided to experts for reference.

**Table 5** Criteria system for the evaluation of scholarship

|  |  |  |  |
| --- | --- | --- | --- |
| First-level criteria | Second-level criteria | Third-level criteria | Remarks |
| Scientific research | Research direction and topics | Innovation | - |
| Theoretical significance | - |
| Application value | - |
| Project research | Participation | - |
| Contribution | - |
| Research methods and achievements | Paper | Assess from multiple aspects such as the impact factor of the journal, author’s order, number of papers and so on |
| Patent | Assess from multiple aspects such as review progress, authorization, number of patents and so on |
| Reward and social work | - | - | Win awards and participate in public welfare activities, scientific and technological services, laboratory management and construction, student work, etc |

The brief introduction of the five students is shown in Table 6. It can be seen that these five students each has his/her own advantage in specific field such as journal or conference paper, patent, competition reward, and services to the college. For example, when comparing *A*3 with *A*5, the paper quality of *A*3 is higher than that of *A*5 with respect to published papers. However, *A*5 has completed a patent while *A*3 does not have. Besides, *A*3 has no competition award, whereas *A*5 has the grand prize of provincial competition. In other words, *A*3 and *A*5 have their own advantages and disadvantages in different aspects. Therefore, the GDM problem contains uncertainty, complexity and fuzziness.

**Table 6** Brief list of achievements by students

|  |  |  |
| --- | --- | --- |
|  | Scientific research situation | Reward situation |
| *A*1 | 1.A SCI paper has been published on ‘Applied Energy’ (Impact factor is 8.5, second author)2.Published an English conference paper (Elsevier: Energy Procedia)3.One patent has been completed (Substantive examination stage)4.Participated in a major provincial science and technology project | -- |
| *A*2 | 1.Published a paper at the 17th management science and Engineering Forum2.One patent has been completed (Accepted stage)3.Participated in a research project of supervisor | 1.Won a corporate sponsored scholarship2.Won the honors of ‘Excellent Graduate Association Chairman’, ‘Excellent Student Cadre’, ‘Excellent League Member’ |
| *A*3 | 1.Published a paper on CSCD core journal ‘Computer Application’ (first author)2.A paper has been submitted for review in ‘IEEE Access’ (Impact factor is 3.7, fourth author)3.Participated in three projects | -- |
| *A*4 | 1.Six patents completed (substantive examination, second inventor)2.A paper has been submitted for review in ‘Engineering Optimization’ (Impact factor is 1.8)3.Participated in writing a book | 1.Won the Special Prize in National college student Challenge Cup provincial competition2.Won the gold prize in the Creative Group of the ‘Internet + Innovation and Entrepreneurship Competition’ |
| *A*5 | 1.Published a paper at the 17th management science and Engineering Forum2.One patent has been completed (Substantive examination stage)3.Participated in two projects | Won the Special Award of Anhui Province in the 8th ‘Challenge Cup’ Anhui College Students’ Extracurricular Academic Science and Technology Works Competition |

The linguistic grades are set as $Ω=\{H\_{1},H\_{2},H\_{3},H\_{4}, H\_{5},H\_{6},H\_{7}\}$ = {*absolutely worse, worse, slightly worse, indifferent, slightly better, better, absolutely better*}. After consultation, the scores of grades are set as *s* = {*s*(*H*1), *s*(*H*2), *s*(*H*3), *s*(*H*4), *s*(*H*5), *s*(*H*6), *s*(*H*7)} = {-1, -0.7, -0.3, 0, 0.3, 0.7, 1} and the consistency threshold is assumed to be *τ*=0.1. Thus, Step 1 is completed.

*4.2. Generating the DPRs from experts*

According to the above criteria system, experts compare candidate students in pairs to form a preliminary impression. Then, they fully discuss and exchange their views on the advantages and disadvantages of each student. After discussion, experts give their DPRs $\{D^{E\_{1}},D^{E\_{2}},D^{E\_{3}},D^{E\_{4}}\}$ independently. The DPR given by expert $E\_{1}$ is shown in Table 7, and the DPRs given by other three experts are presented in the Supplementary Materials. As such, Step 2 is completed.

*4.3. Consistency measurement and automatic adjustment*

First, calculating the original score matrices $S^{t}\left(t=1,2,3,4\right)$ of DPRs $D^{t}\left(t=1,2,3,4\right)$ by Eqs.(1) and (2). Second, the consistent score matrices $\hat{S}^{t}(t=1,2,3,4)$ can be generated by Algorithm Ⅰ. Third, the consistency index of each DPR matrix is to be measured by Eq. (11) respectively. If the consistency threshold is not reached, the DPR is automatically adjusted by algorithm Ⅱ until it satisfies the consistency condition. The specific process of calculation is shown in Supplementary Materials. Thus, Steps 3 and 4 are completed.

**Table 7** DPR given by expert *E*1--$D^{E\_{1}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*5,0.1),(*H*6,0.3),(*H*7,0.6)} | {(*H*4,0.2),(*H*6,0.4),(*H*7,0.3),(*Ω*,0.1)} | {(*H*3,0.3),(*H*4,0.2),(*H*6,0.2), (*H*7,0.3)} | {(*H*4,0.2),(*H*6,0.3),(*H*7,0.5)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.4),(*H*3,0.3),(*H*4,0.2), (*Ω*,0.1)} | {(*H*1,0.3),(*H*2,0.4),(*H*3,0.2),(*H*4,0.1)} | {(*H*3,0.4),(*H*4,0.5), (*Ω*,0.1)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*3,0.3),(*H*4,0.3),(*H*5,0.2),(*Ω*,0.2)} | {(*H*3,0.1),(*H*4,0.2),(*H*5,0.5),(*Ω*,0.2)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*4,0.2),(*H*5,0.3),(*H*6,0.2),(*Ω*,0.3)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

*4.4. Generation of a solution*

By solving Model Ⅱ, we get the experts’ weights and collective DPR under the condition of maximum consensus among experts. The generated weights of four experts are *w*1=0.23282, *w*2=0.27096, *w*3=0.27833, *w*4=0.21789.

**Table 8** The collective DPR matrix--$D^{C}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*1,0.05054),(*H*3,0.02438),(*H*4,0.07673),(*H*5,0.05971),(*H*6,0.22740),(*H*7,0.51248),(*Ω*,0.04875)} | {(*H*3,0.11205),(*H*4,0.16313),(*H*5,0.18417),(*H*6,0.20983),(*H*7,0.21395),(*Ω*,0.11687)} | {(*H*1,0.10535),(*H*2,0.05594),(*H*3,0.09784),(*H*4,0.05072),(*H*5,0.05984),(*H*6,0.22449),(*H*7,0.12824),(*Ω*,0.27758)} | {(*H*1,0.05651),(*H*2,0.02184),(*H*3,0.01781),(*H*4,0.12107),(*H*6,0.17997)(*H*7,0.36142),(*Ω*,0.24139)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.24765),(*H*3,0.23376),(*H*4,0.23913),(*H*5,0.02775),(*H*6,0.05356),(*H*7,0.02775),(*Ω*,0.17040)} | {(*H*1,0.28751),(*H*2,0.28728),(*H*3,0.07537),(*H*4,0.02379),(*H*5,0.02619),(*H*6,0.06744),(*Ω*,0.23242)} | {(*H*2,0.08191),(*H*3,0.21397),(*H*4,0.40870),(*H*5,0.02637),(*H*6,0.05458),(*Ω*,0.21448)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*1,0.08225),(*H*2,0.22468),(*H*3,0.13193),(*H*4,0.13193),(*H*5,0.06921)(*H*6,0.11531),(*Ω*,0.24467)} | {(*H*2,0.11109),(*H*3,0.20594),(*H*4,0.22521),(*H*5,0.22977),(*H*6,0.08428),(*Ω*,0.14370)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*3,0.08117),(*H*4,0.15970),(*H*5,0.38149),(*H*6,0.15783),(*Ω*,0.21981)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

The score matrix and possibility matrix of the collective DPR matrix can then be computed by Eqs.(1), (2) and (5)-(8) as shown in Tables 9 and 10.

**Table 9** Score matrix of the collective DPR matrix--$S^{C}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.58298,0.68047] | [0.26559,0.49933] | [-0.14812,0.40705] | [0.16887,0.65165] |
| *A*2 | [-0.68047,-0.58298] | [0,0] | [-0.34031,0.00049] | [-0.68857,-0.22374] | [-0.28989,0.13906] |
| *A*3 | [-0.49933,-0.26559] | [-0.00049,0.34031] | [0,0] | [-0.42230,0.06704] | [-0.15532,0.13209] |
| *A*4 | [-0.40705,0.14812] | [0.22374,0.68857] | [-0.06704,0.42230] | [0,0] | [-0.019233,0.42039] |
| *A*5 | [-0.65165,-0.16887] | [-0.13906,0.28989] | [-0.13209,0.15532] | [-0.42039,0.01923] | [0,0] |

**Table 10** Possibility matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | 0.5 | 1 | 1 | 0.73320 | 1 |
| *A*2 | 0 | 0.5 | 0.00143 | 0 | 0.32419 |
| *A*3 | 0 | 0.99857 | 0.5 | 0.13700 | 0.45959 |
| *A*4 | 0.26680 | 1 | 0.86300 | 0.5 | 0.95625 |
| *A*5 | 0 | 0.67581 | 0.54041 | 0.04375 | 0.5 |

According to Table 10, we can generate a ranking order of the five students with possibility degrees such that $\begin{matrix} \\A\_{1}\end{matrix}\begin{matrix}0.73320\\≻\end{matrix}\begin{matrix} \\A\_{4}\end{matrix}\begin{matrix}0.95625\\≻\end{matrix}\begin{matrix} \\A\_{5}\end{matrix}\begin{matrix}0.54041\\≻\end{matrix}\begin{matrix} \\A\_{3}\end{matrix}\begin{matrix}0.99857\\≻\end{matrix}\begin{matrix} \\A\_{2}\end{matrix}$, which is a solution to the problem.

*4.5. Discussion and comparative analysis*

In order to illustrate the efficiency of identification rules in Algorithm Ⅱ, we select the judgment of expert *E*4 and compare the adjustment process of using Algorithm Ⅱ with the adjustment process of randomly selecting adjustment elements.



**Fig. 2.** Reduction speed of CI by using Algorithm Ⅱ and random selection

As shown in Fig.2, the consistency index of the original DPR given by expert *E*4 is 0.203452. When using the identification rules of Algorithm Ⅱ to select the adjustment element, it took two iterations to reduce the consistency index to be less than 0.1. Comparatively, when the adjustment element is randomly selected, it took six iterations to reduce the consistency index to be less than 0.1. Obviously, the method of selecting adjustment elements in Algorithm Ⅱ is valid.

There are many kinds of linguistic-based preference relations similar to the concept of DPR, such as hesitant linguistic preference relation (HLPR), probabilistic preference relation (PLPR), distribution linguistic preference relation (DLPR), DLPR with incomplete symbolic proportions (DLPR-I), and so on. In the following, a comparative analysis of our method against other kinds of linguistic preference relations is conducted.

(ⅰ) Zhu et al. [39] proposed the concept of HLPR as an extension of LPR on the basis of hesitant fuzzy linguistic term set (HFLTS) [24]. When HFLTS is used to model the preference, it is assumed that the DM hesitates on several possible linguistic terms and all the possible terms are of equal importance degree. For instance, suppose the linguistic term set is the same as *Ω* of the above example, the preference could be $\{H\_{4},H\_{5},H\_{6}\}$. Obviously, HFLTS can be regarded as a special case of BD, i.e., every linguistic term in an HFLTS is assigned with the same belief degree and global ignorance is always zero. In terms of consistency, they first give the operations of HFLTS which are LPR based essentially and then give the consistency definition based on these operations and transitivity. Consistency index is also defined by calculating deviation between HLPR and its consistent HLPR. As for HLPR with unacceptable consistency, they provide an automatic optimization method and a feedback optimization method respectively to improve the consistency of HLPR.

(ⅱ) To modelling the problems in reality better, Zhang et al. [27] proposed PLPR based on probabilistic linguistic term set (PLTS) [23] which allows DMs to use several linguistic terms whose probabilities can be different. The PLTS can address global ignorance, i.e., the probability sum of all terms is less than one. However, the global ignorance will be proportionally allocated to possible linguistic terms in the normalization process. In contrast, the global ignorance is always preserved in our method no matter in the process of consistency measurement or DPR aggregation, which will preserve the original information to a high extent. Zhang et al. [27] defined the aggregation operation of PLTSs by adding subscripts of linguistic terms and multiplying corresponding probabilities. Similar to HLPR, they defined the additive consistency of PLPR based on the defined operation and transitivity. The consistency index is also defined in their methods by measuring the distance of normalized PLPR and its consistent PLPR. And an automatic optimization algorithm is proposed to improve the consistency of PLPR with unacceptable consistency by iterating part of the original PLPR. Since the subject of his method is not GDM, there is no consensus method for comparison.

(ⅲ) Zhang et al. [40] introduced the concept of DLPR based on LPR. However, global ignorance was not considered because the summation of symbolic proportions that assigned to all linguistic terms equals to 1, which will make the ignorance of DMs in reality cannot be fully reflected. Regarding the measurement of consistency, the distributions on linguistic~~s~~ terms were transformed into expected values. Because the ignorance is not considered in the model, the elements in the expectation matrix (EM) transformed from DLPR are exact numerical values. Then the transfer property of the EM was used as the consistency criterion of DLPR. Specifically, DLPR is considered to be additive consistent if EM is additive transitive, while DLPR is considered to be multiplicative consistent provided that EM is multiplicative transitive. However, consistency measurement method and acceptable consistency are not fully discussed, nor did the adjustment mechanism provided that DLPR is inconsistent. Besides, the aggregation method was based on WA and OWA operators which are conventional aggregation methods. Regarding the degree of consensus among experts, a consensus measurement was proposed based on the distance between experts, followed by a consensus improvement algorithm when consensus threshold is not reached. The algorithm increased the level of consensus by identifying the expert who contributes less to the consensus, and improving their judgment by using weighted average of the expert’s DLPR and remaining experts’ DLPRs.

(ⅳ) Tang et al. [41] proposed the concept of distribution LPR with incomplete symbolic proportions (DLPR-I). As global ignorance is considered in this model, the model is similar to our proposed model to some extent. In their approach, linguistic terms are considered to have different meanings for different DMs. They proposed several numerical scale computation models to personalize numerical scales for each DM. As for the measurement of consistency, they calculated consistency level by expectation-based numerical preference relation (EBNPR) matrix which is transformed from DLPR-I using numerical scales and the conversion method is similar to that in DPR scheme from DPR to the score matrix. Because global ignorance is taken into account, the elements in its EBNPR matrix are also interval numbers. Instead of calculating the consistency measure, their method established an optimization model based on consistency to obtain the range of linguistic term values under maximum consistency for the purpose of personalizing the linguistic value of each DM. Therefore, they provide a research idea different from the conventional research method on the consistency guarantee of the preference relationship matrix. Nevertheless, in aggregating the preference relationship matrices of multiple experts, they used the conventional aggregation method based on WA and OWA operators. However, the degree of consensus among DMs is not fully considered.

**Table 11** Comparison of the proposed method against typical linguistic-based preference relations

|  |  |
| --- | --- |
| Considerations | Methods |
| Zhu & Xu[39] |  Zhang et al.[27] | Zhang et al.[40] | Tang et al.[41] | Proposed |
| Preference representation | HLPR | PLPR | DLPR | DLPR-I | Complete DPR |
| Global ignorance | × | √ | × | √ | √ |
| Consistency measure | √ | √ | √ | √ | √ |
| Consistency adjustment | √ | √ | × | × | √ |
| Aggregation method | LPR based | Linguistic terms subscript multiplying probabilities | WA and OWA based | WA and OWA based | ER algorithm |
| Consensus | -- | -- | √ | × | √ |

Based on the above comparative analysis, it can be seen that although the above four linguistic-based preference relationships have considered distributed reviews on the linguistic term set, the aggregation methods they utilized are based on linear weighting, which are essentially different from the ER combination rule. On the other hand, although both PLPR and DLPR-I take into account incompleteness/ignorance, PLPR evenly distributes ignorance to the potential language terms, whereas DLPR-I assigns ignorance to the subset of language term set. Comparatively, DPR is theoretically different from the other four methods in the treatment of ignorance because it preserves the original ignorance included in experts’ judgment to a high extent.

Table 11 shows the comparison of the proposed complete DPR method against the above four linguistic-based preference relations from the aspect of preference representation, consideration of ignorance, consistency measure and adjustment, aggregation method and consensus.

**5. Concluding remarks**

This paper investigates the GDM problem based on complete DPRs and considers the consistency of individual DPR and the opinion fusion of multiple DPRs given by experts simultaneously. In the decision-making problems which are of great importance or necessity of adequate comparison, the model of using complete DPR will be more appropriate than adjacent DPR. A consistency measurement of complete DPR is further proposed and an algorithm is designed to adjust DPR with unacceptable consistency. The algorithm includes a series of identification and adjustment rules. Compared with the random selection of adjustment elements, the efficiency of identification rule is highlighted. Furthermore, the adjustment rule provides two choices for different decision-making needs: one is a simple and fast process, the other one is the adjustment by solving optimization model to make the adjusted DPR closest to the original judgment of experts. After all the DPRs are acceptable consistent, a nonlinear programming model is constructed to optimize the consensus among experts in which the ER approach is employed to aggregate multiple DPRs to a collective one. The whole decision-making process is illustrated through a GDM problem of evaluating scholarship, which demonstrates the validity and effectiveness of the proposed GDM method. Finally, we compare our method with some typical LPR based methods in detail in order to highlight the characteristics and strengths of the proposed method.

For real decision making problems with a relatively large number of alternatives, DMs may neither compare each pair of alternatives nor the adjacent alternatives. That is to say, DMs are more inclined to give semi-DPR between the complete DPR and the adjacent DPR in such circumstances. Therefore, correlated research about semi-DPR will be conducted in the future. In addition, how to use DPR model in large-scale GDM problems considering social network is also the future research direction.

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**References**

[1] W.D. Cook, M. Kress, Ordinal ranking with intensity of preference, Manage. Sci. 31 (1985) 26–32.

[2] D.S. Hochbaum, A. Levin, Methodologies and algorithms for group-rankings decision, Manage. Sci. 52 (2006) 1394–1408.

[3] D.A. Carrera, R. V. Mayorga, W. Peng, A soft computing approach for group decision making: A supply chain management application, Appl. Soft Comput. J. 91 (2020) 106201.

[4] S. Boroushaki, J. Malczewski, Measuring consensus for collaborative decision-making: A GIS-based approach, Comput. Environ. Urban Syst. 34 (2010) 322–332.

[5] J.W. Gao, F.J. Guo, Z.Y. Ma, X. Huang, X.Z. Li, Multi-criteria group decision-making framework for offshore wind farm site selection based on the intuitionistic linguistic aggregation operators, Energy. 204 (2020) 117899.

[6] E. Triantaphyllou, F.J. Hou, J. Yanase, Analysis of the final ranking decisions made by experts after a consensus has been reached in group decision making, Gr. Decis. Negot. 29 (2020) 271–291.

[7] W.H. Tsai, C.C. Yang, J. Der Leu, Y.F. Lee, C.H. Yang, An integrated group decision making support model for corporate financing decisions, Gr. Decis. Negot. 22 (2013) 1103–1127.

[8] R. Verma, J.M. Merigó, Multiple attribute group decision making based on 2-dimension linguistic intuitionistic fuzzy aggregation operators, Soft Comput. (2020). https://doi.org/10.1007/s00500-020-05026-z

[9] D. Cheng, F.X. Cheng, Z.L. Zhou, Y. Wu, Reaching a minimum adjustment consensus in social network group decision-making, Inf. Fusion. 59 (2020) 30–43.

[10] S. Abootalebi, A. Hadi-Vencheh, A. Jamshidi, An improvement to determining expert weights in group multiple attribute decision making problem, Gr. Decis. Negot. 27 (2018) 215–221.

[11] Ş. Özlü, F. Karaaslan, Some distance measures for type 2 hesitant fuzzy sets and their applications to multi-criteria group decision-making problems, Soft Comput. 24 (2020) 9965–9980.

[12] C. Fu, S.L. Yang, The group consensus based evidential reasoning approach for multiple attributive group decision analysis, Eur. J. Oper. Res. 206 (2010) 601–608.

[13] M. Mohammadi, J. Rezaei, Bayesian best-worst method: A probabilistic group decision making model, Omega. 96 (2019) 102075.

[14] M. Zhou, X.B. Liu, Y.W. Chen, J.B. Yang, Evidential reasoning rule for MADM with both weights and reliabilities in group decision making, Knowledge-Based Syst. 143 (2018) 142–161.

[15] X.B. Liu, J. Pei, L. Liu, H. Cheng, M. Zhou, P.M. Pardalos, Optimization and management in manufacturing engineering-resource collaborative optimization and management through the internet of things, Springer, 2017.

[16] J.B. Yang, D.L. Xu, On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty, IEEE Trans. Syst. Man, Cybern. Part ASystems Humans. 32 (2002) 289–304.

[17] J.B. Yang, D.L. Xu, Evidential reasoning rule for evidence combination, Artif. Intell. 205 (2013) 1–29.

[18] M. Zhou, X.B. Liu, Y.W. Chen, X.F. Qian, J.B. Yang, J. Wu, Assignment of attribute weights with belief distributions for MADM under uncertainties, Knowledge-Based Syst. 189 (2020) 105110.

[19] M. Zhou, Y.W. Chen, X.B. Liu, B.Y. Cheng, J.B. Yang, Weight assignment method for multiple attribute decision making with dissimilarity and conflict of belief distributions, Comput. Ind. Eng. 147 (2020) 106648.

[20] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.

[21] Z.L. Ren, Z.S. Xu, H. Wang, Normal wiggly hesitant fuzzy sets and their application to environmental quality evaluation, Knowledge-Based Syst. 159 (2018) 286–297.

[22] H. Wang, Z.S. Xu, W. Pedrycz, An overview on the roles of fuzzy set techniques in big data processing: Trends, challenges and opportunities, Knowledge-Based Syst. 118 (2017) 15–30.

[23] Q. Pang, H. Wang, Z.S. Xu, Probabilistic linguistic term sets in multi-attribute group decision making, Inf. Sci. (Ny). 369 (2016) 128–143.

[24] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2012) 109–119.

[25] M. Durand, I. Truck, A new proposal to deal with hesitant linguistic expressions on preference assessments, Inf. Fusion. 41 (2018) 176–181.

[26] J. Montserrat-Adell, Z.S. Xu, X.J. Gou, N. Agell, Free double hierarchy hesitant fuzzy linguistic term sets: An application on ranking alternatives in GDM, Inf. Fusion. 47 (2019) 45–59.

[27] Y.X. Zhang, Z.S. Xu, H. Wang, H.C. Liao, Consistency-based risk assessment with probabilistic linguistic preference relation, Appl. Soft Comput. J. 49 (2016) 817–833.

[28] W. Pedrycz, M.L. Song, Analytic hierarchy process (AHP) in group decision making and its optimization with an allocation of information granularity, IEEE Trans. Fuzzy Syst. 19 (2011) 527–539.

[29] T.L. Saaty, Fundamental of Decision Making and Priority Theory with the AHP, RWS Publications, Pittsburgh, 1994.

[30] Z.J. Wang, J. Lin, F. Liu, Axiomatic property based consistency analysis and decision making with interval multiplicative reciprocal preference relations, Inf. Sci. (Ny). 491 (2019) 109–137.

[31] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets Syst. 12 (1984) 117–131.

[32] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating multiplicative preference relations in a multipurpose decision making model based on fuzzy preference relations, Fuzzy Sets Syst. 97 (1998) 33–48.

[33] Z.S. Xu, A survey of preference relations, Int. J. Gen. Syst. 36 (2007) 179–203.

[34] H.M. Zhang, Revisiting multiplicative consistency of interval fuzzy preference relation, Comput. Ind. Eng. 132 (2019) 325–332.

[35] E. Szmidt, J. Kacprzyk, A consensus-reaching process under intuitionistic fuzzy preference relations, Int. J. Intell. Syst. 18 (2003) 837–852.

[36] H.C. Liao, Z.S. Xu, Automatic procedures for group decision making with intuitionistic fuzzy preference relations, J. Intell. Fuzzy Syst. 27 (2014) 2341–2353.

[37] Z.W. Gong, N. Zhang, F. Chiclana, The optimization ordering model for intuitionistic fuzzy preference relations with utility functions, Knowledge-Based Syst. 162 (2018) 174–184.

[38] F.Y. Meng, J. Tang, P. Wang, X.H. Chen, A programming-based algorithm for interval-valued intuitionistic fuzzy group decision making, Knowledge-Based Syst. 144 (2018) 122–143.

[39] B. Zhu, Z.S. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, IEEE Trans. Fuzzy Syst. 22 (2014) 35–45.

[40] G.Q. Zhang, Y.C. Dong, Y.F. Xu, Consistency and consensus measures for linguistic preference relations based on distribution assessments, Inf. Fusion. 17 (2014) 46–55.

[41] X.A. Tang, Q. Zhang, Z.L. Peng, W. Pedrycz, S. Yang, Distribution linguistic preference relations with incomplete symbolic proportions for group decision making, Appl. Soft Comput. J. 88 (2020) 106005.

[42] C. Fu, D.L. Xu, S.L. Yang, Distributed preference relations for multiple attribute decision analysis, J. Oper. Res. Soc. 67 (2016) 457–473.

[43] W.J. Chang, C.P. Sun, X. Hu, Multiple attribute decision making method based on distributed preference relations and its application Chang, Appl. Ｒesearch Comput. 34 (2017) 3693-3697,3707.

[44] X. Hu, W.J. Chang, C.P. Sun, Multi-attribute decision making method based on comparison possibility degree, J. Comput. Appl. 37 (2017) 2223-2228,2286.

[45] Y. Liu, C. Fu, M. Xue, W.J. Chang, S.L. Yang, Interval-valued distributed preference relation and its application to group decision making, PLoS One. 13 (2018) 1–25.

[46] C. Fu, W.J. Chang, M. Xue, S.L. Yang, Multiple criteria group decision making with belief distributions and distributed preference relations, Eur. J. Oper. Res. 273 (2019) 623–633.

[47] S. Genç, F.E. Boran, D. Akay, Z.S. Xu, Interval multiplicative transitivity for consistency, missing values and priority weights of interval fuzzy preference relations, Inf. Sci. 180 (2010) 4877–4891.

[48] F.Y. Meng, S.M. Chen, J. Tang, Group decision making based on acceptable multiplicative consistency of hesitant fuzzy preference relations, Inf. Sci. (Ny). 524 (2020) 77–96.

[49] S.P. Wan, L.G. Zhong, J.Y. Dong, A New Method for Group Decision Making with Hesitant Fuzzy Preference Relations Based on Multiplicative Consistency, IEEE Trans. Fuzzy Syst. 28 (2020) 1449–1463.

[50] E. Herrera-Viedma, F. Herrera, F. Chiclana, M. Luque, Some issues on consistency of fuzzy preference relations, Eur. J. Oper. Res. 154 (2004) 98–109.

[51] J. Krejčí, On additive consistency of interval fuzzy preference relations, Comput. Ind. Eng. 107 (2017) 128–140.

[52] S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets Syst. 1 (1978) 155–167.

[53] A.A. Al Salem, A. Awasthi, Investigating rank reversal in reciprocal fuzzy preference relation based on additive consistency: Causes and solutions, Comput. Ind. Eng. 115 (2018) 573–581.

[54] F. Liu, Z.L. Liu, Y.H. Wu, A group decision making model based on triangular fuzzy additive reciprocal matrices with additive approximation-consistency, Appl. Soft Comput. 65 (2018) 349–359.

[55] T.L. Saaty, Multicriteria Decision Making: The Analytic Hierarchy Process, McGraw-Hill, New York, 1980.

[56] J. Kacprzyk, Group desicion making with a fuzzy linguistic majority, Fuzzy Sets Syst. 18 (1986) 105–118.

[57] J.A. Morente-Molinera, X. Wu, A. Morfeq, R. Al-Hmouz, E. Herrera-Viedma, A novel multi-criteria group decision-making method for heterogeneous and dynamic contexts using multi-granular fuzzy linguistic modelling and consensus measures, Inf. Fusion. 53 (2020) 240–250.

[58] J.M. Tapia García, M.J. Del Moral, M.A. Martínez, E. Herrera-Viedma, A consensus model for group decision making problems with linguistic interval fuzzy preference relations, Expert Syst. Appl. 39 (2012) 10022–10030.

[59] H. Garg, S.M. Chen, Multiattribute group decision making based on neutrality aggregation operators of q-rung orthopair fuzzy sets, Inf. Sci. 517 (2020) 427–447.

[60] R.R. Yager, Families of OWA operators, Fuzzy Sets Syst. 59 (1993) 125–148.

[61] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004) 19–30.

[62] Y.M. Wang, J.B. Yang, D.L. Xu, Environmental impact assessment using the evidential reasoning approach, Eur. J. Oper. Res. 174 (2006) 1885–1913.

[63] D.L. Xu, An introduction and survey of the evidential reasoning approach for multiple criteria decision analysis, Ann. Oper. Res. 195 (2012) 163–187.

[64] M. Zhou, X.B. Liu, J.B. Yang, Y.W. Chen, J. Wu, Evidential reasoning approach with multiple kinds of attributes and entropy-based weight assignment, Knowledge-Based Syst. 163 (2019) 358–375.

[65] J. Wu, L.F. Dai, F. Chiclana, H. Fujita, E. Herrera-Viedma, A minimum adjustment cost feedback mechanism based consensus model for group decision making under social network with distributed linguistic trust, Inf. Fusion. 41 (2018) 232–242.

[66] E. Herrera-Viedma, F. Herrera, F. Chiclana, A consensus model for multiperson decision making with different preference structures, IEEE Trans. Syst. Man, Cybern. Part ASystems Humans. 32 (2002) 394–402.

[67] C. Fu, S.L. Yang, An evidential reasoning based consensus model for multiple attribute group decision analysis problems with interval-valued group consensus requirements, Eur. J. Oper. Res. 223 (2012) 167–176.

[68] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Application, Academic Press, 1980.

[69] F. Chiclana, E. Herrera-Viedma, F. Alonso, S. Herrera, Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity, IEEE Trans. Fuzzy Syst. 17 (2009) 14–23.

[70] X. Li, S.L. Zhang, M. Zhang, H. Liu, Rank of interval numbers based on a new distance measure, J. Xihua Univ. (Natural Sci.) 27 (2008) 87–90.

[71] P. Amenta, A. Lucadamo, G. Marcarelli, On the transitivity and consistency approximated thresholds of some consistency indices for pairwise comparison matrices, Inf. Sci. 507 (2020) 274–287.

[72] H.F. Liu, Z.S. Xu, H.C. Liao, The multiplicative consistency index of hesitant fuzzy preference relation, IEEE Trans. Fuzzy Syst. 24 (2016) 82–93.

[73] S. Siraj, L. Mikhailov, J.A. Keane, Contribution of individual judgments toward inconsistency in pairwise comparisons, Eur. J. Oper. Res. 242 (2015) 557–567.

[74] Z.S. Xu, H.C. Liao, Intuitionistic fuzzy analytic hierarchy process, IEEE Trans. Fuzzy Syst. 22 (2014) 749–761.

[75] Y. Yang, X.X. Wang, Z.S. Xu, The multiplicative consistency threshold of intuitionistic fuzzy preference relation, Inf. Sci. 477 (2019) 349–368.

[76] C. Fu, W.J. Chang, D.L. Xu, S.L. Yang, An evidential reasoning approach based on criterion reliability and solution reliability, Comput. Ind. Eng. 128 (2019) 401–417.

[77] F.Y. Xiao, Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy, Inf. Fusion. 46 (2019) 23–32.

[78] X. Bin Xu, J. Zheng, J.B. Yang, D.L. Xu, Y.W. Chen, Data classification using evidence reasoning rule, Knowledge-Based Syst. 116 (2017) 144–151.

[79] L.L. Chang, Z.J. Zhou, Y.W. Chen, T.J. Liao, Y. Hu, L.H. Yang, Belief rule base structure and parameter joint optimization under disjunctive assumption for nonlinear complex system modeling, IEEE Trans. Syst. Man, Cybern. Syst. 48 (2018) 1542–1554.

**Supplementary Material**

**Supplementary A. DPR matrices given by other three experts in Section 4**

**Table A.1** DPR matrix given by expert *E*2--$D^{E\_{2}}$$D^{E\_{2}}$.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*3,0.1),(*H*4,0.1),(*H*7,0.7),(*Ω*,0.1)} | {(*H*3,0.2),(*H*4,0.2),(*H*7,0.5),(*Ω*,0.1)} | {(*H*1,0.4),(*H*3,0.2),(*H*6,0.1),(*Ω*,0.3)} | {(*H*2,0.2),(*H*4,0.3),(*H*7,0.4),(*Ω*,0.1)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.2),(*H*3,0.2),(*H*4,0.3),(*H*6,0.2),(*Ω*,0.1)} | {(*H*1,0.5),(*H*2,0.3),(*H*6,0.1),(*Ω*,0.1)} | {(*H*2,0.2),(*H*3,0.2),(*H*4,0.3),(*H*5,0.1),(*Ω*,0.2)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*1,0.3),(*H*2,0.2),(*H*6,0.2),(*Ω*,0.3)} | {(*H*2,0.3),(*H*3,0.2),(*H*4,0.3),(*H*6,0.1),(*Ω*,0.1)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*3,0.1),(*H*4,0.2),(*H*5,0.1),(*H*6,0.4),(*Ω*,0.2)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

**Table A.2** DPR matrix given by expert *E*3--$D^{E\_{3}}$.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*1,0.2),(*H*4,0.2),(*H*7,0.5),(*Ω*,0.1)} | {(*H*3,0.2),(*H*4,0.2),(*H*6,0.4),(*Ω*,0.2)} | {(*H*1,0.1),(*H*2,0.2),(*H*6,0.3),(*Ω*,0.4)} | {(*H*1,0.2),(*H*3,0.1),(*H*4,0.1),(*H*7,0.5),(*Ω*,0.1)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.3),(*H*3,0.2),(*H*4,0.2),(*H*5,0.1),(*H*7,0.1),(*Ω*,0.1)} | {(*H*2,0.3),(*H*3,0.1),(*H*5,0.1),(*H*6,0.1),(*Ω*,0.4)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.3),(*H*6,0.2),(*Ω*,0.2)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.2),(*H*6,0.2),(*Ω*,0.3)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.3),(*H*5,0.1),(*H*6,0.2),(*Ω*,0.1)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*3,0.2),(*H*4,0.2),(*H*5,0.2),(*Ω*,0.4)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

**Table A.3** DPR matrix given by expert *E*4--$D^{E\_{4}}$.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*5,0.2),(*H*6,0.8)} | {(*H*5,0.9), (*Ω*,0.1)} | {(*H*3,0.1),(*H*5,0.3), (*Ω*,0.6)} | {(*H*5,0.2), (*H*6,0.8)} |
| *A*2 | - | {(*H*4,1)} | {(*H*3,0.2),(*H*4,0.2), (*Ω*,0.6)} | {(*H*3,0.1),(*H*6,0.6),(*Ω*,0.3)} | {(*H*4,0.4), (*Ω*,0.6)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*2,0.6),(*H*5,0.1), (*Ω*,0.3)} | {(*H*3,0.3), (*H*5,0.4), (*Ω*,0.3)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*5,0.9), (*Ω*,0.1)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

**Supplementary B. Detailed calculation process of illustrative example**

The detailed calculation process of the example in Section 4 is as follows.

(1) Consistency measure and adjustment of expert *E*1

**Table B.1** The original score matrix of $D^{E\_{1}}$--$S^{E\_{1}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.84,0.84] | [0.48,0.68] | [0.35,0.35] | [0.71,0.71] |
| *A*2 | [-0.84,-0.84] | [0,0] | [-0.47,-0.27] | [-0.64,-0.64] | [-0.22,-0.02] |
| *A*3 | [-0.68,-0.48] | [0.27,0.47] | [0,0] | [-0.23,0.17] | [-0.08,0.32] |
| *A*4 | [-0.35,-0.35] | [0.64,0.64] | [-0.17,0.23] | [0,0] | [-0.07,0.53] |
| *A*5 | [-0.71,-0.71] | [0.02,0.22] | [-0.32,0.08] | [-0.53,0.07] | [0,0] |

Calculate the consistent score matrix $\hat{S}^{E\_{1}}$ corresponding to $S^{E\_{1}}$ by Algorithm Ⅰ. The result is shown in the Table B.2. Calculate the consistency index of $S^{E\_{1}}$ by Eq.(11), then we have $CI\left(S^{E\_{1}}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{1}},\hat{S}\_{ik}^{E\_{1}})}{12}$=0.09608<0.1. Therefore, the DPR matrix given by expert *E*1 satisfies the consistency condition and does not need to be improved.

**Table B.2** The consistent score matrix of $S^{E\_{1}}$--$\hat{S}^{E\_{1}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.84,0.84] | [0.37,0.57] | [0.225,0.4672] | [0.4333,0.7656] |
| *A*2 | [-0.84,-0.84] | [0,0] | [-0.47,-0.27] | [-0.5919,-0.1] | [-0.5888,-0.03] |
| *A*3 | [-0.57,-0.37] | [0.27,0.47] | [0,0] | [-0.23,0.17] | [-0.2839,0.6099] |
| *A*4 | [-0.4672,-0.225] | [0.1,0.5919] | [-0.17,0.23] | [0,0] | [-0.07,0.53] |
| *A*5 | [-0.7656,-0.4333] | [0.03,0.5888] | [-0.6099,0.2839] | [-0.53,0.07] | [0,0] |

(2) Consistency measure and adjustment of expert *E*2

**Table B.3** The original score matrix of $D^{E\_{2}}$--$S^{E\_{2}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.57,0.77] | [0.34,0.54] | [-0.69,-0.09] | [0.16,0.36] |
| *A*2 | [-0.77,-0.57] | [0,0] | [-0.16,0.04] | [-0.74,-0.54] | [-0.37,0.03] |
| *A*3 | [-0.54,-0.34] | [-0.04,0.16] | [0,0] | [-0.6,0] | [-0.3,-0.1] |
| *A*4 | [0.09,0.69] | [0.54,0.74] | [0,0.6] | [0,0] | [0.08,0.48] |
| *A*5 | [-0.36,-0.16] | [-0.03,0.37] | [0.1,0.3] | [-0.48,-0.08] | [0,0] |

Calculate the consistent score matrix $\hat{S}^{E\_{2}}$ corresponding to $S^{E\_{2}}$ by Algorithm Ⅰ. And the result is shown in Table B.4.

**Table B.4** The consistent score matrix of $S^{E\_{2}}$--$\hat{S}^{E\_{2}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.57,0.77] | [0.41,0.7792] | [-0.215,0.385] | [-0.1233,0.5356] |
| *A*2 | [-0.77,-0.57] | [0,0] | [-0.16,0.04] | [-0.664,0.04] | [-0.536,-0.06] |
| *A*3 | [-0.7792，-0.41] | [-0.04,0.16] | [0,0] | [-0.6,0] | [-0.52,0.48] |
| *A*4 | [-0.385,0.215] | [-0.04,0.664] | [0,0.6] | [0,0] | [0.08,0.48] |
| *A*5 | [-0.5356，0.1233] | [0.06,0.536] | [-0.48,0.52] | [-0.48,-0.08] | [0,0] |

Calculate the consistency index of $S^{E\_{2}}$ by Eq.(11) such that $CI\left(S^{E\_{2}}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{2}},\hat{S}\_{ik}^{E\_{2}})}{12}$=0.15107>0.1. So the DPR matrix given by expert *E*2 does not satisfy the consistency condition, and it will be automatically adjusted by Algorithm Ⅱ.

Firstly, we calculate $CI\left(S^{E\_{2}}\right)$ when $S\_{15}^{E\_{2}}$ is replaced by $\hat{S}\_{15}^{E\_{2}}$ and find that the result is still greater than 0.1, i.e., we need to select other element instead of $S\_{15}^{E\_{2}} $ to adjust. Then we select the biggest value in the distance matrix shown in Table B.5 which is $dist\left(S\_{14}^{E\_{2}},\hat{S}\_{14}^{E\_{2}}\right)$.

So in the first round, we replace $S\_{14}^{E\_{2}}$ by $\hat{S}\_{14}^{E\_{2}}$. And the consistency index after first round is 0.10585 which is still greater than 0.1. Then the second round adjustment starts. We calculate $CI\left(S^{E\_{2}}\right)$ when $S\_{15}^{E\_{2}}$ is replaced by $\hat{S}\_{15}^{E\_{2}}$ and find that the result is 0.09098 which is smaller than 0.1. So in the second round, we replace $S\_{15}^{E\_{2}}$ by $\hat{S}\_{15}^{E\_{2}}$. The score matrix of expert *E*2 after the second round adjustment is shown in Table B.6 and its consistent score matrix is shown in Table B.7.

**Table B.5** Distance matrix

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | -- | -- | 0.1891 | 0.5320 | 0.1891 |
| *A*2 | -- | -- | -- | 0.3869 | 0.1805 |
| *A*3 | -- | -- | -- | -- | 0.3353 |
| *A*4 | -- | -- | -- | -- | -- |
| *A*5 | -- | -- | -- | -- | -- |

**Table B.6** Score matrix after second round adjustment--$(S^{E\_{2}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.57,0.77] | [0.34,0.54] | [-0.215,0.385] | [0.035,0.6324] |
| *A*2 | [-0.77,-0.57] | [0,0] | [-0.16,0.04] | [-0.74,-0.54] | [-0.37,0.03] |
| *A*3 | [-0.54,-0.34] | [-0.04,0.16] | [0,0] | [-0.6,0] | [-0.3,-0.1] |
| *A*4 | [-0.385,0.215] | [0.54,0.74] | [0,0.6] | [0,0] | [0.08,0.48] |
| *A*5 | [-0.6324,- 0.035] | [-0.03,0.37] | [0.1,0.3] | [-0.48,-0.08] | [0,0] |

**Table B.7** Consistent score matrix corresponding to $(S^{E\_{2}})^{2}$--$(\hat{S}^{E\_{2}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.57,0.77] | [0.41,0.7792] | [-0.215,0.385] | [0.035,0.6324] |
| *A*2 | [-0.77,-0.57] | [0,0] | [-0.16,0.04] | [-0.664,0.04] | [-0.536,-0.06] |
| *A*3 | [-0.7792,-0.41] | [-0.04,0.16] | [0,0] | [-0.6,0] | [-0.52,0.48] |
| *A*4 | [-0.385,0.215] | [-0.04,0.664] | [0,0.6] | [0,0] | [0.08,0.48] |
| *A*5 | [-0.6324,-0.035] | [0. 06,0.536] | [-0.48,0.52] | [-0.48,-0.08] | [0,0] |

The consistency index of score matrix $(S^{E\_{2}})^{2}$ is $CI\left((S^{E\_{2}})^{2}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{2}},\hat{S}\_{ik}^{E\_{2}})}{12}$=0.09098 which is smaller than consistency threshold 0.1. Therefore, the judgment of expert *E*2 reaches the consistency condition after two iterations of Algorithm Ⅱ. And the DPR after two rounds of adjustment is derived by solving optimization model Ⅰ in Step 3 of Algorithm Ⅱ.

**Table B.8** DPR matrix of expert *E*2 after the second round adjustment--$(D^{E\_{2}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*3,0.1),(*H*4,0.1),(*H*7,0.7),(*Ω*,0.1)} | {(*H*3,0.2),(*H*4,0.2),(*H*7,0.5),(*Ω*,0.1)} | {(*H*1,0.26816),(*H*2,0.00033),(*H*6,0.26042),(*H*7,0.17109),(*Ω*,0.3)} | {(*H*2,0.08209),(*H*4,0.22805),(*H*7,0.39116),(*Ω*,0.29870)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.2),(*H*3,0.2),(*H*4,0.3),(*H*6,0.2),(*Ω*,0.1)} | {(*H*1,0.5),(*H*2,0.3),(*H*5,0.1),(*Ω*,0.1)} | {(*H*2,0.2),(*H*3,0.2),(*H*4,0.3),(*H*5,0.1),(*Ω*,0.2)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*1,0.3),(*H*2,0.2),(*H*6,0.2),(*Ω*,0.3)} | {(*H*2,0.3),(*H*3,0.2),(*H*4,0.3),(*H*6,0.1),(*Ω*,0.1)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*3,0.1),(*H*4,0.2),(*H*5,0.1),(*H*6,0.4),(*Ω*,0.2)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

(3) Consistency measure and adjustment of Expert *E*3

**Table B.9** The original score matrix of $D^{E\_{3}}$--$S^{E\_{3}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.2,0.4] | [0.02,0.42] | [-0.43,0.37] | [0.17,0.37] |
| *A*2 | [-0.4.-0.2] | [0,0] | [-0.24,-0.04] | [-0.54,0.26] | [-0.19,0.21] |
| *A*3 | [-0.42,-0.02] | [0.04,0.24] | [0,0] | [-0.29,0.31] | [-0.06,0.14] |
| *A*4 | [-0.37,0.43] | [-0.26,0.54] | [-0.31,0.29] | [0,0] | [-0.4,0.4] |
| *A*5 | [-0.37,-0.17] | [-0.21,0.19] | [-0.14,0.06] | [-0.4,0.4] | [0,0] |

**Table B.10** The consistent score matrix of $S^{E\_{3}}$--$\hat{S}^{E\_{3}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.2,0.4] | [-0.04,0.36] | [-0.305,0.5779] | [-0.2293,0.5497] |
| *A*2 | [-0.4,-0.2] | [0,0] | [-0.24,-0.04] | [-0.4604,0.27] | [-0.5048,0.328] |
| *A*3 | [-0.36,0.04] | [0.04,0.24] | [0,0] | [-0.29,0.31] | [-0.574,0.586] |
| *A*4 | [-0.5779,0.305] | [-0.27,0.4604] | [-0.31,0.29] | [0,0] | [-0.4,0.4] |
| *A*5 | [-0.5497,0.2293] | [-0.328,0.5048] | [-0.586,0.574] | [-0.4,0.4] | [0,0] |

Calculate the consistency measurement of expert *E*3 by Eq.(11) such that $CI\left(E\_{3}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{3}},\hat{S}\_{ik}^{E\_{3}})}{12}$=0.10773>0.1. According to the consistency condition, the DPR matrix given by expert *E*3 does not satisfy the acceptable consistency. Then the original DPR given by expert *E*3 is automatically adjusted using Algorithm Ⅱ.

We calculate $CI\left(S^{E\_{3}}\right)$ when $S\_{15}^{E\_{3}}$ is replaced by $\hat{S}\_{15}^{E\_{3}}$ and find that the result is smaller than 0.1. The score matrix of expert *E*3 after one adjustment is shown as Table B.11.

**Table B.11** Score matrix of experts *E*3 after one adjustment--$(S^{E\_{3}})^{1}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.2,0.4] | [0.02,0.42] | [-0.43,0.37] | [-0.2293,0.5497] |
| *A*2 | [-0.4.-0.2] | [0,0] | [-0.24,-0.04] | [-0.54,0.26] | [-0.19,0.21] |
| *A*3 | [-0.42,-0.02] | [0.04,0.24] | [0,0] | [-0.29,0.31] | [-0.06,0.14] |
| *A*4 | [-0.37,0.43] | [-0.26,0.54] | [-0.31,0.29] | [0,0] | [-0.4,0.4] |
| *A*5 | [-0.5497,0.2293] | [-0.21,0.19] | [-0.14,0.06] | [-0.4,0.4] | [0,0] |

The consistent score matrix corresponding to $(S^{E\_{3}})^{1}$ is as the same as $\hat{S}^{E\_{3}}$ because the change of $S\_{15}^{E\_{3}}$ doesn’t cause change of any element in $\hat{S}^{E\_{3}}$ according to Algorithm Ⅰ. And the consistency measurement of score matrix $(S^{E\_{3}})^{1}$ is $CI\left(E\_{3}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{3}},\hat{S}\_{ik}^{E\_{3}})}{12}$=0.08743 which is smaller than the consistency threshold 0.1. Therefore, the judgment of expert *E*3 reaches the consistency condition after one iteration of Algorithm Ⅱ. And the corresponding new DPR after adjustment is as follows.

**Table B.12** DPR matrix of expert *E*3 after one adjustment--$(D^{E\_{3}})^{1}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*1,0.2),(*H*4,0.2),(*H*7,0.5),(*Ω*,0.1)} | {(*H*3,0.2),(*H*4,0.2),(*H*6,0.4),(*Ω*,0.2)} | {(*H*1,0.1),(*H*2,0.2),(*H*6,0.3),(*Ω*,0.4)} | {(*H*1,0.16786),(*H*3,0.06678),(*H*4,0.02776),(*H*7,0.34810),(*Ω*,0.38950)} |
| *A*2 | - | {(*H*4,1)} | {(*H*2,0.3),(*H*3,0.2),(*H*4,0.2),(*H*5,0.1),(*H*7,0.1),(*Ω*,0.1)} | {(*H*2,0.3),(*H*3,0.1),(*H*5,0.1),(*H*6,0.1),(*Ω*,0.4)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.3),(*H*6,0.2),(*Ω*,0.2)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.2),(*H*6,0.2),(*Ω*,0.3)} | {(*H*2,0.1),(*H*3,0.2),(*H*4,0.3),(*H*5,0.1),(*H*6,0.2),(*Ω*,0.1)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*3,0.2),(*H*4,0.2),(*H*5,0.2),(*Ω*,0.4)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |

(4) Consistency measure and adjustment of Expert *E*4

**Table B.13** The score matrix of $D^{E\_{4}}$--$S^{E\_{4}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.62,0.62] | [0.17,0.37] | [-0.54,0.66] | [0.62,0.62] |
| *A*2 | [-0.62,-0.62] | [0,0] | [-0.66,0.54] | [0.09,0.69] | [-0.6,0.6] |
| *A*3 | [-0.37,-0.17] | [-0.54,0.66] | [0,0] | [-0.69,-0.09] | [-0.27,0.33] |
| *A*4 | [-0.66,0.54] | [-0.69,-0.09] | [0.09,0.69] | [0,0] | [0.17,0.37] |
| *A*5 | [-0.62,-0.62] | [-0.6,0.6] | [-0.33,0.27] | [-0.37,-0.17] | [0,0] |

**Table B.14** The consistent score matrix of $S^{E\_{4}}$--$\hat{S}^{E\_{4}}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.62,0.62] | [-0.04,0.8252] | [0.0671,0.5811] | [-0.15,0.7372] |
| *A*2 | [-0.62,-0.62] | [0,0] | [-0.66,0.54] | [-0.8946,0.45] | [-0.2536,0.7483] |
| *A*3 | [-0.8252,0.04] | [-0.54,0.66] | [0,0] | [-0.69,-0.09] | [-0.52,0.28] |
| *A*4 | [-0.5811,-0.0671] | [-0.45,0.8946] | [0.09,0.69] | [0,0] | [0.17,0.37] |
| *A*5 | [-0.7372,0.15] | [-0.7483,0.2536] | [-0.28,0.52] | [-0.37,-0.17] | [0,0] |

Calculate the consistency measurement of expert *E*4 by Eq.(11) such that $CI\left(E\_{4}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}dist(S\_{ik}^{E\_{4}},\hat{S}\_{ik}^{E\_{4}})}{12}$=0.20345>0.1. According to the consistency condition, the DPR matrix given by expert *E*4 does not satisfy the acceptable consistency. Then the original DPR matrix given by expert *E*4 is automatically adjusted by Algorithm Ⅱ.

The score matrix of expert *E*4 after two rounds of adjustment is shown as follows.

**Table B.15** Score matrix of experts after first round adjustment--$(S^{E\_{4}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.62,0.62] | [0.17,0.37] | [-0.54,0.66] | [-0.15,0.7372] |
| *A*2 | [-0.62,-0.62] | [0,0] | [-0.66,0.54] | [-0.8946,0.45] | [-0.6,0.6] |
| *A*3 | [-0.37,-0.17] | [-0.54,0.66] | [0,0] | [-0.69,-0.09] | [-0.27,0.33] |
| *A*4 | [-0.66,0.54] | [-0.69,-0.09] | [0.09,0.69] | [0,0] | [0.17,0.37] |
| *A*5 | [-0.7372,0.15] | [-0.6,0.6] | [-0.33,0.27] | [-0.37,-0.17] | [0,0] |

**Table B.16** Consistent score matrix --$(\hat{S}^{E\_{4}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | [0,0] | [0.62,0.62] | [-0.04,0.8252] | [-0.3973,0.5355] | [-0.15,0.7372] |
| *A*2 | [-0.62,-0.62] | [0,0] | [-0.66,0.54] | [-0.8946,0.45] | [-0.7382,0.6727] |
| *A*3 | [-0.8252,0.04] | [-0.54,0.66] | [0,0] | [-0.69,-0.09] | [-0.52,0.28] |
| *A*4 | [-0.5355, 0.3973] | [-0.45,0.8946] | [0.09,0.69] | [0,0] | [0.17,0.37] |
| *A*5 | [-0.7372,0.15] | [-0.6727,0.7382] | [-0.28,0.52] | [-0.37,-0.17] | [0,0] |

The consistency measurement of $(S^{E\_{4}})^{2}$ is $CI\left((S^{E\_{4}})^{2}\right)=\frac{\sum\_{i=1}^{3}\sum\_{k=3}^{5}(S\_{ik}^{E\_{4}},\hat{S}\_{ik}^{E\_{4}})}{12}$=0.07853 which is smaller than the consistency threshold 0.1. Therefore, the judgment of expert *E*4 reaches the consistency condition after two iterations of Algorithm Ⅱ. And the DPR after two rounds of adjustment is derived by solving optimization model Ⅰ in Step 3 of Algorithm Ⅱ.

**Table B.17** DPR matrix of expert *E*4 after two round adjustment--$(D^{E\_{4}})^{2}$

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | *A*1 | *A*2 | *A*3 | *A*4 | *A*5 |
| *A*1 | {(*H*4,1)} | {(*H*5,0.2),(*H*6,0.8)} | {(*H*5,0.9), (*Ω*,0.1)} | {(*H*3,0.1),(*H*5,0.3), (*Ω*,0.6)} | {(*H*1,0.05640), (*H*6,0.50000),(*Ω*,0.44360)} |
| *A*2 | - | {(*H*4,1)} | {(*H*3,0.2),(*H*4,0.2), (*Ω*,0.6)} | {(*H*1,0.26570),(*H*6,0.06200),(*Ω*,0.67230)} | {(*H*4,0.4), (*Ω*,0.6)} |
| *A*3 | - | - | {(*H*4,1)} | {(*H*2,0.6),(*H*5,0.1), (*Ω*,0.3)} | {(*H*3,0.3), (*H*5,0.4), (*Ω*,0.3)} |
| *A*4 | - | - | - | {(*H*4,1)} | {(*H*5,0.9), (*Ω*,0.1)} |
| *A*5 | - | - | - | - | {(*H*4,1)} |