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Single Peaked Domains with Tree-Shaped Spectra

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SINGLE PEAKED DOMAINS WITH TREE-SHAPED SPECTRA

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Abstract. This paper provides a characterization of the families of uniquely topped total preorders on a finite set which are single peaked with respect to some tree-shaped spectrum.

Key words: Tree, betweenness, single peakedness, majority rule, coalitional strategy-proofness

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1. Introduction

A single peaked domain on a given alternative space X can be described as a collection D_X of preferences on X whose *top* elements define a *shared compromise-structure* for any population of agents endowed with preferences that are included in D_X.

Such a shared compromise-structure is defined as follows: when comparing a top-alternative x of some preference in D_X to any other alternative y in X *all agents* with preferences in D_X *agree* on the identification of those alternatives in X which lie *between* x and y as a genuine *compromise* between them, i.e. which are *not* strictly worse than *both* x *and* y.[1]

Thus, the very notion of a single-peaked domain ultimately rests on an underlying ternary *betweenness* relation on the relevant alternative space[2]. Arguably, a most common and possibly prototypical example of such a ternary relation is the betweenness induced by a *metric* on

[1]This broad and largely informal description of single peaked domains admits of several distinct specifications, including the case of preference domains that are most often denoted *single plateau* domains, which are *not* covered here. More on this below, especially in the final section concerning related literature.

[2]To be sure, that statement refers to formalization of single peakedness in terms of *ternary spaces* (i.e. sets endowed with a ternary relation). Essentially equivalent formalizations of single peakedness via *interval spaces, ternary algebras* or *incidence structures* are also available. But the approach in terms of ternary spaces is undisputably the most faithful to the original presentation of single peakedness proposed by Black (1948) and Arrow (1963), and is arguably still the most widely used one.

the alternative space as widely employed in standard location models. However, in Black (1948) -the work which first explicitly introduced single peaked preferences- using metric information to define the betweenness relation is regarded as a possible option, but it is *not at all required*. Indeed, the betweenness relation considered by Black is the one induced by an *ordered line,* so that an alternative z lies between alternatives x and y if and only if it is the *median* of the triplet x,y,z: such a betweenness relation defines single peaked preferences on a line, or *line-based single peaked preferences* (the relevant line is also referred to as their *spectrum*). But then, a most obvious generalization of *that* notion of betweenness is *betweenness on a tree* namely on a connected graph T without cycles. Since any two alternatives are connected in T by a unique path (which can also be regarded as a line), just declare z to *lie between* x and y if and only if it lies on the unique path connecting x and y, and take T as the relevant spectrum. Thus, by using the latter betweenness relation, one obtains *single peaked preferences with respect to a tree-shaped spectrum* and the corresponding domains, henceforth also denoted *tree-based single peaked (TSP) domains*.

As a matter of fact, tree-based single peaked preferences were first introduced in Demange (1982) (and also, if only implicitly, in Wendell, McKelvey (1981) for the special *metric*[3] case which is so typical of location theory). Remarkably, it turns out that single peaked preference domains on a finite tree share several significant properties with single peaked preference domains on a line. In particular,

(i) the *core of any proper simple game is non-empty* at any profile of tree-based single peaked preferences (see Demange (1982)[4]);

(ii) thus, in particular, at any tree-based single peaked profile *the set of Condorcet winners of a simple majority game is non-empty*, and some appropriate[5] version of the median voter theorem consequently holds;

(iii) any sincere single peaked preference profile of a TSP domain is a *strong Nash equilibrium* of the game resulting from the combination of such profile with the social choice function which is defined on the

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[3]Metric topped preferences are those induced by distance-minimization from the top (see e.g. Wendell and McKelvey (1981), and Bartholdi and Trick (1986), Trick (1989)). Following the usage introduced by the latter authors, such preferences are now widely also referred to as *narcissistics*.

[4]To be sure, Demange's result concerns connected strict orders or, equivalently, antisymmetric total preorders. But the argument of its proof can be easily extended to 'topped' total preorders.

[5]Depending of course on the exact specification of the notion of single peakedness under consideration.

aforementioned domain and selects *the median of the top alternatives* at any preference profile (i.e. equivalently such a social choice function is *coalitionally strategy-proof* : see Danilov (1994), Vannucci (2016)).[6]

Such remarkable properties shared by all TSP domains largely explain the extensive body of literature devoted to them, and to the related problems of their characterization and recognition. Indeed, in view of the properties listed above TSP domains may be regarded as sets of preference configurations that ensure a very robust type of outcome stability under majoritarian decision rules.

In that connection, the characterization and recognition of TSP domains is a significant task in several respects.

First of all, characterizations of TSP domains (possibly, several and mutually independent ones) help *to assess their plausibility by gauging the size and width* of their collection. Specifically, by singling out some of the key properties of TSP domains, characterizations make it easier to locate them (possibly by assisting in the choice of candidate domains to check[7]), to determine their maximum size, to identify their maximal elements, perhaps even to count them at least for small parameter sizes.

Moreover, if D_X is a TSP domain then it is consistent with the following situation: X is endowed with a shared 'compromise structure' that makes submission of *true information* about private preference rankings a dominant strategy for each agent under majority voting, and the resulting outcome is also immune to the most plausible sort of *coalitional* manipulation. It follows that whenever such a D_X is the range of a submitted ballot-profile under majority voting, then it may be plausibly regarded as *good, reliable* information about the *true* preferences of the relevant 'voters' (which would *not* be the case for a non-TSP domain).

In particular, recognition and characterization(s) of TSP domains can support the *design of new 'hybrid' anonymous voting protocols* that guarantee a suitably adapted version of coalitional strategy-proofness on the full domain of 'topped' total preorders, by combining *conditional* use of the majority rule on TSP (sub)domains and of random

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[6]On the contrary, transitivity of the binary simple majority relation (namely, the dominance relation induced by the simple majority game) is *not* guaranteed by single peakedness on a tree, unless the tree is in fact a line (see e.g. Demange (1982)).

[7]That is so because any characterization amounts to a set of criteria, and a candidate domain that fails any one of them can be immediately discarded, to the effect of easing the selection of promising domains to submit to the recognition test.

dictatorship (or some other coalitionally strategy-proof rule) on other domains.[8]

However, it turns out that -to the best of the author's knowledge- characterizations of *general*[9] TSP domains are not yet available in the literature, while a polynomial recognition algorithm has been proposed only for the special case of TSP domains consisting of linear orders (i.e. *antisymmetric* total preorders).

The aim of the present work is precisely to fill this gap in the literature, providing a characterization of TSP domains of *'topped' total preorders* (i.e. of reflexive, connected, transitive binary relations with a *unique maximum*), with respect to a *very comprehensive notion of single peakedness* requiring that an alternative located between the top and another alternative be just *not worse* (as opposed to *strictly better*) than the latter.[10]

Such a problem is addressed here relying upon two building blocks: (i) an adaptation and extension to general topped total preorders of Trick's Make Tree recognition algorithm for single-peaked linear orders on a tree, which amounts to a specific procedure to build single-peakedness-respecting tree-consistent paths starting from *leaves* (Trick (1989)), and (ii) an application to the specific issue of TSP domains of a recent characterization of finite tree-betweenness relations due to Chvátal, Rautenbach, Schäfer (2011), that in turn relies on the seminal results of Sholander (1952) concerning tree-intervals, with no reference whatsoever to single peakedness.

The present characterization of TSP domains singles out *three*[11] properties of the aforementioned characterization of tree-betweenness that the betweenness relations attached to the output of the adapted version of the Make Tree algorithm do satisfy precisely when the input is

[8]The 'hybrid' qualifier refers both to the combination of two well-known basic voting rules and to the use of a modular arithmetical component to implement a conditional pseudorandom selection between them. The underlying design principle is the following: if the revealed preference profile is a TSP domain then use a lottery to select between simple majority and random dictatorship as a voting rule. Otherwise use random dictatorship.

[9]As opposed to the special case of *line-based* TSP domains (Ballester and Haeringer (2011) and Puppe (2018)), or of TSP domains of *linear orders* (Chatterji, Sen and Zeng (2016)): see the Related Literature section for more details on such contributions.

[10]A related problem is *TSP domain recognition* i.e. finding an algorithm that decides whether D_X is a TSP domain or not.

[11]That is so because the other two properties from the aforementioned characterization of tree-betweenness are embedded in the very definition of the class of tree-admissible betweenness relations employed in the statement of our main result.

a TSP domain. As a result, it shows that an *adapted version* of the specific Make Tree procedure (originally designed to find *one* tree -if any such tree exists- that guarantees single peakedness for a domain of linear orders) can also be deployed to find out *all trees*[12] -if any- that guarantee single peakedness for an *arbitrary domain of topped total preorders.*

2. Tree-based single peaked domains: definitions and preliminaries

Let X be a finite set with $|X| = m$, T_X the set of all binary relations $\succcurlyeq \subseteq X^2$ which are *topped* i.e. with a *unique maximum* $top_X(\succcurlyeq) \in X$. Moreover, let $T_X^* \subseteq T_X$ be the set of all *total preorders* on X having a unique maximum, L_X the set of all *linear orders* (i.e. *antisymmetric* total preorders) on X. The following notation will be used: for any $\succcurlyeq_i \in T_X$, $\succ_i$ and $\sim_i$ denote respectively the asymmetric and symmetric components of $\succcurlyeq_i$, and $\Delta_X := \{(x,x) : x \in X\}$; moreover, for any $D_X = \{\succcurlyeq_1, ..., \succcurlyeq_n\} \subseteq T_X$ and any $x \in X$, $N(D_X) = \{1, ..., n\}$.

A ternary relation $B \subseteq X^3$ is a (nonstrict) **betweenness** on X if and only if for any $x, y, z \in X$ the following two conditions hold:

(B_0)(**Closedness**): for each $x, y, z \in X$, $(x, y, z) \in B$ whenever $y \in \{x, z\}$,

(B_1)(**Symmetry**): for each $x, y, z \in X$, if $(x, y, z) \in B$ then $(z, y, x) \in B$.

A topped $\succcurlyeq_i \in T_X$ is **single peaked with respect to betweenness relation** $B \subseteq X^3$ if for each $i \in N$ and any $x, y, z \in X$, $x = top_X(\succcurlyeq_i)$ and $(x, y, z) \in B$ entail that $z \succ_i y$ does *not* hold. A domain $D_X \subseteq T_X$ is **single peaked** with respect to betweenness B if every $\succcurlyeq_i \in D_X$ is single peaked with respect to B.

The strength and significance of single peakedness depends of course on the strength of the underlying betweenness. For instance, consider the so called *trivial betweenness relation* $B^\emptyset$ on X defined as follows: for any $x, y, z \in X$, $(x, y, z) \in B^\emptyset$ if and only if $y \in \{x, z\}$, i.e. equivalently $B^\emptyset := \{(x, y, z) \in X^3 : |\{x, y, z\}| \leq 2\}$.

It is easily checked that $B^\emptyset$ is indeed a *betweenness* relation since it satisfes properties B_0 and B_1, and that the following claim holds true:

Claim 1. *Let $D_X \subseteq T_X$. Then D_X is a single peaked domain with respect to $B^\emptyset$.*

[12]This property of Trick's Make Tree algorithm is also discussed at length in Peters and Elkind (2016) with reference to domains of linear orders.

The present work is only concerned with single peakedness with respect to tree-betwenness, namely the specific sort of betweenness which is induced by *trees* on X.

A **tree** on X is a simple graph $T=(X,E)$ where the set of edges $E\subseteq\{A\subseteq X:|A|=2\}$ is such that: (i) T is *connected*, namely for any two *distinct* nodes $x,y\in X$, there exists a *simple path* connecting them i.e. a finite sequence $\pi_{\{x,y\}}=(z_1,...,z_k)$ such that $\{z_1,z_k\}=\{x,y\}$, $\{z_i,z_{i+1}\}\in E$, and $\{z_i,z_{i+1}\}\neq\{z_j,z_{j+1}\}$ for each $i,j=1,...,k-1$, with $i\neq j$, and (ii) T is *acyclic* namely T has no *cycle* i.e. no path $\pi_{\{x,y\}}=(z_1,...,z_k)$ such that $z_1=z_k$.

Thus, a simple graph is a tree if and only if every pair of distinct nodes is connected by a (unique) path. As a result -in view of its finiteness- there must be some nodes which belong to just one edge, and are also denoted as **leaves** of the tree.

The **betweenness relation** B_T of a tree $T=(X,E)$ is defined as follows: for any $x,y,z\in X$,

$(x,y,z)\in B_T$ if and only if either $y\in\{x,z\}$ or y *lies on* $\pi_{\{x,z\}}=(z_1,...,z_k)$ namely $y\in\{z_1,...,z_k\}$. A ternary relation B on X is a **tree-betweenness** if and only if there exists a tree $T=(X,E)$ such that $B=B_T$.

We are now ready to introduce the notion of a *single peaked domain with a tree-shaped spectrum*, or *tree-based single peaked domain*.

Definition 1. *(**Tree-Based Single Peaked Domains**) A finite domain $D_X=\{\succcurlyeq_1,...,\succcurlyeq_n\}\subseteq T_X$ is **Tree-based Single Peaked (TSP)** if there exists a tree-betweenness $B\subseteq X^3$ such that every $\succcurlyeq_i\in D_X$ is single peaked with respect to B.*

It is worth emphasizing here that a tree-based single peaked domain may admit *several distinct trees* as spectra, unless it consists of metric topped total preorders (see Trick (1989), Peters and Elkind (2016) and note 3 above).

The ensuing analysis will take advantage of the following characterization of tree-betweenness.

Proposition 1. *(Chvátal, Rautenbach, Schäfer (2011)) A ternary relation $B\subseteq X^3$ is a **tree-betweenness** if and only if it satisfies the following independent conditions: for each $x,y,z\in X$,*

*(B_1)**(Symmetry** (S)): if $(x,y,z)\in B$ then $(z,y,x)\in B$;*

*(B_2)(**Overlapping-Path Tree-Consistency** (OPC)): if $(x,y,z)\in B$, $(y,z,w)\in B$ and $y\neq z$ then $(x,z,w)\in B$;*

(B_3)***(Nested-Path Tree-Consistency** (NPC)): if $(x, y, z) \in B$ and $(x, z, w) \in B$ then $(y, z, w) \in B$;*

(B_4)***(Compromise Availability for Triplets** (CAT)):*

if $B \cap \{(x, y, z), (y, z, x), (z, x, y)\} = \varnothing$ then there exists $u \in X \setminus \{x\}$ such that $(x, u, y) \in B$ and $(x, u, z) \in B$;

(B_5)***(Minimal Closed Acyclicity** (MCA)): $(x, y, z) \in B$ and $(y, x, z) \in B$ if and only if $x = y$.*

Remark 1. The present characterization result is due to Chvátal, Rautenbach and Schäfer (2011) (see their Corollary 5: the descriptive labels are mine). Such a characterization in turn builds upon some earlier work by Sholander on the characterization of *tree-intervals* (also denoted as *tree-segments:* see Sholander (1952)).

Remark 2. Notice that a tree-betweenness B_T does also satisfy B_0 hence it is a special instance of a betweenness relation as defined above. To check this claim, consider any $x, y, z \in X$, $(x, y, z) \in B$ such that $y \in \{x, z\}$. Then, either $x = y$, or $z = y$. In the first case, $(x, y, z) \in B$ by B_5. In the second case, $(z, y, x) \in B$ by B_5 whence $(x, y, z) \in B$ by B_1. Moreover, consider any *partial order* $\leq$ on X. The 'canonical' *order-betweenness* relation $B^{\leq} \subseteq X^3$ is defined in the obvious way, namely

$B^{\leq} := \{(x, y, z) \in X^3\colon\ x \leq y \leq z$ or $z \leq y \leq x$, or $y \in \{x, z\}\}$. It is quite easy -and left to the reader- to check that if $\leq$ is connected hence it is a *linear order* then $B^{\leq}$ (a *line-betweenness*, by definition) does satisfy properties $B_1 - B_5$ i.e. it is indeed a special instance of a tree-betweenness. It should also be noticed -for future reference- that a tree-betweenness B_T on X obviously satisfies a few basic properties including the following:

Idempotence: for any $x, y \in X$, if $(x, y, x) \in B_T$ then $x = y$.

Convexity: for any $x, y, u, v, z \in X$, if $\{(x, u, y), (x, v, y)\} \subseteq B_T$ and $(u, z, v) \in B_T$ then $(x, z, y) \in B_T$.

A betweenness relation that satisfies Idempotence and Convexity is also sometimes denoted as the betweenness relation of a *road-system*[13] or a **road-betweenness** relation. Thus, validity of the following Claim is easily established.

[13]A *road system* is a hypergraph (or set system) $(X, \mathcal{R})$ -i.e. X is a nonempty set and $\mathcal{R} \subseteq \mathcal{P}(X)$, the set of 'roads', is a subset of the power set of X, - such that: (i) $\{x\} \in \mathcal{R}$ for each $x \in X$, and (ii) for every $x, y \in X$ there exists an $S \in \mathcal{R}$ with $\{x, y\} \subseteq S$. The betweenness relation $B_{\mathcal{R}}$ induced by $\mathcal{R}$ is defined by the following rule: $(x, y, z) \in B_{\mathcal{R}}$ if and only if $y \in S$ whenever $\{x, z\} \subseteq S \in \mathcal{R}$ (see Bankston (2013)).

Claim 2. *Every tree-betweenness is a road-betweenness. Moreover, the trivial betweenness* $B^{\emptyset} := \{(x,y,z) \in X^3 : |\{x,y,z\}| \leq 2\}$ *is also a road betweenness.*

Remark 3. Observe that -under the standard notion of simple-graph-betweenness B_G which declares $(x,y,z) \in B_G$ with $z \notin \{x,y\}$ to hold if and only if z lies on a shortest simple path (*geodesic*) of G connecting x and y, Idempotence holds but Convexity may be violated. To see this, just consider the simple connected graph

$$G^* = (X = \{x,y,u,v,z\}, E = \left\{ \begin{array}{c} \{x,u\}, \{x,v\}, \{u,z\}, \{v,z\}, \\ \{u,y\}, \{v,y\}, \{z,y\} \end{array} \right\})$$

where $\{(x,u,y),(x,v,y)\} \subseteq B_{G^*}$ and $(u,z,v) \in B_{G^*}$, yet $(x,z,y) \notin B_{G^*}$.

Next, we provide an adaptation and extension to topped total preorders of a polynomial recognition algorithm for tree-based linear orders due to Trick (1989). Such an adaptation requires a suitable preprocessing of the preference domain as described below.

I: *Data Preprocessing.*

Let $D_X := \{\succcurlyeq_i : i \in N(D_X)\} \subseteq T_X^*$ be the relevant preference domain. To begin with, let us introduce the *linear representation* of D_X, written $\mathcal{L}(D_X)$, which is defined as follows:

$\mathcal{L}(D_X) := \left\{ \{\succcurlyeq_{ij}\}_{j=1,\ldots,k(i)} : i \in N(D_X) \right\}$ where $\{\succcurlyeq_{ij}\}_{j=1,\ldots,k(i)} \subseteq L_X$ is such that $\succcurlyeq_i = \bigcup_{j=1}^{k(i)} \succcurlyeq_{ij}$, for each $i \in N(D_X)$, (namely $\{\succcurlyeq_{ij}\}_{j=1,\ldots,k(i)}$ is the family of all linear orders which are consistent with $\succcurlyeq_i$).

An $\mathcal{L}(D_X)$-*profile* is a family $\succcurlyeq = \{\succcurlyeq_{ij}\}_{i \in N(D_X)}$ such that $j \in \{1,\ldots,k(i)\}$ for any pair ij, and $\mathcal{L}(D_X)^{N(D_X)}$ denotes the set of all $\mathcal{L}(D_X)$-profiles.

Moreover, for any such profile $\succcurlyeq \in \mathcal{L}(D_X)^{N(D_X)}$, and for any $Y \subseteq X$, $i \in N(D_X)$, and any linear order $\succcurlyeq_{ij} \subseteq \succcurlyeq_i$ let us define a set-valued operator $A^Y_{\succcurlyeq_{ij}} : Y \to 2^Y$ by the following rule: for any $x \in Y$,

$$A^Y_{\succcurlyeq_{ij}}(x) := \left\{ \begin{array}{c} \{y \in Y : y \succ_{ij} x\} \text{ if } x \neq top_Y(\succcurlyeq_{ij}) \\ \text{and} \\ \left\{top_{Y \setminus \{x\}}(\succcurlyeq_{ij})\right\} \text{ if } x = top_Y(\succcurlyeq_{ij}) \end{array} \right\}.$$

II: *Informal description of the Adapted Make Tree algorithm*, and definition of **tree-admissible paths.**

The algorithm starts the tree-building process from possible leaves, to be located among elements of the bottom indifference classes of the total preorders of D_X.

Let $D_X \subseteq T_X^*$ with $\mathcal{L}(D_X) := \left\{ \{\succcurlyeq_{ij}\}_{j=1,...,k(i)} : i \in N(D_X) \right\}$ its linear representation, and for each $i \in N(D_X)$, $j \in \{1, ..., k(i)\}$, $h = 1, ..., m$, denote by x_{ij}^h the h-th best alternative of X according to linear order $\succcurlyeq_{ij}$. Then, a *tree-admissible path* $\pi = [x_1, ..., x_p]$ of D_X with respect to $\mathcal{L}(D_X)$*-profile* $\succcurlyeq = \{\succcurlyeq_{ij}\}_{i\in N(D_X)} \in \mathcal{L}(D_X)^{N(D_X)}$,

(with $p \leq m$, and for each $r, s = 1, ..., m, r \neq s$, $x_r \in X$, $x_r \neq x_s$) is defined inductively as follows:

(i) $x_1 \in \{x_{ij}^m : i \in N(D_X)\}$;

(ii) $x_2 \in \bigcap_{i\in N(D_X)} A_{\succcurlyeq ij}^{X\setminus\{x_1\}}(x_1)$;

(iii) $x_{l+1} \in \bigcap_{i\in N(D_X)} A_{\succcurlyeq ij}^{X\setminus\{x_1,...,x_l\}}(x_l)$ for any $l = 1, ..., p-1$.

A sequence $\pi := [x_1, ..., x_p]$ of elements of X is a **tree-admissible path of** D_X joining x_1 and x_p if there exists an $\mathcal{L}(D_X)$*-profile* $\succcurlyeq = \{\succcurlyeq_{ij}\}_{i\in N(D_X)} \in \mathcal{L}(D_X)^{N(D_X)}$ such that π is a tree-admissibile path of D_X with respect to profile $\succcurlyeq$.

III. *The set* $F_A^{D_X}$ *of consistent partial selections of tree-admissible paths with a maximal domain.*

For any $x, y \in X$, $x \neq y$ that are joined by at least one tree-admissible path we denote by $[\pi^A(x, y)]$ the *set of all tree-admissible paths joining* x *and* y*, and by*

$\Pi^A(D_X) := \{[\pi^A(x, y)] : x, y \in X, x \neq y\}$

the collection of all equivalence classes of tree-admissible paths between pairs of distinct elements of X (observe that, by construction, $[\pi^A(x, y)] = [\pi^A(y, x)]$ for any $x, y \in X$. Then,

$$F_A^{D_X} \subseteq \left\{ \begin{array}{c} f : X^2 \setminus \Delta_X \rightharpoonup \Pi^A(D_X) | \text{ if } f(x,y) \text{ is defined} \\ \text{then } f(x,y) = f(y,x) \in [\pi^A(x,y)] \end{array} \right\}$$

denotes the set of all domain-maximal *partial* selections of tree-admissible paths of D_X that are **consistent** i.e. such that for any x, y, z with $x \neq y \neq z \neq x$, if $f(x, z)$ is well-defined $f(x, y) = \pi_1$ and $f(y, z) = \pi_2$ jointly imply $f(x, z) = \pi_1 \circ \pi_2$, and $\{v, z\} \subseteq f(x, y) =$
$= [x = x_1, ..., x_h = v, ..., x_k = z, ..., x_m = y]$ implies $f(v, z) = \{x_h, ..., x_k\}$.

Moreover, $\overline{F}_A^{D_X} \subseteq F^{D_X}$ denotes the set of domain-maximal and consistent partial selections of tree-admissible paths of D_X that are **total** i.e. well-defined functions $f : X^2 \setminus \Delta_X \to \Pi^A(D_X)$.

Now, we can rely on the set $F_A^{D_X}$ of domain-maximal and consistent partial selections of tree-admissible paths to define the *tree-admissible betweenness relations* of a domain $D_X \subseteq T_X^*$ which are to fulfil a key role in the characterization of TSP domains to be presented in the next section.

Definition 2. ***(Tree-admissible betweenness relations of a domain of topped total preorders)*** *Let $D_X \subseteq T_X^*$ be a domain of topped total preorders, and $f \in F_A^{D_X}$. Then, the tree-admissible betweenness of D_X induced by f is the ternary relation $B_{T(f)} \subseteq X^3$ defined as follows: for any $x, z, y \in X$, $(x, y, z) \in B_{T(f)}$ if and only if $y \in f(x, z) \cup \{x, z\}$.*

Observe that by construction a tree-admissible relation $B_{T(f)}$ can be extended to a tree-betweenness, but need *not* be a tree-betweenness itself (see Remark 4 below for some examples).

3. Tree-based single peaked domains: a characterization

We are now ready to state and prove a characterization result for tree-based single peaked domains of total preorders.

Theorem 1. *Let $D_X \subseteq T_X^*$, $F_A^{D_X}$ the set of all domain-maximal and consistent partial selections of tree-admissible paths for D_X, and $\overline{F}_A^{D_X} \subseteq F^{D_X}$ the subset of well-defined functions in F^{D_X}. Then, the following statements are equivalent:*

(i) $\overline{F}_A^{D_X} \neq \varnothing$;

(ii) There exists $f \in F_A^{D_X}$ such that $B_{T(f)}$ satisfies CAT, OPC, NPC;

(iii) D_X is a TSP domain.

Moreover, if (i)-(ii)-(iii) hold then

(iv) $F_A^{D_X} = \overline{F}_A^{D_X}$ and $B_{T(f)}$ satisfies CAT, OPC, NPC for every $f \in F_A^{D_X}$.

Proof. (i)$\Longrightarrow$(ii): Suppose that $f \in F_A^{D_X}$ is a well-defined function $f : X^2 \setminus \Delta_X \to \Pi^A(D_X)$. Then, take any $x, y, z \in X$ such that $B_{T(f)} \cap \{(x, y, z), (y, x, z), (y, z, x))\} = \varnothing$. Hence x, y, z are distinct and $y \notin f(x, z)$, $x \notin f(y, z)$, $z \notin f(y, x)$. By construction, $x \in f(x, y) \cap f(x, z)$. Now, suppose that $f(x, y) \cap f(x, z) = \{x\}$: then $f(y, z) = [y, ..., x, ..., z]$ i.e. $x \in f(y, z)$, a contradiction. Hence, there exists some $u \neq x$ such that $u \in f(x, y) \cap f(x, z)$, namely both $(x, u, y) \in B_{T(f)}$ and $(x, u, z) \in B_{T(f)}$. It follows that $B_{T(f)}$ satisfies CAT.

Next, take any $x, y, z, w \in X$ with $y \neq z$ such that $(x, y, z) \in B_{T(f)}$ and $(y, z, w) \in B_{T(f)}$. Thus, by definition, $y \in \pi := f(x, z)$ and $z \in \pi' := f(y, w)$. But then, $z \in \pi \circ \pi' = f(x, w)$ (by definition of f and its consistency). It follows that $(x,z,w) \in B_{T(f)}$ holds, hence $B_{T(f)}$ satisfies OPC.

Finally, consider any $x,y,z,w \in X$ such that both $(x,y,z) \in B_{T(f)}$ and $(x,z,w) \in B_{T(f)}$ hold. Thus, $y \in \pi := f(x,z)$ and $z \in \pi'' := f(x,w)$. Then, $\{y,z\} \subseteq \pi \circ \pi''$ whence, by construction and consistency of f, $z \in f(y,w)$ i.e. $(y,z,w) \in B_{T(f)}$. It follows that $B_{T(f)}$ also satisfies NPC.

(ii)$\Longrightarrow$(iii): To begin with, notice that by construction D_X is indeed a *single peaked domain with respect to* $B_{T(f)}$. To check this, consider any $(x,y,z) \in B_{T(f)}$ such that $x = top(\succcurlyeq_i)$ for some $\succcurlyeq_i \in D_X$. Clearly, if $y \in \{x,z\}$ there is nothing to prove. So, let suppose that $y \in f(x,z)$. Then, by construction, there exists one and only one tree-admissibile path $f(x,z) = \pi := [x_1,...,x_p]$ of D_X such that $\{x,z\} = \{x_1,x_p\}$ and $y = x_h$ for some h, $1 < h < p$. Then, $y \succcurlyeq_i z$ by definition of $B_{T(f)}$ and construction of π again. Hence, D_X is a TSP domain if $B_{T(f_)}$ is a *tree betweenness.*

To begin with, $B_{T(f_)}$ satisfies CAT, ONC and NPC by hypothesis.

Moreover, for each $x,y,z \in X$, $(x,y,z) \in B_{T(f)}$ iff either $y \in \{x,z\}$ or $y \in \pi := f(x,z) = f(z,x)$ hence in either case $(z,y,x) \in B_{T(f)}$ and Symmetry (S) holds.

Finally, for each $x,y,z \in X$, if $(x,y,z) \in B_{T(f)}$ and $(y,x,z) \in B_{T(f)}$ then the following cases should be distinguished: (a) $y \in \{x,z\}$ and $x \in \{y,z\}$: in this case $x = y$ (and possibly $x = y = z$); (b) $y \in \{x,z\}$ and $x \in f(y,z)$: but by construction $f(y,z)$ entails $y \neq z$ hence $x = y$ also holds; (c) $y \in f(x,z)$ and $x \in \{y,z\}$: since $f(x,z)$ entails $x \neq z$ by construction, from $x \in \{y,z\}$ it follows that $x = y$; (d) $x \in f(y,z)$ and $y \in f(x,z)$: in this case $x \neq y$ implies -by construction of f that $[x,...,y,...,z] = f(x,z) = [x,z]$ whence $x \notin f(y,z)$ by consistency of f, and $[y,...,x,...,z] = f(y,z) = [y,z]$, thus $x \in f(y,z)$, a contradiction . Therefore $x = y$ holds under every possible case. Conversely, if $x = y$ then both $(x,y,z) \in B_{T(f)}$ and $(y,x,z) \in B_{T(f)}$ trivially hold by definition of $B_{T(f)}$. It follows that $B_{T(f)}$ also satisfies Minimal Closed Acyclicity (MCA). Thus, by Proposition 1, $B_{T(f)}$ is indeed a tree betweenness, and D_X is a TSP domain.

(iii)$\Longrightarrow$(i): Suppose that D_X is a TSP domain i.e. there exists a tree $T = (X, E_T)$ such that, for every $i \in N(D_X)$, $\succcurlyeq_i$is single peaked with respect to B_T, the tree-betweenness induced by T. Notice that if $X = \{x,y\}$, $|X| = 2$ then $D_X \subseteq \{(x,y),(y,x)\}$ and $\Pi^A(D_X) := \{\pi^A(x,y) := [x,y]\}$ whence $F_A^{D_X} = \{f :$ with $f(x,y) = f(y,x) = [x,y]\}$. It follows that f is a well-defined function on $X^2 \setminus \Delta_X$, and we are done. Thus, let us assume w.l.o.g. that $|X| \geq 3$. Moreover, for any $Y \subseteq X$

and $y \in Y$, posit $N_b(y, Y) := \{i \in N(D_X) : z \succ_i y$ for each $z \in Y$, $x \neq y\}$ and $N_t(y, Y) := \{i \in N(D_X) : y \succcurlyeq_i z$ for all $z \in Y\}$. Since T is a *finite* tree, it includes -by acyclicity- a non-empty set of *leaves*, namely a non-empty set $\{x_1, ..., x_m\} \subseteq X$ such that for each x_j, $j = 1, ..., m$ there exists a unique $y_j \in X$, $y_j \neq x_j$ with $\{x_j, y_j\} \in E_T$. Moreover, for any y_j, $j = 1, ..., m$ there exists a $k_j \geq 1$ and $z_{j1}, ..., z_{jk_j} \in X \setminus \{x_j, y_j\}$ such that (a) $\{y_j, z_{jh}\} \in E_T$, $h = 1, ..., k_j$, and (b) for every $u \in X \setminus \cup_{j=1}^m \{x_j, y_j\}$ there exists a (finite) path $\pi_u = [z_1, z_2, ..., z_l]$ of T with $z_1 = y_j, z_2 = z_{jh}$ for some $j \in \{1, ..., m\}$ and $h \in \{1, ..., k_j\}$, and $z_l = u$. Then, for any leaf $x_j \in X$ consider $N_b(x_j, X)$ and $N_t(x_j, X)$ as defined above. It is easily checked that $\bigcup_{j=1}^{m} N_b(x_j, X) \neq \varnothing$. Indeed, suppose that $N_b(x_j, X) = \varnothing$. Then, for each $i \in N(D_X)$ there must exist at least one path $\pi^{ji} = \{x_j, ..., z^i\}$ where $z^i \in W(\succcurlyeq_i) := \{z \in X : x \succcurlyeq_i z$ for all $x \in X\}$, and $z^i \in \{x_1, ..., x_m\}$ i.e. z^i is a leaf. That is so because otherwise there would exist one or more paths incident to $W(\succcurlyeq_i)$ and joining x_j to a leaf $x_h \notin W(\succcurlyeq_i)$. But that is a contradiction, since by hypothesis D_X is single-peaked with respect to B_T. It follows that $\bigcup_{j=1}^{m} N_b(x_j, X) \neq \varnothing$, as claimed above.

Now, consider any leaf x_j such that $N_b(x_j, X) \neq \varnothing$ and the unique y_j with $\{x_j, y_j\} \in E_T$. Clearly, $y_j \succcurlyeq_i x_j$ for every $i \in N_b(x_j, X)$. Moreover, $y_j \succcurlyeq_i x_j$ for every $i \in N \setminus (N_b(x_j, X) \cup N_t(x_j, X))$ (that is so, because if $x_j \succ_i y_j$ then the existence of a path of T joining x_j to $top(\succcurlyeq_i)$ -which follows from the fact that T is indeed a tree- contradicts the hypothesis that D_X is a single-peaked domain with respect to B_T). Finally, for any $i \in N_t(x_j, X)$, it must be the case that $y_j \succcurlyeq_i z$ for any $z \in X \setminus \{x_j\}$ (because there exists a unique path π of T joining y_j to any such z, while the concatenation of $\{x_j, y_j\}$ to π gives the unique path of T joining x_j to z: thus, $(x_j, y_j, z) \in B_T$ and $z \succ_i y_j$ would imply that $\succcurlyeq_i$is *not* single-peaked with respect to B_T, a contradiction).

Next, consider subtree $T^{-x_j} := (X \setminus \{x_j\}, E \cap (X \setminus \{x_j\})^2)$, and its leafs which by construction include y_j. By the same argument presented above we may conclude that any path π of tree T is indeed an admissible path, i.e. the function $f_T : X^2 \setminus \Delta_X \rightarrow \{\pi : \pi$ is a path of $T\}$ with $f_T(x, y) = \pi_{xy}$ (where π_{xy} denotes the unique path of T joining x and y, for any $x \neq y$) is by construction consistent, and a well-defined function i.e. $f_T \in \overline{F}_A^{D_X}$.

(i)$\Longrightarrow$(iv) If $\overline{F}_A^{D_X} \neq \varnothing$ then $\overline{F}_A^{D_X} = F_A^{D_X}$ by definition of $F_A^{D_X}$. Thus, it follows from the first part of the present proof that, for any $f \in F_A^{D_X}$, $B_{T(f)}$ satisfies CAT, OPC and NPC. □

Remark 4. The *previous characterization (ii)$\Longleftrightarrow$(iii) is tight***.** Specifically, since tree-betweenness relations are a subclass of road-betweenness relations (see Claim 2 above), we claim that CAT, OPC and NPC are mutually independent properties of road-betweenness relations on X with respect to which certain domains of topped total preorders are single peaked. To see this, consider the following domains of topped total preorders on $X = \{x, v, y, z\}$, with $|X| = 4$ (where $(abcd)$ denotes $a \succ b \succ c \succ d$, and $[abc]$ denotes $a \sim b \sim c$):

(1) $D'_X = \{\succcurlyeq'_1 := (xyvz), \succcurlyeq'_2 := (yzvx), \succcurlyeq'_3 := (zxvy)\}$. It is easily checked that $F_A^{D_X} = \{f^{\emptyset}\}$ where $f^{\emptyset} : X^2 \setminus \Delta_X \to \Pi^A(D_X)$ is *undefined everywhere* on $X^2 \setminus \Delta_X$ (notice that by construction $f^{\emptyset}$ is a domain-maximal and trivially consistent partial selection of tree-admissible paths for D_X).

Hence $B_{T(f)} := \left\{ \begin{array}{c} (a,b,c) : a,b,c \in X, \\ |\{a,b,c\}| \leq 2 \end{array} \right\} = B^{\emptyset}$

trivially satisfies OPC and NPC, but violates CAT

(since e.g. $\{(x,v,y),(v,y,x),(y,x,v)\} \cap B_{T(f)} = \varnothing$ and

$\{(x,z,v),(x,z,y)\} \cap B_{T(f)} = \varnothing$ as well). Moreover, $B_{T(f)}$ is trivially a *road-betweenness* (see Remark 2 above) but of course *not* a tree-betweenness in view of Proposition 1. Observe that D'_X is trivially single peaked with respect to $B_{T(f)}$.

(2) $D''_X = \{\succcurlyeq''_1 := (x[yz]v), \succcurlyeq''_2 := (y[xv]z]), \succcurlyeq''_3 := (v[yz]x), \succcurlyeq''_4 := (z[xv]y)\}$. Now, consider $B'' \subseteq X^3$ defined as follows:

$$B'' := \left\{ \begin{array}{c} (a,b,c) : a,b,c \in X, \\ |\{a,b,c\}| \leq 2 \end{array} \right\} \cup \left\{ \begin{array}{c} (x,y,v),(y,v,z),(v,z,x),(z,x,y) \\ (v,y,x),(z,v,y),(x,z,v),(y,x,z) \end{array} \right\}.$$

B'' trivially satisfies CAT and NPC (since the relevant clauses never apply), but it violates OPC because

$\{(x,y,v),(y,v,z)\} \subseteq B''$, yet $(x,v,z) \notin B''$. Moreover, B'' is a *road-betweenness* (though of course not a tree-betweenness by Proposition 1). Notice that D''_X is single peaked with respect to B''.

(3) $D'''_X = \{\succcurlyeq'''_1 := (x[yv]z), \succcurlyeq'''_2 := (v[xyz]), \succcurlyeq'''_3 := (y[xvz]), \succcurlyeq'''_4 := (z[vy]x)\}$.

Take $B''' \subseteq X^3$ defined as follows:

$$B''' := \left\{ \begin{array}{c} (a,b,c) : a,b,c \in X, \\ |\{a,b,c\}| \leq 2 \end{array} \right\} \cup \{(x,z,v),(x,y,v),(v,z,x),(v,y,x)\}.$$

It is easily checked that B''' trivially satisfies CAT and OPC (since, again, their clauses never apply), but it violates NPC because

$\{(x,y,v),(y,v,z)\} \subseteq B'''$ yet $(x,v,z) \notin B'''$. Moreover, B''' is a *road-betweenness* (though, again, not a tree-betweenness by Proposition 1). Notice that $F_A^{D'''_X} = \varnothing$ because the tree-admissible paths include $f(a,b) = [a,b]$ for every $a,b \in X$, $a \neq b$, such that $\{a,b\} \neq \{x,z\}$, and $f(x,z) \in \{[x,v,z],[x,y,z]\}$ (hence no consistent selection of tree-admissible paths of D'''_X can be total: e.g. if $f(x,z)=[x,v,z]$ then by consistency $f(x,z)=[x,y,z]$ as well, a contradiction). Of course, D'''_X is single peaked with respect to B''' by construction.

Remark 5. Indeed, the tree-admissible paths are computed by an algorithm which is an adaptation to (topped) total preorders of the original Make Tree algorithm proposed by Trick to solve Linear TSP Domain Recognition for *linear orders* (see Trick (1989), Peters and Elkind (2016)). Such an algorithm is known to be solvable by a *polynomial algorithm* of time-complexity $O(n \cdot m^2)$. The adaptation of Make Tree to topped total preorders presented above gives rise to an algorithm of time-complexity $O(n \cdot (m-1)! \cdot m^2)$, since a topped total preorder on X may be represented by a family of linear orders of cardinality ranging from 1 up to $(m-1)!$. The resultant increase in time-complexity is of course considerable, but the problem may be rendered *fixed parameter tractable*[14] through some suitable parameterization (the most obvious option from a mechanism-design perspective being an upper bound on the cardinality of X).

Moreover, it remains to be seen whether alternative representations of single peaked domains in terms of an *incidence structure*[15] can bring some significant computational improvement.

Computational complexity issues however are beyond the scope of the present work, hence the relevant details are best left as a topic for further research.

Remark 6. The characterization offered by Theorem 1 concerns arbitrary tree-based single peaked domains of topped total preorders. No special treatment is given here to the sub-family of *rich*[16] tree-based

[14]The reader is referred to Downey and Fellows (2013), p.15 and p.63 for the relevant basic definitions concerning *fixed parameter tractability.*

[15]A structure consisting of two sets and a binary relation (see note 17 below). The most 'natural' approach seems to be here the following one: take $(\mathcal{P}(X), \mathcal{P}(X), \subseteq)$ as the underlying incidence structure and represent any total preorder on X as the chain (or flag) of that incidence structure- or equivalently of $(\mathcal{P}(X), \subseteq)$- consisting of its (*nested) upper contours.* A domain of topped total preorders is then just a finite family of such flags. Therefore, the domain thus represented is a tree-based single peaked domain if and only if the collection of all elements of its flags satisfy the three axioms claimed by Sholander (1952) to provide a characterization of tree intervals (see also Theorem 1 of Chvátal, Rautenbach and Schäfer (2011) for a full validation of Sholander's partly unproved claims on this matter).

[16]A single peaked domain of topped preorders D_X is *rich* if for each $x \in X$ there exists a preference preorder $\succcurlyeq \in D_X$ such that (i) $x = top(\succcurlyeq)$ and (ii) for each $y \in X$

single peaked domains which play such a significant role in the study of strategy-proofness properties of single peaked preferences (see e.g. Nehring and Puppe (2007), Puppe (2018), Vannucci (2016)). It should be emphasized, however, that every tree-based single peaked domain admits *minimal* (and *maximal) rich* completions to which all of the relevant results just alluded to above do apply.

4. Related literature

As a matter of fact, 'single peakedness' is by now a term commonly used to refer to an *entire family* of distinct if broadly related preference domains. This is not the place for a detailed review of the several distinct notions of single peakedness that have been advanced in the extant literature, and of the relevant results concerning them. But a few basic distinctions are to be recalled and quickly summarized in order to make it possible to locate the significance of the present work, and its marginal contribution.

Indeed, the main distinctions among the available notions of 'single peaked domain' concern:

(i) the *required structure on the ground set X of alternatives* (most typically a *ternary space,* an *interval space*, or an *incidence structure* are deployed, and even a *ternary algebra* might be appropriately invoked[17]);

(ii) the *sort of betweenness relation deployed (closed vs open)*: even if the required structure is agreed to be a *ternary space* (following the original formal presentation of single peakedness[18]), the relevant

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the upper contour of y at $\succcurlyeq$ is precisely the set of all alternatives that lie *between* x and y.

[17]A *ternary space* is a set endowed with a *ternary relation*, its *betweenness* relation. An *interval space* is a set endowed with a symmetric *interval function* attaching to each pair of its elements a set including them, the *closed interval* having the elements of the pair as its extrema. An *incidence structure* is a triplet consisting of two nonempty sets and a *binary reflexive and symmetric relation* defined on their cartesian product, the *incidence* relation. A *ternary algebra* is of course a set endowed with a ternary *operation.* The main ideas if not the details concerning the relevant connections between ternary spaces, ternary algebras and interval spaces are to be found in the seminal Sholander (1952). Following Inada (1964), a generalized version of the original notion of single peakedness with respect to a unique line-betweenness has also been occasionally contemplated, by invoking an *entire family of possibly distinct local line-betweennesses, one for each triplet of alternatives.* That view is partly reflected in the discussion of single peakedness and value restriction of the classic Sen (1970). But such a 'generalized' notion has been shown to be a *special case* of single peakedness with respect to a tree-betweenness (see Demange (1982)).

[18]Such a presentation dates back to the first, 1951 edition of Arrow (1963).

ternary betweenness relation may be taken to be a *closed* (or *nonstrict*) or an *open* (or *strict*) one[19];

(iii) the very *definition of a single peaked domain* which in turn may vary along *several dimensions*, namely:

(a) *comprehensive focus on (total) preorders vs restriction to linear orders* i.e. *antisymmetric* total preorders: comprehensive focus on possibly *not* antisymmetric total preorders raises the possibility of further significant distinctions on the notion of single peaked domain as summarized by points (b) and (c) below[20];

(b) *distinction of single peaked from single plateau preferences vs collapse of both notions* under the common 'single peaked' label: the usage which requires single peaked preferences to have a *unique top element* as opposed to *single plateau* preferences (which may also have two or more top elements) is widely established, but far from being universal[21].

(c) *nonstrictly vs strictly single peaked domains:* an alternative that lies between a top alternative and another one is required to be *not*

It should also be emphasized that, by that time, the study of betweenness relations was already well-established, because (a) it was firmly rooted in the modern axiomatization of Euclidean geometry as pursued and achieved through the joint efforts of Pasch, Peano and Hilbert (see Pambuccian (2011) for an extremely comprehensive historical reconstruction), and (b) had been further bolstered by the rise of lattice theory (as testified by a key paper such as Pitcher and Smiley (1942)).

[19]The open version is the original proposal of Arrow (1963) and may be regarded as more intuitively appealing than the closed one. However, *the closed version is much more convenient* when it comes to clarifying the relationship of the ternary space approach to equivalent approaches to single peakedness via interval spaces, incidence structures or ternary algebras.

[20]The early literature on single peaked domains is typically focused on total preorders (see e.g. Black (1948), Arrow (1963), Moulin (1980) among others). Many later contributions however concern linear orders (see e.g. Demange (1982), Trick (1989), Danilov (1994), Nehring and Puppe (2007), Ballester and Haeringer (2011), Peters and Elkind (2016)). As mentioned in the text, focusing on linear orders makes the further distinctions described in the text below (under points (iii) (b) and (iii) (c)) entirely irrelevant.

[21]Indeed, Black requires uniqueness of top elements for single peaked preferences as opposed to truncated single peaked ones (see Black(1948)), but Arrow's formal definition (Arrow (1963)) allows for *two* distinct top elements. Moreover, the definition of single peaked preferences provided by Fishburn (1973) allows for single plateaus with an *arbitrary* number of top elements.

worse (respectively, *strictly better*) than the latter. Clearly, (nonstrictly) single peaked domains comprise a larger, more comprehensive family than strictly single peaked domains[22].

Characterizations for the special case of *line-based* single peaked domains have been quite recently provided by both Ballester and Haeringer (2011) and Puppe (2018), two significant contributions which are remarkably different in aim and scope. Ballester and Haeringer (2011) is only concerned with preference domains of finite *linear orders*[23]: it provides a characterization of *all line-based single peaked domains of linear orders*[24].

On the contrary, Puppe (2018) covers preference domains of *both* (finite) *linear orders and possibly single-plateau total preorders*, and in the latter case it focuses on *nonstrictly single plateau domains*. On the other hand, the two main characterization results of that work only concern *the full domains of all line-based single peaked linear orders and all line-based nonstrictly single plateau total preorders*, respectively[25].

Concerning the general case of finite *tree-based* single peaked domains, Trick (1989) introduces the Make Tree *recognition algorithm,* a polynomial algorithm that establishes whether any given domain of *linear orders* is actually tree-based single peaked or not.

More recently, Chatterji, Sen and Zeng produced a global-style and *explicitly* mechanism-design oriented characterization[26] of *tree-based*

[22]Single peaked domains of total preorders under a nonstrict interpretation are considered in Moulin (1980), Duggan (2016), Vannucci (2016), Puppe (2018). Conversely, a strict interpretation is proposed by Arrow (1963), Fishburn (1973), and Nehring and Puppe (2007), among others. Dummett and Farquharson (1961) is an early contribution which makes a clear distinction between those two notions of single peakedness for the line-based case, using both of them.

[23]Strictly speaking, Ballester and Haeringer (2011) discuss domains of *strict linear orders* i.e. connected *asymmetric* orders.

[24]The Ballester-Haeringer characterization may be summarized as follows: each triplet of elements of the ground set should include an element which no linear order of the domain ranks as its local minimum for that triplet, and no pair of linear orders of the domain should be mutually reverse on a triplet and rank a fourth alternative above their common medium alternative in the former triplet. Hence, that is in fact a characterization of a *local* sort, since it relies on local properties (concerning arbitrary restrictions to triplets and quartets) of the specific linear orders of a given domain.

[25]Accordingly, the characterization results of Puppe (2018) are indeed of a *global* sort, since they rely on general properties of the relevant domains (such as connectedness and richness, maximal width), over and above properties of their constituents (i.e. universal existence of Condorcet winners).

[26]To be sure, *every* characterization of single peaked domains is arguably motivated by some sort of mechanism-design issue. That is so because the main

single peaked domains of linear orders within the class of '*connected*'[27] domains (Chatterji, Sen and Zeng (2016)). Specifically, they prove that (finite) 'connected' tree-based single peaked domains of linear orders are *the only* (finite) 'connected' domains of linear orders which admit a random social choice function that is strategy-proof, top-only, ex-post efficient and satisfies a certain '*compromise property*' requiring assignment of a positive probability to an unanimous second-best choice when top choices are maximally polarized between two alternatives.

However, to the best of the present author's knowledge, no characterization is available in the previous literature for the *general* case of arbitrary domains of *nonstrictly single peaked topped total preorders* with a tree-shaped spectrum.

Thus, the present work fills a gap in the literature, and contributes to the study of single peaked domains along the following lines:

(i) it provides a *local-style* characterization of *arbitrary* domains of *nonstrict single peaked of uniquely topped total preorders*: hence it covers a family of *very comprehensive* single peaked domains while *retaining the original requirement of peak-uniqueness*;

(ii) the characterization offered here relies on the mutual adaptation and combination of two largely unrelated pieces of previous work: a polynomial *recognition algorithm* for a specialized class of tree-based single peaked domains of *linear orders* and a quite recent, entirely *general characterization of tree-betweenness relations* as a class of ternary relations. The result is a sort of dual axiomatic/algorithmic characterization which might be usefully deployed to design new ballot-sensitive aggregation/voting mechanisms[28] with nice coalitional-strategy-proofness properties.

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source of interest in such domains is precisely the fact that they support reliance on majority-like aggregation and voting protocols. Hence any characterization of single peaked domains may be helpful for mechanism-design tasks at least in two ways. Namely, by assisting the mechanism-designing agency (α) to gauge the plausibility of such domains, and (β) to define admissible ballot spaces. In particular, local-style characterizations may be especially helpful with reference to point (β) above.

[27]A domain of linear orders is said to be *'connected'* if for any two alternatives there is a finite sequence of linear orders of the domain such that: (a) the first (respectively, last) order of the sequence has the first (respectively, the second) alternative as its top element, and (b) any two adjacent orders of the sequence are the same except for a permutation of their first two elements.

[28]We refer the reader to note 8 of the Introduction.

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