Preserving or removing special players: what keeps your payoff unchanged in TU-games?

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Preserving or removing special players: what keeps your payoff unchanged in TU-games? *

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Abstract

If a player is removed from a game, what keeps the payoff of the remaining players unchanged? Is it the removal of a special player or its presence among the remaining players? This article answers this question in a complement study to Kamijo and Kongo [9]. We introduce axioms of invariance from player deletion in presence of a special player. In particular, if the special player is a nullifying player (resp. dummifying player), then the equal division value (resp. equal surplus division value) is characterized by the associated axiom of invariance plus efficiency and balanced cycle contributions. There is no type of special player from such a combination of axioms that characterizes the Shapley value.

Keywords: weighted division values, equal division, weighted surplus division values, equal surplus division, Shapley value, null player, nullifying player, dummifying player, invariance from player deletion in presence of a special player.

JEL Classification number: C71.

1 Introduction

This article studies the class of cooperative games with transferable utilities (TU-games henceforth) and variable player sets. For this class, a natural question to ask is whether eliminating a player from a game influences the payoff of the other players.

A first interesting way to tackle this problem is investigated by Kamijo and Kongo [9], who ask the question of which kind of players *can be removed* from a TU-game in order to preserve the payoff of the remaining players in the induced sub-game. If the considered value is the Shapley value (Shapley [12]), then it is known since Derks and Haller [5] that removing null players from a TU-game, *i.e.* players with null contribution to coalitions, does not affect the payoff of the other players. Kamijo and Kongo [9] prove that for the Equal Division value and the Solidarity value (Nowak and Radzik [10]), two other types of players can be deleted from a TU-game without altering the payoff of the other players: the proportional players and the quasi-proportional players, respectively. These types of players are the basis of three corresponding axioms of invariance. By combining each of these axioms of invariance with Efficiency and Balanced cycle contributions (see Kamijo and Kongo [8]), the authors provide comparable characterizations of the Shapley value, the Equal Division and the Solidarity value. Roughly speaking, the Balanced cycle contributions requires, for any ordering of the players, that the sum of the claims from each player against his predecessor is balanced with the sum of the claims from each player against his successor.

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In this article we consider a complementary way to evaluate the influence of removing a player on the payoff of the remaining players. More specifically, we also explore the consequence of taking away a player from a TU-game, but we ask the question of which kind of players *needs to be preserved* in the sub-game in order to keep the payoff of the remaining players unchanged. When any player is deleted from a TU-game, we show that if both the original game and the sub-game contain a nullifying player (resp. a dummifying player), *i.e.*, a player belonging to coalitions with null (resp. additive) worths only, then the payoff of all remaining players are not affected if they are computed according to the equal division value (resp. the equal surplus division value). It turns out that combining each of the corresponding axioms of invariance with Efficiency and Balanced cycle contributions provides a characterization of both the Equal Division and Equal Surplus Division values. Such a result is not possible if the presence of a null player is required: there exists no value satisfying Efficiency and Invariance from a player deletion in presence of a null player.

Our study offers many insights and enables to connect various articles in the literature.

Firstly, we adopt the general approach of Kamijo and Kongo [9] based on different Q-related players, which encompass null players, nullifying players, and dummifying players with null stand-one worths, among other types of players. Under this approach, we provide a necessary and sufficient condition on the function Q under which there exists a unique value that satisfies Efficiency, Balanced cycle contributions, and the associated axiom of Invariance from player deletion in presence of a Q-related player.

Secondly, while the proportional player is newly introduced by Kamijo and Kongo [9], we rest on the already existing types of nullifying and dummifying players. In fact, among the values satisfying Efficiency, Additivity and Equal treatment, the Shapley value, the Equal Division value and the Equal Surplus Division value are the unique values assigning a zero payoff to null players, a zero payoff to nullifying players and their stand-alone worth to dummifying players, respectively (see van den Brink [14] and Casajus and Huettner [2]). As such, our characterizations are closer to the aforementioned characterizations of the Equal Division value and the Equal Surplus Division value than the characterization of the Equal Division value in Kamijo and Kongo [9]. On one side, the payoffs recommended by the Shapley value are stabilized in a sub-game by deleting a specific player while, on the other side, the payoffs specified by the Equal Division and Equal Surplus Division values are stabilized by preserving a specific player in a sub-game

Thirdly, our axioms of invariance are also useful to provide characterizations of the so-called Weighted Division values and Weighted Surplus Division values. A Weighted Division value splits efficiently the worth achieved by the grand coalition according to an exogenously given weight vector summing up to unity. Similarly a Weighted Surplus Division value assigns to each player his stand-alone worth plus an exogenously given share of what remains of the worth of the grand coalition. Among the efficient and linear values, the class of all Weighted Division values is characterized by Invariance from a player deletion in presence of a nullifying player, while the class of all Weighted Surplus Division values is characterized by Invariance from a player deletion in presence of a dummifying player.

Fourthly, we provide some impossibility results, which complete the comparison with Kamijo and Kongo [9]. On the one hand, we prove that the Shapley value cannot be characterized under our approach, in the sense that there does not exist any Q-related player for which the Shapley value satisfies the associated axiom of Invariance from player deletion in presence of a Q-related player. On the other hand, we prove that the Equal Surplus Division value cannot be characterized under the approach of Kamijo and Kongo [9] by using Invariance from dummifying player deletion. This is a consequence of a more general result showing that within the class of values satisfying Efficiency and Balanced cycle contributions, if a value satisfies invariance from player deletion in *presence* of a Q-related for a fixed function Q, then there is no value satisfying Invariance from Q-related player deletion.

With respect to the close literature, Weighted Division values have been popularized by Kalai and Samet [7], who introduce and characterize the Weighted Shapley values. See also Radzik [11] for recent developments. The class of Weighted Division values is studied by van den Brink [15] and Béal *et al.* [1]. Other articles studying the Equal Division value and the Equal Surplus Division value on variable player sets are due to Chun and Park [4] and van den Brink *et al.* [13], while van den Brink and Funaki [16] investigate the same values by imposing fixed player sets.

The rest of the article is organized as follows. Section 2 provides the definitions and notations. Our axioms of invariance are introduced by means of Q-related players in Section 3, where useful properties are demonstrated. The characterizations of the Weighted Division values, Weighted Surplus Division values, Equal Division and Equal Surplus Division values are contained in Section 4. Section 5 deepens the comparison with Kamijo and Kongo [9] with some impossibility results. Section 6 concludes.

2 Definitions and notations

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of $\mathcal{U}$. A TU-game is a pair (N, v) where $N \in U$ and $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of the coalition. For any non-empty coalition S, let s be the cardinality of S. The **sub-game** of (N, v) induced by $S \subseteq N$ is denoted by (S, v), where it is obvious that v is restricted to 2^S.

The **null game** $(N, \mathbf{0})$ on N is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For $(N, v), (N, w) \in \mathbb{V}$ and $c \in \mathbb{R}$, the games $(N, v + w)$ and $(N, c \cdot v)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(c \cdot v)(S) = c \cdot v(S)$ for all $S \subseteq N$. For $\emptyset \subsetneq T \subseteq N$, the game (N, e_T) given by $e_T(S) = 1$ if $S = T$ and $e_T(S) = 0$ for $S \neq T$ is called the **standard game** induced by T. For $\emptyset \subsetneq T \subseteq N$, the game (N, u_T) given by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ if $S \not\supseteq T$ is called the **unanimity game** induced by T.

Player $i \in N$ is **null** in $(N, v) \in \mathbb{V}$ if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$ such that $S \ni i$. Player $i \in N$ is **nullifying** in $(N, v) \in \mathbb{V}$ if $v(S) = 0$ for all $S \subseteq N$ such that $S \ni i$. Player $i \in N$ is **dummifying** in $(N, v) \in \mathbb{V}$ if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ such that $S \ni i$. Two players $i, j \in N$ are **equal** in $(N, v) \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **value** on $\mathbb{V}$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any $(N, v) \in \mathbb{V}$. We consider the following values. For each $N \in U$, let $\Delta^N := \left\{\omega^N \in \mathbb{R}^N : \sum_{i \in N} \omega_i^N = 1\right\}$. For all $\omega = (\omega^N : \omega^N \in \Delta^N)_{N \in U}$, define the **$\omega$-Weighted Division value** WD^ω given by:

$$\mathrm{WD}_i^\omega(N, v) = \omega_i^N \cdot v(N) \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The class of all Weighted Division values is denoted by $\mathcal{WD}$.

The **Equal Division value** is the weighted division value given by:

$$\mathrm{ED}_i(N, v) = \frac{v(N)}{n} \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The **ω-Weighted Surplus Division value** WSD^ω is given by:

$$\mathrm{WSD}_i^\omega(N, v) = v(\{i\}) + \omega_i^N \cdot \left(v(N) - \sum_{j \in N} v(\{j\}) \right) \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The class of all weighted division values is denoted by $\mathcal{WSD}$.

The **Equal Surplus Division value** is the weighted surplus division value given by

$$\mathrm{ESD}_i(N,v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n} \qquad \text{for all } (N,v) \in \mathbb{V} \text{ and } i \in N.$$

In the definitions of the Weighted Division and Weighted Surplus Division values, the constants ω_i^N, $i \in N$, $N \in U$, are exogenously given, *i.e.* they coincide for two games (N,v) and (N,w) associated with the same player set N.

The **Shapley value** (Shapley [12]) is given by:

$$\mathrm{Sh}_i(N,v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)! \cdot s!}{n!} \cdot \big(v(S \cup \{i\}) - v(S)\big) \qquad \text{for all } (N,v) \in \mathbb{V} \text{ and } i \in N.$$

In this article, we invoke the following axioms.

Efficiency For all $(N,v) \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(N,v) = v(N)$.

Additivity For all $(N,v), (N,w) \in \mathbb{V}$, $\varphi(N, v+w) = \varphi(N,v) + \varphi(N,w)$.

Linearity For all $(N,v), (N,w) \in \mathbb{V}$ and $c \in \mathbb{R}$, $\varphi(c \cdot v + w) = c \cdot \varphi(v) + \varphi(w)$.

Equal treatment For all $(N,v) \in \mathbb{V}$, all $i,j \in N$ who are equal in (N,v), $\varphi_i(N,v) = \varphi_j(N,v)$.

Null player For all $(N,v) \in \mathbb{V}$, all $i \in N$ who is null in (N,v), $\varphi_i(N,v) = 0$.

Nullifying player For all $(N,v) \in \mathbb{V}$, all $i \in N$ who is nullifying in (N,v), $\varphi_i(N,v) = 0$.

Dummifying player For all $(N,v) \in \mathbb{V}$, all $i \in N$ who is dummifying in (N,v), $\varphi_i(N,v) = v(\{i\})$.

Balanced cycle contributions For all $(N,v) \in \mathbb{V}$, all ordering $(i_1, \ldots, i_p, \ldots, i_n)$ on N,

$$\sum_{p=1}^{n} \Big(\varphi_{i_p}(N,v) - \varphi_{i_p}\big(N \setminus \{i_{p-1}\}, v\big) \Big) = \sum_{p=1}^{n} \Big(\varphi_{i_p}(N,v) - \varphi_{i_p}\big(N \setminus \{i_{p+1}\}, v\big) \Big),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$. It can be readily seen that the condition described in Balanced cycle contributions is equivalent to:

$$\sum_{p=1}^{n} \varphi_{i_p}\big(N \setminus \{i_{p-1}\}, v\big) = \sum_{p=1}^{n} \varphi_{i_p}\big(N \setminus \{i_{p+1}\}, v\big).$$

The Nullifying player property, the Dummifying player property and Balanced contributions are introduced by van den Brink [14], Casajus and Huettner [2], and Kamijo and Kongo [9], respectively.

Proposition 1 *(Kamijo and Kongo [9]) If a value φ on $\mathbb{V}$ satisfies Linearity and Equal treatment, then it satisfies Balanced cycle contributions.*

3 Invariance to player deletion in presence of Q-related players

The axioms of invariance defined in this section compare the payoff of the players in a TU-game before and after one of them has been taken away, subject to the presence of a special type of player. In order to formulate a general class of such axioms, we rely on a definition introduced in Kamijo and Kongo [9]. Let $Q : \mathbb{V} \longrightarrow \mathbb{R}$, where $Q(\emptyset, v) = 0$ for all $(\emptyset, v) \in \mathbb{V}$, be a function which assigns to each TU-game $(N, v) \in \mathbb{V}$ a real number. Given a function Q and a TU-game $(N, v) \in \mathbb{V}$, player $q \in N$ is a **Q-related player** if $v(S \cup \{q\}) = Q(S, v)$ for all $S \subseteq N \setminus \{q\}$. Observe that if a player $q \in N$ is a Q-related player in a game (N, v), then he or she is still Q-related in the sub-games (S, v), $S \subsetneq N$, $S \ni i$. Furthermore, $v(\emptyset \cup \{q\}) = Q(\emptyset, v) = 0$ so that $(\{q\}, v) = (\{q\}, \mathbf{0})$. The function Q is called **$\mathcal{U}$-additive** if for all $(N, v) \in \mathbb{V}$ and all non-empty $S \subsetneq N$, it holds that $Q(N, v) = Q(S, v) + Q(N \setminus S, v)$. The function Q is called **Null for one-player null games** if $Q(\{i\}, \mathbf{0}) = 0$ for all $i \in \mathcal{U}$.

A general definition of our axioms of invariance can be stated as follows.

Invariance from player deletion in presence of a Q-related player For all $(N, v) \in \mathbb{V}$, all Q-related player $q \in N$ in (N, v), all $j \in N \setminus \{q\}$ and all $i \in N \setminus \{j\}$, $\varphi_i(N, v) = \varphi_i(N \setminus \{j\}, v)$.

Because the deleted player j is chosen in $N \setminus \{q\}$, the Q-related player q still participates to the resulting sub-game. Therefore, the axiom simply states that the payoff of the players are not affected in a sub-game if the Q-related player q still belongs to the smaller player set in the sub-game. Observe also that if a TU-game contains more than one Q-related player, then such players can be removed, provided that at least one of them still belongs to the induced sub-game.

In this article, we call upon the axioms of invariance from player deletion in *presence* of a null player, a nullifying player and a dummifying player, respectively. While the null and nullifying players can be obtained through Q-related players for some function Q as shown below, this is not the case with the dummifying player. In fact, two dummifying players can have different (non-null) stand-alone worths, whereas the definition of Q imposes $v(\{q\}) = 0$ for all Q-related player q in a TU-game (N, v). Nevertheless, by restricting to zero-dummifying players, *i.e.* dummifying players with null stand-alone worths only, it becomes possible to construct an associated Q function. More specifically, these types of players are obtained by choosing the functions $Q^{\mathbf{N}}$, $Q^{\mathbf{Nf}}$, and $Q^{\mathbf{0-Df}}$ respectively, where for all $(N, v) \in \mathbb{V}$:

$$Q^{\mathbf{N}}(N, v) = v(N),$$
$$Q^{\mathbf{Nf}}(N, v) = 0,$$
$$Q^{\mathbf{0-Df}}(N, v) = \textstyle\sum_{i \in N} v(\{i\}).$$

We start by providing three useful results involving invariance from player deletion in presence of a Q-related player.

Proposition 2 *If a value φ on $\mathbb{V}$ satisfies Efficiency and Invariance from player deletion in presence of a Q-related player, then for all $(N, v) \in \mathbb{V}$ and all Q-related player $q \in N$ in (N, v), it holds that $\varphi_q(N, v) = v(\{q\}) = 0$.*

Proof. Consider a value φ on $\mathbb{V}$ that satisfies Efficiency and Invariance from player deletion in presence of a Q-related player. Pick any TU-game $(N, v) \in \mathbb{V}$ and any Q-related player $q \in N$ in (N, v). For all $S \subsetneq N \setminus \{q\}$, q remains Q-related in the sub-game $(S \cup \{q\}, v)$.

Then, $n-1$ successive applications of Invariance from player deletion in presence of a Q-related player yields $\varphi_q(N,v) = \varphi_q(\{q\},v)$. By Efficiency and the definition of a Q-related player, $\varphi_q(\{q\},v) = v(\{q\}) = 0$. ■

We obtain the following corollary.

Corollary 1 *Let φ be a value on $\mathbb{V}$ that satisfies Efficiency. The following implications hold.*

(i) If φ satisfies Invariance from player deletion in presence of a null player, then it also satisfies Null player.

(ii) If φ satisfies Invariance from player deletion in presence of a nullifying player, it also satisfies Nullifying player.

(iii) If φ satisfies Invariance from player deletion in presence of a zero-dummifying player, then it also satisfies Zero-dummifying player.

The following result establishes that there exists exactly one value that satisfies Efficiency, Invariance from player deletion in presence of a Q-related player and Balanced cycle contributions if and only if Q is an $\mathcal{U}$-additive function and Null for one-player null games.

Proposition 3

(i) There exists at most one value φ on $\mathbb{V}$ that satisfies Efficiency, Invariance from player deletion in presence of a Q-related player and Balanced cycle contributions.

(ii) Such a value exists if and only if Q is an $\mathcal{U}$-additive function and Null for one-player null games.

Proof. Part (i). Consider a value φ on $\mathbb{V}$ satisfying the three axioms. For any $(N,v) \in \mathbb{V}$ be such that $n=1$, efficiency uniquely determines $\varphi(N,v)$. So, consider any $(N,v) \in \mathbb{V}$ such that $n \geq 2$. From (N,v) construct the TU-game (M,w) such that $M = N \cup \{q\}$ where $q \in \mathcal{U} \setminus N$, and, for all $S \subseteq M$:

$$w(S) = \begin{cases} v(S) & \text{if } S \subseteq N, \\ Q(S \setminus \{q\}, v) & \text{if } q \in S. \end{cases} \tag{1}$$

By construction, q is a Q-related player in (M,w). Next, consider the ordering $(1,\ldots,n)$ on N. For any integer $p \in \{2,\ldots,n\}$, construct the associated orderings on M by placing player q in position p, *i.e.* the orderings $(1,\ldots,p-1,q,p,p+1,\ldots,n)$ for $p \in \{2,\ldots,n\}$. By Balanced cycle contributions, we obtained, for all $p \in \{2,\ldots,n\}$,

$$\begin{aligned} & \varphi_1(M \setminus \{n\}, w) + \cdots + \varphi_{p-1}(M \setminus \{p-2\}, w) + \varphi_q(M \setminus \{p-1\}, w) \\ & + \varphi_p(M \setminus \{q\}, w) + \varphi_{p+1}(M \setminus \{p\}, w) + \cdots + \varphi_n(M \setminus \{n-1\}, w) \\ = \; & \varphi_1(M \setminus \{2\}, w) + \cdots + \varphi_{p-1}(M \setminus \{q\}, w) + \varphi_q(M \setminus \{p\}, w) \\ & + \varphi_p(M \setminus \{p+1\}, w) + \varphi_{p+1}(M \setminus \{p+2\}, w) + \cdots + \varphi_n(M \setminus \{1\}, w). \end{aligned} \tag{2}$$

In order to simplify (2), we point out the following three facts.
(a) By construction, we have $(M \setminus \{q\}, w) = (N,v)$ so that $\varphi_p(M \setminus \{q\}, w) = \varphi_p(N,v)$ and $\varphi_{p-1}(M \setminus \{q\}, w) = \varphi_{p-1}(N,v)$.
(b) From Proposition 2 and definition (1) of w, we obtain $\varphi_q(M \setminus \{p-1\}, w) = w(\{q\}) = 0$ and $\varphi_q(M \setminus \{p\}, w) = w(\{q\}) = 0$.

(c) For all $i \in N$ and all $j \in N \setminus \{i\}$, we have:

$$\begin{aligned}\varphi_i(M \setminus \{j\}, w) &= w(M \setminus \{j\}) - \sum_{p \in M \setminus \{i,j\}} \varphi_p(M \setminus \{j\}, w) \\ &= Q(M \setminus \{j,q\}, v) - \sum_{p \in M \setminus \{i,j\}} \varphi_p(M \setminus \{i,j\}, w) \\ &= Q(N \setminus \{j\}, v) - w(M \setminus \{i,j\}) \\ &= Q(N \setminus \{j\}, v) - Q(N \setminus \{i,j\}, v).\end{aligned}$$

The first and third equalities follow from Efficiency, the second equality follows from Invariance from player deletion in presence of a Q-related, and the substitutions of $w(M \setminus \{j\})$ by $Q(N \setminus \{j\}, v)$ and $w(M \setminus \{i,j\})$ by $Q(N \setminus \{i,j\}, v)$ come from the fact that $q \in M \setminus \{i,j\}$ and the definition (1) of w.

Using (a), (b) and (c), we can rewrite equation (2) as:

$$\begin{aligned} & \big[Q(N \setminus \{n\}, v) - Q(N \setminus \{1,n\}, v)\big] + \cdots + \big[Q(N \setminus \{p-2\}, v) - Q(N \setminus \{p-2, p-1\}, v)\big] \\ & \varphi_p(N, v) + [Q(N \setminus \{p\}, v) - Q(N \setminus \{p, p+1\}, v)] \\ & + \cdots + \big[Q(N \setminus \{n-1\}, v) - Q(N \setminus \{n-1, n\}, v)\big] \\ = \;& \big[Q(N \setminus \{2\}, v) - Q(N \setminus \{1,2\}, v)\big] + \cdots + \varphi_{p-1}(N, v) \\ & + \big[Q(N \setminus \{p+1\}, v) - Q(N \setminus \{p, p+1\}, v)\big] + \big[Q(N \setminus \{p+2\}, v) - Q(N \setminus \{p+1, p+2\}, v)\big] \\ & + \cdots + \big[Q(N \setminus \{1\}, v) - Q(N \setminus \{1,n\}, v)\big]. \end{aligned}$$

Rearranging terms and adding the equation generated by Efficiency, we get the following linear system of n equations with n unknowns:

$$\begin{cases} \varphi_p(N, v) - \varphi_{p-1}(N, v) = Q(N \setminus \{p-1\}, v) - Q(N \setminus \{p\}, v), & \text{for all } p \in \{2, \ldots, n\}, \\ \sum_{p \in N} \varphi_p(N, v) = v(N). \end{cases} \tag{3}$$

Since these n equations are linearly independent, the linear system (3) possesses a unique solution φ^Q, which ensures the uniqueness (but not the existence) of a value satisfying Efficiency, Invariance from player deletion in presence of a Q-related player and Balanced cycle contributions.

Par (ii). Consider the unique solution φ^Q of the linear system (3), and assume it satisfies Efficiency, Invariance from player deletion in presence of a Q-related player and Balanced cycle contributions. To show: Q is $\mathcal{U}$-additive and Null for one-player null games.

Pick any $(N, v) \in \mathbb{V}$ and consider the TU-game (M, w) constructed from (N, v) as in part (i) of this proof (see (1)). Recall that q is a Q-related player in (M, w) so that $\varphi_q(S, w) = 0$ for all $S \subseteq M$ such that $q \in S$ (see the proof of Proposition 2). In particular, $\varphi_q(M, w) = 0$, and by Efficiency in (M, w), this implies that

$$\sum_{i \in N} \varphi_i^Q(M, w) = w(M) = Q(M \setminus \{q\}, w) = Q(N, v). \tag{4}$$

Now, pick any non-empty $S \subsetneq N$, and consider the TU-games $(S \cup \{q\}, w)$ and $((N \setminus S) \cup \{q\}, w)$. Exactly as above, we get:

$$\sum_{i \in S} \varphi_i^Q(S \cup \{q\}, w) = Q(S, v) \quad \text{and} \quad \sum_{i \in N \setminus S} \varphi_i^Q((N \setminus S) \cup \{q\}, w) = Q(N \setminus S, v).$$

By successive applications of invariance from player deletion in presence of a Q-related player, we have, for all $i \in S$, $\varphi_i^Q(S \cup \{q\}, w) = \varphi_i^Q(M, w)$. Similarly, for all $i \in N \setminus S$, we get

$\varphi_i^Q((N \setminus S) \cup \{k\}, w) = \varphi_i^Q(M, w)$. Thus:

$$\sum_{i \in S} \varphi_i^Q(M, w) = Q(S, v) \quad \text{and} \quad \sum_{i \in N \setminus S} \varphi_i^Q(M, w) = Q(N \setminus S, v). \tag{5}$$

Combining (4) and (5), we obtain:

$$Q(N, v) = \sum_{i \in N} \varphi_i^Q(M, w) = \sum_{i \in S} \varphi_i^Q(M, w) + \sum_{i \in N \setminus S} \varphi_i^Q(M, w) = Q(S, v) + Q(N \setminus S, v).$$

Because (N, v) and S has been chosen arbitrarily in $\mathbb{V}$ and 2^N respectively, we conclude that Q is $\mathcal{U}$-additive.

By $\mathcal{U}$-additivity of Q, the linear system (3) rewrites:

$$\begin{cases} \varphi_p(N, v) - \varphi_{p-1}(N, v) = Q(\{p\}, v) - Q(\{p-1\}, v) & \text{for all } p \in \{2, \ldots, n\} \\ \sum_{p \in N} \varphi_p(N, v) = v(N). \end{cases} \tag{6}$$

Solving this linear system, we get:

$$\varphi_i^Q(N, v) = Q(\{i\}, v) + \frac{v(N) - \sum_{p \in N} Q(\{p\}, v)}{n} \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N. \tag{7}$$

With this expression at hand, we can verify whether Q is Null for one-player null games. Pick any $q \in \mathcal{U}$ and any TU-game $(N, v) \in N$ such that q is Q-related in (N, v). Note that such a TU-game exists. For instance, construct such a TU-game in same manner as the TU-game (M, w) defined in Part (i) of this proof. For all $j \in N \setminus \{q\}$ and all $i \in N \setminus \{j\}$, the definitions of a Q-related player and $\mathcal{U}$-additivity of Q imply that:

$$\begin{aligned} & \varphi_i^Q(N, v) - \varphi_i^Q(N \setminus \{j\}, v) \\ = \; & \frac{1}{n}\Big(v(N) - \sum_{p \in N} Q(\{p\}, v)\Big) - \frac{1}{n-1}\Big(v(N \setminus \{j\}) - \sum_{p \in N \setminus \{j\}} Q(\{p\}, v)\Big) \\ = \; & \frac{1}{n}\Big(Q(N \setminus \{q\}) - \sum_{p \in N} Q(\{p\}, v)\Big) - \frac{1}{n-1}\Big(Q(N \setminus \{j, q\}) - \sum_{p \in N \setminus \{j\}} Q(\{p\}, v)\Big) \\ = \; & \frac{1}{n}\Big(\sum_{p \in N \setminus \{q\}} Q(\{p\}, v) - \sum_{p \in N} Q(\{p\}, v)\Big) - \frac{1}{n-1}\Big(\sum_{p \in N \setminus \{j,q\}} Q(\{p\}, v) - \sum_{p \in N \setminus \{j\}} Q(\{p\}, v)\Big) \\ = \; & -\frac{1}{n} Q(\{q\}, v) + \frac{1}{n-1} Q(\{q\}, v) \\ = \; & \frac{1}{n(n-1)} Q(\{q\}, v) \\ = \; & \frac{1}{n(n-1)} Q(\{q\}, \mathbf{0}). \end{aligned} \tag{8}$$

The last equality comes for the definition of a Q-related player: $v(\emptyset \cup \{q\}) = Q(\emptyset, v) = 0$ so that the TU-game $(\{q\}, v)$ is equal to null game $(\{q\}, \mathbf{0})$. By Invariance from player deletion in presence of a Q-related player, we obtain $Q(\{q\}, \mathbf{0}) = 0$. Conclude that Q is Null for one-player null games.

Conversely, assume that Q that is $\mathcal{U}$-additive and is Null for one-player null games. Consider φ^Q the unique solution (7) of the linear system (6). By (6), φ^Q is Efficient. Using equality (8) and the fact that Q is Null for one-player null games, we conclude that φ^Q satisfies Invariance from player deletion in presence of a Q-related player. It remains to verify whether φ^Q satisfies Balanced cycle contributions. Pick any $(N, v) \in \mathbb{V}$ and any ordering $(i_1, \ldots, i_p, \ldots, i_n)$ on N. Then, we have:

$$\begin{aligned} & \sum_{p=1}^{n} \varphi^Q_{i_p}(N \setminus \{i_{p-1}\}, v) \\ = \; & \sum_{p=1}^{n} Q(\{i_p\}, v) + \sum_{p=1}^{n} \frac{v(N \setminus \{i_{p-1}\}) - \sum_{j \in N \setminus \{i_{p-1}\}} Q(\{j\}, v)}{n-1} \\ = \; & \sum_{p=1}^{n} Q(\{i_p\}, v) + \frac{1}{n-1} \cdot \sum_{j \in N} v(N \setminus \{j\}) - \frac{1}{n-1} \cdot \sum_{p=1}^{n} (n-1) \cdot Q(\{i_p\}, v) \\ = \; & \frac{1}{n-1} \cdot \sum_{j \in N} v(N \setminus \{j\}), \end{aligned}$$

which is independent of the considered ordering. This proves that φ^Q satisfies Balanced cycle contributions. ■

Note that Kamijo and Kongo [9] do not manage to provide a necessary and sufficient condition under which there exists a unique value satisfying Efficiency, Balanced cycle contributions and Invariance from *Q-related player deletion*. Furthermore, contrary to Theorem 3 in Kamijo and Kongo [9], we do not need induction on the number of players to prove the existence and uniqueness result. The logical independence of the axioms used in Proposition 3 as well as in all axiomatic characterizations in the article is demonstrated in the Appendix. Since $Q^{\mathbf{N}}$ is not $\mathcal{U}$-additive, we obtain the following interesting corollary of Proposition 3.

Proposition 4 *There does not exist any value on $\mathbb{V}$ satisfying Efficiency, Balanced cycle contributions and Invariance from player deletion in presence of a null player.*

In the appendix, we even prove this result without requiring that φ satisfies Balanced cycle contributions. This result echoes Theorem 4 in Kamijo and Kongo [9] stating that there is no value that satisfies Efficiency and Invariance from nullifying player *deletion*.

4 Characterizations

We begin this section with comparable axiomatic characterizations of the Equal Division value and the Equal Surplus Division value. These characterizations exploit the result contained in Proposition 3.

Proposition 5 *Consider any value φ on $\mathbb{V}$ that satisfies Efficiency, and Balanced cycle contributions. Then the following results hold:*

(i) The value φ satisfies Invariance from player deletion in presence of a nullifying player if and only if $\varphi = ED$.

(ii) The value φ satisfies Invariance from player deletion in presence of a dummifying player if and only if $\varphi = ESD$.

Proof. Part (i). Because ED satisfies Efficiency and Equal treatment, it satisfies Balance cycle contributions by Proposition 1. Now, pick any $(N, v) \in \mathbb{V}$ containing a nullifying player, say q.

Then, $v(N) = v(N \setminus \{j\}) = 0$ for all $j \in N \setminus \{q\}$. It follows that $\mathrm{ED}_i(N \setminus \{j\}, v) = \mathrm{ED}_i(N, v)$ for all $i \in N \setminus \{j\}$, which means that ED satisfies Invariance from player deletion in presence of a nullifying player. Conversely, consider any value $\varphi \in \mathbb{V}$ satisfying the three axioms. The function $Q^{\mathbf{Nf}}$ is $\mathcal{U}$-additive and Null for one player null games. By Proposition 3 (see (7)), $\varphi = \varphi^{Q^{\mathbf{Nf}}} = \mathrm{ED}$.

Part (ii). The value ESD satisfies Efficiency and Equal treatment. By Proposition 1, ESD satisfies Balance cycle contributions. Pick any TU-game (N, v) such that $q \in N$ is dummifying. For each $S \subseteq N$ such that $S \ni q$, it holds that $v(S) = \sum_{i \in S} v(\{i\})$. Fix any player $j \in N \setminus \{q\}$ and any $i \in N \setminus \{j\}$. By definition of ESD, we have:

$$\mathrm{ESD}_i(N, v) = v(\{i\}) + \frac{\Big(v(N) - \sum_{p \in N} v(\{p\})\Big)}{n} = v(\{i\}),$$

where the last equality follows from the fact that q is a dummifying player in (N, v). In the sub-game $(N \setminus \{j\}, v)$, player q obviously remains dummifying. Therefore, exactly as above:

$$\mathrm{ESD}_i(N \setminus \{j\}, v) = v(\{i\}) + \frac{\Big(v(N \setminus \{j\}) - \sum_{p \in N \setminus \{j\}} v(\{p\})\Big)}{n-1} = v(\{i\}).$$

This proves that ESD satisfies Invariance from player deletion in presence of a dummifying player. Conversely, consider any value φ on $\mathbb{V}$ satisfying the three axioms. It is obvious that any zero-dummifying player is also a dummifying player. It follows that if a value satisfies Invariance from player deletion in presence of a dummifying player, then it also satisfies Invariance from player deletion in presence of a zero-dummifying player. This also implies that the statement of Proposition 3 still holds if Invariance from player deletion in presence of a Q-related player is replaced by Invariance from player deletion in presence of a dummifying player. This completes the proof. ■

The rest of this section offer comparable axiomatic characterizations of the Weighted Division values and the Weighted Surplus-Division values by substituting Balance cycle contributions for Linearity. Indeed, these values satisfy Linearity but not Balance cycle contributions.

Proposition 6 *Consider any value φ on $\mathbb{V}$ that satisfies Efficiency and Linearity. Then, the following results hold:*

(i) The value φ satisfies Invariance from player deletion in presence of a dummifying player if and only if $\varphi \in \mathcal{WSD}$.

(ii) The value φ satisfies Invariance from player deletion in presence of a nullifying player if and only if $\varphi \in \mathcal{WD}$.

Proof. Part (i). For all $\omega = (\omega^N : \omega^N \in \Delta^N)_{N \in U}$, the value $\mathrm{WSD}^\omega \in \mathcal{WSD}$ satisfies Efficiency and Linearity. Proceeding as in the proof of Proposition 5, we can show that WSD^ω satisfies Invariance from player deletion in presence of a dummifying player.

Conversely, consider any value φ on $\mathbb{V}$ satisfying the three axioms, and any TU-game $(N, v) \in \mathbb{V}$. If $n = 1$, then Efficiency characterizes φ. So, assume that $n \geq 2$. It is well known that (N, v) admits a linear representation in terms of standard games:

$$(N, v) = \sum_{S \in 2^N, S \neq \emptyset} v(S)(N, e_S). \tag{9}$$

Choose any $S \subseteq N$ such that $1 < s < n$. Observe that all the players in $N \setminus S$ are dummifying in (N, e_S). Pick any $j \in S$. Invariance from player deletion in presence of a dummifying player yields:

$$\varphi_i(N, e_S) = \varphi_i(N \setminus \{j\}, e_S) \qquad \text{for all } i \in N \setminus \{j\}. \tag{10}$$

Since $(N \setminus \{j\}, e_S) = (N \setminus \{j\}, \mathbf{0})$, Linearity implies $\varphi_i(N \setminus \{j\}, \mathbf{0}) = 0$ and, in turn, by (10), $\varphi_i(N, e_S) = 0$ for all $i \in N \setminus \{j\}$. Finally, applying Efficiency in (N, e_S) yields $\varphi_j(N, e_S) = 0$ too. Therefore, representation (9) enables to rewrite φ, as:

$$\varphi_i(N, v) = \sum_{p \in N} v(\{p\}) \cdot \varphi_i(N, e_{\{p\}}) + v(N) \cdot \varphi_i(N, e_N) \qquad \text{for all } i \in N. \tag{11}$$

It remains to consider standard TU-games $(N, e_{\{p\}})$, $p \in N$, and (N, e_N). Note that such TU-games contain no dummifying player. Nonetheless, we can decompose these TU-games in the following way. For any $p \in N$, we have:

$$(N, e_{\{p\}}) = (N, u_{\{p\}}) - \sum_{S \ni p : 1 < s < n} (N, e_S) - (N, e_N). \tag{12}$$

We have already shown that $\varphi_i(N, e_S) = 0$ for all S such that $1 < s < n$ and all $i \in N$. Furthermore, in the unanimity TU-game $(N, u_{\{p\}})$ induced by $\{p\}$, $p \in N$, all players in N are dummifying since $(N, u_{\{p\}})$ is an additive TU-game. By Invariance from player deletion in presence of a dummifying player and Linearity, we get:

$$\varphi_i(N, u_{\{p\}}) = \varphi_i(N \setminus \{p\}, u_{\{p\}}) = \varphi_i(N \setminus \{p\}, \mathbf{0}) = 0 \qquad \text{for all } i \in N \setminus \{p\},$$

Applying Efficiency in $(N, u_{\{p\}})$ implies $\varphi_p(N, u_{\{p\}}) = 1$. As a consequence, using equations (12) and the linearity of φ, we obtain:

$$\varphi_i(N, e_{\{p\}}) = \begin{cases} 1 - \varphi_i(N, e_N) & \text{if } i = p, \\ -\varphi_i(N, e_N) & \text{if } i \in N \setminus \{p\}. \end{cases} \tag{13}$$

Inserting (13) into (11) yields, for all $i \in N$:

$$\begin{aligned} \varphi_i(N, v) &= v(\{i\}) \cdot (1 - \varphi_i(N, e_N)) + \sum_{p \in N \setminus \{i\}} \big(v(\{p\}) \cdot (-\varphi_i(N, e_N))\big) + v(N) \cdot \varphi_i(N, e_N) \\ &= v(\{i\}) - \sum_{p \in N} v(\{p\}) \cdot \varphi_i(N, e_N) + v(N) \cdot \varphi_i(N, e_N) \\ &= v(\{i\}) + \varphi_i(N, e_N) \cdot \left(v(N) - \sum_{p \in N} v(\{p\}) \right). \end{aligned}$$

Now, set $\omega_i^N := \varphi_i(N, e_N)$ for all $N \in U$ and all $i \in N$. Conclude by Efficiency that $\varphi = \text{WSD}^{\omega}$.

Part (ii). Any Weighted Division value satisfies Efficiency and Linearity. Because a nullifying player remains nullifying in any sub-game induced by any coalition he or she belongs to, any Weighted Division value also satisfies Invariance from player deletion in presence of a nullifying player. Conversely, consider any value φ on $\mathbb{V}$ satisfying the three axioms. Similarly as for part (i), it is easy to show that, for all $i \in N$ and all S such that $1 \leq s < n$, $\varphi_i(N, e_S) = 0$ since any such TU-game contains at least one nullifying player. Contrary to the proof of part (i), remark that this is also true for standard TU-games induced by singleton coalitions: each of these TU-games contains nullifying players but does not contain dummifying players. Now, set $\omega_i^N := \varphi_i(N, e_N)$ for all $i \in N$ and conclude by Linearity and representation (9) that $\varphi \in \mathcal{WD}$. ■

Proposition 5 can be read as a way to single out the Equal Surplus Division value among the class of all Weighted Surplus Division values and the Equal Division value among the class of all Weighted Division values by replacing, in Proposition 6 Linearity by Balanced Cycle Contributions. Another way to do so is to replace, in Proposition 6, Linearity by Additivity, and add Equal Treatment, as shown in the following proposition.

Proposition 7 *Consider a value φ on $\mathbb{V}$ that satisfies Efficiency, Additivity and Equal treatment. Then, the following results hold:*

(i) The value φ satisfies Invariance from player deletion in presence of a dummifying player if and only if $\varphi =$ ESD.

(ii) The value φ satisfies Invariance from player deletion in presence of a nullifying player if and only if $\varphi =$ ED.

Proof. It is enough to prove the uniqueness part of (i) and (ii).

Part (i). Consider any φ on $\mathbb{V}$ satisfying Efficiency, Additivity, Equal treatment and Invariance from player deletion in presence of a dummifying player. Pick any (N, v) such that there is a dummifying player $q \in N$ in (N, v). Repeated applications of Invariance from player deletion in presence of a dummifying player and Efficiency yield $\varphi_q(N, v) = \varphi_q(\{q\}, v) = v(\{q\})$, which proves that φ satisfies the Dummifying player property. Then, the result follows from Theorem 2 in Casajus and Huettner [2].

Part (ii), consider any φ on $\mathbb{V}$ satisfying Efficiency, Additivity, Equal treatment and Invariance from player deletion in presence of a nullifying player. Pick any $(N, v) \in \mathbb{V}$ containing a nullifying player. By Corollary 1, φ satisfies the Nullifying player property. Then, the result follows from Theorem 3.1 in van den Brink [14]. ■

5 Further comparisons with Kamijo and Kongo [9]

The first result of this section is an impossibility result: the Shapley value cannot be characterized by invoking an axiom of Invariance from player deletion in presence of a Q-related player.

Proposition 8 *There does not exist a function Q such that the Shapley value satisfies Invariance from player deletion in presence of a Q-related.*

Proof. Proposition 4 proves that there does not exist any value φ on $\mathbb{V}$ satisfying Efficiency, Balanced cycle contributions and Invariance from player deletion in presence of a null player. Since the Shapley value satisfies the first two axioms, it is enough to prove that if the Shapley value satisfies Invariance from player deletion in presence of a Q-related player for some function Q, then it must be that $Q = Q^{\mathbf{N}}$. So, assume that the Shapley value satisfies Invariance from player deletion in presence of a Q-related player for some function Q. We prove that $Q = Q^{\mathbf{N}}$ by induction on n.

Initial step. Suppose that $n = 0$. By definition of function Q, $Q(\emptyset, v) = 0 = v(\emptyset)$.

Induction hypothesis. Suppose that $Q(N, v) = v(N)$ for all N such that $n < p$.

Induction step. Consider any game $(N, v) \in \mathbb{V}$ such that $n = p$. Pick any $q \in \mathcal{U} \setminus N$, and construct the augmented TU-game (M, w) as in (1). Player q is a Q-related player in (M, w). By Proposition 2 and the assumption that the Shapley value satisfies Invariance from player deletion in presence of a Q-related, we have:

$$\mathrm{Sh}_q(M, w) = w(\{q\}) = Q(\emptyset, w) = 0.$$

Combining the definition of the Shapley value with definition (1) and the induction hypothesis, we obtain:

$$\begin{aligned}
0 &= \mathrm{Sh}_q(M,w) \\
&= \sum_{S\subseteq M\setminus\{q\}} \frac{(n+1-s-1)!\cdot s!}{(n+1)!}\big(w(S\cup\{q\})-w(S)\big) \\
&= \sum_{S\subseteq N} \frac{(n-s)!\cdot s!}{(n+1)!}\big(Q(S,v)-v(S)\big) \\
&= \frac{1}{n+1}\big(Q(N,v)-v(N)\big),
\end{aligned}$$

which forces $Q(N,v)=v(N)$, as desired. Conclude that $Q=Q^{\mathbf{N}}$. ■

The Shapley value does not satisfy the close axiom of Invariance from a player deletion in presence of a dummy player either, even if the dummy player cannot be obtained by a Q function.

Kamijo and Kongo [9] do not characterize the ESD value through an axiom of Invariance from a Q-player deletion. It turns out that the Equal Surplus Division cannot be characterized by invoking the following axiom of Invariance from Dummifying player deletion. This result is the analog of Theorem 4 in Kamijo and Kongo [9], which states that no value satisfies Invariance from Nullifying player deletion and Efficiency at the same time.

Invariance from dummifying player deletion For all (N,v) and all dummifying player $q\in N$ in (N,v), and all $i\in N\setminus\{q\}$, $\varphi_i(N,v)=\varphi_i(N\setminus\{q\})$.

In fact, the aforementioned statement is a consequence of a more general result based on the following axiom defined by Kamijo and Kongo [9].

Invariance from Q-player deletion For all (N,v) and all Q-player $q\in N$ in (N,v), and all $i\in N\setminus\{q\}$, $\varphi_i(N,v)=\varphi_i(N\setminus\{q\})$.

We show that if Q is $\mathcal{U}$-additive and Null for one-player null games, no value can satisfy Efficiency and Invariance from Q-related player deletion. In other words, for a given Q, within the class of values satisfying Efficiency and Balanced cycle contributions, if a value satisfies Invariance from player deletion in presence of a Q-related, then there is no value satisfying Invariance from Q-related player deletion.

Proposition 9 *Suppose that Q is $\mathcal{U}$-additive and Null for one-player null games. Then, there does not exist any value on $\mathbb{V}$ satisfying Efficiency and Invariance from Q-related player deletion.*

Proof. By way of contradiction, assume that a value φ on $\mathbb{V}$ satisfies the two axioms for some Q and assume further that Q is $\mathcal{U}$-additive and Null for one-player null games. Consider any $(N,e_{N\setminus\{i,j\}})\in\mathbb{V}$ such that $n\geq 4$, and $i\neq j$. Pick any player $q\in\{i,j\}$. On the one hand, by the properties of Q, we have:

$$e_{N\setminus\{i,j\}}(S\cup\{q\})=0=\sum_{p\in S}Q(\{p\},\mathbf{0})=Q(S,\mathbf{0})\qquad\text{for all } S\subseteq N\setminus\{q\}. \tag{14}$$

On the other hand, the $\mathcal{U}$-additivity of Q, the hypothesis $n\geq 4$, the definition of the TU-games $(S,e_{N\setminus\{i,j\}})$ for all $S\subseteq N\setminus\{q\}$ imply:

$$Q(S,e_{N\setminus\{i,j\}})=\sum_{p\in S}Q(\{p\},e_{N\setminus\{i,j\}})=\sum_{p\in S}Q(\{p\},\mathbf{0})=Q(S,\mathbf{0})\qquad\text{for all } S\subseteq N\setminus\{q\}. \tag{15}$$

Combining (14) and (15), we have:

$$e_{N\setminus\{i,j\}}(S\cup\{q\}) = Q(S, e_{N\setminus\{i,j\}}) \qquad \text{for all } S \subseteq N\setminus\{q\},$$

which proves that both i and j and Q-related players in $(N, e_{N\setminus\{i,j\}})$. Two successive applications of Invariance from Q-related player deletion yield:

$$\varphi_p(N, e_{N\setminus\{i,j\}}) = \varphi_p(N\setminus\{i\}, e_{N\setminus\{i,j\}}) = \varphi_p(N\setminus\{i,j\}, e_{N\setminus\{i,j\}}) \qquad \text{for all } p\in N\setminus\{i,j\}. \quad (16)$$

By (16) and Efficiency applied to $(N\setminus\{i,j\}, e_{N\setminus\{i,j\}})$:

$$\sum_{p\in N\setminus\{i,j\}} \varphi_p(N, e_{N\setminus\{i,j\}}) = \sum_{p\in N\setminus\{i,j\}} \varphi_p(N\setminus\{i,j\}, e_{N\setminus\{i,j\}}) = 1. \quad (17)$$

Apply Efficiency in $(N, e_{N\setminus\{i,j\}})$ and deduce from (17) that:

$$\varphi_i(N, e_{N\setminus\{i,j\}}) + \varphi_j(N, e_{N\setminus\{i,j\}}) = -1. \quad (18)$$

On the other hand, by (16), $\varphi_p(N, e_{N\setminus\{i,j\}}) = \varphi_p(N\setminus\{q\}, e_{N\setminus\{i,j\}})$ for all $q\in\{i,j\}$ and $p\in N\setminus\{q\}$. In particular, we have $\varphi_i(N, e_{N\setminus\{i,j\}}) = \varphi_i(N\setminus\{j\}, e_{N\setminus\{i,j\}})$ and $\varphi_j(N, e_{N\setminus\{i,j\}}) = \varphi_j(N\setminus\{i\}, e_{N\setminus\{i,j\}})$. Taking into account this fact, equality (17), and applying Efficiency in $(N\setminus\{j\}, e_{N\setminus\{i,j\}})$, we obtain:

$$\begin{aligned} 0 &= \sum_{p\in N\setminus\{j\}} \varphi_p(N\setminus\{j\}, e_{N\setminus\{i,j\}}) \\ &= \sum_{p\in N\setminus\{i,j\}} \varphi_p(N, e_{N\setminus\{i,j\}}) + \varphi_i(N, e_{N\setminus\{i,j\}}) \\ &= 1 + \varphi_i(N, e_{N\setminus\{i,j\}}). \end{aligned}$$

Thus, $\varphi_i(N, e_{N\setminus\{i,j\}}) = -1$. Proceeding in a similar way by deleting i instead of j in $(N, e_{N\setminus\{i,j\}})$, we find $\varphi_j(N, e_{N\setminus\{i,j\}}) = -1$, which contradicts (18). ■

6 Conclusion

Few questions remain unanswered in our study. Firstly, is it possible to use our approach to characterize other well-known values or class of values such as the Consensus value (Ju, Borm and Ruys [6]) and the Egalitarian Shapley values (Casajus and Huettner [3])? Secondly, is it possible to enlarge the definition of function Q to enable a characterization of the Shapley value through Invariance from player deletion in presence of a Q-related player? Thirdly, do our results still hold, or can we obtain new characterizations, by considering sub-classes of the class of all TU-games?

Appendix

In order to show the logical independence of the axioms used in our characterizations, for all functions Q and all games $(N,v)\in\mathbb{V}$, define $A^Q(N,v)$ as the set of players who are Q-related in (N,v). Note that for all Q, there exists some $(N,v)\in\mathbb{V}$ such that $A^Q(N,v)=\emptyset$. In fact, for any TU-game $(N,v)\in\mathbb{V}$ and any Q-related player $i\in N$ in (N,v), we have $v(\{i\})=0$ by definition of a Q-related player. Thus, a TU-game (N,v) such that $v(\{i\})\neq 0$ for all (N,v) has an empty $A^Q(N,v)$ for all functions Q. Similarly, we will denote $A^{\mathbf{Df}}(N,v)$ the set of dummifying players in (N, v), and again there exists some (N, v) such that $A^{\mathbf{Df}}(N, v) = \emptyset$.

Proposition 3

Consider any function $Q : \mathbb{V} \longrightarrow \mathbb{R}$ such that Q is $\mathcal{U}$-additive and Null for one-player null-games. The Shapley value satisfies Efficiency and Balanced cycle contributions but not Invariance from player deletion in presence of a Q-related player by Proposition 8. The Null value satisfies Balanced cycle contributions and Invariance from player deletion in presence of a Q-related player but not Efficiency. Next, construct the value Sh^Q given by:

$$\mathrm{Sh}^Q(N, v) = \begin{cases} \mathrm{Sh} & \text{if } A^Q(N, v) = \emptyset, \\ \varphi^Q & \text{if } A^Q(N, v) \neq \emptyset, \end{cases} \tag{19}$$

where φ^Q is the solution of the linear system given by (6). The value Sh^Q satisfies Efficiency. Since a Q-player in a TU-game $(N, v) \in \mathbb{V}$ remains Q-related in the sub-games induced by all coalitions he or she belongs to, Sh^Q also satisfies Invariance from player deletion in presence of a Q-related player since φ^Q satisfies this axiom by Proposition 3. Finally, it is easy to check that Sh^Q does not satisfy Balanced cycle contributions.

Proposition 4

As shown below, there is some redundancy in the axioms used in Proposition 4 since the result still holds if it is not required that the value φ satisfies Balanced cycle contributions.

Proposition 10 *There does not exist any value on $\mathbb{V}$ satisfying Efficiency and Invariance from player deletion in presence of a null player.*

Proof. By way of contradiction, suppose that a value φ on $\mathbb{V}$ satisfies the two axioms. Let $N \in U$ be such that $n \geq 3$. Consider any $S \in 2^N$ with $2 \leq s < n$, and the unanimity game (N, u_S). The players in $N \setminus S$ are null. Now, pick any player $j \in S$. Firstly, $(N \setminus \{j\}, u_S) = (N \setminus \{j\}, \mathbf{0})$. Secondly, by Invariance from player deletion in presence of a null player, we get $\varphi_p(N, u_S) = \varphi_p(N\setminus\{j\}, \mathbf{0})$ for all $p \in N\setminus\{j\}$. Since all players are null in $(N\setminus\{j\}, \mathbf{0})$, Proposition 2 yields $\varphi_p(N \setminus \{j\}, \mathbf{0}) = 0$ for all $p \in N \setminus \{j\}$, and so $\varphi_p(N, u_S) = 0$ for all $p \in N \setminus \{j\}$. By Efficiency, we necessarily have $\varphi_j(N, u_S) = 1$. Thirdly, by considering $i \in S \setminus \{j\}$ instead of $j \in S$, and proceeding as above, we also obtain $\varphi_i(N, u_S) = 1$, a contradiction. ■

Propositions 5, 6, 7

The Shapley value satisfies Efficiency, Additivity, Linearity and Balanced cycle contributions but neither Invariance from player deletion in presence of a nullifying player nor Invariance from player deletion in presence of a dummifying player. The Null value satisfies Additivity, Linearity, Balanced cycle contributions, Invariance from player deletion in presence of a nullifying player and Invariance from player deletion in presence of a dummifying player but not Efficiency. The value $\mathrm{Sh}^{Q^{\mathbf{Nf}}}$ given by (19) for $Q = Q^{\mathbf{Nf}}$ satisfies Efficiency and Invariance from player deletion in presence of a nullifying player but neither Linearity, nor Additivity, nor Balanced cycle contributions. The value $\mathrm{Sh}^{\mathbf{Df}}$ given by $\mathrm{Sh}^{\mathbf{Df}}(N, v) = \mathrm{Sh}(N, v)$ if $A^{\mathbf{Df}}(N, v) = \emptyset$, and $\mathrm{Sh}^{\mathbf{Df}}(N, v) = \mathrm{ESD}(N, v)$ if $A^{\mathbf{Df}}(N, v) \neq \emptyset$ satisfies Efficiency and Invariance from player deletion in presence of a dummifying player but neither Linearity, nor Additivity, nor Balanced cycle contributions. Any value $\varphi \in \mathcal{WD} \setminus \{\mathrm{ED}\}$ satisfies Efficiency, Additivity and Invariance from player deletion in presence of a nullifying player but not Equal treatment. Any value $\varphi \in \mathcal{WDS} \setminus \{\mathrm{ESD}\}$ satisfies Efficiency, Additivity and Invariance from player deletion in presence of a dummifying player but not Equal treatment.

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