# A mathematical model of counterflow superfluid turbulence describing heat waves and vortex-density waves 

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#### Abstract

The interaction between vortex density waves and high-frequency second sound in counterflow superfluid turbulence is examined, incorporating diffusive and elastic contributions of the vortex tangle. The analysis is based on a set of evolution equations for the energy density, the heat flux, the vortex line density, and the vortex flux, the latter being considered here as an independent variable, in contrast to previous works. The latter feature is crucial in the transition from diffusive to propagative behavior of vortex density perturbations, which is necessary to interpret the details of high-frequency second sound.


## 1 Introduction

The study of vortex tangles in superfluid helium II has received much attention during the last decades, both because of its own interest and as a first step to understand the classical turbulence. Many theoretical and experimental studies on superfluid arguments have enhanced the knowledge on these intricate phenomena: experimental studies have allowed direct results and confirmed theoretical intuition, while, on the other hand, theoretical studies are important not only as a guide for experiments but also as an explanation of the experimental results [1][3].

It is known that the presence of a heat flow in superfluid helium II causes the formation of quantized vortex lines, which move inside the superfluid until a stationary situation is reached, and whose presence is usually investigated by second sound waves [1]-[3]. But, these waves interact with the vortex lines, causing not only a modification of their velocity and an attenuation of it, but also a modification of the vortex lines profile and of their motion. In

[^0]the last years, the study of non-stationary and inhomogeneous turbulent states has attracted much attention [4]-[7]. The vortex lines and their evolution are investigated by second sound waves, so that it is necessary to analyze in depth their mutual interactions. In particular, high-frequency second sound may be of special interest to probe small length scales in the tangle, which is necessary in order to explore, for instance, the statistical properties of the vortex loops of several sizes. In fact, the reduction of the size of space averaging is one of the active frontiers in second sound techniques applied to turbulence, but at high-frequencies, the response of the tangle to the second sound is expected to be qualitatively different than at low frequencies, as its perturbations may change from diffusive to propagative behavior [4]-9].

In a previous paper [4] a thermodynamical model of inhomogeneous superfluid turbulence was built up with the aim to study the mutual interactions between second sound and the vortex tangle. The fundamental fields of the model were the density $\rho$, the velocity $\mathbf{v}$, the internal energy density $E$, the heat flux $\mathbf{q}$ and the average vortex line length per unit volume $L$. In a successive paper [9, starting from this model, a semiquantitative expression for the vortex diffusion coefficient was obtained and the interaction between second sound and the tangle in the high-frequency regime was studied. In both these works, for the sake of simplicity, the diffusion flux of vortices $\mathbf{J}$ was considered as a dependent variable, collinear with the heat flux $\mathbf{q}$, which is proportional to the counterflow velocity $\mathbf{V}$, defined as $\mathbf{V}=\mathbf{v}_{n}-\mathbf{v}_{s}, \mathbf{v}_{n}$ and $\mathbf{v}_{s}$ being the velocities of the normal and superfluid components, respectively.

But, in general, this feature is not strictly verified because the vortices move with a velocity $\mathbf{v}_{L}$, which is not collinear with the counterflow velocity. Indeed, using the vortex filaments model proposed by Schwarz in [10]-[12], which describes the vortex line by a vectorial function $\mathbf{s}(\xi, t), \xi$ being the arc-length measured along the curve of the vortex filament, the velocity of the vortex element is [1]-3]

$$
\begin{equation*}
\mathbf{v}_{L}=\mathbf{v}_{s l}+\alpha \mathbf{s}^{\prime} \times\left(\mathbf{V}-\mathbf{v}_{\mathbf{i}}\right)-\alpha^{\prime} \mathbf{s}^{\prime} \times\left[\mathbf{s}^{\prime} \times\left(\mathbf{V}-\mathbf{v}_{\mathbf{i}}\right)\right] \tag{1.1}
\end{equation*}
$$

where $\alpha$ and $\alpha^{\prime}$ are temperature-dependent friction coefficients between the normal fluid and the vortex line, $\mathbf{s}^{\prime}$ the unit vector tangent along the vortex line at a given point, and $\mathbf{v}_{s l}=\mathbf{v}_{s}+\mathbf{v}_{\mathbf{i}}$ the "local superfluid velocity", sum of the superfluid velocity at large distance from any vortex line and of the "self-induced velocity", a flow due to all the other vortices including other parts of the same vortex, induced by the curvature of all these lines. In (1.1) the prime indicates the derivative with respect to the arc-length $\xi$, that is $s^{\prime} \equiv \partial s / \partial \xi$. In the "local induction approximation", the self-induced velocity $\mathbf{v}_{\mathbf{i}}$ is approximated by [1]-3]

$$
\begin{equation*}
\mathbf{v}_{\mathbf{i}}{ }^{(l o c)}=\tilde{\beta}\left[\mathbf{s}^{\prime} \times \mathbf{s}^{\prime \prime}\right]_{s=s_{0}}, \quad \text { with } \quad \tilde{\beta}=\frac{\kappa}{4 \pi} \ln \left(\frac{c}{a_{0} L^{1 / 2}}\right), \tag{1.2}
\end{equation*}
$$

with $c$ a constant of the order of unity, $\kappa$ the quantum of vorticity, given by $\kappa=h / m, h$ Planck's constant and $m$ the mass of a helium atom, $a_{0}$ the size of the vortex core and $\mathbf{s}^{\prime \prime}=\partial^{2} s / \partial \xi^{2}$ the curvature vector. The intensity of $\mathbf{v}_{\mathbf{i}}$ is $\left|\mathbf{v}_{\mathbf{i}}\right|=\tilde{\beta} / R$, with $R$ the curvature radius of the vortex line. The coefficient $\tilde{\beta}$ is linked to the internal energy per unit length of the vortex line (the tension of the vortex line) by the relation $\epsilon_{V}=\rho_{s} \kappa \tilde{\beta}$.

Using the local induction approximation, equation (1.1) can be written as

$$
\begin{equation*}
\mathbf{v}_{L}=\left(1-\alpha^{\prime}\right) \tilde{\beta} \mathbf{s}^{\prime} \times \mathbf{s}^{\prime \prime}+\alpha \tilde{\beta} \mathbf{s}^{\prime \prime}+\mathbf{v}_{s}+\alpha \mathbf{s}^{\prime} \times \mathbf{V}-\alpha^{\prime} \mathbf{s}^{\prime} \times\left(\mathbf{s}^{\prime} \times \mathbf{V}\right) \tag{1.3}
\end{equation*}
$$

Consider now a mesoscopic portion of turbulent superfluid, contained in a small volume $\Lambda$, which contains a small vortex tangle. In the following, the vortex velocity $\left\langle\mathbf{v}_{L}\right\rangle$, averaged in $\Lambda$, will be denoted with $\mathbf{v}_{\text {tangle }}=\mathbf{v}_{\text {tan }}$.

Integrating over the volume $\Lambda$, recalling that in counterflow situations, i.e. for $\rho_{n} \mathbf{v}_{n}+$ $\rho_{s} \mathbf{v}_{s}=0$, it is $\mathbf{v}_{s}=-\left(\rho_{n} / \rho\right) \mathbf{V}$, and supposing that in $\Lambda$ the counterflow velocity is constant, one gets

$$
\begin{equation*}
\mathbf{v}_{t a n}=\left(1-\alpha^{\prime}\right) \tilde{\beta}<\mathbf{s}^{\prime} \times \mathbf{s}^{\prime \prime}>+\alpha \tilde{\beta}<\mathbf{s}^{\prime \prime}>-\frac{\rho_{n}}{\rho} \mathbf{V}+\alpha<\mathbf{s}^{\prime}>\times \mathbf{V}+\alpha^{\prime}<\mathbf{U}-\mathbf{s}^{\prime} \mathbf{s}^{\prime}>\cdot \mathbf{V} \tag{1.4}
\end{equation*}
$$

where $<\cdot>$ stands for the averaged value of the vector in $\Lambda$.
Suppose that in the small volume $\Lambda$ the vortex tangle is homogeneous. If the considered volume $\Lambda$ is sufficiently far from the walls, and therefore does not contain pinned vortices, one can suppose $\left\langle\mathrm{s}^{\prime}\right\rangle=0$ and $\left\langle\mathrm{s}^{\prime \prime}\right\rangle=0$, obtaining

$$
\begin{equation*}
\mathbf{v}_{t a n}=<\mathbf{v}_{L}>=-\frac{\rho_{n}}{\rho} \mathbf{V}+\frac{2}{3} \alpha^{\prime} \boldsymbol{\Pi}^{s} \cdot \mathbf{V}+\left(1-\alpha^{\prime}\right) \tilde{\beta} \mathbf{I} c_{1} \tag{1.5}
\end{equation*}
$$

where the tensor $\boldsymbol{\Pi}^{s}=\frac{3}{2}\left\langle\mathbf{U}-\mathbf{s}^{\prime} \mathbf{s}^{\prime}\right\rangle$ was studied in [13, while the vector $\mathbf{I}=\frac{\left\langle\mathbf{s}^{\prime} \times \mathbf{s}^{\prime \prime}\right\rangle}{\left\langle\mathbf{s}^{\prime \prime} \mid\right\rangle}$ and the scalar $c_{1}=\frac{\langle | \mathrm{s}^{\prime \prime} \mid>}{\Lambda L^{3 / 2}}$ were introduced by Schwarz [12] in a microscopic approach to the dynamics of superfluid vortex tangles. From (1.5) one sees that $\mathbf{v}_{\text {tan }} \| \mathbf{V}$ only when $\boldsymbol{\Pi}^{s}=\mathbf{U}$, i.e. when the distribution of $\mathbf{s}^{\prime}$ in the tangle is isotropic, and the anisotropy vector $\mathbf{I}$ is collinear to the counterflow velocity V. However, experiments and numerical simulations show that these hypotheses are only approximately verified.

In the hypothesis of isotropy of the tangle $\left(\boldsymbol{\Pi}^{s}=\mathbf{U}\right)$, the assumption $\mathbf{v}_{\text {tan }}=0$ implies $\mathbf{I} \| \mathbf{V}$. But, in general, one could have $\mathbf{v}_{\tan } \neq 0$ and $\mathbf{I}$ not collinear to $\mathbf{V}$, which means $\mathbf{v}_{t a n}=\bar{A} \mathbf{V}+\bar{B} \mathbf{I}$, with $\bar{A}$ and $\bar{B}$ suitable coefficients coming from the relation (1.5). Therefore, the hypothesis $\mathbf{v}_{\text {tan }}$ collinear with $\mathbf{V}$ is not in exact agreement with the microscopic results of the vortex filament model. For this reason, the aim of this paper is to build up a model of inhomogeneous counterflow superfluid turbulence, in which the flux of the vortex lines, which is parallel to $\mathbf{v}_{\text {tan }}$, is taken as an additional independent variable.

In Section 2 a relaxational generalization of the diffusion equation for vortex lines is proposed, analogous to the well known Maxwell-Cattaneo generalization of Fick, Fourier, Ohm and Newton-Stokes classical transport laws [14, 15]. There, the corresponding generalization of the entropy in order to achieve compatibility between the relaxational transport laws and the second law of thermodynamics in a simplified model is studied, in which only the line density $L$ and its diffusion flux $\mathbf{J}$ are fundamental variables. This simplified model allows us to understand the contribution of $\mathbf{J}$ to the nonequilibrium entropy but does not describe the interaction between vortex density waves and second sound. For this reason, in Section 3 a more general analysis, including as independent variables of the theory not only the line density $L$ and the diffusion flux of vortices, but also the energy and the heat flux is undertaken. The mathematical formalism is physically motivated to explore the interactions between heat waves and vortex-density waves. In Section 4, the physical meaning of the several coefficients appearing in the model are examined, and in Section 5 wave propagation in this more complete model in uncoupled and coupled situations is studied and the results are compared with those obtained in [4, 9].

## 2 Simple formulation of vortex-density waves

This Section aims to provide a simplified view of the behavior of turbulent vortices in inhomogeneous situations, with special emphasis on the transition from diffusive behavior at low
frequencies to propagative behavior at high frequencies. This example provides the physical motivation for the wider treatment presented in Section 3.

In this Section the constitutive equation for the diffusive flux of vortex lines is generalized, by including relaxational effects due to their inertia. Thus, one will be interested in the evolution of inhomogeneous vortex tangles or in the response to external perturbations inducing local changes in the vortex line density. Usually, in counterflow situation (i.e. under vanishing barycentric velocity) one considers homogeneous vortex tangles and the evolution equation of $L$ is assumed to be the well-known Vinen's equation [1]-[3], [16]

$$
\begin{equation*}
\frac{d L}{d t}=\alpha_{v} V L^{3 / 2}-\beta_{v} \kappa L^{2} \equiv \sigma_{L} \tag{2.1}
\end{equation*}
$$

where $\sigma_{L}$ stands for the net vortex production per unit volume and time and $\alpha_{v}$ and $\beta_{v}$ are numerical coefficients.

When one assumes inhomogeneous vortex tangles, with $L$ differing from point to point, a further contribution should be added to (2.1), thus becoming

$$
\begin{equation*}
\frac{d L}{d t}=-\nabla \cdot \mathbf{J}+\sigma_{L} . \tag{2.2}
\end{equation*}
$$

In (2.2), it appears a new quantity, $\mathbf{J}$, the flux of vortex lines. In principle, one could expect that $\mathbf{J}$ will be related to the gradient of $L$, in analogy with the well-known Fick's law for matter diffusion. In fact, in [4] a thermodynamic formalism leading to such a transport equation is studied. In some occasions, however, inertial effects may be relevant, and they will contribute to the constitutive law for $\mathbf{J}$ with a relaxation term.

The aim of this Section is to write an evolution equation for $\mathbf{J}$ which is compatible with the second law of thermodynamics. The dependence of the constitutive relations on the temperature $T$ and on the heat flux $\mathbf{q}$ will be neglected. In the following Section this simplified hypothesis will be abandoned. To achieve a consistent thermodynamic formalism, with a positive definite entropy production, the entropy must be extended by including $\mathbf{J}$ in the set of independent variables, as in extended thermodynamics. The corresponding generalized Gibbs equation is [14]

$$
\begin{equation*}
\rho d s=-T^{-1} \mu^{L} d L-T^{-1} \tilde{\alpha} \mathbf{J} \cdot d \mathbf{J}, \tag{2.3}
\end{equation*}
$$

where $s$ is the entropy per unit mass, $\tilde{\alpha}$ a coefficient that will be identified below, and $\mu^{L}$ the vortex chemical potential. Equation (2.3) can be written in terms of time derivatives as

$$
\begin{equation*}
\rho \dot{s}=-T^{-1} \mu^{L} \dot{L}-T^{-1} \tilde{\alpha} \mathbf{J} \cdot \dot{\mathbf{J}} . \tag{2.4}
\end{equation*}
$$

Substituting (2.2) in (2.4) one has

$$
\begin{equation*}
\rho \dot{s}+\nabla \cdot\left(-T^{-1} \mu^{L} \mathbf{J}\right)=\mathbf{J} \cdot\left[-\nabla\left(T^{-1} \mu^{L}\right)-T^{-1} \tilde{\alpha} \dot{\mathbf{J}}\right]-T^{-1} \mu^{L} \sigma_{L} \tag{2.5}
\end{equation*}
$$

where $-T^{-1} \mu^{L} \mathbf{J}$ may be interpreted as the vortex contribution to the entropy flux, and the term on the right-hand side as the entropy production. To ensure the positive character of the diffusion contribution to the latter, one writes

$$
\begin{equation*}
\mathbf{J}=\gamma\left[-\nabla\left(T^{-1} \mu^{L}\right)-T^{-1} \tilde{\alpha} \dot{\mathbf{J}}\right] \tag{2.6}
\end{equation*}
$$

with $\gamma$ a positive phenomenological coefficient. A relaxation time $\tau_{J}$ may be defined as $\tau_{J}=$ $\gamma T^{-1} \tilde{\alpha}$. With this identification of the relaxation time, (2.6) may be written as

$$
\begin{equation*}
\tau_{J} \dot{\mathbf{J}}+\mathbf{J}=-D \nabla L \tag{2.7}
\end{equation*}
$$

with $D$ a vortex diffusion coefficient identified as $D=\gamma T^{-1} \frac{\partial \mu^{L}}{\partial L}$. By using dimensional analysis, $\tau_{J}$ can be taken of the form $\tau_{J}=\left(\gamma_{1} \kappa L\right)^{-1}$, where $\gamma_{1}$ is a dimensionless phenomenological coefficient. For fast variations of $\mathbf{J}$ - namely, in high-frequency experiments - the first term on the left-hand side of (2.7) becomes dominant.

Combination of (2.7) with (2.2), neglecting second-order terms in $\nabla L$, yields

$$
\begin{equation*}
\tau_{J} \ddot{L}+\dot{L}=D \nabla^{2} L+\tau_{J} \dot{\sigma}_{L}+\sigma_{L} . \tag{2.8}
\end{equation*}
$$

For low values of the frequency, the first term of the equation (2.7) is negligible and one has a reaction-diffusion equation, whereas if the frequency is high, the first term is dominant and (2.8) yields a wave equation for $L$.

This brief presentation, based on the simplest version of the so-called Extended Thermodynamics [14], has been enough to give us a simple introduction to the topic studied below, namely, the transition from a diffusive behavior to an undulatory behavior of the vortex tangle and to the new physical features related with $\mathbf{J}$ as an independent variable. However, the evolution equation for $\mathbf{J}$ is introduced as an additional equation to the previous system proposed in 4], ignoring the couplings with other possible phenomena, mainly, high-frequency second sound. Indeed, in this case, the vortex tangle will not behave as a diffusive system but as an elastic matrix, and the dispersion relation for second sound will be changed with respect to the previous model studied in [4]. It is need to know in detail these changes in the dispersion relation, because of the instrumental importance of second sound as a probe for the vortex tangle.

## 3 Balance equations and constitutive theory

To deal with sufficient generality with the interactions between second sound and the dynamics of the vortex tangle, one builds up a thermodynamical model of inhomogeneous counterflow superfluid turbulence, which chooses as fundamental fields the energy density $E$, the heat flux $\mathbf{q}$, the averaged vortex line length per unit volume $L$, and the vortex diffusion flux $\mathbf{J}$. Because experiments in counterflow superfluid turbulence in the linear regime are characterized by a zero value of the barycentric velocity $\mathbf{v}$, in this paper one does not consider $\mathbf{v}$ as independent variable. In a more complete model $\mathbf{v}$, and $\rho$ will be also fundamental fields.

It is known that the heat flux $\mathbf{q}$ is linked to the counterflow $\mathbf{V}$ through the relation $\mathbf{q}=\rho_{s} T s \mathbf{V}$, and here one prefers choosing $\mathbf{q}$ as variable because it is the macroscopic variable appearing in the balance equation for the energy, and it may be controlled in the experiments.

Consider the following balance equations

$$
\left\{\begin{array}{l}
\partial_{t} E+\partial_{k} q_{k}=0  \tag{3.1}\\
\partial_{t} q_{i}+\partial_{k} J_{i k}^{q}=\sigma_{i}^{q} \\
\partial_{t} L+\partial_{k} J_{k}=\sigma_{L} \\
\partial_{t} J_{i}+\partial_{k} F_{i k}=\sigma_{i}^{\mathbf{J}}
\end{array}\right.
$$

where $\partial_{t}$ stands for $\partial / \partial t$ and $\partial_{k}$ for $\partial / \partial x_{k}, E$ is the specific energy per unit volume of the superfluid component plus the normal component plus the vortex lines, $J_{i j}^{q}$ the flux of the heat
flux, $J_{i}$ the flux of vortex lines, and $F_{i j}$ the flux of the flux of vortex lines; $\sigma_{i}^{q}, \sigma_{L}$ and $\sigma_{i}^{\mathbf{J}}$ are the respective production terms. Since in this work one is interested to study the linear propagation of the second sound and vortex waves, the convective terms have been neglected.

If one supposes that the fluid is isotropic, the constitutive equations for the fluxes $J_{i j}^{q}$ and $F_{i j}$, to the first order in $q_{i}$ and $J_{i}$, can be expressed in the form

$$
\begin{align*}
J_{i k}^{q} & =\beta(E, L) \delta_{i k}, \\
F_{i k} & =\psi(E, L) \delta_{i k} . \tag{3.2}
\end{align*}
$$

Restrictions on these relations are obtained imposing the validity of the second law of thermodynamics, applying Liu's procedure [15, 17]. This method requires the existence of a scalar function $S$ and a vector function $J_{k}^{S}$ of the fundamental fields, namely the entropy per unit volume and the entropy flux per unit volume respectively, such that the following inequality

$$
\begin{array}{r}
\partial_{t} S+\partial_{k} J_{k}^{S}-\Lambda^{E}\left[\partial_{t} E+\partial_{k} q_{k}\right]-\Lambda_{i}^{q}\left[\partial_{t} q_{i}+\partial_{k} J_{i k}^{q}-\sigma_{i}^{q}\right]  \tag{3.3}\\
\quad-\Lambda^{L}\left[\partial_{t} L+\partial_{k} J_{k}-\sigma_{L}\right]-\Lambda_{i}^{J}\left[\partial_{t} J_{i}+\partial_{k} F_{i k}-\sigma_{i}^{J}\right] \geq 0,
\end{array}
$$

is satisfied for arbitrary fields $E, q_{i}, L$ and $J_{i}$. In this inequality, which expresses the second law of thermodynamics, $S$ and $J_{k}^{S}$ are objective functions of the fundamental fields. In order to make the theory internally consistent, one must consider for $S$ and $J_{k}^{S}$ approximate constitutive relations to second order in $q_{i}$ and $J_{i}$

$$
\begin{equation*}
S=S_{0}(E, L)+S_{1}(E, L) q^{2}+S_{2}(E, L) J^{2}+S_{3}(E, L) q_{i} J_{i}, \quad J_{k}^{s}=\phi^{q}(E, L) q_{k}+\phi^{J}(E, L) J_{k} \tag{3.4}
\end{equation*}
$$

The quantities $\Lambda^{E}, \Lambda_{i}^{q}, \Lambda^{L}$ and $\Lambda_{i}^{J}$ are Lagrange multipliers, which are also objective functions of $E, q_{i}, L$ and $J_{i}$; in particular, one puts

$$
\begin{gather*}
\Lambda^{E}=\Lambda^{E}\left(E, L, q_{i}, J_{i}\right)=\Lambda_{0}^{E}(E, L)+\Lambda_{1}^{E}(E, L) q^{2}+\Lambda_{2}^{E}(E, L) J^{2}+\Lambda_{3}^{E}(E, L) q_{i} J_{i} \\
\Lambda^{L}=\Lambda^{L}\left(E, L, q_{i}, J_{i}\right)=\Lambda_{0}^{L}(E, L)+\Lambda_{1}^{L}(E, L) q^{2}+\Lambda_{2}^{L}(E, L) J^{2}+\Lambda_{3}^{L}(E, L) q_{i} J_{i} \\
\Lambda_{i}^{q}=\lambda_{11} q_{i}+\lambda_{12} J_{i} \quad \text { and } \quad \Lambda_{i}^{J}=\lambda_{21} q_{i}+\lambda_{22} J_{i} \tag{3.5}
\end{gather*}
$$

with $\lambda_{m n}=\lambda_{m n}(E, L)$. The constitutive theory is obtained imposing in (3.3) that the coefficients of all derivatives vanish. Imposing that the coefficients of the time derivatives are zero, one obtains

$$
\begin{equation*}
d S=\Lambda^{E} d E+\Lambda_{i}^{q} d q_{i}+\Lambda^{L} d L+\Lambda_{i}^{J} d J_{i} \tag{3.6}
\end{equation*}
$$

Note that $S=\rho s$, therefore this equation generalizes equation (2.3) when energy and heat flux variations are taken into account. In the same way, imposing that the coefficients of space derivatives vanish, one finds

$$
\begin{equation*}
d J_{k}^{S}=\Lambda^{E} d q_{k}+\Lambda_{i}^{q} d J_{i k}^{q}+\Lambda^{L} d J_{k}+\Lambda_{i}^{J} d F_{i k} . \tag{3.7}
\end{equation*}
$$

Substituting now (3.2), (3.4) and (3.5) in (3.6+3.7), one gets

$$
\begin{gather*}
S_{1}=\frac{1}{2} \lambda_{11}, \quad S_{2}=\frac{1}{2} \lambda_{22}, \quad S_{3}=\lambda_{12}=\lambda_{21},  \tag{3.8}\\
\phi^{q}=\Lambda_{0}^{E}, \quad \phi^{J}=\Lambda_{0}^{L}, \tag{3.9}
\end{gather*}
$$

$$
\begin{array}{cc}
d S_{0}=\Lambda_{0}^{E} d E+\Lambda_{0}^{L} d L, & d S_{1}=\Lambda_{1}^{E} d E+\Lambda_{1}^{L} d L \\
d S_{2}=\Lambda_{2}^{E} d E+\Lambda_{2}^{L} d L, & d S_{3}=\Lambda_{3}^{E} d E+\Lambda_{3}^{L} d L \\
d \phi^{q}=\lambda_{11} d \beta+\lambda_{21} d \psi, & d \phi^{J}=\lambda_{12} d \beta+\lambda_{22} d \psi \tag{3.12}
\end{array}
$$

In particular, one obtains to the second order in $\mathbf{q}$ and $\mathbf{J}$ the following expressions the entropy and for the entropy flux

$$
\begin{equation*}
S=S_{0}+\frac{1}{2} \lambda_{11} q^{2}+\frac{1}{2} \lambda_{22} J^{2}+\lambda_{12} q_{i} J_{i}, \quad J_{k}^{S}=\Lambda_{0}^{E} q_{k}+\Lambda_{0}^{L} J_{k} . \tag{3.13}
\end{equation*}
$$

It remains the following residual inequality for the entropy production

$$
\begin{equation*}
\sigma^{S}=\Lambda_{i}^{q} \sigma_{i}^{q}+\Lambda^{L} \sigma_{L}+\Lambda_{i}^{J} \sigma_{i}^{J} \geq 0 \tag{3.14}
\end{equation*}
$$

Now, the relations obtained are analyzed in detail. One first introduces a generalized temperature as the reciprocal of the first-order part of the Lagrange multiplier of the energy

$$
\begin{equation*}
\Lambda_{0}^{E}=\left[\frac{\partial S_{0}}{\partial E}\right]_{L}=\frac{1}{T} \tag{3.15}
\end{equation*}
$$

Observe that, in the laminar regime (when $L=0$ ), $\Lambda_{0}^{E}$ reduces to the inverse of the absolute temperature of thermostatics. In the presence of a vortex tangle the quantity (3.15) depends also on the line density $L$.

As in [4], writing equation (3.10) 1 as

$$
\begin{equation*}
d S_{0}=\frac{1}{T} d E+\Lambda_{0}^{L} d L=\frac{1}{T} d E-\frac{\mu_{0}^{L}}{T} d L \tag{3.16}
\end{equation*}
$$

one can identify the quantity $-\Lambda_{0}^{L} / \Lambda_{0}^{E}=-T \Lambda_{0}^{L}$ with the chemical potential of vortex lines (near equilibrium)

$$
\begin{equation*}
-T \Lambda_{0}^{L}=\mu^{L} \tag{3.17}
\end{equation*}
$$

From (3.16) one obtains the integrability condition

$$
\begin{equation*}
\frac{\partial E}{\partial L}=T^{2} \frac{\partial}{\partial T}\left(-\frac{\mu_{0}^{L}}{T}\right) . \tag{3.18}
\end{equation*}
$$

Neglecting in (3.6) second order terms in $\mathbf{q}$ and $\mathbf{J}$, and using relations (3.8), (3.15) and (3.17), the following expression for the entropy density $S$ is obtained

$$
\begin{equation*}
d S=\frac{1}{T} d E-\frac{\mu^{L}}{T} d L+\lambda_{11} q_{i} d q_{i}+\lambda_{22} J_{i} d J_{i}+\lambda_{12}\left(J_{i} d q_{i}+q_{i} d J_{i}\right) . \tag{3.19}
\end{equation*}
$$

Consider now equations (3.12), which one rewrites using (3.9) and (3.17) as

$$
\begin{equation*}
d\left(\frac{1}{T}\right)=\lambda_{11} d \beta+\lambda_{21} d \psi, \quad d\left(-\frac{\mu^{L}}{T}\right)=\lambda_{12} d \beta+\lambda_{22} d \psi \tag{3.20}
\end{equation*}
$$

From these equations, one obtains the following relations

$$
\begin{gather*}
d \beta=\frac{\lambda_{22}}{\lambda_{11} \lambda_{22}-\lambda_{12}^{2}} d\left(\frac{1}{T}\right)+\frac{\lambda_{12}}{\lambda_{11} \lambda_{22}-\lambda_{12}^{2}} d\left(\frac{\mu^{L}}{T}\right),  \tag{3.21}\\
d \psi=-\frac{\lambda_{12}}{\lambda_{11} \lambda_{22}-\lambda_{12}^{2}} d\left(\frac{1}{T}\right)-\frac{\lambda_{11}}{\lambda_{11} \lambda_{22}-\lambda_{12}^{2}} d\left(\frac{\mu^{L}}{T}\right), \tag{3.22}
\end{gather*}
$$

from which, putting

$$
\begin{equation*}
\frac{\partial \beta}{\partial T}=\xi, \quad \frac{\partial \beta}{\partial L}=\chi, \quad \frac{\partial \psi}{\partial T}=\eta, \quad \frac{\partial \psi}{\partial L}=\nu \tag{3.23}
\end{equation*}
$$

one obtains

$$
\begin{align*}
\lambda_{11} \chi+\lambda_{21} \nu=0, & \lambda_{11} \xi+\lambda_{21} \eta & =-\frac{1}{T^{2}}  \tag{3.24}\\
\lambda_{12} \xi+\lambda_{22} \eta=\frac{\partial}{\partial T}\left(-\frac{\mu_{0}^{L}}{T}\right), & \lambda_{12} \chi+\lambda_{22} \nu & =\frac{\partial}{\partial L}\left(-\frac{\mu_{0}^{L}}{T}\right), \tag{3.25}
\end{align*}
$$

and also

$$
\begin{gather*}
\xi=\frac{1}{N}\left[-\frac{1}{T^{2}} \lambda_{22}+\lambda_{12} \frac{\partial}{\partial T}\left(\frac{\mu_{0}^{L}}{T}\right)\right], \quad \eta=\frac{1}{N}\left[\frac{1}{T^{2}} \lambda_{12}-\lambda_{11} \frac{\partial}{\partial T}\left(\frac{\mu_{0}^{L}}{T}\right)\right],  \tag{3.26}\\
\chi=\frac{1}{T} \frac{\lambda_{12}}{N} \frac{\partial \mu_{0}^{L}}{\partial L}, \quad \nu=-\frac{1}{T} \frac{\lambda_{11}}{N} \frac{\partial \mu_{0}^{L}}{\partial L}, \tag{3.27}
\end{gather*}
$$

where $N=\lambda_{11} \lambda_{22}-\lambda_{12}{ }^{2}$, and $\nu$ is a positive coefficient because it is the square of the velocity of the vortex wave, as it will be shown in the next section.

Finally, one obtains for the entropy flux

$$
\begin{equation*}
J_{k}^{s}=\frac{1}{T} q_{k}-\frac{\mu_{0}^{L}}{T} J_{k}, \tag{3.28}
\end{equation*}
$$

which is analogous to the usual expression of the entropy flux in the presence of a mass flux and heat flux, but with the second term related to vortex transport rather than to mass transport.

Observe that the expression of the entropy flux (3.28) obtained in this Section is in agreement with (2.5) when the dependence on the heat flux is neglected. In the same way, comparing the expression of the entropy (3.19) with the generalized Gibbs equation (2.3), proposed in the simplified model in Section 2, and keeping in mind equation (2.6) one gets

$$
\lambda_{22}=-\tilde{\alpha} T^{-1}=-\frac{\tau_{J}}{\gamma},
$$

thus furnishing a physical meaning of the coefficient $\lambda_{22}$ appearing in previous equations.
Finally, substituting the constitutive equations (3.2) in system (3.1), and using the relations (3.21 3.27), the following system of field equations is obtained

$$
\left\{\begin{array}{l}
\partial_{t} E+\partial_{j} q_{j}=0  \tag{3.29}\\
\partial_{t} q_{i}+\xi \partial_{i} T+\chi \partial_{i} L=\sigma_{i}^{q} \\
\partial_{t} L+\partial_{j} J_{j}=\sigma^{L} \\
\partial_{t} J_{i}+\eta \partial_{i} T+\nu \partial_{i} L=\sigma_{i}^{J}
\end{array}\right.
$$

The coefficients $\gamma$ and $\eta$ describe cross effects linking the dynamics of $\mathbf{q}$ and $\mathbf{J}$ with $L$ and $T$, respectively. Thus, they are expected to settle an interaction between heat waves and vortex
waves, whose study is one of the aims of the present work. The production terms $\sigma$ must also be specified. Regarding $\sigma_{i}^{q}$, since only counterflow situation is considering, a simplified expression, already noted in literature [18, 19], is assumed

$$
\begin{equation*}
\vec{\sigma}^{\mathbf{q}}=-\frac{1}{3} \kappa B_{H V} L \boldsymbol{\Pi}^{s} \cdot \mathbf{q}, \tag{3.30}
\end{equation*}
$$

where $B_{H V}=\frac{2 \rho}{\rho_{n}} \alpha$ is the Hall-Vinen coefficient [1] and $\boldsymbol{\Pi}^{s}=\frac{3}{2}<\mathbf{U}-\mathbf{s}^{\prime} \mathbf{s}^{\prime}>$ is the symmetric tensor mentioned in Section 1. If one assumes isotropy in the plane $y z$, this tensor $\boldsymbol{\Pi}^{s}$ can be written as [13, 20]

$$
\boldsymbol{\Pi}^{s}=\frac{3}{2}\left(\begin{array}{ccc}
2 a & 0 & 0  \tag{3.31}\\
0 & 1-a & 0 \\
0 & 0 & 1-a
\end{array}\right)
$$

where $0 \leq a \leq \frac{1}{3}$ is a parameter characterizing the anisotropy of the tangle such that < $s_{y}^{\prime 2}>=<s_{z}^{\prime 2}>=a$ and $\left.<s_{x}^{\prime 2}\right\rangle=1-2 a$. If the tangle is completely anisotropic, as in the case of a regular array produced by rotation, then $a=0$, whereas if it is isotropic then $a=\frac{1}{3}$. This term describes a friction force when $\mathbf{q}$ is orthogonal to vortex lines and null force when it is parallel to them. A further dissipative term, proportional to the binormal vector I introduced in (1.5), could be added to right-hand side in (3.30) (see the Appendix of Ref. [4]), but it is neglected here because its contribute is small compared to the right-hand side of (3.30). For the production term $\sigma^{L}$, one chooses the Vinen's production and destruction terms, equation (2.1), which one can write in terms of $L$ and of the absolute value of $q$ using the relation $\mathbf{q}=\rho_{s} T s \mathbf{V}$

$$
\begin{equation*}
\sigma^{L}=-B L^{2}+A q L^{3 / 2} \tag{3.32}
\end{equation*}
$$

where $A=\alpha_{v} / \rho_{s} T s$ and $B=\beta_{v} \kappa$. For the production term of vortex line diffusion, one assumes the following relaxational expression (see the relation below equation (2.7))

$$
\begin{equation*}
\vec{\sigma}^{J}=-\gamma_{1} \kappa L \mathbf{J}=-\frac{\mathbf{J}}{\tau_{J}} \tag{3.33}
\end{equation*}
$$

where the positive coefficient $\gamma_{1}$ can depend on the temperature $T$; with this expression, in isothermal situations, one would have a diffusion coefficient given by $D=\tau_{J} \nu=-\frac{\tau_{J}}{T \lambda_{22}} \frac{\partial \mu^{L}}{\partial L}$. Note that in (3.30) and (3.33) one has assumed that the respective production terms of $\mathbf{q}$ and $\mathbf{J}$ depend on $\mathbf{q}$ and $\mathbf{J}$, respectively, but not on both variables. In more general terms, one could assume that both production terms depend on the two fields $\mathbf{q}$ and $\mathbf{J}$ simultaneously.

Analyzes, now, the entropy production (3.14) which, with the expressions of the production terms defined above, becomes
$\sigma^{S}=-\lambda_{11} \varpi\left(\boldsymbol{\Pi}^{s} \cdot \mathbf{q}\right)_{i} q_{i}-\lambda_{12}\left(\varpi\left(\boldsymbol{\Pi}^{s} \cdot \mathbf{q}\right)_{i}+\gamma_{1} \kappa q_{i}\right) J_{i}-\lambda_{22} \gamma_{1} \kappa J_{i}^{2}-\frac{\mu^{L}}{T}\left(-B L+A q L^{1 / 2}\right) \geq 0$,
where $\varpi=\frac{1}{3} \kappa B_{H V}$ and $\gamma$ is the positive phenomenological constant defined in (2.6). Looking at the expression (3.34), one notes that the entropy production $\sigma^{S}$ is positive when a suitable choice of the coefficients $\lambda_{11}$ and $\lambda_{12}$ is made. In particular, assuming that $\boldsymbol{\Pi}^{s}=\mathbf{U}, \sigma^{S}$ in (3.34) is a quadratic form on the variables $|q|, L^{1 / 2}$ and $|J|$. Therefore, (3.34) is verified if the matrix

$$
\left(\begin{array}{ccc}
-\lambda_{11} \varpi & -\frac{1}{2 T} \mu^{L} A & -\frac{1}{2} \lambda_{12}\left(\varpi+\gamma_{1} \kappa\right)  \tag{3.35}\\
-\frac{1}{2 T} \mu^{L} A & \frac{1}{T} \mu^{L} B & 0 \\
-\frac{1}{2} \lambda_{12}\left(\varpi+\gamma_{1} \kappa\right) & 0 & -\lambda_{22} \gamma_{1} \kappa
\end{array}\right)
$$

is semidefinite positive. This implies that the coefficient $\lambda_{11}$ has to be negative and, being $\partial \mu_{0}^{L} / \partial L>0$, from the relation (3.27) one deduces that $N$ is positive.

Observe that the field equations for $\mathbf{q}$ and $\mathbf{J}$ can be written also as

$$
\begin{gather*}
\frac{\partial q_{i}}{\partial t}-\frac{\lambda_{22}}{N T^{2}} \frac{\partial T}{\partial x_{i}}+\frac{\lambda_{21}}{N} \frac{\partial}{\partial x_{i}}\left(\frac{\mu^{L}}{T}\right)=-\frac{1}{2} \kappa B_{H V} L\left[(3 a-1) q_{1} \delta_{1 i}+(1-a) q_{i}\right]  \tag{3.36}\\
\frac{\partial J_{i}}{\partial t}+\frac{\lambda_{12}}{N T^{2}} \frac{\partial T}{\partial x_{i}}-\frac{\lambda_{11}}{N} \frac{\partial}{\partial x_{i}}\left(\frac{\mu^{L}}{T}\right)=-\gamma_{1} \kappa L J_{i} . \tag{3.37}
\end{gather*}
$$

Comparing the equation (3.37) with (2.6), one deduces that they can be identified with each other if one puts $\lambda_{12}=0, \lambda_{21}=0, \lambda_{22}=-T^{-1} \tilde{\alpha}$ and $\gamma_{1}^{-1}=T^{-1} \gamma \tilde{\alpha} \kappa L$. Observe also that under this hypothesis equation (3.36) becomes

$$
\frac{\partial q_{i}}{\partial t}+\xi \frac{\partial T}{\partial x_{i}}=\sigma_{i}^{q}
$$

where $\xi=-\frac{1}{\lambda_{11} T^{2}}$. This latter equation is identical to that used in Refs. [18, 19], where the fields $L$ and $J_{i}$ were considered as dependent variables.

## 4 Physical meaning of the coefficients of proposed equations

In order to determine the physical meaning of the coefficients appearing in equations (3.29)(3.33), concentrate first the attention on the equations for $L$ and $\mathbf{J}$,

$$
\begin{gather*}
\partial_{t} L+\partial_{i} J_{i}=\sigma^{L}  \tag{4.1}\\
\partial_{t} J_{i}+\eta \partial_{i} T+\nu \partial_{i} L=\sigma^{J_{i}}=-\gamma_{1} \kappa L J_{i} . \tag{4.2}
\end{gather*}
$$

Supposing that $\mathbf{J}$ varies very slowly, Eq. (4.2) gets the form

$$
\begin{equation*}
\mathbf{J}=-\frac{\eta}{\gamma_{1} \kappa L} \nabla T-\frac{\nu}{\gamma_{1} \kappa L} \nabla L . \tag{4.3}
\end{equation*}
$$

Substituting it in (4.1), one obtains

$$
\begin{equation*}
\partial_{t} L=\frac{\eta}{\gamma_{1} \kappa L} \nabla^{2} T+\frac{\nu}{\gamma_{1} \kappa L} \nabla^{2} L+\sigma^{L} . \tag{4.4}
\end{equation*}
$$

It is then seen that the coefficient $\frac{\nu}{\gamma_{1} \kappa L} \equiv D_{1}$ represents the diffusion coefficient of vortices as already introduced in (2.8) in a simpler setting. Coefficient $\frac{\eta}{\gamma_{1} \kappa L} \equiv D_{2}$ may be interpreted as a thermodiffusion coefficient of vortices because it links the temperature gradient to vortex diffusion. In other terms, this implies a drift of the vortex tangle. Detailed measurements have indeed shown [1, pag.216] a slow drift of the tangle towards the heater; this indicates that $\eta<0$ and small. The hypothesis $\eta=0$ corresponds to $D_{2}=0$, i.e. the vortices do not diffuse in response to a temperature gradient. Now, focus the attention on the equations of $T$ and $\mathbf{q}$

$$
\begin{gather*}
\rho c_{V} \partial_{t} T+\rho \epsilon_{L} \partial_{t} L+\partial_{i} q_{i}=0,  \tag{4.5}\\
\partial_{t} q_{i}+\xi \partial_{i} T+\chi \partial_{i} L=\vec{\sigma}^{\mathbf{q}}=-\frac{1}{3} \kappa B_{H V} L \boldsymbol{\Pi}^{s} \cdot \mathbf{q} . \tag{4.6}
\end{gather*}
$$

Supposing $\partial_{t} q_{i}$ negligible in (4.6), one gets

$$
\begin{equation*}
\left(\boldsymbol{\Pi}^{s} \cdot \mathbf{q}\right)_{i}=-\frac{3 \xi}{\kappa B_{H V} L} \nabla T-\frac{3 \chi}{\kappa B_{H V} L} \nabla L \tag{4.7}
\end{equation*}
$$

that is

$$
\begin{equation*}
q_{i}=-\frac{3 \xi}{\kappa B_{H V} L}\left(\boldsymbol{\Pi}^{s}\right)^{-1} \nabla T-\frac{3 \chi}{\kappa B_{H V} L}\left(\boldsymbol{\Pi}^{s}\right)^{-1} \nabla L . \tag{4.8}
\end{equation*}
$$

The first term in (4.8) may be identified as a tensorial thermal diffusivity, and the second one is analogous to Soret diffusion term, which describes a coupling between heat flux and concentration gradient in usual fluids mixtures; here, instead of the concentration of a chemical species, one has a vortex density gradient.

Substituting (4.8) in (4.5), one gets

$$
\begin{equation*}
\rho c_{V} \partial_{t} T+\rho \epsilon_{L} \partial_{t} L=-\frac{3 \xi}{\kappa B_{H V} L}\left(\boldsymbol{\Pi}^{s}\right)^{-1} \nabla^{2} T-\frac{3 \chi}{\kappa B_{H V} L}\left(\boldsymbol{\Pi}^{s}\right)^{-1} \nabla^{2} L, \tag{4.9}
\end{equation*}
$$

and assuming isotropy one gets

$$
\begin{equation*}
\partial_{t} T=\frac{1}{\rho c_{V}}\left[\frac{\rho \epsilon_{L} \eta}{\gamma_{1} \kappa L}-\frac{3 \xi}{\kappa B_{H V} L}\right] \nabla^{2} T+\frac{1}{\rho c_{V}}\left[\frac{\rho \epsilon_{L} \nu}{\gamma_{1} \kappa L}-\frac{3 \chi}{\kappa B_{H V} L}\right] \nabla^{2} L, \tag{4.10}
\end{equation*}
$$

From the relations (4.4) and (4.10), and from the positive character of the vortex diffusion coefficient and of thermal conductivity one deduces

$$
\begin{equation*}
\frac{\nu}{\gamma_{1}}>0 \tag{4.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{3 \xi}{\kappa B_{H V} L}-\frac{\rho \epsilon_{L} \eta}{\gamma_{1} \kappa L}<0 . \tag{4.12}
\end{equation*}
$$

Thus, despite of the high number of coefficients appearing in (3.29)-(3.33), one has been able to provide a physical interpretation for many of them, which would allow for their respective measurements in suitable experiments.

## 5 Interaction of second sound and vortex density waves

In this Section wave propagation in counterflow vortex tangles is studied, with the aim to discuss the physical effects of the interaction between high-frequency second sound and vortex waves. Expressing the energy $E$ in terms of $T$ and $L$, the system (3.29) becomes

$$
\left\{\begin{array}{l}
\rho c_{V} \partial_{t} T+\rho \epsilon_{L} \partial_{t} L+\partial_{j} q_{j}=0,  \tag{5.1}\\
\partial_{t} q_{i}+\xi \partial_{i} T+\chi \partial_{i} L=\sigma_{i}^{q} \\
\partial_{t} L+\partial_{j} J_{j}=\sigma^{L} \\
\partial_{t} J_{i}+\eta \partial_{i} T+\nu \partial_{i} L=\sigma^{J_{i}}
\end{array}\right.
$$

where $c_{V}=\partial_{T} E$ is the specific heat at constant volume and $\epsilon_{L}=\partial_{L} E$. These equations are analogous to those proposed in [4] except for the choice of $J_{i}$ : in fact here $J_{i}$ is assumed to
be an independent field whereas in [4] $J_{i}$ was assumed as dependent on $q_{i}$. However, at high frequency, $J_{i}$ will become dominant and will play a relevant role, as shown in the following.

A stationary solution of the system (5.1), with the expressions of the production terms (3.30 3.33), is

$$
\begin{gather*}
\mathbf{q}=\mathbf{q}_{0}=\left(q_{01}, 0,0\right), \quad L=L_{0}=\frac{A^{2}}{B^{2}} q_{01}^{2}  \tag{5.2}\\
T=T_{0}(\mathbf{x})=T^{*}-\frac{\kappa B_{H V}}{\xi} L_{0} a q_{01} x_{1}, \quad \mathbf{J}_{0}=\left(\frac{\kappa B_{H V}}{\xi \gamma_{1} \kappa} a q_{01}, 0,0\right) \tag{5.3}
\end{gather*}
$$

with $q_{01}>0$.
The quantities (3.30), (3.32) and (3.33) can be approximated around the stationary solutions in the following way

$$
\begin{align*}
\sigma_{i}^{\mathbf{q}} & \simeq-\frac{1}{2} \kappa B_{H V}\left[(3 a-1) \delta_{i 1}+(1-a)\right]\left(q_{i 0}\left(L-L_{0}\right)+L_{0} q_{i}\right)  \tag{5.4}\\
\sigma^{L} & \simeq-\left[2 B L_{0}-\frac{3}{2} A L_{0}^{1 / 2} q_{01}\right]\left(L-L_{0}\right)+A L_{0}^{3 / 2} \hat{\mathbf{q}}_{0} \cdot\left(\mathbf{q}-\mathbf{q}_{0}\right) \tag{5.5}
\end{align*}
$$

and

$$
\begin{equation*}
\vec{\sigma}^{J} \simeq-\gamma_{1} \kappa L_{0} \mathbf{J}-\gamma_{1} \kappa\left(L-L_{0}\right) \mathbf{J}_{0} \tag{5.6}
\end{equation*}
$$

where the subscript 0 denotes the stationary values for $\mathbf{q}, L$ and $\mathbf{J}$.
Now, consider the propagation of harmonic plane waves of the four fields of the equation (5.1) in the following form

$$
\left\{\begin{array}{l}
T=T_{0}(\mathbf{x})+\tilde{T} e^{i(K \mathbf{n} \cdot \mathbf{x}-\omega t)}  \tag{5.7}\\
\mathbf{q}=\mathbf{q}_{0}+\tilde{\mathbf{q}} e^{i(K \mathbf{n} \cdot \mathbf{x}-\omega t)} \\
L=L_{0}+\tilde{L} e^{i(K \mathbf{n} \cdot \mathbf{x}-\omega t)} \\
\mathbf{J}=\mathbf{J}_{0}+\tilde{\mathbf{J}} e^{i(K \mathbf{n} \cdot \mathbf{x}-\omega t)}
\end{array}\right.
$$

where $K=k_{r}+i k_{s}$ is the wave number, $\omega$ the real frequency, $\mathbf{n}$ the unit vector along the direction of the wave propagation, and the oversigned quantities denote small amplitudes of the fields, whose product can be neglected.

Substituting (5.7) in the system (5.1), the following equations for the small amplitudes are obtained

$$
\left\{\begin{array}{l}
-\omega\left[\rho c_{V}\right]_{0} \tilde{T}-\omega\left[\rho \epsilon_{L}\right]_{0} \tilde{L}+K \tilde{\mathbf{q}} \cdot \mathbf{n}=0  \tag{5.8}\\
{\left[-\omega-\frac{i}{2} \kappa B_{H V} L_{0}\left((3 a-1) \mathbf{c}_{1} \mathbf{c}_{1}+1-a\right)\right] \tilde{\mathbf{q}}+\xi_{0} K \tilde{T} \mathbf{n}} \\
\quad-\left(-\chi_{0} K \mathbf{n}+i a \kappa B_{H V} \mathbf{q}_{0}\right) \tilde{L}=0 \\
{\left[-\omega-i\left(2 B L_{0}-\frac{3}{2} A L_{0}^{1 / 2} q_{01}\right)\right] \tilde{L}+K \tilde{\mathbf{J}} \cdot \mathbf{n}+i A L_{0}^{3 / 2} \tilde{q}_{1}=0} \\
\left(-\omega-i \gamma_{1} \kappa L_{0}\right) \tilde{\mathbf{J}}+\eta_{0} K \mathbf{n} \tilde{T}+\left(\nu_{0} K \mathbf{n}-i \gamma_{1} \kappa \mathbf{J}_{0}\right) \tilde{L}=0
\end{array}\right.
$$

where $\mathbf{c}_{1}$ is their unit vector along the first axis $x_{1}$ and $\mathbf{c}_{1} \mathbf{c}_{1}$ is the dyadic product. Note that the subscript 0 refers to the unperturbed state; in what follows, this subscript will be dropped out to simplify the notation.

## First case: n parallel to $\mathrm{q}_{0}$

Now, impose the condition that the direction of the wave propagation $\mathbf{n}$ is parallel to the heat flux $\mathbf{q}_{\mathbf{0}}$, namely $\mathbf{n}=(1,0,0)$. Through these conditions the system (5.8) becomes

$$
\left\{\begin{array}{l}
-\omega \rho c_{V} \tilde{T}+K \tilde{q}_{1}-\omega \rho \epsilon_{L} \tilde{L}=0  \tag{5.9}\\
\xi K \tilde{T}-\left(\omega+i a \kappa B_{H V} L\right) \tilde{q}_{1}-\left(-\chi K+i \kappa B_{H V} a q_{1}\right) \tilde{L}=0 \\
i A L^{3 / 2} \tilde{q}_{1}-\left(\omega+i \tau_{L}^{-1}\right) \tilde{L}+K \tilde{J}_{1}=0 \\
\eta K \tilde{T}+\left(\nu K-i \gamma_{1} \kappa J_{1}\right) \tilde{L}+\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{1}=0 \\
\\
\left(-\omega-\frac{i}{2} \kappa B_{H V} L(1-a)\right) \tilde{q}_{2}=0 \\
\left(-\omega-\frac{2}{2} \kappa B_{H V} L(1-a)\right) \tilde{q}_{3}=0 \\
\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{2}=0 \\
\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{3}=0
\end{array}\right.
$$

where

$$
\tau_{L}^{-1}=\left(2 B L-\frac{3}{2} A L^{1 / 2} q_{1}\right)
$$

Note that the transversal modes, those corresponding to the four latter equations, evolve independently with respect to the longitudinal ones, corresponding to the four former equations.

One will limit the study to the case in which $\omega$ and the modulus of the wave number $K$ assume values high enough to make considerable simplification in the system. Indeed, it is for high values of the frequency that the wave behavior of the vortex tangle can be evidenced because the first term in (5.15) will become relevant, as shown in Section 2. Note that the assumption $|K|=\left|k_{r}+i k_{s}\right|$ large refers to a large value of its real part $k_{r}$, which is related to the speed of the vortex wave, whereas the imaginary part $k_{s}$, corresponding to the attenuation factor of the wave, will be assumed small.

This problem is studied into two steps: first assuming $|K|$ and $\omega$ extremely high to neglect all terms which do not depend on them. Then, the solution so obtained is perturbed in order to evaluate the influence of the neglected terms on the velocity and the attenuation of high-frequency waves.

Step I: Under the mentioned assumptions the system (5.9) becomes

$$
\left\{\begin{array}{l}
-\omega \rho c_{V} \tilde{T}+k_{r} \tilde{q}_{1}-\omega \rho \epsilon_{L} \tilde{L}=0  \tag{5.10}\\
\xi k_{r} \tilde{T}-\omega \tilde{q}_{1}+\chi k_{r} \tilde{L}=0 \\
-\omega \tilde{L}+k_{r} \tilde{J}_{1}=0 \\
\eta k_{r} \tilde{T}+\nu k_{r} \tilde{L}-\omega \tilde{J}_{1}=0 \\
-\omega \tilde{q}_{2}=0, \quad-\omega \tilde{q}_{3}=0, \quad-\omega \tilde{J}_{2}=0, \quad-\omega \tilde{J}_{3}=0
\end{array}\right.
$$

Denoting with $w=\omega / k_{r}$ the speed of the wave, the following dispersion relation is obtained

$$
\begin{equation*}
w^{4}-\left[V_{2}^{2}+\nu-\frac{\eta}{\rho c_{V}}\left(\rho \epsilon_{L}-\frac{\chi}{\nu}\right)\right] w^{2}+V_{2}^{2} \nu=0 \tag{5.11}
\end{equation*}
$$

where $V_{2}=\left(-\lambda_{11} T^{2} \rho c_{V}\right)^{-1 / 2}$ is the second sound speed in the absence of vortex tangle [4, 18, 19] and from (3.24b) it is related to the coefficient $\xi$ by the relation $\xi=V_{2}^{2} \rho c_{V}-\lambda_{12} \eta / \lambda_{11}$. Further, if one assumes that the coefficient $\eta$ is zero

$$
\begin{equation*}
\eta=0 \quad \Rightarrow \quad \frac{\lambda_{12}}{\lambda_{11}}=T^{2} \frac{\partial}{\partial T}\left(\frac{\mu^{L}}{T}\right)=\frac{2 S_{3}}{S_{2}}=-\frac{\chi}{\nu}, \tag{5.12}
\end{equation*}
$$

then the dispersion relation (5.11) has the solutions

$$
\begin{equation*}
w_{1,2}= \pm V_{2}, \quad w_{3,4}= \pm \sqrt{\nu} \tag{5.13}
\end{equation*}
$$

to which correspond the propagation modes shown in Table $\mathbb{1}$.

| $w_{1,2}= \pm V_{2}$ | $w_{3,4}= \pm \sqrt{\nu}$ |
| :--- | :--- |
|  | $\tilde{T}=-\frac{1}{\rho c_{V}}\left(\frac{\chi-\nu \rho \epsilon_{L}}{V_{2}^{2}-\nu}\right) \psi$ |
| $\tilde{T}=\psi$ | $\tilde{T}$ |
| $\tilde{q}_{1}= \pm V_{2} \rho c_{V} \psi$ | $\tilde{q}_{1}= \pm \frac{\sqrt{\nu}\left(\rho \epsilon_{L} V_{2}^{2}-\chi\right)}{V_{2}^{2}-\nu} \psi$ |
| $\tilde{L}^{2}=0$ | $\tilde{L}=\psi$ |
| $\tilde{J}_{1}=0$ | $\tilde{J}_{1}= \pm \sqrt{\nu} \psi$ |

Table 1: Modes corresponding to second sound velocity and vortex waves, respectively.
As one sees from the first column of Table $\mathbb{1}_{1}$ under the hypothesis (5.12) the high-frequency wave of velocity $w_{1,2}= \pm V_{2}$ is a temperature wave (i.e. the second sound) in which the two quantities $\tilde{L}$ and $\tilde{J}_{1}$ are zero, whereas in the second column the high-frequency wave of velocity $w_{3,4}= \pm \sqrt{\nu}$ is a wave in which all fields vibrate. The latter result is logic because when the vortex wave is propagated in the superfluid helium, temperature $T$ and heat flux $q_{1}$ cannot remain constant. This behavior is different from that obtained in [9], because using that model in the second sound also the line density $L$ vibrates. In fact, there the flux of vortices $\mathbf{J}$ was chosen proportional to $\mathbf{q}$, so that vibrations in the heat flux (second sound) produce vibrations in the vortex tangle. Experiments on high-frequency second sound are needed to confirm this new result.

Step II: Suppose that the terms of the system (5.9), which don't appear in the system (5.10), and the term $\eta$ are small enough to be considered as perturbations of the velocity $w$ of the wave and of the attenuation term $k_{s}$ of the wave number $K$. Substituting the following assumptions

$$
\bar{w}=\frac{\omega}{k_{r}}=w+\delta \quad \text { and } \quad K=k_{r}+i k_{s}
$$

in the system (5.9), one find the expression (5.13), at the zeroth order in $\delta$ and $k_{s}$, whereas at the first order in $\delta$ and $k_{s}$, one obtains

$$
\begin{align*}
& \bar{w}_{1,2}=\left(1-\frac{\eta}{2 \rho c_{V}\left(w_{1,2}^{2}-w_{3,4}^{2}\right)}\left(\rho \epsilon_{L}-\frac{\chi}{w_{3,4}^{2}}\right)\right) w_{1,2}  \tag{5.14}\\
& \bar{w}_{3,4}=\left(1+\frac{\eta}{2 \rho c_{V}\left(w_{1,2}^{2}-w_{3,4}^{2}\right)}\left(\rho \epsilon_{L}-\frac{\chi}{w_{3,4}^{2}}\right)\right) w_{3,4} \tag{5.15}
\end{align*}
$$

and

$$
\begin{gather*}
k_{s}^{(1,2)}=\frac{a \kappa L B_{H V}}{2 w_{1,2}}+\frac{A L^{3 / 2}\left(w_{1,2}^{2} \rho \epsilon_{L}-\chi\right)}{2\left(w_{1,2}^{2}-w_{3,4}^{2}\right)},  \tag{5.16}\\
k_{s}^{(3,4)}=\frac{\kappa L \gamma_{1}+\tau_{L}^{-1}}{2 w_{3,4}}-\frac{A L^{3 / 2}\left(w_{1,2}^{2} \rho \epsilon_{L}-\chi\right)}{2\left(w_{1,2}^{2}-w_{3,4}^{2}\right)}+\frac{J_{1} \kappa \gamma_{1}}{2 w_{3,4}^{2}} . \tag{5.17}
\end{gather*}
$$

Observe that in this approximation all thermodynamical fields vibrate simultaneously and the attenuation coefficients $k_{s}$ are influenced by the choice of $\mathbf{J}$ as independent variable, as one easily sees by comparing expressions (5.16) 5.17) with those obtained in [9]. Looking at these results, in particular the two speeds (5.14) 5.15), one sees that these velocities are not modified when one makes the simplified hypothesis that the coefficient $\eta$ is equal to zero. In [9] it was observed that the second sound velocity is much higher than that of the vortex waves, so that the small quantity $\eta$ should influence the two velocities (5.14](5.15) in a different way: negligible for the second sound velocity but relevant for the vortex waves. Regarding the attenuation coefficients (5.16) 5.17 ), one sees that the first term in (5.16) is identical to that obtained in [18], when the vortices are considered fixed. The new term, proportional to $A$, comes from the interaction between second sound and vortex waves.

It is to note that the first term (5.16) produces an attenuation both to forward waves and to backward waves, while the second term contributes to the two kinds of waves in a opposite way, according to the sign of this term. Detailed measurements of the attenuation of second sound in directions parallel and orthogonal to the heat flux could allow us to establish the presence and the sign of this term.

Note also that the second term of the dissipative coefficient $k_{s}^{(1,2)}$ is the same as the third term of $k_{s}^{(3,4)}$, but with an opposite sign. This means that this term contributes to the attenuation of the two waves in opposite ways; and its contribution depends also on whether the propagation of forward waves or of backward waves is considered. The first term of $k_{s}^{(3,4)}$ produces always an attenuation of the wave, while the behavior of the third term is analogous to the first one.

## Second case: $n$ orthogonal to $q_{0}$

In order to make a more detailed comparison with the model studied in [4, 9, one proceeds to analyze another situation, in which the direction of the wave propagation is perpendicular to the heat flux, that is, for example, assuming $\mathbf{n}=(0,0,1)$. This choice simplifies the system (5.8) in the following form

$$
\left\{\begin{array}{l}
-\omega \rho c_{V} \tilde{T}+K \tilde{q}_{3}-\omega \rho \epsilon_{L} \tilde{L}=0  \tag{5.18}\\
\left(-\omega-i \kappa B_{H V} L a\right) \tilde{q}_{1}-i \kappa B_{H V} a q_{1} \tilde{L}=0 \\
\xi K \tilde{T}-\left(\omega+\frac{i}{2} \kappa B_{H V} L(1-a)\right) \tilde{q}_{3}+\chi K \tilde{L}=0 \\
i A L^{3 / 2} \tilde{q}_{1}-\left(\omega+i \tau_{L}^{-1}\right) \tilde{L}+K \tilde{J}_{3}=0 \\
\eta K \tilde{T}+\nu K \tilde{L}+\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{3}=0 \\
\\
\left(-\omega-\frac{i}{2} \kappa B_{H V} L(1-a)\right) \tilde{q}_{2}=0 \\
-i \gamma_{1} \kappa J_{1} \tilde{L}+\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{1}=0 \\
\left(-\omega-i \gamma_{1} \kappa L\right) \tilde{J}_{2}=0
\end{array}\right.
$$

Note that, in contrast with what was seen before, but in agreement with the corresponding situation of the model described in [4, 9], here the transversal and the longitudinal modes in general do not evolve independently, as shown from the first five equations. However, one will see that this is the case if high-frequency waves are considered.

As in the previous situation, assume that the values of the frequencies $\omega$ and of the real part of the wave number, $k_{r}$, are high enough, such that the system (5.18) may be simplified in the following form

$$
\left\{\begin{array}{l}
-\omega \rho c_{V} \tilde{T}+k_{r} \tilde{q}_{3}-\omega \rho \epsilon_{L} \tilde{L}=0  \tag{5.19}\\
-\omega \tilde{q}_{1}=0 \\
\xi k_{r} \tilde{T}-\omega \tilde{q}_{3}+\chi k_{r} \tilde{L}=0 \\
-\omega \tilde{L}+k_{r} \tilde{J}_{3}=0 \\
\eta k_{r} \tilde{T}+\nu k_{r} \tilde{L}-\omega \tilde{J}_{3}=0 \\
-\omega \tilde{q}_{2}=0 \quad-\omega \tilde{J}_{1}=0 \quad-\omega \tilde{J}_{2}=0
\end{array}\right.
$$

Note that in this special case, as in the previous case and in [9, only the longitudinal modes are present, so that the dispersion relation assumes the form

$$
\begin{equation*}
w\left(w^{4}-\left[V_{2}^{2}+\nu-\frac{\eta}{\rho c_{V}}\left(\rho \epsilon_{L}+\frac{\lambda_{12}}{\lambda_{11}}\right)\right] w^{2}+V_{2}^{2} \nu\right)=0 \tag{5.20}
\end{equation*}
$$

which is similar to equation (5.11).
Now, the arguments suggested are the same to the previous situation, in fact, under the hypothesis (5.12), the dispersion relation (5.20) takes the form

$$
\begin{equation*}
w\left(w^{2}-\nu\right)\left(w^{2}-V_{2}^{2}\right)=0 \tag{5.21}
\end{equation*}
$$

where $V_{2}$ is the second sound velocity and $\sqrt{\nu}$ is the velocity of the vortex density waves in helium II. The conclusions which one achieves here are the same to those of the previous situation. Indeed, the modes corresponding to the solutions (5.21) are showed in Table 2, which, apart from the first column, are identical to those shown in Table $\mathbb{1}$.

$$
\begin{array}{|l|l|l|}
w_{0}=0 & w_{1,2}= \pm V_{2} & w_{3,4}= \pm \sqrt{\nu} \\
\hline \hline & & \\
\tilde{q}_{1}=\psi & \tilde{q}_{1}=0 & \tilde{q}_{1}=0 \\
\tilde{T}=0 & \tilde{T}=\psi & \tilde{T}=-\frac{1}{\rho c_{V}}\left(\frac{\chi-\nu \rho \epsilon_{L}}{V_{2}^{2}-\nu}\right) \psi \\
\tilde{q}_{3}=0 & \tilde{q}_{3}= \pm V_{2} \rho c_{V} \psi & \tilde{q}_{3}= \pm \frac{\sqrt{\nu}\left(\rho \epsilon_{L} V_{2}^{2}-\chi\right)}{V_{2}^{2}-\nu} \psi \\
\tilde{L}=0 & \tilde{L}=0 & \tilde{L}=\psi \\
\tilde{J}_{3}=0 & \tilde{J}_{3}=0 & \tilde{J}_{3}= \pm \sqrt{\nu} \psi
\end{array}
$$

Table 2: Modes corresponding to null velocity, second sound velocity, and vortex waves, respectively.
Now, the same procedure than in the previous situation is followed, that is one supposes that all the quantities of the system (5.18), which don't appear in the system (5.19), are small enough compared to the other terms of the same system. Further, one also assume that the coefficient $\eta$ is not zero, but it has small enough values to be considered as a small perturbation to the physical system. Therefore, one assumes

$$
\bar{w}=\frac{\omega}{k_{r}}=w+\delta \quad \text { and } \quad K=k_{r}+i k_{s},
$$

and substituting them in the dispersion relation of the system (5.18), one finds the relation (5.21), at the zeroth order in $\delta$ and $k_{s}$, and the following two expressions at the first order in $\delta$ and $k_{s}$

$$
\begin{equation*}
\bar{w}_{1,2}=\left(1-\frac{\eta}{2 \rho c_{V}\left(w_{1,2}^{2}-w_{3,4}^{2}\right)}\left(\rho \epsilon_{L}-\frac{\chi}{w_{3,4}^{2}}\right)\right) w_{1,2} \tag{5.22}
\end{equation*}
$$

$$
\begin{equation*}
\bar{w}_{3,4}=\left(1+\frac{\eta}{2 \rho c_{V}\left(w_{1,2}^{2}-w_{3,4}^{2}\right)}\left(\rho \epsilon_{L}-\frac{\chi}{w_{3,4}^{2}}\right)\right) w_{3,4}, \tag{5.23}
\end{equation*}
$$

and

$$
\begin{align*}
k_{s}^{(1,2)} & =\frac{(1-a) \kappa L B_{H V}}{4 w_{1,2}}  \tag{5.24}\\
k_{s}^{(3,4)} & =\frac{\tau_{L}^{-1}+\kappa L \gamma_{1}}{2 w_{3,4}} \tag{5.25}
\end{align*}
$$

As regards the expression (15.24) for the dissipative term $k_{S}^{(1,2)}$, note that it is the same as the expression obtained when the vortices are assumed fixed [13, 20], whereas the attenuation term $k_{s}^{(3,4)}$ is the same as the second term of $k_{s}^{(3,4)}$ of the first case ( $\mathbf{n}$ parallel to $\mathbf{q}_{\mathbf{0}}$ ).

## 6 Conclusions

The previous hydrodynamical model of inhomogeneous turbulent vortex tangle [4, 9 , has been generalized, by adding the vortex flux $\mathbf{J}$ to the set of the independent variables $E, \mathbf{q}$ and $L$. In this new model, $\mathbf{J}$ is no longer described by a usual constitutive equation but it has its own dynamical equation.

A set of evolution equations for $E, \mathbf{q}, L$, and $\mathbf{J}$ subject to the restrictions of the second law of thermodynamics are studied and used to analyze the behavior of second sound and vortex waves, with special emphasis on their mutual couplings. This mathematical analysis may be useful in the interpretation of high-frequency second sound experiments, which play a key role as a tool for the probe of small spatial scales of turbulent vortex tangles. Indeed, the time derivative of $\mathbf{J}$ becomes relevant at high enough frequencies; in such a regime, the behavior of vortices becomes undulatory instead of being diffusive - the behavior assumed in the previous studies [4, 9].

An interesting result is found from the comparison between the results of the wave propagation parallel and orthogonal to the heat flux. In fact, when the waves propagate orthogonal to the heat flux, the presence of the vortex tangle always causes an attenuation of the waves. But, when the propagation of the wave is collinear to the heat flux other terms are present. These terms have a positive or negative contribution depending on whether the direction of the wave is the same or opposite to the direction of the heat flux.

Now, compare the results of the perturbed situation of the first case, when $\mathbf{n}$ is parallel to $\mathbf{q}$, with those obtained in [4, [9]. The results of the comparison regarding the second case, $\mathbf{n}$ normal to $\mathbf{q}$, are the same to those of the first case. As in [4, 9], in this case one has the propagation of two kinds of waves, namely heat waves and vortex waves, which cannot be considered as propagating independently from each other. In fact, the uncoupled situation (equation (5.13)), in which the propagation of the second sound is not influenced by the fluctuations of the vortices, is no more the case when the quantities $N_{1}=a \kappa B_{H V} L, N_{2}=\kappa B_{H V} a q_{1}, N_{3}=A L^{3 / 2}$, $N_{4}=\gamma_{1} \kappa J_{1}, N_{5}=\gamma_{1} \kappa L, \tau_{L}^{-1}$ and $\eta$, appearing in the system (5.9), are considered. Indeed, from (5.14 5.15) and from the results of [9] one makes in evidence that heat and vortex waves cannot be considered separately, that is as two different waves, but as two different features of the same phenomena. Of course, the results obtained here are more exhaustive than those of [4, 9]: in fact, comparing the velocities at the first order of approximation in both models, one deduces that the expressions (5.14) 5.15) depend not only on the velocities of heat waves and
vortex waves, as in [4, 9, but also on the coefficient $\eta$, which comes from the equation (5.1d) of the vortex flux $\mathbf{J}$, and whose physical meaning is a thermodiffusion coefficient of vortices. The fourth equation of the system (5.9) shows that the vortex flux $\tilde{J}_{1}$ is not proportional to the heat flux, as it was assumed in [4, 9], but it satisfies an equation in which also the fields $\tilde{L}$ and $\tilde{T}$, through $\eta$, are present.

It is to note that the attenuation of the second sound depends on the relative direction of the wave with respect to the heat flux: in some experiments this dependence was shown for parallel and orthogonal directions [21]. These results were explained assuming an anisotropy of the tangle of vortices, which corresponds to the presence of the parameter $a$ in (3.31). But, looking at the expressions (5.16) and (5.24) of the attenuation of the second sound in the high-frequency regime, one notes that these expressions are not equal after assuming $a=1 / 3$ (isotropy of the tangle). In particular, the term

$$
\begin{equation*}
\frac{A L^{3 / 2}\left(w_{1,2}^{2} \rho \epsilon_{L}-\chi\right)}{2\left(w_{1,2}^{2}-w_{3,4}^{2}\right)} \tag{6.26}
\end{equation*}
$$

in (5.16) causes a dependence of the attenuation depending on whether the wave direction agrees with the direction of the heat flux $\mathbf{q}$ or not. This term is absent if the wave propagates orthogonal to the heat flux. Note, in contrast, that the propagation speeds (5.14) and (5.15) for propagation direction $\mathbf{n}$ parallel to $\mathbf{q}$ coincide with (5.22) and (5.23), respectively, for propagation direction $\mathbf{n}$ normal to $\mathbf{q}$. Thus, the behavior of speed of propagation is isotropic and does not depend on the isotropy or anisotropy of the tangle.

In conclusion, it could be that an anisotropy of the behavior of high-frequency second sound does not necessarily imply an actual anisotropy of the tangle in pure counterflow regime, but only a different behavior of the second sound due to the interaction with the vortex waves. This may be of interest if one wants to explore the degree of isotropy at small spatial scales. Of course, some more experiments are needed in order to establish the presence and the sign of these additional terms.

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