Minimal Extending Sets in Tournaments

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Tournament solutions play an important role within social choice theory and the mathematical social sciences at large. In 2011, Brandt proposed a new tournament solution called the *minimal extending set* (*ME*) and an associated graph-theoretic conjecture. If the conjecture had been true, *ME* would have satisfied a number of desirable properties that are usually considered in the literature on tournament solutions. However, in 2013, the existence of an enormous counter-example to the conjecture was shown using a non-constructive proof. This left open which of the properties are actually satisfied by *ME*. It turns out that *ME* satisfies idempotency, irregularity, and inclusion in the iterated Banks set (and hence the Banks set, the uncovered set, and the top cycle). Most of the other standard properties (including monotonicity, stability, and computational tractability) are violated, but have been shown to hold for all tournaments on up to 12 alternatives and all random tournaments encountered in computer experiments.

1 Introduction

Many problems in the mathematical social sciences can be addressed using tournament solutions, i.e., functions that associate with each connex and asymmetric relation on a set of alternatives a non-empty subset of the alternatives. Tournament solutions are most prevalent in social choice theory, where the binary relation is typically assumed to be given by the simple majority rule (e.g., Moulin, 1986; Laslier, 1997). Other application areas include multi-criteria decision analysis (e.g., Arrow and Raynaud, 1986; Bouyssou et al., 2006), zero-sum games (e.g., Fisher and Ryan, 1995; Laffond et al., 1993; Duggan and Le Breton, 1996), and coalitional games (e.g., Brandt and Harrenstein, 2010).

Examples of well-studied tournament solutions are the Copeland set, the uncovered set, and the Banks set. A common benchmark for tournament solutions is which desirable properties they satisfy (see, e.g., Laslier, 1997; Brandt et al., 2016, for an overview of tournament solutions and their axiomatic properties).

In 2011, Brandt (2011) proposed a new tournament solution called the *minimal extending set* (*ME*) and an associated graph-theoretic conjecture, which weakens a 20-year-old conjecture by Schwartz (1990). Brandt's conjecture is closely linked to the axiomatic properties of ME in the sense that if the conjecture had held, ME would have satisfied virtually all desirable properties that are usually considered in the literature on tournament solutions. In particular, it would have been the only tournament solution known to simultaneously satisfy stability and irregularity. In 2013, however, the existence of a counter-example with about 10^{104} alternatives was shown.[1] The proof is non-constructive and uses the probabilistic method (Brandt et al., 2013). This counter-example also disproves Schwartz's conjecture and implies that the *tournament equilibrium set*—a tournament solution proposed by Schwartz (1990)—violates most desirable axiomatic properties.[2]

This left open which of the properties are actually satisfied by ME. In this paper, we resolve these open questions. In particular, we show that ME fails to satisfy monotonicity, stability, and computational tractability while it does satisfy a strengthening of idempotency, irregularity, and inclusion in the (iterated) Banks set.[3] Our negative theorems for monotonicity and stability are based on the non-constructive existence proof by Brandt et al. (2013). Concrete tournaments for which ME violates any of these properties therefore remain unknown.

2 Preliminaries

A *tournament* T is a pair $(A, \succ)$, where A is a finite set of alternatives and $\succ$ a binary relation on A, usually referred to as the *dominance relation*, that is both asymmetric ($a \succ b$ implies not $b \succ a$) and connex ($a \neq b$ implies $a \succ b$ or $b \succ a$). Thus, the dominance relation is generally irreflexive (not $a \succ a$). Intuitively, $a \succ b$ signifies that alternative a is preferable to alternative b and we denote this by an edge from a to b in our figures. The dominance relation can be extended to sets of alternatives by writing $A \succ B$ when $a \succ b$ for all $a \in A$ and $b \in B$. We also write $a \succ B$ for $\{a\} \succ B$. Moreover, for a subset of alternatives $B \subseteq A$, we will sometimes consider the restriction of the dominance relation $\succ_B = \succ \cap (B \times B)$ and write $T|_B$ for $(B, \succ_B)$. The *order* $|T|$ of a tournament $T = (A, \succ)$ refers to the cardinality of A. The set of all linear orders on some set A is denoted by $\mathcal{L}(A)$. Define the set of all transitive subsets of a tournament T as $\mathcal{B}_T = \{Q \subseteq A : \succ_Q \in \mathcal{L}(Q)\}$ whereas $\mathcal{B}_T(a) = \{Q \in \mathcal{B}_T : a \succ Q\}$ denotes the set of all transitive subsets that a dominates. In such a case, a *extends* Q, implying $Q \cup \{a\} \in \mathcal{B}_T$.

A *tournament solution* is a function S that maps a tournament $T = (A, \succ)$ to a nonempty subset of its alternatives. We write $S(B)$ instead of $S(T|_B)$ whenever the tournament T is clear from the context.

[1] The bound is $\binom{2^{15}}{30} < 10^{104}$. The weaker bound of 10^{136} mentioned by Brandt et al. (2013) stems from the estimate $\binom{2^{15}}{30} < 2^{15^{30}}$.

[2] A significantly smaller counter-example for Schwartz's conjecture consisting of only 24 alternatives was subsequently found by Brandt and Seedig (2013). However, this counter-example does not constitute a counter-example to Brandt's conjecture.

[3] Previously, the two statements on computational tractability and inclusion in the Banks set were only known to hold if the (now disproved) conjecture had been true.

Choosing from a *transitive* tournament is straightforward because every transitive tournament—and all of its subtournaments—possess a unique maximal element. In other words, the core of the problem of choosing from a tournament is the potential intransitivity of the dominance relation. Clearly, every tournament contains transitive subtournaments. For example, all subtournaments of order one or two are trivially transitive. Based on these observations, it seems natural to consider *inclusion-maximal* transitive subtournaments and collect their maximal elements in order to define a tournament solution. This tournament solution is known as the *Banks set*.[4]

Formally, the Banks set $BA(T)$ of a tournament is defined as

$$BA(T) = \{a \in A : \exists B \in \mathcal{B}_T(a) \text{ such that } \nexists b\colon b \succ B \cup \{a\}\}.$$

In many cases, the Banks set contains all alternatives of a tournament. Since there are tournaments T for which $BA(BA(T)) \subsetneq BA(T)$, one can define a series of more discriminating tournament solutions by letting $BA^1(T) = BA(T)$ and $BA^k = BA(BA^{k-1}(T))$ for all $k > 1$. The *iterated Banks set* $BA^\infty(T)$ of a tournament T is then defined as

$$BA^\infty(T) = \bigcap_{k \in \mathbb{N}} BA^k(T).$$

Due to the finiteness of T, $BA^\infty(T) = BA^{|T|}(T)$, and BA^∞ is a well-defined tournament solution.

Generalizing an idea by Dutta (1988), which in turn is based on the well-established notion of von-Neumann-Morgenstern stable sets in cooperative game theory, Brandt (2011) proposed another method for refining a tournament solution S by defining minimal sets that satisfy a natural stability criterion with respect to S.[5] A subset of alternatives $B \subseteq A$ is called *S-stable* for tournament solution S if

$$a \notin S(B \cup \{a\}) \text{ for all } a \in A \setminus B.$$

Since $S(B \cup \{a\}) = \{a\}$ if $B = \emptyset$, it follows that S-stable sets can never be empty. It has turned out that BA-stable sets, so-called *extending sets*, are of particular interest because they are strongly related to Schwartz's tournament equilibrium set and because they can be used to define a tournament solution that potentially satisfies a number of desirable properties. An extending set is *inclusion-minimal* if it does not contain another extending set. Since the number of alternatives is finite, inclusion-minimal extending sets are guaranteed to exist. The union of all inclusion-minimal extending sets defines the tournament solution ME (Brandt, 2011), i.e.,

$$ME(T) = \bigcup\{B : B \text{ is } BA\text{-stable and no } C \subsetneq B \text{ is } BA\text{-stable }\}.$$

[4] Banks's original motivation was slightly different as his aim was to characterize the set of outcomes under sophisticated voting in the amendment procedure (Banks, 1985).

[5] A well-known example is the *minimal covering set*, which is the unique minimal set that is stable with respect to the uncovered set (Dutta, 1988).

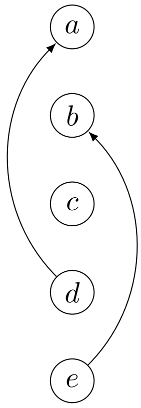

Figure 1: In this tournament, $ME(T) = \{a, b, d\}$ whereas $BA(T) = \{a, b, c, d\}$. Omitted edges point downwards.

Example 1. Consider the tournament T in Figure 1. It is easy to verify that the maximal transitive sets in T are $\{a, b, c\}$, $\{a, e, b\}$, $\{a, c, e\}$, $\{b, c, d\}$, $\{c, d, e\}$, and $\{d, a, e\}$. $\{e, b\}$ (the only nontrivial transitive subset with e as maximal element) is extended by a. Therefore, we have $BA(T) = \{a, b, c, d\}$.

We claim that $ME(T) = \{a, b, d\}$. To this end, let B be any extending set of T. Assume that $a \notin B$. Since B is non-empty and stable with respect to a, it must be the case that $d \in B$. Then, b has to be contained in B as well because no alternative could extend $\{b, d\}$. But then B cannot be stable with respect to a as there exists no alternative that could extend $\{a, b\}$. Therefore, $a \in B$ and immediately $d \in B$ (as nothing could extend $\{d, a\}$) and $b \in B$ (as nothing could extend $\{b, d\}$). It turns out that $\{a, b, d\}$ is already an extending set because $c \notin BA\{a, b, c, d\} = \{a, b, d\}$ and $e \notin BA\{a, b, d, e\} = \{a, b, d\}$. So, $\{a, b, d\}$ is the unique minimal extending set of T.

Note that $ME(T)$ is strictly contained in $BA(T)$. Tournament T is the smallest tournament for which this is the case (Brandt et al., 2015). For this particular tournament, $ME(T)$ and $BA^\infty(T)$ coincide.

We will show in Section 4.3 that

$$ME(T) \subseteq BA^\infty(T) \subseteq BA(T)$$

holds for all tournaments T and both inclusions may be strict.[6]

3 Minimal extending sets

Minimal extending sets satisfy a number of interesting properties.

[6]An analogous inclusion chain is known for the uncovered set, the iterated uncovered set, and the minimal covering set (see, e.g., Laslier, 1997).

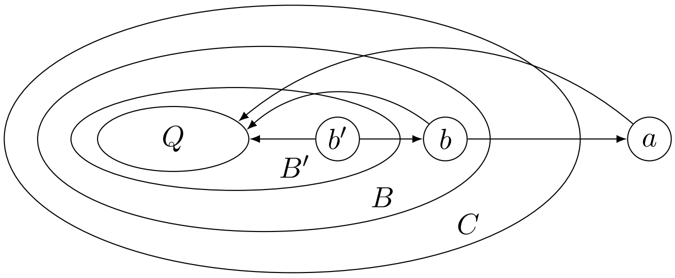

Figure 2: Minimal extending sets remain minimal extending in subtournaments (Lemma 1).

First, it is obvious that an extending set remains an extending set when outside alternatives are removed. Moreover, when the set was a *minimal* extending set, it is still minimal in the reduced tournament.

Lemma 1. *Let $T = (A, \succ)$ be a tournament, $B \subseteq A$ an extending set in T, and $C \subseteq A$ such that $B \subseteq C$. Then, B is also an extending set in $T|_C$. Moreover, if B is a minimal extending set in T, then B is also a minimal extending set in $T|_C$.*

Proof. First assume that B is an extending set in T. To prove that B is an extending set also in $T|_C$, consider an arbitrary $a \in C \setminus B$. Then, $a \in A \setminus B$ and, because B is an extending set in T, also $a \notin BA(B \cup \{a\})$. The result then follows immediately.

Now let B be a *minimal* extending set in T. As we have just seen, B is also an extending set in $T|_C$. To see that B is then also a *minimal* extending set in $T|_C$, assume for contradiction that there is a $B' \subsetneq B$ such that B' is extending in $T|_C$. As B' is not extending in T, there exist a set $Q \subseteq B'$ and an alternative $a \in A \setminus B'$ such that $Q \cup \{a\}$ is maximal in $\mathcal{B}_{T|_{B' \cup \{a\}}}$ and $a \succ Q$. Having assumed that B' is extending in $T|_C$, it follows that $a \in A \setminus C$. See Figure 2 for an illustration of the situation. However, since B is an extending set in T, there is a $b \in B$ with $b \succ Q \cup \{a\}$. As $Q \cup \{a\}$ is maximal in $\mathcal{B}_{T|_{B' \cup \{a\}}}$, it follows that $b \in B \setminus B'$. Now observe that $Q \cup \{b\} \in \mathcal{B}_{T|_{B' \cup \{b\}}}$ and recall that B' is extending in $T|_C$. Accordingly, there is a $b' \in B'$ with $b' \succ Q \cup \{b\}$. Now either $a \succ b'$ or $b' \succ a$. In either case $Q \cup \{a\}$ is not maximal in $\mathcal{B}_{T|_{B' \cup \{a\}}}$ and a contradiction entails. $\square$

Lemma 1 implies that minimal extending sets satisfy what is usually called *internal S-stability*: For every minimal extending set B, $BA(B) = B$.

Secondly, from the definition of extending sets, it is immediate that a minimal extending set is unaffected by modifying the dominance relation among outside alternatives.

Lemma 2. *Let $T = (A, \succ)$ and $T' = (A, \succ')$ be tournaments such that $B \subseteq A$ is a minimal extending set in T and for all $a \in A$ and $b \in B$, $a \succ' b$ if and only if $a \succ b$. Then B is a minimal extending set in T'.*

Brandt (2011) conjectured that every tournament contains a unique minimal extending set and showed that, if the conjecture holds, ME satisfies a large number of desirable properties. In 2013, however, Brandt et al. (2013) disproved the conjecture by a non-constructive argument showing the existence of tournaments with more than one minimal extending set.

Lemma 3 (Brandt et al., 2013)**.** *There is a tournament with more than one minimal extending set.*

The existence proof consists of two steps. First, Brandt et al. (2013) have shown that there is a tournament T^{CF} with two disjoint "chain-free" subsets of alternatives A_1 and A_2 such that for every transitive set B_1 in $\mathcal{B}_{A_1}$, there is an a_2 in A_2 such that $a_2 \succ B_1$ and, conversely, every $B_2 \in \mathcal{B}_{A_2}$ is extended by an $a_1 \in A_1$.

Second, two isomorphic copies of T^{CF} on alternative sets X and Y are combined into T^{ME} with disjoint subsets X_1, X_2 and Y_1, Y_2. The dominance relation between X and Y is depicted in Figure 3. Since X and Y both are extending sets and disjoint, T^{ME} contains multiple minimal extending sets.

Some of our proofs require the existence of a tournament with exactly two minimal extending sets that are disjoint. It is unknown whether the tournaments described by Brandt et al. (2013) satisfy this property. To this end, we first show that every tournament with multiple minimal extending sets contains a subtournament that can be partitioned into exactly two minimal extending sets and that contains no other minimal extending sets.[7]

Lemma 4. *Let $T = (A, \succ)$ be a tournament with multiple minimal extending sets. Then, there is $A' \subseteq A$ such that $T|_{A'}$ contains exactly two minimal extending sets B_1 and B_2 with $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = A'$.*

Proof. Let $T = (A, \succ)$ be a tournament with distinct minimal extending sets B_1, B_2 such that $C = B_1 \cap B_2 \neq \emptyset$. First, we show how to find a subtournament whose minimal extending sets are mutually disjoint. Due to minimality of B_1 (and B_2), C is not an extending set. Hence, there have to be $Q \subseteq C$ and $a \in A \setminus C$ such that $a \succ Q$ and

[7]The proof is based on an argument by Brandt (2011, Lemma 2).

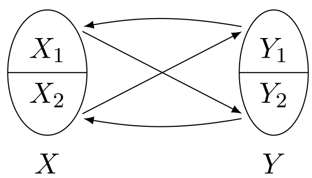

Figure 3: The structure of the tournament T^{ME} with two disjoint extending sets X and Y as described by Brandt et al. (2013). The two subtournaments $T^{ME}|_X$ and $T^{ME}|_Y$ are isomorphic. Their exact order is unknown and may be as large as 10^{104}.

$Q \cup \{a\}$ is maximal in $\mathcal{B}_{C\cup\{a\}}$ and cannot be extended by an alternative in C. Define $B_1' = \{b \in B_1 : b \succ Q\}$ and $B_2' = \{b \in B_2 : b \succ Q\}$.

Assume without loss of generality that $a \notin B_1$. Then there has to be a $b_1 \in B_1$ that extends $\{a\} \cup Q$ because B_1 is an extending set, i.e., B_1' is not empty. To show that B_1' and B_2' are disjoint, assume for contradiction that there is a $b \in B_1' \cap B_2'$. It is easy to check that no matter whether $a \succ b$ or $b \succ a$, $Q \cup \{a, b\} \in \mathcal{B}_{C\cup\{a\}}$ and thus $Q \cup \{a\}$ is not maximal in $\mathcal{B}_{C\cup\{a\}}$. Hence, $B_1' \cap B_2' = \emptyset$ and by stability of B_2, there has to be a $b_2 \in B_2$ that extends $Q \cup \{b_1\}$, i.e., B_2' is non-empty as well. The situation is depicted in Figure 4.

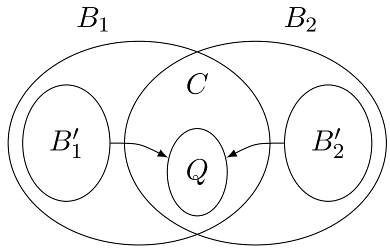

Figure 4: Relevant subsets in the argument to construct a tournament with disjoint minimal extending sets B_1', B_2', given a tournament with overlapping minimal extending sets B_1, B_2 (Lemma 4).

Next, we show that B_1' and B_2' are extending sets in $T|_{B_1'\cup B_2'\cup Q}$. To this end, consider $a' \in B_2'$ and R a maximal transitive subset of $B_1' \cup Q$ such that $a' \succ R$. It is easy to see that $Q \subseteq R$ due to $B_1' \succ Q$, $B_2' \succ Q$, and maximality of R. As B_1 is an extending set in T, there has to be a $c \in B_1$ that extends $R \cup \{a'\}$. By $c \succ Q \subseteq R$, c is contained in B_1', i.e., B_1' (and analogously B_2') is an extending set in $T|_{B_1'\cup B_2'\cup Q}$. Due to Lemma 1, B_1' and B_2' are also extending sets in $T' = T_{B_1'\cup B_2'}$, which is of strictly smaller order than T.

Repeated application yields a subset $A' \subseteq A$ such that $T|_{A'}$ only contains mutually disjoint minimal extending sets. Now let B_1 and B_2 be two minimal extending sets of $T|_{A'}$. By Lemma 1, B_1 and B_2 are still minimal extending sets in $T|_{B_1\cup B_2}$. □

The following insight will later prove useful when reasoning about *ME*.

Corollary 1. *Let $T^* = (A, \succ^*)$ be a smallest tournament with multiple minimal extending sets. Then T^* contains exactly two minimal extending sets that partition A.*

For the remainder of this article, let T^* be such a tournament of minimal order with multiple extending sets.[8] Interestingly, the size of this tournament is unknown. Exhaustive analysis has found no such tournament with up to 12 alternatives. Hence, the size of a minimal counter-example must be at least 13 and at most 10^{104}.

[8]The proof of Lemma 4 shows that T^* contains exactly three extending sets: the two minimal extending sets and the set of all alternatives.

4 Properties of ME

We analyze ME with respect to two different types of properties: dominance-based properties and choice-theoretic properties. Both serve as important benchmarks for the evaluation of decision-theoretic and choice-theoretic concepts. We furthermore investigate ME's relationship to other tournament solutions. Finally, we also give lower and upper bounds on the computational complexity of deciding whether an alternative is in ME for a given tournament.

4.1 Dominance-based properties

In this section, we consider two properties that are based on the dominance relation. The first property is called *monotonicity* and corresponds to a well-established standard condition in social choice theory. It prescribes that a chosen alternative should still be chosen if it is reinforced. Formally, a tournament solution S satisfies monotonicity if $a \in S(T)$ implies $a \in S(T')$ for all tournaments $T = (A, \succ)$, $T' = (A, \succ')$, and $a \in A$ such that $\succ_{A\setminus\{a\}} = \succ'_{A\setminus\{a\}}$ and for all $b \in A \setminus \{a\}$, $a \succ b$ implies $a \succ' b$. Equivalently, monotonicity can be defined by requiring that unchosen alternatives remain unchosen when they are weakened.

The second property, *independence of unchosen alternatives*, states that the choice set should be unaffected by changes in the dominance relation between unchosen alternatives. Formally, a tournament solution S is independent of unchosen alternatives if $S(T) = S(T')$ for all tournaments $T = (A, \succ)$ and $T' = (A, \succ')$ such that for all $a \in S(T)$, $b \in A$, $a \succ b$ if and only if $a \succ' b$.

BA satisfies monotonicity, but violates independence of unchosen alternatives, and BA^∞ violates both properties. As it turns out, this is also the case for ME.

Theorem 1. *ME does not satisfy monotonicity and independence of unchosen alternatives.*

Proof. Consider $T^* = (A, \succ^*)$ from Corollary 1 with its two (disjoint) minimal extending sets B_1 and B_2 and alternatives $b_1 \in B_1, b_2 \in B_2$ with $b_2 \succ^* b_1$. Let $T^*_{b_1} = (A, \succ')$ be the modified tournament such that $T^*|_{A\setminus\{b_1\}} = T^*_{b_1}|_{A\setminus\{b_1\}}$ and $b_1 \succ' B_1 \setminus \{b_1\}$, i.e., b_1 now dominates all other alternatives in B_1. By Lemma 2, B_2 is still a minimal extending set in $T^*_{b_1}$. By minimality of T^*, $T^*_{b_1}$ can only have at most one more minimal extending set, which furthermore, by Lemma 4, has to be B_1. However, B_1 itself is no extending set in $T^*_{b_1}$ as no alternative in B_1 extends $\{b_2, b_1\}$. Hence, $ME(T^*_{b_1}) = B_2$, i.e., the strengthened alternative b_1 is no longer contained in ME.

Tournaments $T^*_{b_1}$ and T^* are also witness to the fact that independence of unchosen alternatives is violated. □

An interesting aspect of minimal extending sets is that, by Lemma 2, they satisfy a local variant of independence of unchosen alternatives, a property that their union (ME) fails to satisfy. Minimal extending sets also satisfy a local variant of monotonicity: a minimal extending set is unaffected by weakening outside alternatives.

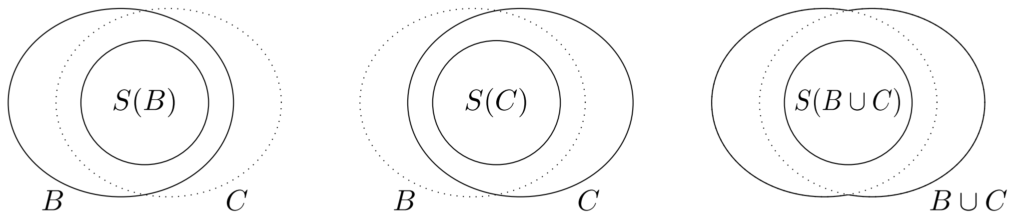

Figure 5: A stable tournament solution S chooses a set from $B \cup C$ (right) if and only if it chooses the same set from both B (left) and C (middle).

4.2 Choice-theoretic properties

An important class of properties concerns the consistency of choice and relate choices from different subtournaments of the same tournament to each other. A relatively strong property of this type is *stability* (or self-stability) (Brandt and Harrenstein, 2011), which requires that a set is chosen from two different sets of alternatives if and only if it is chosen from the union of these sets (see Figure 5).[9]

Definition 1. A tournament solution S is *stable* if for all tournaments $(A, \succ)$ and for all non-empty subsets $B, C, X \subseteq A$ with $X \subseteq B \cap C$,

$$X = S(B) = S(C) \quad \text{if and only if} \quad X = S(B \cup C).$$

Stability is a demanding property that is neither satisfied by BA nor by BA^∞. Three well-known tournament solutions that are stable are the top cycle, the minimal covering set, and the bipartisan set. Stability is closely connected to rationalizability (Brandt and Harrenstein, 2011) and together with monotonicity implies a weak notion of strategyproofness (Brandt, 2015).

Stability can be factorized into conditions $\widehat{\alpha}$ and $\widehat{\gamma}$ by considering each implication in the above equivalence separately. The former is also known as Chernoff's *postulate* 5* (Chernoff, 1954), the *strong superset property* (Bordes, 1979), *outcast* (Aizerman and Aleskerov, 1995), and the *attention filter axiom* (Masatlioglu et al., 2012).[10] A tournament solution S satisfies $\widehat{\alpha}$, if for all non-empty sets of alternatives B and C,

$$S(B \cup C) \subseteq B \cap C \text{ implies } S(B \cup C) = S(B) = S(C).$$

Equivalently, S satisfies $\widehat{\alpha}$ if for all sets of alternatives B and C,

$$S(B) \subseteq C \subseteq B \text{ implies } S(B) = S(C).$$

[9] The term self-stability originates from the fact that a tournament solution S is (self-)stable if and only if it returns the unique minimal S-stable set for every tournament T (cf. Brandt and Harrenstein, 2011, Th. 3).

[10] We refer to Monjardet (2008) for a more thorough discussion of the origins of this condition.

A tournament solution S satisfies $\widehat{\gamma}$, if for all sets of alternatives B and C,

$$S(B) = S(C) \text{ implies } S(B \cup C) = S(B) = S(C).$$

For a finer analysis, we split $\widehat{\alpha}$ and $\widehat{\gamma}$ into two conditions (Brandt and Harrenstein, 2011, Remark 1).

Definition 2. A tournament solution S satisfies

- $\widehat{\alpha}_{\subseteq}$ if for all B, C, it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \subseteq S(B)$,[11]
- $\widehat{\alpha}_{\supseteq}$ if for all B, C, it holds that $S(B) \subseteq C \subseteq B$ implies $S(C) \supseteq S(B)$,
- $\widehat{\gamma}_{\subseteq}$ if for all B, C, it holds that $S(B) = S(C)$ implies $S(B) \subseteq S(B \cup C)$, and
- $\widehat{\gamma}_{\supseteq}$ if for all B, C, it holds that $S(B) = S(C)$ implies $S(B) \supseteq S(B \cup C)$.

Obviously, for any tournament solution S we have

$$\begin{aligned} S \text{ satisfies stability} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\alpha} \text{ and } \widehat{\gamma}, \text{ and} \\ S \text{ satisfies } \widehat{\alpha} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\alpha}_{\subseteq} \text{ and } \widehat{\alpha}_{\supseteq}, \text{ and} \\ S \text{ satisfies } \widehat{\gamma} &\quad \text{if and only if} \quad S \text{ satisfies } \widehat{\gamma}_{\subseteq} \text{ and } \widehat{\gamma}_{\supseteq}. \end{aligned}$$

A tournament solution is *idempotent* if the choice set is invariant under repeated application of the solution concept, i.e., $S(S(A)) = S(A)$ for all tournaments $T = (A, \succ)$. It is easily seen that $\widehat{\alpha}_{\supseteq}$ is stronger than idempotency since $S(T|_{S(T)}) \supseteq S(T)$ implies $S(T|_{S(T)}) = S(T)$.
Figure 6 shows the logical relationships between stability and its weakenings.

Theorem 2. *ME satisfies (i) $\widehat{\alpha}_{\supseteq}$ but neither (ii) $\widehat{\alpha}_{\subseteq}$ nor (iii) $\widehat{\gamma}_{\supseteq}$.*

Proof. We show each statement separately. For (i), let $T = (A, \succ)$ be a tournament with minimal extending sets $B_1, \ldots, B_k$ and let $C \subseteq A$ such that $ME(T) = B_1 \cup \ldots \cup B_k \subseteq C$. By Lemma 1, every B_i is still a minimal extending set in $T' = T|_C$. Hence, $ME(T') \supseteq ME(T)$.

For (ii), consider again $T^* = (A, \succ^*)$ from Corollary 1 with its two minimal extending sets B_1 and B_2. We create a larger tournament T^*_x by adding an alternative x such that $B_1 \succ x$ and $x \succ B_2$. This tournament is depicted in Figure 7. Clearly, B_1 still is a minimal extending set and we claim that there is no other. Assume for contradiction that there is another minimal extending set $B' \neq B_1$ in T^*_x. If $x \notin B'$ then B' is also a minimal extending set in T^* by Lemma 1. As T^* has no minimal extending sets besides B_1 and B_2, it follows that $B' = B_2$. But B_2 cannot be an extending set in T^*_x because for any $b_2 \in B_2$ there is no $b_2' \in B_2$ that extends $\{x, b_2\}$. If, on the other hand, $x \in B'$, consider $T_{B_1 \cup B'}$. Since $B_1 \setminus B' \succ x$, x does not contribute anything to B' being an extending set; it is dominated by all outside alternatives. Hence, $B' \setminus \{x\}$ is a minimal extending set in $T' = T_{B_1 \cup B' \setminus \{x\}}$. Moreover, $B' \cap B_1 \neq \emptyset$ since otherwise $\{b_1, x\}$ cannot

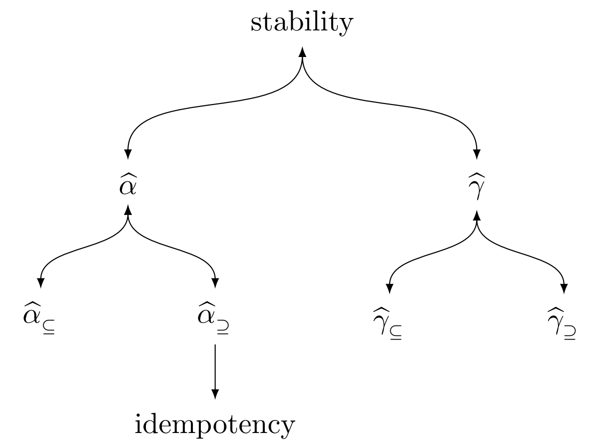

Figure 6: Logical relationships between choice-theoretic properties.

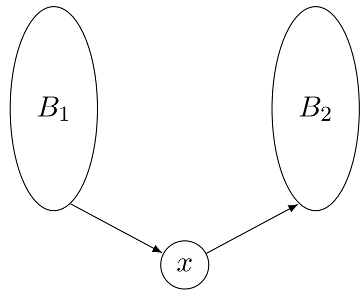

Figure 7: Structure of the tournament T^*_x used to show that ME violates $\widehat{\alpha}_{\subseteq}$. Without alternative x, this is a minimal tournament T^* among those with multiple minimal extending sets (Theorem 2).

be extended from within B' for all $b_1 \in B_1$ due to $x \succ B_2$. Therefore T' contains two overlapping minimal extending sets and $|T'| \leq |T^*|$. This contradicts Corollary 1.

For (iii), consider $T^* = (A, \succ^*)$ with its minimal extending sets B_1 and B_2. For all $b \in B_2$, let $T^*_b = (B_1 \cup \{b\}, \succ^*_{B_1 \cup \{b\}})$. By Lemma 1, B_1 is still a minimal extending set in all T^*_b. There cannot be another minimal extending set B_2 (containing b) because otherwise T^*_b would have multiple extending sets, contradicting the minimality of T^*. Therefore $ME(T^*_b) = B_1$ for all $b \in B_2$. Note that $A = \bigcup_{b \in B_2} B_1 \cup \{b\}$. If $\widehat{\gamma}_{\supseteq}$ held for ME, then repeated application of $\widehat{\gamma}_{\supseteq}$ would imply $ME(T^*) \subseteq ME(T^*_b) = B_1$. However, $ME(T^*) = A$, a contradiction, which concludes the proof. $\square$

It is open whether ME satisfies $\widehat{\gamma}_{\subseteq}$. Nevertheless, the fact that ME violates $\widehat{\alpha}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ immediately implies that it violates stability.

Corollary 2. *ME does not satisfy $\widehat{\alpha}$ and $\widehat{\gamma}$ and is therefore not stable.*

[11] $\widehat{\alpha}_{\subseteq}$ has also been called the *weak superset property* or the *Aïzerman property* (e.g., Laslier, 1997; Brandt, 2009).

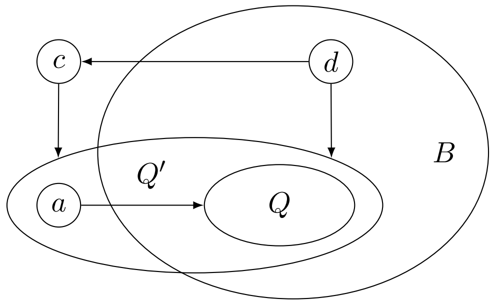

Figure 8: Extending sets remain extending sets when removing Banks losers (Lemma 5).

From Theorem 2, we also get that, in contrast to BA, ME is idempotent.

Corollary 3. *ME satisfies idempotency.*

Interestingly, minimal extending sets themselves are stable sets. They satisfy a *local* version of stability,[12] namely

- removing alternatives outside of a minimal extending set has no effect because of Lemma 1 (local $\widehat{\alpha}$), and
- a set that is a minimal extending set in several tournaments is also a minimal extending set in the union of these by the definition of minimal extending sets (local $\widehat{\gamma}$).

4.3 Relationships to other tournament solutions

Besides the axiomatic properties of ME, we are also interested in its set-theoretic relationships to other tournament solutions. Assuming that every tournament has only one minimal extending set, Brandt (2011) showed that ME always selects subsets of BA and subsets of the minimal covering set. Under an even stronger conjecture, he also proved that ME always selects supersets of the tournament equilibrium set (Schwartz, 1990). Since the conjectures turned out to be incorrect (Lemma 3), these questions are open again. We can now answer one of these in the affirmative, namely that ME indeed chooses from BA. To this end, we prove a more general statement about the relationship between extending sets and BA.[13]

Lemma 5. *For all tournaments T, $ME(T) \subseteq ME(BA(T))$.*

[12] Interestingly, this is *not* the case for TEQ-retentive sets, which are otherwise quite similar to extending sets.

[13] It even holds that $ME(T) = ME(BA(T))$ for all tournaments T, but this stronger statement is not required for Theorems 3 and 4.

Proof. Let $T = (A, \succ)$ be a tournament. It suffices to show for every extending set $B \subseteq A$ in T, $B \cap BA(T)$ is also an extending set in $T|_{BA(T)}$. Let $a \in A \setminus B$ and $Q \in \mathcal{B}_{T|_B}(a)$ such that Q is inclusion-maximal in $\mathcal{B}_{T|_B}(a)$. Since B is an extending set, there has to be some inclusion-maximal set $Q' \in \mathcal{B}(T|_{B\cup\{a\}})$ with $Q \cup \{a\} \subsetneq Q'$. Let b be the maximal element of Q', i.e., $b \succ Q' \setminus \{b\}$. We claim that $b \in BA(T)$ (and therefore b extends $Q \cup \{a\}$ in $T|_{BA(T)}$). Suppose otherwise, then there exists $c \in A$ such that $c \succ Q'$. Since Q' is inclusion-maximal in $T|_{B\cup\{a\}}$, $c \notin B$. By virtue of B being an extending set, there has to be some $d \in B$ such that $d \succ (Q' \setminus \{a\}) \cup \{c\}$. Since Q is inclusion-maximal in $\mathcal{B}_{T|_B}(a)$, $a \not\succ d$ and consequently $d \succ a$ and $d \succ Q'$. This, however, contradicts the inclusion-maximality of Q' in $T|_{B\cup\{a\}}$ as illustrated in Figure 8. Hence, $b \in B \cap BA(T)$ and b extends $Q \cup \{a\}$, which concludes the proof. $\square$

Since $ME(BA(T)) \subseteq BA(T)$, Lemma 5 implies that ME is contained in BA.

Theorem 3. *For all tournaments T, $ME(T) \subseteq BA(T)$.*

Moreover, by repeated application of Lemma 5 and the finiteness of the set of alternatives, it follows that ME is contained in BA^∞.

Theorem 4. *For all tournaments T, $ME(T) \subseteq BA^\infty(T)$.*

It is easy to construct tournaments T in which $ME(T) \neq BA^\infty(T)$ (see Figure 9).

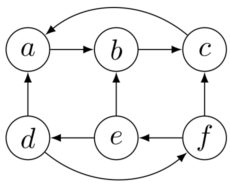

Figure 9: In this tournament, $ME(T) = \{a, b, c\}$ whereas $BA^\infty(T) = \{a, b, c, d, e, f\}$. Omitted edges point downwards.

From ME's inclusion in BA, it follows immediately that ME always chooses subsets of the uncovered set and the top cycle. It remains open whether the tournament equilibrium set is always contained in ME.

A tournament $T = (A, \succ)$ is *regular* if $|\{z : x \succ z\}| = |\{z : y \succ z\}|$ for all $x, y \in A$. A tournament solution is *regular* if $S(T) = A$ for all regular tournaments $T = (A, \succ)$. Finally, a tournament solution is *irregular* if it fails to be regular. Laslier (1997, Theorem 7.1.3) showed that BA is irregular by constructing a corresponding tournament on 45 alternatives. By Theorem 3, ME also has to be irregular. In Figure 10, we give a smallest regular tournament in which ME and BA can be seen to be irregular. The tournament is of order 13 and was found by exhaustively checking all tournaments of increasing order where the tournaments were generated with the help of McKay's *nauty* package (McKay and Piperno, 2013).

Theorem 5. *ME is irregular.*

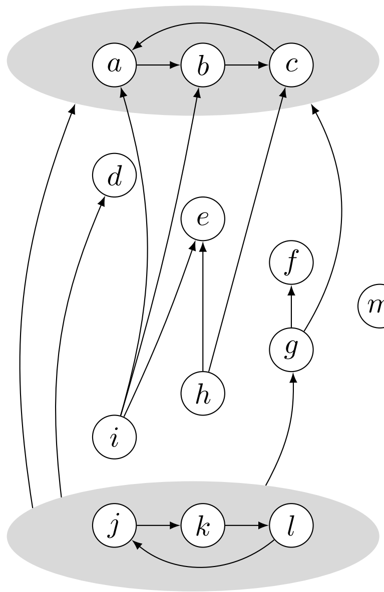

Figure 10: A regular tournament on 13 vertices. Omitted edges point downwards. Alternative m is not in BA and thereby not in ME. This is the smallest tournament in which ME (and BA) is irregular.

4.4 Computational complexity

An important property of every tournament solution is whether it can be computed efficiently. This is typically phrased as a decision problem that asks whether a given alternative is contained in the choice set of a given tournament. Hardness of the decision problem implies hardness of computing the tournament solution because if there was an efficient algorithm for computing the choice set, this algorithm could be used to efficiently solve the decision problem. While it is known that the decision problem for BA is NP-hard (and therefore computationally intractable) (Woeginger, 2003), this has no immediate consequence on the complexity of the problem for ME. For example, Hudry (2004) has pointed out that individual alternatives in BA can be found efficiently (in linear time). We show that computing ME is indeed NP-hard.

Theorem 6. *Deciding whether an alternative in a tournament is contained in ME is NP-hard.*

Proof. We reduce from the NP-complete problem 3SAT using the same construction (but a different proof) that was used to show the hardness of computing the tournament equilibrium set (Brandt et al., 2010). For φ a formula in 3-conjunctive normal form given by

$$(x_1^1 \vee x_1^2 \vee x_1^3) \wedge \cdots \wedge (x_m^1 \vee x_m^2 \vee x_m^3),$$

Brandt et al. (2010) construct a tournament $T_\varphi = (U \cup C, \succ)$ with $C = \{c_0, c_1, \ldots, c_{4m-3}\}$ and $U = \bigcup_{1 \le i \le 4m-3} U_i$, where

$$U_i = \begin{cases} \{u_i^1, u_i^2, u_i^3\} & \text{if } i \text{ is odd,} \\ \{u_i\} & \text{if } i \text{ is even.} \end{cases}$$

Furthermore, the dominance relation $\succ$ is defined such that, for all i, j, k, and k' with $0 \le i, j \le 4m-3$ and $1 \le k < k' \le m$, the following conditions hold:

(*i*) $c_i \succ c_j$ if and only if $i > j$,

(*ii*) $c_i \succ U_j$ if and only if $i \ne j$

(*iii*) $u_i^1 \succ u_i^2 \succ u_i^3 \succ u_i^1$ whenever i is odd,

(*iv*) $u_{4k-1}^l \succ u_{4k-3}^{l'}$ whenever $l \ne l' \quad (l, l' \in \{1, 2, 3\})$,

(*v*) $u_{4k'-3}^{l'} \succ u_{4k-3}^l$ whenever $x_k^l = \neg x_{k'}^{l'}$ or $x_{k'}^{l'} = \neg x_k^l \quad (l, l' \in \{1, 2, 3\})$,

(*vi*) $u_i \succ u_j$ whenever $i < j$, for all $u_i \in U_i$ and $u_j \in U_j$ to which (*iv*) and (*v*) do not apply.

If $i = 4k-3$ for some $1 \le k \le 4m-3$, we say that U_i is a set of *clause alternatives*, whereas, if $i = 4k-1$ for some $1 \le k \le 4m-3$, U_i is said to be a set of *channeling alternatives*. A clause alternative u and a channeling alternative u' are said to be *aligned* whenever $u = u_{4k-3}^l$ and $u' = u_{4k-1}^{l'}$ for some $k \ge 0$ and $l = l'$. If i even, the alternative in U_i is called a *separating alternative*. A complete example of such a tournament T_φ is shown in Figure 11. We prove that

$$\varphi \text{ is satisfiable} \quad \text{if and only if} \quad c_0 \in ME(T_\varphi).$$

First assume that φ is not satisfiable. Then, by a simple argument (Brandt et al., 2010, proof of Theorem 2) we have $c_0 \notin BA(T_\varphi)$. From Theorem 3, it also follows that $c_0 \notin ME(T_\varphi)$.

For the opposite direction, assume that φ is satisfiable and let α be the witnessing assignment. We prove that $c_0 \in ME(T_\varphi)$. Let

$$U^- = \bigcup_{1 \le k \le m} \{u_{4k-3}^j : x_k^j \text{ is false in } \alpha\} \quad \text{and} \quad U^+ = U \setminus U^-.$$

We introduce, moreover, for every i with $1 \le i \le 4m-3$, the following notations:

$$\begin{aligned} U_i^+ &= U^+ \cap U_i & \qquad U_{\ge i} &= U_i \cup \cdots \cup U_{4m-3}, \\ C_{\ge i} &= C_i \cup \cdots \cup C_{4m-3} & \qquad U_{\ge i}^+ &= U_i^+ \cup \cdots \cup U_{4m-3}^+. \end{aligned}$$

We say that a subset $V \subseteq U$ is *i-leveled* if $V \cap U_j \ne \emptyset$ for all j with $4m-3 \ge j \ge i$. As α is a satisfying assignment, we then have that U^+ is 1-leveled. Moreover, for every i with

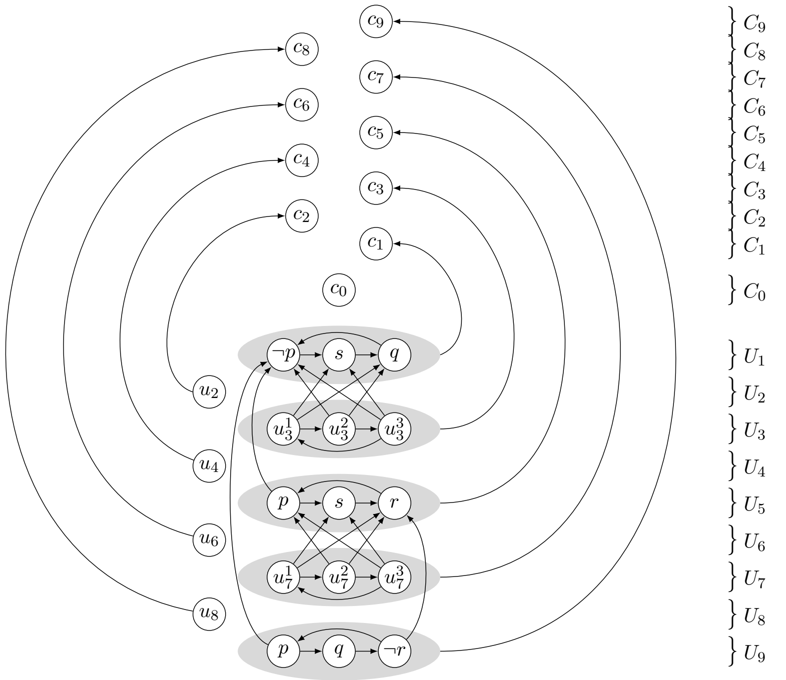

Figure 11: The tournament T_φ for $\varphi = (\neg p \vee s \vee q) \wedge (p \vee s \vee r) \wedge (p \vee q \vee \neg r)$ as used in the proof of Theorem 6. Every alternative u^l_{4k-3} is denoted by the literal x^l_k it represents and omitted edges point downwards. Furthermore, e.g., U_5 is a set of clause alternatives, u_6 is a separating alternative, and u^2_7 is a channeling alternative.

$4m-3 \geq i \geq 1$ and all $u \in U^+_i$, there is a transitive i-leveled set $X_u \subseteq U^+_{\geq i}$ including u as maximal element. To see this, observe that we can always choose $X_u \subseteq U^+_{\geq i}$ such that X includes u, chooses exactly one alternative from U_j for every $j \geq i$, and all clause and channeling alternatives in X are aligned. The latter can always be guaranteed as all channeling alternatives are included in U^+. Moreover, α sets each propositional variable either to true or to false, but not both, and, therefore, clause (v) above does not apply to any pair of clause alternatives u^l_{4k-3} and $u^{l'}_{4k'-3}$ in X_u. Hence, for all $u, u' \in X_u$ with $u \in U_j$, $u' \in U_{j'}$, it holds that $u \succ u'$ whenever $j < j'$. It can thus easily be appreciated that X_u has to be transitive.

Let B be a minimal extending set of T_φ. We first show by induction on i that for all i with $4m-3 \geq i \geq 1$ we have

$$C_{\geq i} \cup U^+_{\geq i} \subseteq B.$$

For the basis, i.e., if $i = 4m-3$, we show that $c_{4m-3} \in B$ as well as that $U_{4m-3} \subseteq B$. To prove the former, first assume for a contradiction that $B \cap C = \emptyset$. Then, $B \subseteq U$ and $B \neq \emptyset$. Observe, however, that $c_0 \succ U$. Accordingly, there is some $u \in U \cap B$ with $c_0 \succ u$ but no $u' \in U$ with $u' \succ c_0$ and $u' \succ u$. Having assumed that B is an extending set, it follows that $c_0 \in B$, a contradiction. We may conclude that there is some $c \in C$

with $c \in B$. If $c = c_{4m-3}$, we are done. Otherwise, $c_{4m-3} \succ c$. Observe that there is no alternative $a \in C \cup U$ with both $a \succ c_{4m-3}$ and $a \succ c$. It follows that $c_{4m-3} \in B$.

Second, we show that for each $u \in U_{4m-3}$ we have $u \in B$. Consider an arbitrary $u \in U_{4m-3}$. Without loss of generality we may assume that $u = u^1_{4m-3}$ and, for a contradiction, that $u^1_{4m-3} \notin B$. Also consider u^2_{4m-3} and observe that $u^2_{4m-3} \succ c_{4m-3}$. If $u^2_{4m-3} \notin B$, then u^1_{4m-3} is the *only* alternative a in $C \cup U$ with both $a \succ u^2_{4m-3}$ and $a \succ c_{4m-3}$. As $c_{4m-3} \in B$ by the previous argument and B is an extending set, it then follows that $u^1_{4m-3} \in B$. If, on the other hand, $u^2_{4m-3} \in B$, then $\{u^2_{4m-3}, c_{4m-3}\} \subseteq B$. Now observe that both $u^1_{4m-3} \succ u^2_{4m-3}$ and $u^1_{4m-3} \succ c_{4m-3}$. Moreover, there is no alternative $a \in C \cup U$ with $a \succ \{u^1_{4m-3}, u^2_{4m-3}, c_{4m-3}\}$. As B is an extending set, it again follows that $u^1_{4m-3} \in B$.

For the induction step, we may assume that $C_{\geq i} \cup U^+_{\geq i} \subseteq B$ (induction hypothesis) and it suffices to show that $\{c_{i-1}\} \cup U^+_{i-1} \subseteq B$.

To see that $c_{i-1} \in B$ recall that U^+ is i-leveled and, hence, there is some $u \in U^+_i$. Consider this u along with the set $X_u \subseteq U^+_{\geq i}$. By the induction hypothesis we have that $X_u \subseteq B$. We first show that there is some $c \in \{c_{i-1}, \ldots, c_0\}$ with $c \in B$. To see this, assume for contradiction that $B \cap \{c_{i-1}, \ldots, c_0\} = \emptyset$. Then, in particular, $c_0 \notin B$. Moreover, $c_0 \succ X_u$. Observe that $c_0 \succ u'$ for every $u' \in U$ and, therefore, no alternative in U extends $\{c_0\} \cup X_u$. As X_u is i-leveled, moreover, there is no $c \in C_{\geq i}$ that extends $\{c_0\} \cup X_u$. Having assumed that $B \cap \{c_{i-1}, \ldots, c_0\} = \emptyset$, it follows that B is not extending, a contradiction. Accordingly, there is some $c \in B \cap \{c_{i-1}, \ldots, c_0\}$. If $c = c_{i-1}$ we are done. Otherwise, note that $c_{i-1} \succ \{c\} \cup X_u$. Observe, however, that there is no alternative $a \in C \cup U$ with $a \succ \{c_{i-1}, c\} \cup X_u$. As $\{c\} \cup X_u \subseteq B$ and B is extending, it follows that $c_{i-1} \in B$ as well.

Now consider arbitrary U^+_{i-1} and let u_{i-1} be an arbitrary alternative in U^+_{i-1}. It remains to be proved that $u_{i-1} \in B$. One of the following holds:

(i) $U_{i-1} = \{u_{i-1}\}$, where u_{i-1} is a separating alternative.

(ii) U_{i-1} is a set of clause alternatives.

(iii) U_{i-1} is a set of channeling alternatives.

If *(i)*, observe that u_{i-1} is the maximal element of the transitive set $\{c_{i-1}\} \cup X_{u_{i-1}}$. As $X_{u_{i-1}} \setminus \{u_{i-1}\} \subseteq U^+_{\geq i}$, by the induction hypothesis and the previous argument, we have that $\{c_{i-1}\} \cup (X_{u_{i-1}} \setminus \{u_{i-1}\}) \subseteq B$. Observe, however, that in this case, there is no alternative $a \in C \cup U$ that extends $\{c_{i-1}\} \cup X_{u_{i-1}}$, i.e., such that $a \succ \{c_{i-1}\} \cup X_{u_{i-1}}$. As B is an extending set, it follows that $u_{i-1} \in B$.

If *(ii)*, let $U_{i-1} = \{u^1_{i-1}, u^2_{i-1}, u^3_{i-1}\}$. Without loss of generality we may assume that $u_{i-1} = u^1_{i-1}$ and observe that u_{i-1} is the maximal element of the transitive set $\{c_{i-1}\} \cup X_{u_{i-1}}$. By the induction hypothesis and the first part of the induction step, $\{c_{i-1}\} \cup (X_{u_{i-1}} \setminus \{u_{i-1}\}) \subseteq B$. As $X_{u_{i-1}}$ is i-leveled and U_{i-1} is a set of clause alternatives, there is some $u'' \in X_{u_{i-1}} \cap U_{i+1}$. By construction, moreover, $u'' = u^1_{i+1}$. Again, there is no alternative $a \in C \cup U$ that extends $\{c_{i-1}\} \cup X_{u_{i-1}}$. In particular, u^3_{i-1} does not extend $\{c_{i-1}\} \cup X_{u_{i-1}}$. To see this, recall that $u^1_{i+1} \in X_{i+1}$ and that $u^1_{i+1} \succ u^3_{i-1}$ by construction of T_φ. As B is an extending set, it follows that $u_{i-1} \in B$.

Finally, if *(iii)*, we have $U_{i-1} = U^+_{i-1} = \{u^1_{i-1}, u^2_{i-1}, u^3_{i-1}\}$. Without loss of generality, we may again assume that $u_{i-1} = u^1_{i-1}$ and consider u^2_{i-1}. Observe that u^2_{i-1} is the maximal element of the transitive set $\{c_{i-1}\} \cup X_{u^2_{i-1}}$. Moreover, by the induction hypothesis and the first part of the induction step, we have $\{c_{i-1}\} \cup (X_{u^2_{i-1}} \setminus \{u^2_{i-1}\}) \subseteq B$. Now, u^1_{i-1} is the *only* alternative in $C \cup U$ extending $\{c_{i-1}\} \cup X_{u^2_{i-1}}$. Either $u^2_{i-1} \in B$ or $u^2_{i-1} \notin B$. As B is an extending set, in either case it follows that $u^1_{u-1} \in B$.

If $u^2_{i-1} \notin B$, then With B being an extending set, it then follows that $u^1_{i-1} \in B$. If, on the other hand, $u^2_{i-1} \in B$, then $\{c_{i-1}\} \cup X_{u^2_{i-1}} \subseteq B$ and u^1_{i-1} is the only alternative that extends $\{c_{i-1}\} \cup X_{u^2_{i-1}}$. Again, by B being an extending set, it follows that $u^1_{i-1} \in B$.

To conclude the proof, let $u \in U^+_1$ and consider also X_u. Observe that $c_0 \succ X_u$. As we have seen above, $X_u \subseteq B$. Recall that X_u is 1-leveled and observe that there is no alternative $a \in C \cup U$ that extends $\{c_0\} \cup X_u$. As B is extending, it follows that $c_0 \in B$, as desired. $\square$

We do not expect the problem to be in NP. The best known upper bound is the complexity class Σ^P_3.[14]

The proof of Theorem 6 effectively shows the NP-hardness of computing any tournament solution that is sandwiched between *BA* and *ME* and therefore, by Theorem 4, also the NP-hardness of computing BA^∞.

5 Conclusion and Discussion

We have analyzed the axiomatic as well as computational properties of the tournament solution *ME*. Results were mixed. In conclusion, *ME*

(i) is not monotonic,

(ii) is not independent of unchosen alternatives,

(iii) satisfies $\widehat{\alpha}_{\supseteq}$ and idempotency,

(iv) does not satisfy $\widehat{\alpha}_{\subseteq}$ and $\widehat{\gamma}_{\supseteq}$ and is not stable,

(v) satisfies irregularity,

(vi) is contained in the (iterated) Banks set,

(vii) is NP-hard to compute, and

(viii) satisfies composition-consistency.

[14] This follows from the definition. An alternative x is in $ME(T)$ if and only if there is an extending set X that contains x and does not contain a smaller extending subset. Verifying whether a set is an extending set is in coNP, verifying whether it is a minimal extending set is in Π^P_2.

Statement (*viii*) was shown by Brandt (2011), the others were shown in this paper. Two relationships of *ME* with other tournament solutions are still open: it is unknown whether the tournament equilibrium set (Schwartz, 1990) is always contained in *ME* and whether *ME* is always contained in the minimal covering set (Dutta, 1988).

These results, together with recent results about the tournament equilibrium set—which satisfies irregularity but also fails stability—, suggest that stability and irregularity may be incompatible to some extent. Only few tournament solutions are known to be stable and all of them are regular. We intend to further pursue this question in future work.

We observed that many of the properties that are violated by *ME* are nevertheless satisfied by *individual* minimal extending sets (see Table 1). It is an interesting question whether extending sets could perhaps still be used as the basis for choice in tournaments. Selecting *one* of the extending sets in a way such that the axioms considered in this paper are still satisfied appears to be problematic because the proofs for *ME*'s violation of monotonicity and independence of unchosen alternatives could easily be adapted.

Alternatively, one could consider *tournament correspondences*, i.e., functions that associate with each tournament a *set of sets* of alternatives. One can straightforwardly extend the definitions of dominance-based and choice-theoretic properties to the class of tournament correspondences. Then, the tournament correspondence that returns all minimal extending sets would indeed constitute a very attractive solution concept. Moreover, this correspondence would essentially be single-valued because tournaments with multiple minimal extending sets seem to be extremely rare.

property	BA	$\supseteq$	BA^∞	$\supseteq$	ME	$\supseteq$	minimal extending sets
monotonicity	✓		–		–		✓
independence of unchosen alternatives	–		–		–		✓
stability ($\widehat{\alpha}$ and $\widehat{\gamma}$)	–		–		–		✓
idempotency	–		✓		✓		✓
inclusion in the Banks set	✓		✓		✓		✓
irregularity	✓		✓		✓		✓
composition-consistency	✓		✓		✓		✓
efficient computability	–		–		–		–

Table 1: Comparison of BA, BA^∞, ME, and minimal extending sets as tournament correspondences. For the latter, we considered local variants of the properties as defined in Section 4.

As a matter of fact, *ME*'s violation of monotonicity, stability, and independence of unchosen alternatives crucially depends on the existence of tournaments with more than one minimal extending set. This existence was only settled recently and the size of known tournaments of this type is enormous. Through exhaustive search, we have found that in all tournaments of order 12 or less, minimal extending sets are unique, implying that for up to 12 alternatives, *ME* satisfies virtually all desirable properties and is a strict refinement of both the Banks set and the minimal covering set. We have also searched billions of random tournaments with up to 30 alternatives and never encountered a tournament with multiple extending sets.

Hence, it is fair to say that *ME* satisfies the considered properties for all practical purposes. This, in turn, may be interpreted as a criticism of the axiomatic method in general: For what does it mean if a tournament solution (or any other mathematical object) in principle violates some desirable properties, but no concrete example of a violation is known and will perhaps ever be known?

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