SEPARATING SIGNALING EQUILIBRIA UNDER RANDOM RELATIONS BETWEEN COSTS AND ATTRIBUTES

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Abstract. We identify conditions for separating signaling equilibria where attributes are randomly related to costs. Under discrete attributes, a necessary and sufficient condition is the ordering of the cost distributions conditional on attributes by the Monotone Likelihood Ratio Property (MLRP). An equivalent condition is the monotone ordering of the cost elasticities of these distributions. Under a continuum of attributes, a necessary and sufficient condition is ordering by the cost elasticity of the cost density functions with respect to the original probability measure and with respect to a probability measure modified by the “attribute payoff function.” This condition is the equivalent, under a continuum of attributes, to the condition of ordering by the MLRP. We, thus, introduce the definition of Generalized MLRP which orders posterior distributions induced by distributions as well as by particular realizations. We examine a “Spence labor market” in which education costs are randomly related to productivity.

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1 Introduction

The seminal Spence/Riley signaling model assumes that costs and attributes are related by a one-to-one correspondence. [(Spence (1973), Riley (1975)]. Thus, in equilibrium, separation by costs implies separation by attributes. In many real-world situations, however, there is no one-to-one correspondence between costs and attributes. Costs of acquiring education are related not only to one's productivity but also, for example, to one's wealth level, family support, and athletic and performing arts talents. When costs and attributes are randomly related, perfect separation by (unobservable) attributes is impossible. Depending on the random relation between costs and attributes, however, a separating-by-costs[1] signaling equilibrium might arise. In such equilibria, though separation by attribute works "on average," there will be some high-attribute individuals (with high costs) who receive lower payoffs than those of some low-attribute individuals (with low costs). In this paper, we describe conditions for such equilibria under both, a discrete attributes and a continuum of attributes.

To simplify the exposition, we conduct our analysis within Spence's (1973) "job market" framework but assume a random relation between costs and attributes. Employees privately draw at random signaling costs (per year costs of acquiring education) and attributes (constant marginal productivities). The probability distributions are common knowledge. Employers publicly announce a wage schedule that is strictly increasing in employees' education level (number of years of acquired education). Employees signal their attributes to employers by choosing signals (education levels) that maximize their net income (wages minus their total signaling

[1] Throughout the paper, we use the term separating to mean fully separating. This paper's results, however, hold for partially separating equilibria as well. When the appropriate boundary conditions hold, we would have full separation

costs [costs of acquiring education]). In a separating signaling equilibrium employees with lower costs choose higher education levels. Because every level of attribute (cost) might be associated with every level of cost (attribute), full separation by attributes is impossible. The finest possible separation is by costs, thus, we actually consider a separating-by-cost signaling equilibrium. In a fully separating-by-cost signaling equilibrium, employees with a certain education level have a distribution of productivities. This is in contrast with Spence's equilibrium where a certain education level induces a certain productivity level. Thus, Spence's equilibrium is a special case of our equilibrium, where the ranges of the distributions of attributes induced by costs has degenerated to single atoms.

A separating-by-cost signaling equilibrium exists if there exists a wage schedule such that: i) lower cost employees choose higher education levels (separation), ii) employees maximize their net income (employees rationality), and iii) employers receive on average the productivities implied/induced by the wage levels (employers rationality).

We first examine the case of a continuum of costs and a binary attribute, but conjecture that the analysis carries to a model with a finite number of possible attribute values. We find that a necessary condition for a separating-by-cost signaling equilibrium is ordering by First-Order Stochastic Dominance[2] (FOSD) of the cost distributions conditional on attributes. A necessary and sufficient condition is the ordering of these distributions by the monotone likelihood ratio property (MLRP). An equivalent necessary and sufficient condition is the monotone ordering of the cost elasticities of these distributions. See Milgrom (1981) for an examination of the informational role of the MLRP.

[2] See definition of stochastic dominance in, for example, Huang and Lintzenberger (1988), Chapter 2.

We then examine the case where both costs and attributes can take a continuum of values. We demonstrate that a necessary and sufficient condition for separating-by-cost signaling equilibria is the ordering by the cost elasticities of the cost density functions with respect to the original probability measure and with respect to a probability measure modified by the "attribute payoff function." This condition is the equivalent, under a continuum of attributes, to the condition of ordering by the MLRP of the cost distributions conditional on discrete attributes. We thus introduce the definition of Generalized MLRP. While the MLRP ranks posterior distributions induced by realizations from a certain domain, the GMLRP ranks posterior distributions induced by distributions over a certain domain. Clearly if the inducing distributions degenerate to single atom, GMLRP becomes MLRP. The GMLRP is thus necessary and sufficient for the separating equilibrium under a continuum of attributes.

We present our latter results in terms of the ratio of the probability distribution functions of the cost under the original and a modified probability measure. This not only allows for a concise presentation, but also has an intuitive appeal: we change the original measure by multiplying each attribute realization by its "full information" reward/wage. Thus, both employees cost distributions and their “relevance” to employers determine the nature of the equilibrium.

The intuition behind the equilibrium conditions is appealing. Clearly, in equilibrium, employees who receive higher pay should provide higher productivity. Thus, it is necessary that the distributions of productivities conditional on costs are ordered by FOSD. However, because FOSD is an aggregate measure, it does not prevent localized “cheating.” The MLRP, however, induces the ordering of the density functions point by point, induces the ordering by FSOD, and facilitates the equilibrium. The MLRP role here is similar to that in the applications of Milgrom (1981). If, however, the distributions of costs are independent of those of the attributes, there will be, of course, no information revealed and thus, no separating equilibrium.

For expositional convenience, this analysis examines the case in which only one signal is available regarding the unobservable attribute. All the model's results, however, hold for the case of multiple signals. Please see Riley (2001) for additional references regarding signaling models.

Section 2 describes the model and results under discrete attributes, Section 3 describes the model and results under a continuum of attributes, Section 4 concludes.

2 Equilibrium under Discrete Attributes

There are two groups of employees, each composing a q_i $(0 < q_i < 1,\ i = 1,2)$ fraction of the population. Group *i* is characterized by the attribute θ_i, where $\theta_1 < \theta_2$, a realization of a positive binary random variable θ. The attribute, θ, is the constant marginal productivity of labor, later referred to simply as productivity. Employees draw a per unit signaling cost *c* (later referred to as signaling cost), $c > 0,\ c \in [\underline{c}, \overline{c}]$, from a twice continuously differentiable probability distribution conditional on their productivity type, $f_i(c),\ i = 1,2$. Employees know their own productivity type and realized cost, but these are unobservable to others. An alternative assumption that employees know their realized costs but not their productivity types might be less economically appealing but equally supports the paper results. All probability distributions are common knowledge.

A representative employer who pays the competitive wage wishes to discriminate among employees based on their productivity. He draws inferences about employees' productivity from their signals, education level measured in years, and sets employees wages to be a function of the perceived productivity given an education level.

Let the level of a signal that an employee sends be s, $s > 0$, $s \in [\underline{s}, \overline{s}]$, and $b(\cdot)$ a strictly monotonic and twice continuously differentiable signaling function that maps costs, c, into signals s. In a separating-by-cost signaling equilibrium, employees with cost c signal $s = b(c)$. Let $W(s)$ denote the wage offered to employees who signal s. Then, this wage is

$$W(s) = \theta_1 p[\theta_1 \mid b^{-1}(s)] + \theta_2 p[\theta_2 \mid b^{-1}(s)], \tag{2.1}$$

where $p[\theta_i \mid b^{-1}(s)],\ i = 1,2$, is the posterior distribution of θ given a signal s under the signaling rule b. Clearly, this wage rule is rational only if it induces employees with cost c to signal there "true type" $s = b(c)$.

Let $V(\hat{s}, s)$ be the value function of employees who signal $\hat{s}$ and have signaling costs c. Thus,

$$V(\hat{s}, c) = W(\hat{s}) - \hat{s}c = W(\hat{s}) - \hat{s}b^{-1}(s). \tag{2.2}$$

The first term on the right-hand side of Equation (2.2) is the wage offered to the employee, and the second term is the total signaling cost. Employees choose signal level $\hat{s}$ to maximize their value function. Note that in a separating signaling equilibrium, employees choose a signal $\hat{s}$ which is equal to $s = b(c)$, thus, in this case, $b^{-1}(\hat{s}) = b^{-1}(s) = c$.

We aim to construct a separating-by-cost signaling equilibrium, or, a differentiable equilibrium where 1) rational employees choose signals that maximize their value function, 2) rational employers offer wages that equal employees' conditional marginal productivity, and 3) employees with lower costs acquire higher education levels (monotone signaling or separation). We impose the condition of employer's rationality by setting the wage function as in Equation (2.1). Given employees' value function, we now need to identify a signaling rule *b* that induces the satisfaction of the two other equilibrium properties, separation and employees' rationality. We summarize the results in the following lemma, proposition, and corollaries.

Lemma 2.1. *A necessary condition for the existence of a differentiable separating-by-cost signaling equilibrium is the existence of a signaling rule b(c), that obeys the system,*

$$b'(c) = \frac{\alpha}{c} \frac{f_1^{'}(c) \quad f_2^{'}(c)}{[q_1 f_1(c) + q_2 f_2(c)]^2} \left[\frac{f_2^{'}(c)}{f_2(c)} - \frac{f_1^{'}(c)}{f_1(c)} \right], \ \forall c, \tag{2.3}$$

$$b(\underline{c}) \leq \overline{s}, \tag{2.4}$$

$$b'(c) < 0, \ \forall c, \tag{2.5}$$

where

$$\alpha \triangleq q_1 q_2 (\theta_2 - \theta_1). \tag{2.6}$$

Proof. See Appendix. □

We prove Lemma 2.1 by postulating a signaling rule, performing a Bayes revision switching from the signal space to the cost space, and requesting that the first and second order conditions, of the employee's value function maximization problem, are satisfied at the signal level that corresponds (by the signaling rule) to the realized cost.[3]

[3] Holt (1980) used a similar proof technique to develop an optimal bidding rule.

Proposition 2.1. *A separating-by-*cost *signaling equilibrium exists if and only if the distributions of the per-unit signaling cost conditional on the attribute obey the strict MLRP*:

$$\frac{f_1^{'}(c)}{f_1(c)} > \frac{f_2^{'}(c)}{f_2(c)},\ \forall c. \tag{2.7}$$

Proof. The MLRP above is necessary and sufficient for the strict monotonicity of the signaling function $b(c)$, defined in Lemma 2.1. This establishes separation. The first order conditions of the maximization of employees' value function were established in Lemma 2.1. The above MLRP is also necessary and sufficient for the second-order conditions of the maximization of employees' value function. This is demonstrated in the Appendix. The first and second order conditions of employees' maximization problem, together, are necessary and sufficient for employees' optimum. Thus, because employers' condition is satisfied by the construction of the wage function W in Equation (2.1), The MLRP is necessary and sufficient for a separating-by-cost signaling equilibrium. □

The following corollary will be useful for a comparison with Section 3 results.

Corollary 2.1. *A separating-by-cost signaling equilibrium exists if and only if the cost elasticity of the cost density function conditional on the low realization of the attribute is greater than the cost elasticity of the cost density function conditional on the high realization of the attribute, or*

$$\frac{\dfrac{\partial f_1(c)}{\partial c}}{\dfrac{f_1(c)}{c}} > \frac{\dfrac{\partial f_2(c)}{\partial c}}{\dfrac{f_2(c)}{c}},\ \forall c. \tag{2.8}$$

Proof. The MLRP in Proposition 2.1 implies the relation in (1). □

Corollary 2.2. *A separating-by-cost signaling equilibrium exists only if the distribution of c conditional on* θ_1 *dominates the distribution of c conditional on* θ_2 *by first-order stochastic dominance.*

Proof. The MLRP in Proposition 2.1, implies this result as well; see, for example, Lehmann (1959, p. 74). □

Corollary 2.3. *A separating-by-cost signaling equilibrium exists if and only if* $\forall c_0, c_1,\ c_0 < c_1,\ \ p(\theta \mid c = c_0)$ *dominates* $p(\theta \mid c = c_1)$ *by first-order stochastic dominance.*

Proof. The MLRP in Proposition 2.1 is necessary and sufficient for the first-order stochastic dominance stated in Corollary 2.3. See, for example, Milgrom (1981, Proposition 2). □

Note that the result of Corollary 2.3 guarantees that employees with a higher education level are more productive on average.

3 Equilibrium Under a Continuum of Attributes

We will now present a version of model in the previous section under a continuum of attributes. We will emphasize those features of the model that are different here and assume that all else is as in the previous section. We now assume that employees first draw a realization of productivity from a real valued positive random variable θ distributed according to a probability measure μ_θ with a corresponding density function $f(\theta)$. Then, employees draw a per-unit signaling cost

c, $c > 0$, $c \in [\underline{c}, \overline{c}]$, from a positive, twice continuously differentiable probability distribution function, conditional on the realization of the attribute,[4] $f(c \mid \theta)$.

Because employers cannot separate employees by the realizations of their attribute θ, they try to separate employees by their signaling costs. Suppose that employers act as if employees use a strictly monotonic and twice continuously differentiable signaling rule $b(\cdot)$ to map a cost c into a signal s, $s \in [\underline{s}, \overline{s}]$. If an employee sends a signal s, the employer uses the inverse signaling function $b^{-1}(\cdot)$ to infer the cost and, therefore, faces the posterior distribution $f[\theta \mid b^{-1}(s)]$. Then, the wage offered to employees, corresponding to their expected marginal productivity, as in Equation (2.1), is

$$W(s) = \int \theta f[\theta \mid b^{-1}(s)] d\theta. \tag{3.1}$$

Similar to Equation (2.2), the value function of employees who signal s and have signaling costs c, is[5]

$$V(s,c) = W(s) - sc. \tag{3.2}$$

The first term on the right-hand side of Equation (3.2) is the employee's wage, and the second term is the signaling cost. Employees choose signal level s to maximize their value function. In a separating-by-cost signaling equilibrium, employees signal truthfully, i.e., $b^{-1}(s) = c$.

We now proceed with an analysis "along the equilibrium path:" we use the Spence/Riley conditions for a signaling equilibrium to identify the conditions for a signaling equilibrium under random costs. We offer an alternative derivation of this

4 For notational brevity, we suppress the subscripts of the probability distribution functions. The arguments will define the functions, i.e., $f(c \mid \theta) \triangleq f_{c|\theta}(c \mid \theta)$, etc.

5 To simplify the exposition and notation we assume a linear signaling cost structure where the signaling cost is sc. Our results hold for a general signaling cost function $C(s,c)$ under appropriate assumptions about the function's partial derivatives.

section's results, following the proof procedure of the previous section in the Appendix.

Let the separating-by-cost equilibrium wage for a given cost level be $N(c)$. Then, along the Spence/Riley work we define a separating-by-cost signaling equilibrium as a differentiable equilibrium with mapping of costs to signals $c \to b(c)$, and signals to wages $s \to W(s)$, such that

(C.1) $b'(c) < 0,\ \forall c$ Separation by cost: signals are a strictly monotonic function of costs[6]

(C.2) $W'[b(c)] = c,\ W''[b(c)] < 0,\ \forall c$ Employees rationality: Employees maximize their value function.[7]

(C.3) $W[b(c)] = N(c)$ Employers rationality: the wage rule $W(\cdot)$ assigns the competitive wage $N(\cdot)$.

The Spence/Riley model however, does not explicitly allow for random relation between attributes and costs. Thus, it is impossible to identify the necessary and sufficient conditions for signaling equilibria under random costs from Conditions (C.1)-(C.3) alone. We must use the following analysis.

Let the cumulative distribution function of attribute conditional on cost be $F(\theta \mid c)$. Therefore, $N(c)$, the competitive wage equals the expected wage conditional on the cost level *c*, or

$$N(c) = \int \theta dF(\theta \mid c),\ \forall c. \tag{3.3}$$

[6] To simplify the exposition, we assume mathematical conditions for separation by cost that are more restricting than necessary. For example, we can allow b'(c) = 0 for some c.

[7] These are the necessary and sufficient conditions for employees value maximization under the cost structure specified in Equation (3.2). Different cost structures would induce corresponding conditions.

Note that $N(c)$ is a function of both the “attribute payoff function,” in our case simply θ, and the probability distribution of attributes conditional on costs. Note that all the paper results hold for the case where the “attribute payoff function” is some monotonic function of the attribute, say M_θ. In this case, the last equation becomes $N(c) = \int M_\theta dF(\theta \mid c),\ \forall c$.

Differentiating Condition (C.3) and substituting Condition (C.2) yields the restriction

$$b'(c) = \frac{N'(c)}{c},\ \forall c \tag{3.4}$$

Therefore, in order to satisfy Condition (C.1), $N'(c)$ must be negative. From Equation (3.3), this implies that

$$\int \theta dF_1(\theta) < \int \theta dF_0(\theta),\ \forall c_0, c_1,\ c_0 < c_1, \tag{3.5}$$

where $F_i(\theta) \triangleq F(\theta \mid c = c_i),\ i = 0,1$.

We will now derive the necessary and sufficient conditions for the equilibrium. We start with the first-order conditions. Differentiating $N(c)$, defined in Equation (3.3), yields

$$N'(c) = \frac{\partial}{\partial c}\left[\int \theta dF(\theta \mid c)\right], \tag{3.6}$$

and since $f(c)$ is positive, by Bayes rule

$$N'(c) = \frac{\partial}{\partial c}\left[\frac{\int \theta f(c \mid \theta) d\mu_\theta}{f(c)}\right]. \tag{3.7}$$

We define a new probability measure μ_θ^*, $\mu_\theta^* \triangleq \frac{\theta\mu_\theta}{\int \theta d\mu_\theta}$, and denote densities under μ_θ^* as $f^*(\cdot)$. In particular, $f^*(c) = \int f[c \mid \theta] d\mu_\theta^*$. Thus,

$$N'(c) = \frac{\partial}{\partial c}\left[\int \theta d\mu_\theta \frac{\int f(c \mid \theta) d\mu_\theta^*}{f(c)}\right], \tag{3.8}$$

and if E is the expectation operator under μ_θ, $\mathrm{E}[g(\cdot)] \triangleq \int g(\cdot) d\mu_\theta$, we have

$$N'(c) = \mathrm{E}[\theta]\frac{\partial}{\partial c}\left[\frac{f^*(c)}{f(c)}\right]. \tag{3.9}$$

Performing the differentiation on the right-hand side of Equation (3.9) yields

$$N'(c) = \mathrm{E}[\theta]\frac{f^*(c)}{f(c)}\left[\frac{f^{*\prime}(c)}{f^*(c)} - \frac{f'(c)}{f(c)}\right], \ \forall c, \tag{3.10}$$

and substituting Equation (3.10) in Equation (3.4) we get

$$b'(c) = \frac{\mathrm{E}[\theta]}{c} \frac{f^*(c)}{f(c)} \left[\frac{f^*{}'(c)}{f^*(c)} - \frac{f'(c)}{f(c)} \right], \ \forall c. \tag{3.11}$$

Under the appropriate boundary conditions, for Condition (C.1) to hold, the third multiplicand in Equation (3.11) has to be negative. It is easy to show that the requirement that this third multiplicand is negative is equivalent to the ordering of the cost elasticity of the cost density functions with respect to the original and the new probability measures. If we divide each term of the multiplicand by c each term becomes the cost elasticity of the density function under the corresponding probability measure. That is,

$$\frac{f^*{}'(c)}{f^*(c)} < \frac{f'(c)}{f(c)} \Leftrightarrow \frac{\frac{\partial f^*(c)}{\partial (c)}}{\frac{f^*(c)}{c}} < \frac{\frac{\partial f(c)}{\partial (c)}}{\frac{f(c)}{c}}, \ \forall c. \tag{3.12}$$

Let subscripts denote partial derivative with respect to the subscript argument. We will now examine the second-order conditions, or when $V_{11}[b(c),c] = W''[b(c)] < 0, \ \forall c$ in order to establish sufficiency. Recall that, the first-order conditions for employees' value maximization problem are $V_1[b(c),c] = 0, \ \forall c$. Thus, differentiating with respect to c we have,

$$V_{11}[b(c),c]b'(c) + V_{12}[b(c),c] = 0, \ \forall c, \tag{3.13}$$

and because, from the definition of V in Equation (3.2), $V_{12} = -1$, Equation (3.13) becomes

$$V_{11}[b(c),c] = W''[b(c)] = \frac{1}{b'(c)} \ , \ \forall c. \tag{3.14}$$

Therefore, the fulfillment of Condition (C.1), that requires $b'(c) < 0, \ \forall(c)$, is necessary and sufficient for the fulfillment of the second-order conditions $V_{11}[b(c),c] = W''[b(c)] < 0, \ \forall c$. Because the ordering by cost elasticity of the cost density functions is necessary and sufficient for Condition (C.1) to hold, this ordering is necessary and sufficient for the second-order sufficient conditions of a separating-by-cost signaling equilibrium.

We have now proved the following proposition.

Proposition 1. *A necessary condition for the existence of a separating-by-cost signaling equilibrium is the existence of a signaling rule* $b(\cdot)$*, which satisfies*

$$b'(c) = \frac{\mathrm{E}[\theta]}{c} \frac{f^*(c)}{f(c)} \left[\frac{f^*{}'(c)}{f^*(c)} - \frac{f'(c)}{f(c)} \right], \ \forall c, \tag{3.15}$$

$$b'(c) < 0, \ \ \forall c, \tag{3.16}$$

where, for economic reasons, we require

$$b(\underline{c}) \le \overline{s}, \tag{3.17}$$

where we define $f(c) \triangleq \mathrm{E}[f(c \mid \theta)]$, *and where* $\mathrm{E}[f(\cdot)] \triangleq \int f(\cdot) d\mu_\theta$ *is the expectation operator under* μ_θ. *Similarly,* $f^*(c) \triangleq \mathrm{E}^*[f(c \mid \theta)]$, *where* E^* *is the expectation operator under a new probability measure* μ_θ^*, *defined by* $\mu_\theta^* \triangleq \dfrac{\theta \mu_\theta}{\int \theta d\mu_\theta}$. □

We now formalize the role of the ordering by cost elasticity of the cost density functions.

Proposition 2. *A separating-by-cost signaling equilibrium exists if and only if there exists an ordering by the cost elasticity of the distributions of the per-unit signaling cost with respect to the original probability measure* μ_θ *and with respect to the modified probability measure* μ_θ^*. *In particular, this equilibrium exists if and only if*

$$\frac{f^{*\prime}(c)}{f^*(c)} < \frac{f'(c)}{f(c)} \ , \ \forall c. \tag{3.18}$$

Proof. 1) The ordering by cost elasticity is necessary and sufficient for separation [Condition (C.1)]: Equation (3.15) is an ordinary differential equation with a unique solution for a given boundary condition. Equation (3.15) implies that $b'(c) < 0, \ \forall c,$ if and only if Inequality (3.18) is satisfied. Inequality (3.18) implies the ordering by cost elasticity. 2) From Equation (3.14), the ordering by cost elasticity is necessary and sufficient for employees' rationality. 3) From Equation (3.10) the ordering by cost elasticity is necessary and sufficient for employers' rationality. Because the ordering by cost elasticity is necessary and sufficient for the three equilibrium conditions, it is necessary and sufficient for equilibrium. □

The ordering of cost densities under a continuum of attributes corresponds to the ordering by the MLRP under discrete attributes, both in form and implications. While the MLRP ranks posterior distributions induced by particular *realizations* taken from a certain domain, the ranking under a continuum of attributes is of posterior distributions induced by *distributions* over a certain domain. Clearly, if the inducing distributions degenerate to a single atom, the ranking under a continuum of attributes becomes MLRP. We, thus, suggest to call the ranking under a continuum of attributes Generalized MLRP (GMLRP). This makes the MLRP and GMLRP, the necessary and sufficient conditions for separating signaling equilibria under discrete and a continuum of attributes, respectfully.

4 Summary and Conclusion

We used Spence's job market framework to identify conditions for a separating signaling equilibrium where costs and attributes are randomly related. The ordering of the distributions of costs conditional on attributes by First Order Stochastic Dominance (FOC) is necessary for a separating equilibrium. As an aggregate measure, ordering by FOC is not sufficient to prevent localized "cheating," thus not sufficient for a separating equilibrium. The ordering of the costs distributions conditional on attributes by the Monotone Likelihood Ratio Property (MLRP), a measure defined over densities, is necessary and sufficient for a separating equilibrium under discrete attributes, and the ordering by the cost elasticity of the cost density functions is necessary and sufficient under a continuum of attributes. The latter condition corresponds to an MLRP that ranks posterior distributions induced by distributions rather than the usual MLRP that ranks posterior distributions induced by realizations. We thus suggest the definition of the Generalized MLRP (GMLRP). The

GMLRP, then, is the necessary and sufficient condition for a separating signaling equilibrium under a continuum of attributes.

Our model results and intuition carry over to other signaling models. It is relevant to all signaling situations where costs and attributes might not be deterministically related.

APPENDIX A

Proof of Lemma 2.1

Restating Equation (2.1),

$$W(\hat{s}) = \theta_1 p[\theta_1 \mid b^{-1}(\hat{s})] + \theta_2 p[\theta_2 \mid b^{-1}(\hat{s})]. \tag{A.1}$$

By Bayes Rule,

$$p[\theta_i \mid b^{-1}(\hat{s})] = \frac{f_i[b^{-1}(\hat{s})]q_i}{f_1[b^{-1}(\hat{s})]q_1 + f_2[b^{-1}(\hat{s})]q_2}\ , \quad i = 1,2. \tag{A.2}$$

Substituting Equation (A.2) into Equation (A.1) yields

$$W(\hat{s}) = \frac{\theta_1 f_1[b^{-1}(\hat{s})]q_1 + \theta_2 f_2[b^{-1}(\hat{s})]q_2}{f_1[b^{-1}(\hat{s})]q_1 + f_2[b^{-1}(\hat{s})]q_2}, \tag{A.3}$$

or

$$W(\hat{s}) = \theta_1 + (\theta_2 - \theta_1)\ \frac{q_2 f_2[b^{-1}(\hat{s})]}{q_1 f_1[b^{-1}(\hat{s})] + q_2 f_2[b^{-1}(\hat{s})]}. \tag{A.4}$$

In equilibrium, employees maximize the value function V defined in Equation (2.2). They set $\partial V / \partial \hat{s} = 0$. Therefore, $W'(\hat{s}) - b^{-1}(s) = 0$, or, substituting Equation (A.4)

$$\alpha \frac{f_1[b^{-1}(\hat{s})]f_2'[b^{-1}(\hat{s})] - f_1'[b^{-1}(\hat{s})]f_2[b^{-1}(\hat{s})]}{\{q_1 f_1[b^{-1}(\hat{s})] + q_2 f_2[b^{-1}(\hat{s})]\}}\ b^{-1}{}'(\hat{s}) - b^{-1}(s) = 0, \tag{A.5}$$

where $\alpha \triangleq q_1 q_2(\theta_2 - \theta_1)$. Note that $\alpha > 0$. Equation (A5) should be satisfied at $\hat{s} = s$; and since in equilibrium $b^{-1}(s) = c$, we have $b[b^{-1}(s)] = s$. Taking the derivative of both sides with respect to s yields

$$b'[b^{-1}(s)]b^{-1}{}'(s) = 1, \tag{A.6}$$

or

$$b'(c)b^{-1}{}'(s) = 1\,, \tag{A.7}$$

or

$$b^{-1}{}'(s) = 1/b'(c)\,. \tag{A.8}$$

Substituting $\hat{s} = s$, then $b^{-1}(s) = c$, and Equation (A.8) into Equation (A.5) yields

$$\alpha \frac{f_1(c)f_2'(c) - f_1'(c)f_2(c)}{[q_1 f_1(c) + q_2 f_2(c)]^2}\ \frac{1}{b'(c)} - c = 0\,, \tag{A.9}$$

or

$$b'(c) = \left(\frac{\alpha}{c}\right) \frac{f_1(c)f_2(c)}{[q_1 f_1(c) + q_2 f_2(c)]^2} \left[\frac{f_2'(c)}{f_2(c)} - \frac{f_1'(c)}{f_1(c)}\right]. \tag{A.10}$$

Equation (A.10) is an ordinary differential equation which has a unique solution that satisfies the boundary condition $b(\underline{c}) = \bar{s}$. □

Completing the Proof Proposition 2.1: The Second-Order Conditions

To complete the proof of Proposition 2.1 we need to show that the MLRP is necessary and sufficient for the second order conditions of employees' value function maximization problem.

For the second-order condition, we differentiate employees' value function again with respect to $\hat{s}$ and evaluate at s. From Equation (A.5),

$$
\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{\alpha f_1 f_2}{(q_1 f_1 + q_2 f_2)^2}
$$

$$
\left\{\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right]b^{-1\prime\prime}(s)+\left[\frac{f_2''}{f_2}-\frac{f_1''}{f_1}\right][b^{-1\prime}(s)]^2-2\frac{q_1 f_1'+q_2 f_2'}{q_1 f_1+q_2 f_2}\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right][b^{-1\prime}(s)]^2\right\}, \tag{A.11}
$$

where we omitted the arguments of *f*. Taking the derivative of both sides of Equation (A.6), we obtain

$$
b^{-1}{}''(s) = -b''(c)/b'^3(c). \tag{A.12}
$$

We rewrite Equation (A.11) by substituting $\hat{s} = s$, then $b^{-1}(s) = c$, and Equations (A.8) and (A.12)

$$
\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{\alpha f_1 f_2}{(q_1 f_1 + q_2 f_2)^2}\;\frac{1}{b'^2(c)}
$$

$$
\left[-\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right]\frac{b''(c)}{b'(c)}+\left[\frac{f_2''}{f_2}-\frac{f_1''}{f_1}\right]-2\frac{q_1 f_1'+q_2 f_2'}{q_1 f_1+q_2 f_2}\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right]\right]. \tag{A.13}
$$

We also rewrite Equation (A.9) as

$$
b'(c) = \frac{\alpha}{c}\;\frac{f_1 f_2' - f_1' f_2}{(q_1 f_1 + q_2 f_2)^2}. \tag{A.14}
$$

Differentiating Equation (A.14),

$$
b''(c) = \frac{\alpha}{c}\frac{f_1 f_2}{(q_1 f_1 + q_2 f_2)^2}\left[-\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right]\frac{1}{c}+\left[\frac{f_2''}{f_2}-\frac{f_1''}{f_1}\right]-2\frac{q_1 f_1'+q_2 f_2'}{q_1 f_1+q_2 f_2}\left[\frac{f_2'}{f_2}-\frac{f_1'}{f_1}\right]\right]. \tag{A.15}
$$

We define $a_1 \triangleq \dfrac{\alpha f_1 f_2}{(q_1 f_1 + q_2 f_2)^2}$, $a_2 \triangleq \dfrac{f_2'}{f_2} - \dfrac{f_1'}{f_1}$, $a_3 \triangleq \dfrac{f_2''}{f_2} - \dfrac{f_1''}{f_1}$, $a_4 \triangleq 2\dfrac{q_1 f_1' + q_2 f_2'}{q_1 f_1 + q_2 f_2}$, and

rewrite Equations (A.13), (A.14) and (A.15), respectively, as

$$\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{a_1}{b'^2(c)} \left\{ -a_2 \left[\frac{b''(c)}{b'(c)} \right] + a_3 - a_2 a_4 \right\} , \tag{A.16}$$

$$b'(c) = \frac{1}{c} a_1 a_2 , \tag{A.17}$$

$$b''(c) = a_1 \frac{1}{c} \left[a_3 - a_2 a_4 - a_2 \frac{1}{c} \right] \tag{A.18}$$

Substituting Equations (A.17) and (A.18) into Equation (A.16),

$$\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{c}{a_1 a_2} . \tag{A.19}$$

Since $c > 0$ and $a_1 > 0$, the second-order conditions, i.e., $\left.\dfrac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} < 0$, are satisfied if and only if $a_2 < 0$, which implies the strict MLRP. □

APPENDIX B: ALTERNATIVE DERIVATION OF SECTION 3 RESULTS

We now offer an alternative derivation of Section 3 results, independent of the one in the body of the paper. Here we do not use the Spence/Riley conditions. The analysis follows that of Section 2. We assume that employers assign markups as if employees signal "truthfully" using a strictly monotonic and twice continuously differentiable signaling rule $b(\cdot)$. We then determine the implications of this assumption on the underlying distributions of costs and attributes. This will lead to the identification of necessary and sufficient conditions on these distributions under which the employers assumption that employees use a strictly monotonic signaling rule is self fulfilling. In other words, the conditions under which the wage function that employers use induces employees to signal truthfully when maximizing their value function.

We use the paper's definitions and notation. If an employee sends a signal $\hat{s}$, an employer uses the inverse signaling function $b^{-1}(\cdot)$ to infer its cost type and, therefore, faces the posterior distribution $f[\theta \mid b^{-1}(\hat{s})]$. With $W(\hat{s})$ representing the wage offered by employers to employees who signal $\hat{s}$, we have

$$W(\hat{s}) = \int \theta f[\theta \mid b^{-1}(\hat{s})] d\theta . \tag{B.1}$$

Let $V(\hat{s}, c)$ be the value function of employees who signal $\hat{s}$ and have signaling costs c. Then

$$V(\hat{s}, c) = W(\hat{s}) - \hat{s}c = W(\hat{s}) - \hat{s}b^{-1}(s) . \tag{B.2}$$

Employees choose signal level $\hat{s}$ to maximize their value function. In a separating-by-cost signaling equilibrium, employees signal truthfully, i.e., $\hat{s} = s$, and hence, $b^{-1}(\hat{s}) = c$.

Recalling that $f(c)$ is positive, we apply Bayes' rule to Equation (B.1). This yields

$$W(\hat{s}) = \int \theta f[\theta \mid b^{-1}(\hat{s})] d\theta = \frac{\int \theta f[b^{-1}(\hat{s}) \mid \theta] d\mu_\theta}{f[b^{-1}(\hat{s})]} . \tag{B.3}$$

We define a new probability measure μ_θ^*, $\mu_\theta^* \triangleq \frac{\theta\mu_\theta}{\int \theta d\mu_\theta}$, and denote densities under μ_θ^* as $f^*(\cdot)$.

In particular, $f^*[b^{-1}(\hat{s})] \triangleq \int f[b^{-1}(\hat{s}) \mid \theta] d\mu_\theta^*$. Then, if $\mathrm{E}[g(\cdot)] \triangleq \int g(\cdot) d\mu_\theta$ is the expectation operator under μ_θ

$$W(\hat{s}) = \int \theta d\mu_\theta \frac{\int f[b^{-1}(\hat{s}) \mid \theta] d\mu_\theta^*}{f[b^{-1}(\hat{s})]} = \mathrm{E}[\theta] \frac{f^*[b^{-1}(\hat{s})]}{f[b^{-1}(\hat{s})]}. \qquad \text{(B.4)}$$

In equilibrium, employees maximize their value function *V* defined in Equation (B.2). The first-order condition for this optimization (FOC) is

$$\frac{\partial V}{\partial \hat{s}} = W'(\hat{s}) - b^{-1}(s) = 0. \qquad \text{(B.5)}$$

Substituting Equation (B.4) into the FOC yields

$$\mathrm{E}[\theta] \frac{f^{*\prime}[b^{-1}(\hat{s})] f[b^{-1}(\hat{s})] - f^*[b^{-1}(\hat{s})] f'[b^{-1}(\hat{s})]}{f^2[b^{-1}(\hat{s})]} b^{-1\prime}(\hat{s}) - b^{-1}(s) = 0. \qquad \text{(B.6)}$$

Equation (B.6) should be satisfied at $\hat{s} = s$; and since in equilibrium $b^{-1}(s) = c$, we have $b[b^{-1}(s)] = s$. Therefore, $b^{-1}{}'(s) = 1/b'(c)$, and $b^{-1}{}''(s) = -b''(c)/b'^3(c)$. We rewrite Equation (B.6) as

$$b'(c) = \frac{\mathrm{E}[\theta]}{c} \frac{f^{*\prime}(c) f(c) - f^*(c) f'(c)}{f^2(c)} = \frac{\mathrm{E}[\theta]}{c} \frac{f^*(c)}{f(c)} \left[\frac{f^{*\prime}(c)}{f^*(c)} - \frac{f'(c)}{f(c)} \right]. \qquad \text{(B.7)}$$

Equation (B.7) is an ordinary differential equation which has a unique solution that satisfies a boundary condition, say $b(\underline{c}) = \overline{s}$. This proves Equation (3.15). Inequality

(3.16) is, clearly, necessary for separation. We now turn to the second-order condition. Omitting $\mathrm{E}[\theta]$ and the arguments of *f* and $b^{-1}(\cdot)$,

$$\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{\partial}{\partial \hat{s}}\left[\frac{f^*}{f}\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]b^{-1\prime}\right]\Bigg|_{\hat{s}=s} = \frac{f^*}{f}\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]^2 b^{-1\prime 2}$$
$$+\frac{f^*}{f}\left[\left[\frac{f^{*\prime\prime}}{f^*}-\frac{f''}{f}\right]-\left[\frac{f^{*\prime 2}}{f^{*2}}-\frac{f'^2}{f^2}\right]\right]b^{-1\prime 2}+\frac{f^*}{f}\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]b^{-1\prime\prime}. \tag{B.8}$$

Changing arguments to *c* and substituting for the derivative functions of b^{-1} yields

$$\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = \frac{f^*}{f}\left[\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]^2+\left[\frac{f^{*\prime\prime}}{f^*}-\frac{f''}{f}\right]-\left[\frac{f^{*\prime 2}}{f^{*2}}-\frac{f'^2}{f^2}\right]\right]\frac{1}{b'^2(c)}$$
$$+\frac{f^*}{f}\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]\left[-\frac{b''(c)}{b'^3(c)}\right]. \tag{B.9}$$

Differentiating Equation (B.7) gives

$$b''(c) = \frac{1}{c}\frac{f^*}{f}\left[\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]^2+\left[\frac{f^{*\prime\prime}}{f^*}-\frac{f''}{f}\right]-\left[\frac{f^{*\prime 2}}{f^{*2}}-\frac{f'^2}{f^2}\right]\right]$$
$$-\frac{1}{c^2}\frac{f^*}{f}\left[\frac{f^{*\prime}}{f^*}-\frac{f'}{f}\right]. \tag{B.10}$$

We then define $a_1 \triangleq \frac{f^*}{f}$, $a_2 \triangleq \frac{f^{*\prime}}{f^*}-\frac{f'}{f}$, $a_3 \triangleq \frac{f^{*\prime\prime}}{f^*}-\frac{f''}{f}$, $a_4 \triangleq \frac{f^{*2}}{f^{*2}}-\frac{f'^2}{f^2}$, and rewrite Equations (B.9), (B.7) and (B.10), respectively, as

$$\left.\frac{\partial^2 V}{\partial \hat{s}^2}\right|_{\hat{s}=s} = a_1\left[a_2^2+(a_3-a_4)\right]\frac{1}{b'^2(c)}+a_1a_2\left[-\frac{b''(c)}{b'^3(c)}\right], \tag{B.11}$$

$$b'(c) = \frac{1}{c} a_1 a_2 , \tag{B.12}$$

$$b''(c) = \frac{1}{c} \quad a_1 \left[a_2^2 + (a_3 - a_4) - \frac{1}{c} a_2 \right]. \tag{B.13}$$

Substituting Equations (B.12) and (B.13) into Equation (B.11), we get

$$\left. \frac{\partial^2 V}{\partial \hat{s}^2} \right|_{\hat{s}=s} = \frac{c}{a_1 a_2} . \tag{B.14}$$

Because $c > o$, and $a_1 > 0$, we get that $\left. \frac{\partial^2 V}{\partial \hat{s}^2} \right|_{\hat{s}=s} < 0$ if and only if $a_2 < 0$, which implies ordering by cost elasticity of the cost densities. □

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