# New criterion of asymptotic stability for delay systems with time-varying structures and delays 1 

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#### Abstract

In this paper, we study asymptotic stability of the zero solution of a class of differential systems governed by a scalar differential inequality with time-varying structures and delays. We establish a new generalized Halanay inequality for the asymptotic stability of the zero solution for such systems under more relaxed conditions than the existing ones. We also apply the theoretical results to the analysis of self synchronization in networks of delayed differential systems and obtained a more general sufficient condition for self synchronization.


Key words: Delay systems; Time-varying systems; Neural networks; Synchronization

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## 1 Introduction

The applications of delay differential equations can be found in many areas including control systems, neural networks, and many others. And a fundamental problem in these applications is to determine the stability of the solutions, which has been analyzed for decades. For example, in order to analyze the asymptotic stability of the zero solution of the following delay-differential equations with fixed delay $\tau>0$,

$$
\begin{equation*}
\dot{x}(t)=-a x(t)+b x(t-\tau), \tag{1}
\end{equation*}
$$

Halanay(1966) proved the following inequality which was later called Halanay inequality.
Proposition 1. Let $x(t)>0, t \in \mathbb{R}$, be a differentiable scalar function of $t$ that satisfies

$$
\begin{array}{r}
\dot{x}(t) \leq-a x(t)+b \sup _{t-\tau \leq s \leq t} x(s), \quad t \geq t_{0} \\
x(t)=\psi(t), \quad t \leq t_{0} \tag{3}
\end{array}
$$

with $a>b>0$ being constants and $\psi(t) \geq 0$ continuous and bounded for $t \leq t_{0}$, then there exist $k>0$ and $\gamma>0$ such that $x(t) \leq k e^{-\gamma\left(t-t_{0}\right)}$. Hence $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Later on, this inequality has been extended to more general types of delay differential equations. For example, in Baker \& Tang (1996),Wen, Yu \& Wang (2008), it has been proved

Proposition 2. Let $x(t)>0, t \in \mathbb{R}$ be a differentiable scalar function that satisfies

$$
\begin{array}{r}
\dot{x}(t) \leq-a(t) x(t)+b(t) \sup _{q(t) \leq s \leq t} x(s), \quad t \geq t_{0} \\
x(t)=\psi(t), \quad t \leq t_{0} \tag{5}
\end{array}
$$

where $\psi(t)>0$ is bounded and continuous for $t \leq t_{0}, a(t), b(t) \geq 0$ for $t \geq t_{0}, 0<q(t) \leq t$ and $q(t) \rightarrow \infty$ as $t \rightarrow \infty$. If there exists $\sigma>0$ such that

$$
\begin{equation*}
-a(t)+b(t) \leq-\sigma<0, t \geq t_{0} \tag{6}
\end{equation*}
$$

then (i) $x(t) \leq \sup _{-\infty<s \leq t_{0}}|\psi(s)|$, (ii) $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

In Mohamad \& Gopalsamy (2000), the authors consider continuous and discrete time Halanaytype inequalities and further generalize the results of Baker \& Tang (1996) to the case of distributed delays.

Proposition 3. (Theorem 2.2 of Mohamad \& Gopalsamy, 2000) Let $x(t), t \in \mathbb{R}$ be a nonnegative function that satisfies

$$
\begin{array}{r}
\dot{x}(t) \leq-a(t) x(t)+b(t) \int_{0}^{\infty} K(s) x(t-s) d s, \quad t>t_{0} \\
x(t)=|\varphi(t)| \quad t \leq t_{0}
\end{array}
$$

where $\varphi(s), s \in\left(-\infty, t_{0}\right], a(t)$ and $b(t), t \in \mathbb{R}$ are nonnegative continuous and bounded functions; the delay kernel $K(\cdot):[0, \infty) \rightarrow[0, \infty)$ satisfies

$$
\int_{0}^{\infty} K(s) e^{\alpha s} d s<\infty
$$

for some positive number $\alpha$. Suppose further that

$$
\begin{equation*}
a(t)-b(t) \int_{0}^{\infty} K(s) d s \geq \sigma, \quad t \in \mathbb{R} \tag{7}
\end{equation*}
$$

where $\sigma=\inf _{t \in \mathbb{R}}\left[a(t)-b(t) \int_{0}^{\infty} K(s) d s\right]>0$. Then there exists a positive number $\tilde{\alpha}$ such that

$$
x(t) \leq\left(\sup _{s \leq t_{0}} x(s)\right) e^{-\tilde{\alpha}\left(t-t_{0}\right)}, \quad t>t_{0}
$$

In Baker (2010), the author made some refinement on the decay rate of their pervious works.
Generalized Halanay inequalities have also been developed in the stability analysis of delay differential systems. For example, in Chen (2001), Lu \& Chen (2004), the authors proposed some variants of the Halanay inequality to solve the global stability of delayed Hopfield neural networks.

Particularly, in Chen \& Lu (2003), Lu \& Chen (2004), the following periodic and almost periodic integro-differential systems

$$
\begin{align*}
& \frac{d u_{i}(t)}{d t}=-d_{i}(t) u_{i}(t)+\sum_{j=1}^{n} a_{i j}(t) g_{j}\left(u_{j}(t)\right)+\sum_{j=1}^{n} \int_{0}^{\infty} f_{j}\left(u_{j}\left(t-\tau_{i j}(t)-s\right)\right) d_{s} K_{i j}(t, s)+I_{i}(t) \\
& i=1,2, \ldots, n \tag{8}
\end{align*}
$$

where $d_{s} K_{i j}(t, s)$ are Lebesgue-Stieltjes measures for each $t$, are discussed.
As a direct consequence of the main Theorem in Lu \& Chen (2004), we have
Proposition 4. Suppose that $\left|g_{j}(x)\right| \leq G_{j}|x|$ and $\left|f_{j}(x)\right| \leq F_{j}|x|$. If there exist positive constants $\xi_{1}, \xi_{2}, \cdots, \xi_{n}, \alpha$ such that for all $t>0$ and $i=1,2, \cdots, n$,

$$
\begin{equation*}
-\xi_{i}\left(d_{i}(t)-\alpha\right)+\sum_{j=1}^{n} \xi_{j} G_{j}\left|a_{i j}(t)\right|+\sum_{j=1}^{n} \xi_{j} F_{j} e^{\alpha \tau_{i j}(t)} \int_{0}^{\infty} e^{\alpha s}\left|d_{s} K_{i j}(t, s)\right| \leq 0 \tag{9}
\end{equation*}
$$

then for any solution $u(t)=\left[u_{1}(t), \cdots, u_{n}(t)\right], t>0$ of the system (8) with $I_{i}(t)=0, i=1, \cdot, n$, we have

$$
\begin{equation*}
\max _{i=1, \cdots, n}\left|u_{i}(t)\right| \leq \max _{i=1, \cdots, n} \max _{-\tau \leq s \leq 0}\left(e^{\alpha s}\left|u_{i}(s)\right|\right) e^{-\alpha t} . \tag{10}
\end{equation*}
$$

In particular, when $n=1, d_{s} K_{11}(t, s)=b(t) \delta(s), \tau_{11}(t)=\tau(t)$, we have
Proposition 5. (also see Chen, 2001) Suppose $-(a(t)-\alpha)+|b(t)| e^{\alpha \tau(t)} \leq 0$, then for any continuous scalar function $x(t) \geq 0$ that satisfies

$$
\left\{\begin{array}{l}
\dot{x}(t) \leq-a(t) x(t)+|b(t)| \sup _{s \geq 0} x(t-s), \quad t>0  \tag{11}\\
x(t)=|\varphi(t)|, \quad t \leq 0
\end{array}\right.
$$

we have

$$
\begin{equation*}
|x(t)| \leq \max _{-\tau \leq s \leq 0}\left(e^{\alpha s}|\phi(s)|\right) e^{-\alpha t} \tag{12}
\end{equation*}
$$

Instead, when $n=1, d_{s} K_{11}(t, s)=b(t) k(s) d s, \tau_{11}(t)=0$, we have
Proposition 6. Suppose $-(a(t)-\alpha)+b(t) \int_{0}^{\infty} e^{\alpha s} K(s) d s \leq 0$, then for any continuous scalar function $x(t)$ satisfying

$$
\left\{\begin{array}{l}
\dot{x}(t) \leq-a(t) x(t)+b(t) \int_{0}^{\infty} K(s) x(t-s) d s, \quad t>0  \tag{13}\\
x(s)=|\varphi(s)|, \quad t \leq 0
\end{array}\right.
$$

we have

$$
\begin{equation*}
|x(t)| \leq \max _{-\tau \leq s \leq 0}\left(e^{\alpha s}|\phi(s)|\right) e^{-\alpha t} \tag{14}
\end{equation*}
$$

For more recent works, refer to Liu, Lu \& Chen (2011) and Gil' (2013). In all the above mentioned works, there is a basic requirement: $a(t)>b(t)$ for all $t$. This requirement is not satisfied in many real systems. For example, it is well known that a system switching among several subsystems can be stable even not all the subsystems are stable. So it is necessary, if possible, to further generalize the Halanay inequality so that it can be used to more general cases.

In this paper, we will first generalize the differential inequalities with bounded time-varying delays under more relaxed requirements, say, without $a(t)>b(t)$ for all $t$. Then, we provide two applications of the theoretical results. First, we apply the theoretical results to the analysis of self
synchronization in neural networks. Based on our new generalized Halanay inequality, we proved new sufficient conditions for self synchronization in neural networks with bounded time-varying delays. Then, we investigate periodic solutions of neural networks with periodic coefficients and time delays. Under more relaxed requirement, we proved new sufficient conditions for the existence and exponential stability of the periodic solutions of such neural networks.

The rest of the paper is organized as follows. In Section 2, the new generalized Halanay inequality is proposed and proved; two applications of the theoretical results are given in Section 3. Numerical examples with simulations are given in Section 4, the paper is concluded in Section 5.

## 2 Generalized Halanay inequality

Consider a scalar function $x(t)$ governed by the inequality

$$
\left\{\begin{align*}
D^{+}|x(t)| & \leq-a(t)|x(t)|+b(t) \sup _{t-\tau_{\max } \leq s \leq t}|x(t-s)|, \quad t \geq 0  \tag{15}\\
x(s) & =\phi(s), s \in\left[-\tau_{\max }, 0\right]
\end{align*}\right.
$$

where $D^{+}$represents the upper right Dini derivative, $a(\cdot): \mathbb{R}^{+} \mapsto \mathbb{R}^{+}, b(\cdot): \mathbb{R}^{+} \mapsto \mathbb{R}$ are piecewise continuous and uniformly bounded, i.e., there exists $M_{a}>0, M_{b}>0$ such that $0<a(t) \leq M_{a}$, $|b(t)| \leq M_{b}, \phi(s) \geq 0$ is the initial value, and $\tau(\cdot): \mathbb{R}^{+} \mapsto\left(0, \tau_{\max }\right]$ is the time-varying delay with $\tau_{\max }$ being the upper bound.

For a fixed $\eta>0$ and $0 \leq t_{1}<t_{2}<\infty$, denote the set $S_{\eta}\left(t_{1}, t_{2}\right)=\left\{t \in\left(t_{1}, t_{2}\right): a(t)-|b(t)|>\right.$ $\eta\}$, and the set $S_{-}\left(t_{1}, t_{2}\right)=\left\{t \in\left(t_{1}, t_{2}\right): a(t)<|b(t)|\right\}, S_{+}\left(t_{1}, t_{2}\right)=\left(t_{1}, t_{2}\right) \backslash\left(S_{\eta}\left(t_{1}, t_{2}\right) \cup\right.$ $\left.S_{-}\left(t_{1}, t_{2}\right)\right)$. It is obvious that $S_{\eta}\left(t_{1}, t_{2}\right), S_{-}\left(t_{1}, t_{2}\right), S_{+}\left(t_{1}, t_{2}\right)$ are composed of a series of intervals, and $0 \leq a(t)-|b(t)| \leq \eta$ on $S_{+}\left(t_{1}, t_{2}\right)$. Let $\mu_{s}\left(t_{1}, t_{2}\right)=\mu\left(S_{s}\left(t_{1}, t_{2}\right)\right)$ be the Lebesgue measure of the set $S_{s}\left(t_{1}, t_{2}\right)$, where $s=\eta,{ }^{\prime}+{ }^{\prime},{ }^{\prime}-{ }^{\prime}$. And the parameter $\delta=1-\frac{\eta}{2 M_{a}} \in(0,1)$ will also be used later.

Before stating main results, we summarize the basic conditions into the following
Definition 1 ( $\eta$-condition). A function pair $\{a(\cdot), b(\cdot)\}$ with $0<a(\cdot) \leq M_{a},|b(\cdot)| \leq M_{b}$ is said to satisfy the $\eta$-condition, if there exist $t_{0} \geq 0,0<C^{*}<\eta / 2$ and an integer $N>0$ such that

$$
\begin{equation*}
\sum_{k=0}^{+\infty} \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)=\infty \tag{i}
\end{equation*}
$$

(ii)

$$
\begin{equation*}
\limsup _{k \rightarrow \infty} \frac{\left[e^{M_{b} \mu_{-}\left(t_{k}, t_{k+1}\right)}-1\right] e^{M_{a} \mu_{+}(N+1) \tau_{\max }}}{\min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\}}=C^{*} \tag{17}
\end{equation*}
$$

where $t_{k}=t_{0}+k(N+1) \tau_{\max }$, and $t_{k}^{-}=t_{k}-\tau_{\max }$.
Remark 1. If $0 / 0$ appears on the left-hand side of (17), then we explain it as 0 . Thus (17) will always hold when the numerator on its left-hand side is zero.

Now, we state the main result which can be called a "Generalized Halanay Inequality".
Theorem 1. For any given $a(\cdot), b(\cdot)$ in (15), if there exists $\eta>0$ such that $\{a(\cdot), b(\cdot)\}$ satisfies the $\eta$-condition, then the zero solution of any system governed by (15) is asymptotically stable, i.e., from any initial value $\phi(s), s \in\left[-\tau_{\max }, 0\right]$, there exists $K>1$ such that the solution $x(t)$ satisfies

$$
|x(t)| \leq K \max _{-\tau_{\max } \leq s \leq 0}|\phi(s)|,
$$

for all $t \geq 0$, and $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, if there exists $\epsilon>0$ such that $\mu_{\eta}\left(t_{k}, t_{k+1}\right) \geq \epsilon$ for each $k$, then the convergence is exponential, i.e., there exists $\widetilde{K}>0, \alpha>0$ such that

$$
|x(t)| \leq \widetilde{K} \max _{-\tau_{\max } \leq s \leq 0}|\phi(s)| e^{-\alpha t}
$$

As a direct consequence of Theorem 1, we have
Corollary 1. If $\mu_{-}(0,+\infty)=0$ and there exists $\eta>0$ such that $\mu_{\eta}(0,+\infty)=+\infty$, then the zero solution of any system governed by (15) is asymptotically stable.

Remark 2. In previous works, it was always assumed that for all $t \geq 0, a(t)-|b(t)|>\eta$. Theorem 1 indicates that even the condition $a(t)-|b(t)|>\eta$ is not satisfied for some $t>0$, the zero solution of any system governed by (15) can still be asymptotically stable if only (16) and (17) are satisfied. Corollary 1 indicates that in case $a(t)-|b(t)| \geq 0$, only if the measure of the set satisfying $a(t)-|b(t)| \geq \eta$ is infinite, then the " 0 " is a stable equilibrium point.

In the proof of Theorem [1, instead of proving $|x(t)| \rightarrow 0$, we prove the following maximal function

$$
\begin{equation*}
M_{0}(t)=\sup _{t-\tau_{\max } \leq s \leq t}|x(s)| \tag{18}
\end{equation*}
$$

tends to zero as $t \rightarrow \infty$. The basic idea for the proof is to establish a uniform estimation for $M_{0}(t)$ on an interval $\left(t_{1}, t_{2}\right)$ when $S_{+}\left(t_{1}, t_{2}\right), S_{-}\left(t_{1}, t_{2}\right)$ and $S_{\eta}\left(t_{1}, t_{2}\right)$ coexist. This will be proved by induction.

Before proving Theorem 1 , we need to make some preparations by establishing some lammas.
Lemma 1. $M_{0}(t)$ is nonincreasing on the set $S_{+}(0,+\infty)$ as well as on the set $S_{\eta}(0,+\infty)$ for any given $\eta>0$.

Proof. Given $t_{1} \in S_{+}(0,+\infty) \cup S_{\eta}(0,+\infty)$, then $a\left(t_{1}\right) \geq\left|b\left(t_{1}\right)\right|$. If $\left|x\left(t_{1}\right)\right|<M_{0}\left(t_{1}\right)$, then from the continuity of $x(t)$, there exists $t_{2}>t_{1}$ such that $|x(t)| \leq M_{0}\left(t_{1}\right)$ on $\left[t_{1}, t_{2}\right]$, which implies that $M_{0}(t)$ is nonincreasing at $t_{1}$. Otherwise, $\left|x\left(t_{1}\right)\right|=M_{0}\left(t_{1}\right)$. We have:

$$
\left\{D^{+}|x(t)|\right\}_{t=t_{1}} \leq-a\left(t_{1}\right)\left|x\left(t_{1}\right)\right|+\left|b\left(t_{1}\right)\right| M_{0}\left(t_{1}\right) \leq-\left[a\left(t_{1}\right)-b\left(t_{1}\right)\right]\left|x\left(t_{1}\right)\right| \leq 0
$$

This also implies that $M_{0}(t)$ is nonincreasing at $t_{1}$. The proof is completed.
Lemma 2. Given any $t_{1}<t_{2}$, we have

$$
M_{0}\left(t_{2}\right) \leq M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}
$$

Proof. Let $t_{1} \leq \underline{t}_{1}<\bar{t}_{1}<\underline{t}_{2}<\bar{t}_{2}<\cdots<\underline{t}_{s}<\bar{t}_{s} \leq t_{2}$ such that $S_{-}\left(t_{1}, t_{2}\right)=\bigcup_{i=1}^{s}\left(\underline{t}_{i}, \bar{t}_{i}\right)$. From Lemma 1, we see that $M_{0}(t)$ can increase only on $S_{-}\left(t_{1}, t_{2}\right)$. Thus, $M_{0}\left(\underline{t}_{1}\right) \leq M_{0}\left(t_{1}\right)$. On the other hand, if $M_{0}(t)$ is increasing at $t^{*} \in\left(\underline{t}_{1}, \bar{t}_{1}\right)$,

$$
\left\{D^{+} M_{0}(t)\right\}_{t=t^{*}}=\left\{D^{+}|x(t)|\right\}_{t=t^{*}} \leq-a(t)|x(t)|+|b(t)| M_{0}(t) \leq M_{b} M_{0}(t)
$$

This implies that

$$
M_{0}\left(\bar{t}_{1}\right) \leq M_{0}\left(\underline{t}_{1}\right) e^{M_{b}\left(\bar{t}_{1}-\underline{t}_{1}\right)} \leq M_{0}\left(t_{1}\right) e^{M_{b}\left(\bar{t}_{1}-\underline{t}_{1}\right)} .
$$

Similarly, we have

$$
M_{0}\left(\bar{t}_{2}\right) \leq M_{0}\left(\underline{t}_{2}\right) e^{M_{b}\left(\bar{t}_{2}-\underline{t}_{2}\right)} \leq M_{0}\left(\bar{t}_{1}\right) e^{M_{b}\left(\bar{t}_{2}-\underline{t}_{2}\right)} \leq M_{0}\left(t_{1}\right) e^{M_{b}\left[\left(\bar{t}_{1}-\underline{t}_{1}\right)+\left(\bar{t}_{2}-\underline{t}_{2}\right)\right]} .
$$

Repeating this process, finally we can have:

$$
M_{0}\left(t_{2}\right) \leq M_{0}\left(\bar{t}_{s}\right) \leq M_{0}\left(t_{1}\right) e^{M_{b} \sum_{i=1}^{s}\left(\bar{t}_{i}-\underline{t}_{i}\right)}=M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)} .
$$

The proof is completed.

Lemma 3. For any $t_{1}<t_{2}$,

1. if $\left(t_{1}, t_{2}\right)=S_{+}\left(t_{1}, t_{2}\right)$, then

$$
\begin{equation*}
\left|x\left(t_{2}\right)\right| \leq M_{0}\left(t_{1}\right)-\left[M_{0}\left(t_{1}\right)-\left|x\left(t_{1}\right)\right|\right] e^{-M_{a}\left(t_{2}-t_{1}\right)} \tag{19}
\end{equation*}
$$

2. if $\left(t_{1}, t_{2}\right)=S_{\eta}\left(t_{1}, t_{2}\right)$, then for any $\widetilde{M} \geq M_{0}\left(t_{1}\right)$,

$$
\begin{equation*}
\left|x\left(t_{2}\right)\right| \leq \max \left\{\delta \widetilde{M},\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu\left(S_{\eta}\left(t_{1}, t_{2}\right)\right) \widetilde{M}\right\} \tag{20}
\end{equation*}
$$

3. if $\left(t_{1}, t_{2}\right)=S_{-}\left(t_{1}, t_{2}\right)$, then

$$
\left|x\left(t_{2}\right)\right| \leq\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t_{2}-t_{1}\right)}-1\right] .
$$

Proof. 1. $\left(t_{1}, t_{2}\right)=S_{+}\left(t_{1}, t_{2}\right)$; In this case, by Lemma 1, $M_{0}(t)$ is nonincreasing on $\left(t_{1}, t_{2}\right)$. Then,

$$
D^{+}|x(t)| \leq-a(t)|x(t)|+|b(t)| M_{0}\left(t_{1}\right)
$$

By some calculations, we have

$$
\left|x\left(t_{2}\right)\right| \leq M_{0}\left(t_{1}\right)-\left[M_{0}\left(t_{1}\right)-\left|x\left(t_{1}\right)\right|\right] e^{-M_{a}\left(t_{2}-t_{1}\right)}
$$

2. $\left(t_{1}, t_{2}\right)=S_{\eta}\left(t_{1}, t_{2}\right)$; From Lemma 1, $M_{0}(t)$ is nonincreasing on $\left(t_{1}, t_{2}\right)$. For any $t \in\left(t_{1}, t_{2}\right)$, if $|x(t)| \geq \delta \widetilde{M} \geq \delta M_{0}(t)$ for $t \in\left(t_{1}, t_{2}\right)$, then

$$
D^{+}|x(t)| \leq-a(t)|x(t)|+|b(t)| \widetilde{M} \leq-a(t) \delta \widetilde{M}+|b(t)| \widetilde{M}=-[\delta a(t)-|b(t)|] \widetilde{M} \leq-\frac{\eta}{2} \widetilde{M}
$$

Thus,

$$
\begin{aligned}
\left|x\left(t_{2}\right)\right| & \left.\leq \max \left\{\delta \widetilde{M},\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu\left(S_{\eta}\left(t_{1}, t_{2}\right)\right) \widetilde{M}\right)\right\} \\
& =M_{0}\left(t_{1}\right)-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu\left(S_{\eta}\left(t_{1}, t_{2}\right)\right) M_{0}\left(t_{1}\right)\right\}\right]
\end{aligned}
$$

3. $\left(t_{1}, t_{2}\right)=S_{-}\left(t_{1}, t_{2}\right)$; If $M_{0}(t) \leq M_{0}\left(t_{1}\right)$ on $\left(t_{1}, t_{2}\right)$, then

$$
\begin{equation*}
D^{+}|x(t)| \leq-a(t)|x(t)|+|b(t)| M_{0}\left(t_{1}\right) \leq M_{b} M_{0}\left(t_{1}\right) \tag{21}
\end{equation*}
$$

Thus,

$$
\left|x\left(t_{2}\right)\right| \leq\left|x\left(t_{1}\right)\right|+M_{b} M_{0}\left(t_{1}\right)\left(t_{2}-t_{1}\right) \leq\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t_{2}-t_{1}\right)}-1\right]
$$

Otherwise, let $t^{*}=\inf \left\{t \in\left(t_{1}, t_{2}\right):\left|x\left(t^{*}\right)\right|=M_{0}\left(t_{1}\right)\right\}$.
Then for $t \in\left(t_{1}, t^{*}\right)$, we have

$$
\begin{equation*}
D^{+}|x(t)| \leq-a(t)|x(t)|+|b(t)| M_{0}\left(t_{1}\right) \leq M_{b} M_{0}\left(t_{1}\right) \tag{22}
\end{equation*}
$$

which implies

$$
M_{0}\left(t_{1}\right)=\left|x\left(t^{*}\right)\right| \leq\left|x\left(t_{1}\right)\right|+M_{b} M_{0}\left(t_{1}\right)\left(t^{*}-t_{1}\right) \leq\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t^{*}-t_{1}\right)}-1\right] .
$$

Therefore, (noting Lemma 24), we have:

$$
\begin{aligned}
\left|x\left(t_{2}\right)\right| & \leq M_{0}\left(t_{2}\right) \leq M_{0}\left(t^{*}\right) e^{M_{b}\left(t_{2}-t^{*}\right)} \leq M_{0}\left(t_{1}\right) e^{M_{b}\left(t_{2}-t^{*}\right)} \\
& =M_{0}\left(t_{1}\right)+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t_{2}-t^{*}\right)}-1\right] \\
& \leq\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t^{*}-t_{1}\right)}-1\right]+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t_{2}-t^{*}\right)}-1\right] \\
& \leq\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t^{*}-t_{1}\right)}-1\right]+M_{0}\left(t_{1}\right) e^{M_{b}\left(t^{*}-t_{1}\right)}\left[e^{M_{b}\left(t_{2}-t^{*}\right)}-1\right] \\
& =\left|x\left(t_{1}\right)\right|+M_{0}\left(t_{1}\right)\left[e^{M_{b}\left(t_{2}-t_{1}\right)}-1\right] .
\end{aligned}
$$

The estimation given in the following Lemma is the key step of the proof of main Theorem.
Lemma 4. For any $t_{1}<t_{2}$,

$$
\begin{equation*}
\left|x\left(t_{2}\right)\right| \leq M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)} . \tag{23}
\end{equation*}
$$

Proof. We prove this lemma by induction.
Step 1. We verify the initial case that $\left(t_{1}, t_{2}\right)$ is contained in only one of $S_{+}\left(t_{1}, t_{2}\right), S_{\eta}\left(t_{1}, t_{2}\right)$ and $S_{-}\left(t_{1}, t_{2}\right)$. There are totally three cases corresponding to those considered in Lemma 3.

1. $\left(t_{1}, t_{2}\right)=S_{+}\left(t_{1}, t_{2}\right)$. In this case, (23) reduces to

$$
\begin{aligned}
\left|x\left(t_{2}\right)\right| & \leq M_{0}\left(t_{1}\right)-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)} \\
& =M_{0}\left(t_{1}\right)\left[\left(1-e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)}\right]+\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right\} e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)} .\right.
\end{aligned}
$$

This is obvious from case 1 of Lemma 3.
2. $\left(t_{1}, t_{2}\right)=S_{\eta}\left(t_{1}, t_{2}\right)$. In this case, (23) reduces to

$$
\begin{aligned}
\left|x\left(t_{2}\right)\right| & \leq M_{0}\left(t_{1}\right)-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] \\
& =\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}
\end{aligned}
$$

This can be obtained by case 2 of Lemma 3 by letting $\widetilde{M}=M_{0}\left(t_{1}\right)$.
3. $\left(t_{1}, t_{2}\right)=S_{-}\left(t_{1}, t_{2}\right)$. In this case, (23) reduces to

$$
\begin{aligned}
\left|x\left(t_{2}\right)\right| & \leq M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right\}\right] \\
& =\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right\}+M_{0}\left(t_{1}\right)\left[e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-1\right]
\end{aligned}
$$

This is also obvious from case 3 of Lemma 3.

Step 2. Assume that (23) holds for a given $\left(t_{1}, t_{2}\right)$, and consider $\left(t_{1}, t_{3}\right)$, where $t_{3}>t_{2}$ and $\left(t_{2}, t_{3}\right)$ is contained in only one of $S_{+}\left(t_{2}, t_{3}\right), S_{\eta}\left(t_{2}, t_{3}\right)$ and $S_{-}\left(t_{2}, t_{3}\right)$. There are also three cases.

1. Case 1. $\left(t_{2}, t_{3}\right)=S_{+}\left(t_{2}, t_{3}\right)$. In this case, from Lemma 3, we have

$$
\begin{aligned}
& \left|x\left(t_{3}\right)\right| \leq M_{0}\left(t_{2}\right)-\left[M_{0}\left(t_{2}\right)-\left|x\left(t_{2}\right)\right|\right] e^{-M_{a} \mu_{+}\left(t_{2}, t_{3}\right)} \\
& =M_{0}\left(t_{2}\right)\left[1-e^{-M_{a} \mu_{+}\left(t_{2}, t_{3}\right)}\right]+\left|x\left(t_{2}\right)\right| e^{-M_{a} \mu_{+}\left(t_{2}, t_{3}\right)} \\
& \leq M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}\left[1-e^{-M_{a} \mu_{+}\left(t_{2}, t_{3}\right)}\right]+\left\{M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}\right. \\
& \left.-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)}\right\} e^{-M_{a} \mu_{+}\left(t_{2}, t_{3}\right)} \\
& =M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)} \\
& =M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)} .
\end{aligned}
$$

2. Case 2. $\left(t_{2}, t_{3}\right)=S_{\eta}\left(t_{2}, t_{3}\right)$. In this case, let $\widetilde{M}=\max \left\{M_{0}\left(t_{1}\right), M_{0}\left(t_{2}\right)\right\}$, then $M_{0}\left(t_{1}\right) \leq$ $\widetilde{M} \leq M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}=M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}$.

From Lemma 3, we have

$$
\begin{aligned}
\left|x\left(t_{3}\right)\right| \leq & \max \left\{\delta \widetilde{M},\left|x\left(t_{2}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{2}, t_{3}\right) \widetilde{M}\right\} \\
\leq & \max \left\{\delta M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)},\left|x\left(t_{2}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{2}, t_{3}\right) M_{0}\left(t_{1}\right)\right\} \\
\leq & \max \left\{\delta M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}, M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\right.\right.\right. \\
& \left.\left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)}-\frac{\eta}{2} \mu_{\eta}\left(t_{2}, t_{3}\right) M_{0}\left(t_{1}\right)\right\} \\
\leq & \max \left\{\delta M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}, M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\right.\right.\right. \\
& \left.\left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)}\right\} \\
= & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\min \left\{(1-\delta) M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)},\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\right.\right.\right. \\
& \left.\left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)}\right\} \\
\leq & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\min \left\{(1-\delta) M_{0}\left(t_{1}\right),\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\right.\right.\right. \\
& \left.\left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)}\right\} \\
\leq & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\min \left\{(1-\delta) M_{0}\left(t_{1}\right), M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\right.\right. \\
& \left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right\} e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)} \\
= & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right), \max \left\{\delta M_{0}\left(t_{1}\right),\right.\right.\right. \\
& \left.\left.\left.\left|x\left(t_{1}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)} \\
= & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right.\right. \\
& \left.\left.-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)}
\end{aligned}
$$

3. Case 3. $\left(t_{2}, t_{3}\right)=S_{-}\left(t_{2}, t_{3}\right)$. In this case, from Lemma 3, we have

$$
\begin{aligned}
\left|x\left(t_{3}\right)\right| \leq & \left|x\left(t_{2}\right)\right|+M_{0}\left(t_{2}\right)\left[e^{M_{b} \mu_{-}\left(t_{2}, t_{3}\right)}-1\right] \\
\leq & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right.\right. \\
& \left.\left.-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{2}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{2}\right)}+M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{2}\right)}\left[e^{M_{b} \mu_{-}\left(t_{2}, t_{3}\right)}-1\right] \\
= & M_{0}\left(t_{1}\right) e^{M_{b} \mu_{-}\left(t_{1}, t_{3}\right)}-\left[M_{0}\left(t_{1}\right)-\max \left\{\delta M_{0}\left(t_{1}\right),\left|x\left(t_{1}\right)\right|\right.\right. \\
& \left.\left.-\frac{\eta}{2} \mu_{\eta}\left(t_{1}, t_{3}\right) M_{0}\left(t_{1}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{1}, t_{3}\right)}
\end{aligned}
$$

The proof is completed.

Based on the estimation (23) given in Lemma 4, we are to prove Theorem (1.

## Proof of Theorem 1.

For $t \in\left[t_{1}^{-}, t_{1}\right]$, from Lemma 4, and noting $\left|x\left(t_{0}\right)\right| \leq M_{0}\left(t_{0}\right)$, we have

$$
\begin{aligned}
|x(t)| & \leq M_{0}\left(t_{0}\right) e^{M_{b} \mu_{-}\left(t_{0}, t\right)}-\left[M_{0}\left(t_{0}\right)-\max \left\{\delta M_{0}\left(t_{0}\right),\left|x\left(t_{0}\right)\right|-\frac{\eta}{2} \mu_{\eta}\left(t_{0}, t\right) M_{0}\left(t_{0}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{0}, t\right)} \\
& \leq M_{0}\left(t_{0}\right) e^{M_{b} \mu_{-}\left(t_{0}, t\right)}-\left[M_{0}\left(t_{0}\right)-\max \left\{\delta M_{0}\left(t_{0}\right), M_{0}\left(t_{0}\right)-\frac{\eta}{2} \mu_{\eta}\left(t_{0}, t\right) M_{0}\left(t_{0}\right)\right\}\right] e^{-M_{a} \mu_{+}\left(t_{0}, t\right)} \\
& =M_{0}\left(t_{0}\right)\left[e^{M_{b} \mu_{-}\left(t_{0}, t\right)}-\min \left\{1-\delta, \frac{\eta}{2} \mu_{\eta}\left(t_{0}, t\right)\right\} e^{-M_{a} \mu_{+}\left(t_{0}, t\right)}\right] \\
& =M_{0}\left(t_{0}\right)\left[e^{M_{b} \mu_{-}\left(t_{0}, t\right)}-\frac{\eta}{2} \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{0}, t\right)\right\} e^{-M_{a} \mu_{+}\left(t_{0}, t\right)}\right]
\end{aligned}
$$

This implies

$$
\begin{aligned}
M_{0}\left(t_{1}\right) & \leq M_{0}\left(t_{0}\right)\left[e^{M_{b} \mu_{-}\left(t_{0}, t_{1}\right)}-\frac{\eta}{2} \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{0}, t_{1}^{-}\right)\right\} e^{-M_{a} \mu_{+}\left(t_{0}, t_{1}\right)}\right] \\
& \leq M_{0}\left(t_{0}\right)\left[e^{M_{b} \mu_{-}\left(t_{0}, t_{1}\right)}-\frac{\eta}{2} \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{0}, t_{1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }}\right]
\end{aligned}
$$

Repeating this process, we have

$$
M_{0}\left(t_{m}\right) \leq M_{0}\left(t_{0}\right) \prod_{k=0}^{m-1}\left[e^{M_{b} \mu_{-}\left(t_{k}, t_{k+1}\right)}-\frac{\eta}{2} \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }}\right]
$$

Under the condition (17), for a given $C \in\left(C^{*}, \eta / 2\right)$, we can choose $k^{*}$ large enough such that for all $k \geq k^{*}$,

$$
\frac{\left[e^{M_{b} \mu_{-}\left(t_{k}, t_{k+1}\right)}-1\right] e^{M_{a}(N+1) \tau_{\max }}}{\min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\}} \leq C
$$

which implies

$$
e^{M_{b} \mu_{-}\left(t_{k}, t_{k+1}\right)} \leq 1+C \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }}
$$

Thus for $m>k^{*}$,

$$
\begin{equation*}
M_{0}\left(t_{m}\right) \leq M_{0}\left(t_{k^{*}}\right) \prod_{k=k^{*}-1}^{m-1}\left[1-\left(\frac{\eta}{2}-C\right) \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }}\right] \tag{24}
\end{equation*}
$$

By the condition (16), we have

$$
\begin{equation*}
\sum_{k=0}^{+\infty} \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }}=\infty \tag{25}
\end{equation*}
$$

Thus

$$
\lim _{m \rightarrow+\infty} M_{0}\left(t_{m}\right)=0
$$

For $t \in\left[t_{m}^{-}, t_{m}\right],|x(t)| \leq M_{0}\left(t_{m}\right)$, and for $t \in\left[t_{m}, t_{m+1}^{-}\right]$, we have the estimation that

$$
\begin{equation*}
|x(t)| \leq M_{0}\left(t_{m}\right) e^{M b N \tau_{\max }} . \tag{26}
\end{equation*}
$$

Therefore, we have

$$
\lim _{t \rightarrow+\infty}|x(t)|=0
$$

On the other hand, when $a(\cdot), b(\cdot)$ are given, we can find a fixed $k^{*}$ for (24). Thus let $K^{\prime}=$ $\max _{1 \leq k \leq k^{*}} M_{0}\left(t_{k}\right) / M_{0}\left(t_{0}\right)$, we have

$$
M_{0}\left(t_{m}\right) \leq K^{\prime} M_{0}(0)=\max _{-\tau_{\max } \leq s \leq 0} K^{\prime}|\phi(s)|
$$

for each $m$. Let $K=K^{\prime} e^{M \tau_{b} N \tau_{\text {max }}}$, then

$$
|x(t)| \leq K \max _{-\tau_{\max } \leq s \leq 0}|\phi(s)|
$$

for each $t$.

Furthermore, if there exists $\epsilon>0$ such that $\mu_{\eta}\left(t_{k}, t_{k+1}\right) \geq \epsilon$, then $1-\left(\frac{\eta}{2}-C\right) \min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\} e^{-M_{a}(N+1) \tau_{\max }} \leq 1-\left(\frac{\eta}{2}-C\right) \min \left\{\frac{1}{M_{a}}, \epsilon\right\} e^{-M_{a}(N+1) \tau_{\max }} \triangleq \lambda_{0}<1$. Then from (24) we have for $m \geq k^{*}$,

$$
M_{0}\left(t_{m}\right) \leq M_{0}\left(t_{k^{*}}\right) \lambda_{0}^{m-k^{*}} .
$$

Thus for $t \in\left[t_{m}, t_{m+1}\right]$, we have the estimation

$$
\begin{aligned}
|x(t)| & \leq M_{0}\left(t_{m}\right) e^{M_{b}(N+1) \tau_{\max }} \leq M_{0}\left(t_{k^{*}}\right) e^{M_{b}(N+1) \tau_{\max }} \lambda_{0}^{m-k^{*}} \leq K^{\prime} M_{0}\left(t_{0}\right) e^{M_{b}(N+1) \tau_{\max }} \lambda_{0}^{-\left(k^{*}+1\right)} e^{(m+1) \ln \lambda_{0}} \\
& =K^{\prime} M_{0}\left(t_{0}\right) e^{M_{b}(N+1) \tau_{\max }} \lambda_{0}^{-\left(k^{*}+1\right)} e^{\frac{\ln \lambda_{0}}{(N+1) \tau_{\max }}(m+1)(N+1) \tau_{\max }} \\
& \leq K^{\prime} M_{0}\left(t_{0}\right) e^{M_{b}(N+1) \tau_{\max }} \lambda_{0}^{-\left(k^{*}+1\right)} e^{\frac{\ln \lambda_{0}}{(N+1) \tau_{\max }} t},
\end{aligned}
$$

where the last inequality comes from the fact that $\frac{\ln \lambda_{0}}{(N+1) \tau_{\max }}<0$ and $t \leq(m+1)(N+1) \tau_{\max }$. Let $\alpha=-\frac{\ln \lambda_{0}}{(N+1) \tau_{\max }}>0$, and $\widetilde{K}=\max \left\{\max _{t_{0} \leq t \leq t_{k^{*}}} \frac{|x(t)|}{M_{0}(t)}, K^{\prime} e^{M_{b}(N+1) \tau_{\max }} \lambda_{0}^{-\left(k^{*}+1\right)}\right\} e^{\alpha k^{*}(N+1) \tau_{\max }}$, then

$$
|x(t)| \leq \widetilde{K} \max _{-\tau_{\max } \leq s \leq 0}|\phi(s)| e^{-\alpha t}
$$

The proof is completed.

## 3 Applications

In this section, we will give two applications the theoretical results, including self synchronization in a class of neural networks with time varying delays, and the existence and exponential stability of periodic solutions of a class of neural networks with periodic coefficients and delays.

### 3.1 Self synchronization of neural networks

First, we apply the theoretical results obtained in previous sections to the self synchronization analysis of neural networks. In Liu, Lu \& Chen(2011), we have discussed almost sure self synchronization in neural networks with randomly switching connections without time delays. In this paper, we discuss self synchronization in neural networks with bounded time-varying delays.

To be more general, consider the following Volterra functional differential systems

$$
\begin{equation*}
\frac{d x_{i}(t)}{d t}=-d_{i}(t) x_{i}(t)+f_{i}\left(x_{1}, \cdots, x_{n}, x_{1}\left(t-\tau_{i 1}(t)\right), \cdots, x_{n}\left(t-\tau_{i n}(t)\right), t\right)+I_{i}(t), \quad i=1, \cdots, n, \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
& \left|\frac{\partial f_{i}\left(u_{1}, \cdots, u_{n}, v_{1}, \cdots, v_{n}, t\right)}{\partial u_{j}}\right| \leq A_{i j}(t)  \tag{i}\\
& \left|\frac{\partial f_{i}\left(u_{1}, \cdots, u_{n}, v_{1}, \cdots, v_{n}, t\right)}{\partial v_{j}}\right| \leq B_{i j}(t)
\end{align*}
$$

(ii) $\tau_{i j}(t) \leq \tau_{\max }, \quad i, j=1,2, \cdots, n$.

Before state our main result, we first extend the $\eta$-condition to the case of $n$ function pairs.
Definition 2 (Common $\eta$-condition). Given $n$ function pairs $\left\{a_{i}(\cdot), b_{i}(\cdot)\right\}_{i=1}^{n}$ with $0<a_{i}(\cdot) \leq M_{a}$, $\left|b_{i}(\cdot)\right| \leq M_{b}$, we say they satisfy the common $\eta$-condition if the function pair $\{a(\cdot), b(\cdot)\}$ satisfies the $\eta$-condition, where $a(t)=a_{i_{t}}(t), b(t)=b_{i_{t}}(t)$, with $i_{t}$ satisfying $a_{i_{t}}(t)-\left|b_{i_{t}}(t)\right|=$ $\min _{j}\left\{a_{j}(t)-\left|b_{j}(t)\right|\right\}$ for each $t$.

Then we have

Theorem 2. Suppose that $0<d_{i}(t)-\sum_{j=1}^{n} A_{i j}(t) \leq M_{a}, \sum_{j=1}^{n} B_{i j}(t) \leq M_{b}$ for any $t \geq 0$. If $\left\{d_{i}(\cdot)-\sum_{j=1}^{n} A_{i j}(\cdot), \sum_{j=1}^{n} B_{i j}(\cdot)\right\}_{i=1}^{n}$ satisfy the common $\eta$-condition for a constant $\eta>0$, then the network (27) will reach outer self synchronization, i.e., for any two initial values $\phi(s), \psi(s) \in \mathbb{R}^{n}$, $s \in\left[-\tau_{\max }, 0\right]$, the trajectories with initial values $\phi(s), \psi(s)$ respectively will satisfy

$$
\lim _{t \rightarrow \infty}\|x(t)-y(t)\|=0
$$

Proof. Let $\tau(t)=\max _{i, j=1, \cdots, n} \tau_{i j}(t), z_{i}(t)=\left|x_{i}(t)-y_{i}(t)\right|$, and $z(t)=\max _{i}\left\{z_{i}(t)\right\}$, then we have

$$
\begin{aligned}
& D^{+} z_{i}(t)=-\operatorname{sign}\left(x_{i}(t)-y_{i}(t)\right) d_{i}(t)\left[x_{i}(t)-y_{i}(t)\right]+\operatorname{sign}\left(x_{i}(t)-y_{i}(t)\right)\left[f _ { i } \left(x_{1}(t), \cdots, x_{n}(t)\right.\right. \\
& \left.\left.x_{1}\left(t-\tau_{i 1}(t)\right), \cdots, x_{n}\left(t-\tau_{i n}(t)\right), t\right)-f_{i}\left(y_{1}(t), \cdots, y_{n}(t), y_{1}\left(t-\tau_{i 1}(t)\right), \cdots, y_{n}\left(t-\tau_{i n}(t)\right), t\right)\right] \\
& \leq-d_{i}(t) z_{i}(t)+\sum_{j=1}^{n} A_{i j}(t) z_{j}(t)+\sum_{j=1}^{n} B_{i j}(t) z_{j}\left(t-\tau_{i j}(t)\right)
\end{aligned}
$$

which implies

$$
D^{+} z(t) \leq-\left[d_{i_{t}}(t)-\sum_{j=1}^{n} A_{i t j}(t)\right] z(t)+\sum_{j=1}^{n} B_{i_{t j}}(t) \sup _{t-\tau(t) \leq s \leq t} z(s)
$$

It is easy to see that $\left\{d_{i_{t}}(\cdot)-\sum_{j=1}^{n} A_{i_{t} j}(\cdot), \sum_{j=1}^{n} B_{i_{t} j}\right\}$ satisfies the $\eta$-condition with the same $\eta$ if $\left\{d_{i}(\cdot)-\sum_{j=1}^{n} A_{i j}(\cdot), \sum_{j=1}^{n} B_{i j}(\cdot)\right\}$ satisfies the $\eta$-condition, by Theorem 1 , we conclude

$$
\begin{equation*}
\lim _{t \rightarrow \infty} z(t)=0 \tag{28}
\end{equation*}
$$

which implies

$$
\lim _{t \rightarrow \infty}\|x(t)-y(t)\|=0
$$

In particular, if $I_{i}(t) \equiv 0$ and $f_{i}(0, \cdots, 0)=0$ for $i=1, \cdots, n$. Then $x=0$ is a equilibrium. As a direct consequence of Theorem 2, we have

Corollary 2. Under the conditions in Theorem 2, if $I_{i}(t) \equiv 0$, and $f_{i}(0, \cdots, 0)=0$ for each $i$, then the equilibrium 0 of (27) is globally asymptotically stable.

### 3.2 Periodic neural networks

As another application, we discuss periodic neural networks, which can be described as

$$
\begin{equation*}
\frac{d u_{i}(t)}{d t}=-d_{i}(t) u_{i}(t)+\sum_{j=1}^{n} a_{i j}(t) g_{j}\left(u_{j}(t)\right)+\sum_{j=1}^{n} b_{i j}(t) f_{j}\left(u_{j}\left(t-\tau_{i j}(t)\right)\right)+I_{i}(t) \tag{29}
\end{equation*}
$$

where $d_{i}(t)>d_{i}>0, a_{i j}(t), b_{i j}(t), \tau_{i j}(t)>0, I_{i}(t): \mathbb{R}^{+} \rightarrow \mathbb{R}$ are continuously periodic functions with period $\omega>0, i, j=1,2, \ldots, n$.

By using a maximum function and the Brouwer fixed point theorem, it was proved in $\mathrm{Lu} \&$ Chen(2004) that

Proposition 7. Under the conditions that for $i=1, \cdots, n,\left|g_{i}(x+h)-g_{i}(x)\right| \leq G_{i}|h|, \mid f_{i}(x+h)-$ $f_{i}(x)\left|\leq F_{i}\right| h \mid$ and $-d_{i}(t)+\sum_{j=1}^{n} G_{j}\left|a_{i j}(t)\right|+\sum_{j=1}^{n} F_{j}\left|b_{i j}(t)\right|<-\eta$, the system (29) has an $\omega$-periodic solution $x(t)$, and there exists $\alpha>0$ such that for any solution $u(t)=\left[u_{1}(t), \cdots, u_{n}(t)\right]$, we have

$$
\begin{equation*}
\|u(t)-x(t)\|=O\left(e^{-\alpha t}\right), \quad t \rightarrow \infty \tag{30}
\end{equation*}
$$

Obviously, if the requirement $-d_{i}(t)+\sum_{j=1}^{n} G_{j}\left|a_{i j}(t)\right|+\sum_{j=1}^{n} F_{j}\left|b_{i j}(t)\right|<-\eta, i=1,2, \cdots, n$ is not satisfied, then the Brouwer fixed point theorem is no longer applicable. Here, as application of the theoretical results, we can prove the same result without this requirement. First, we make the following assumption.

Assumption 1. 1. $d_{i}(t)>d_{i}>0, a_{i j}(t), b_{i j}(t), 0<\tau_{i j}(t) \leq \tau_{\max }, I_{i}(t)$ are continuously periodic functions of $t$ with period $\omega>0, i, j=1,2, \ldots, n$;
2. There exist $G_{i}>0, F_{i}>0, i=1, \cdots, n$, such that $\left|g_{i}(x+h)-g_{i}(x)\right| \leq G_{i}|h|, \mid f_{i}(x+h)-$ $f_{i}(x)\left|\leq F_{i}\right| h \mid$ for each $h \in \mathbb{R} ;$
3. There exists $M_{a}>0, M_{b}>0$ such that $0 \leq d_{i}(t)-\sum_{j=1}^{n} G_{j}\left|a_{i j}(t)\right| \leq M_{a}, \sum_{j=1}^{n} F_{j}\left|b_{i j}(t)\right| \leq M_{b}$;

For some $\eta>0$, denote $\bar{S}_{\eta}=\left\{t \in[0, \omega]: \quad d_{i}(t)-\sum_{j=1}^{n}\left[G_{j}\left|a_{i j}(t)\right|+F_{j}\left|b_{i j}(t)\right|\right] \geq \eta, \quad i=\right.$ $1, \cdots, n\}$, and $\bar{\mu}_{\eta}=\mu\left(\bar{S}_{\eta}\right)$. Denote $\bar{S}_{-}=\left\{t \in[0, \omega]: d_{i}(t)-\sum_{j=1}^{n}\left[G_{j}\left|a_{i j}(t)\right|+F_{j}\left|b_{i j}(t)\right|\right]<\right.$ 0 for some $i\}$, and $\bar{\mu}_{-}=\mu\left(\bar{S}_{-}\right)$. Let $p$ be the smallest integer such that $\tau_{\max } \leq p \omega$.

Theorem 3. Under Assumption 1, if there exists some $\eta>0, N>0$ such that $\mu\left(\bar{S}_{\eta}\right)>0$ and

$$
\begin{equation*}
\frac{\left[e^{M_{b} p(N+1) \bar{\mu}_{-}}-1\right] e^{M_{a} p(N+1) \omega}}{\min \left\{\frac{1}{M_{a}}, p N \bar{\mu}_{\eta}\right\}}<\frac{\eta}{2} \tag{31}
\end{equation*}
$$

then the periodic neural network (29) has an $\omega$-periodic solution $x(t)$. Furthermore, for any solution $u(t)=\left[u_{1}(t), \cdots, u_{n}(t)\right]$, we have

$$
\begin{equation*}
\|u(t)-x(t)\|=O\left(e^{-\alpha t}\right), \quad t \rightarrow \infty \tag{32}
\end{equation*}
$$

Proof. First, we prove the existence of an $\omega$ period solution $x(t)$. Let $u(t)=\left[u_{1}(t), \cdots, u_{n}(t)\right]^{\top}$ be an arbitrary solution of (29). Let $\bar{u}(t)=u(t)-u(t-\omega)$, then we have:

$$
\begin{aligned}
& \frac{d \bar{u}_{i}(t)}{d t}=-d_{i}(t) \bar{u}_{i}(t)+\sum_{j=1}^{n} a_{i j}(t)\left[g_{j}\left(u_{j}(t)\right)-g_{j}\left(u_{j}(t-\omega)\right)\right] \\
& \quad+\sum_{j=1}^{n} b_{i j}(t)\left[f_{j}\left(u_{j}\left(t-\tau_{i j}(t)\right)\right)-f_{j}\left(u_{j}\left(t-\tau_{i j}(t)-\omega\right)\right)\right]
\end{aligned}
$$

Let $v(t)=\max _{i}\left\{\left|\bar{u}_{i}(t)\right|\right\}$, and denote $i_{t}$ the index such that $v(t)=\left|\bar{u}_{i_{t}}(t)\right|$, then we have:

$$
D^{+} v(t) \leq-d_{i_{t}}(t) v(t)+\sum_{j=1}^{n}\left|G_{j} a_{i_{t} j}(t)\right| v(t)+\sum_{j=1}^{n} F_{j}\left|b_{i_{t} j}(t)\right| \sup _{0 \leq s \leq p w} v(t-s)
$$

From definition, we have $S_{\eta}(0, \omega) \supseteq \bar{S}_{\eta}$ and $S_{-}(0, \omega) \subseteq \bar{S}_{-}$. Let $\tau_{\max }=p \omega$, from Theorem 11, $v(t)$ will converge to zero exponentially. Then for any given $t^{*}$, the sequence $\left\{\sum_{m=1}^{k} \bar{u}\left(t^{*}+k \omega\right)\right\}$, $k=1,2,3, \cdots$ is a Cauchy sequence. Thus there exists $x\left(t^{*}\right) \in \mathbb{R}^{n}$ such that $\lim _{k \rightarrow \infty} \sum_{m=1}^{k} \bar{u}\left(t^{*}+\right.$ $k \omega)=x\left(t^{*}\right)$. From the definition of $\bar{u}(t)$, this implies

$$
\lim _{k \rightarrow \infty} u\left(t^{*}+k \omega\right)=x\left(t^{*}\right)
$$

Since $\lim _{k \rightarrow \infty} u\left(t^{*}+k \omega\right)=\lim _{k \rightarrow \infty} u\left(t^{*}+\omega+k \omega\right)$, it is easy to see that $x\left(t^{*}\right)=x\left(t^{*}+\omega\right)$.
For any $t_{1}>0, t_{2}>0$,

$$
\begin{aligned}
x_{i}\left(t_{2}\right)-x_{i}\left(t_{1}\right)= & \lim _{k \rightarrow \infty}\left[u_{i}\left(t_{2}+k \omega\right)-u_{i}\left(t_{1}+k \omega\right)\right] \\
= & \lim _{k \rightarrow \infty} \int_{t_{1}}^{t_{2}}\left[-d_{i}(t) u_{i}(t+k \omega)+\sum_{j=1}^{n} a_{i j}(t) g_{j}\left(u_{j}(t+k \omega)\right)\right. \\
& \left.+\sum_{j=1}^{n} b_{i j}(t) f_{j}\left(u_{j}\left(t+k \omega-\tau_{i j}(t)\right)\right)\right] d t \\
= & \int_{t_{1}}^{t_{2}}\left[-d_{i}(t) x_{i}(t)+\sum_{j=1}^{n} a_{i j}(t) g_{j}\left(x_{j}(t)\right)\right. \\
& \left.+\sum_{j=1}^{n} b_{i j}(t) f_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)\right] d t
\end{aligned}
$$

Therefore, $x_{i}(t)$ is absolutely continuous and

$$
\frac{d x_{i}(t)}{d t}=-d_{i}(t) x_{i}(t)+\sum_{j=1}^{n} a_{i j}(t) g_{j}\left(x_{j}(t)\right)+\sum_{j=1}^{n} b_{i j}(t) f_{j}\left(x_{j}\left(t-\tau_{i j}(t)\right)\right)
$$

Thus, $x(t)$ is a periodic solution of (29).

Again, let $u(t)=\left[u_{1}(t), \cdots, u_{n}(t)\right]^{\top}$ be an arbitrary solution of (29). Denote $\tilde{u}(t)=u(t)-$ $x(t)$, and $\tilde{v}(t)=\max _{i}\left\{\left|\tilde{u}_{i}(t)\right|\right\}$. Then, using an argument similar as above, we can show that $\tilde{v}(t)$ tends to zero exponentially. This implies

$$
\begin{equation*}
\|u(t)-x(t)\|=O\left(e^{-\alpha t}\right), \quad t \rightarrow \infty \tag{33}
\end{equation*}
$$

for some $\alpha>0$. The proof is completed.

From Theorem 3, we can have the following corollary.
Corollary 3. Under Assumption [1, if $d_{i}(t)-\sum_{j=1}^{n}\left|a_{i j}(t)\right| G_{j}-\sum_{j=1}^{n}\left|b_{i j}(t)\right| F_{j} \geq 0$, and there exists $\eta>0$ such that $\mu\left(\bar{S}_{\eta}\right)>0$, then the periodic neural network (29) has an $\omega$-period solution which is exponentially asymptotically stable.

## 4 Numerical Examples

In this section, we provide two simple examples with simulation to illustrate the theoretical results.

### 4.1 Delay differential system

We consider the following delay differential system:

$$
\begin{equation*}
\dot{x}(t)=-a(t) x(t)+b(t) x(t-\tau(t)) . \tag{34}
\end{equation*}
$$

Here we take $\tau(t)=t-\lfloor t\rfloor$, where $\lfloor t\rfloor$ denotes the largest integer that is no greater than $t$. Let $a(t) \equiv 1$, and $b(t)$ be a step function such that

$$
b(t)=\left\{\begin{aligned}
0.8, & t \in[2 k, 2 k+0.5] \\
1.2, & t \in[2 k+1,2 k+1.002] \\
1, & \text { otherwise }
\end{aligned}\right.
$$

Thus, $\tau_{\max }=1, M_{a}=1$, and $M_{b}=1.2$. We take $\eta=0.2$, then $S_{\eta}(0,+\infty)=\cup_{k=0}^{+\infty}[2 k, 2 k+0.5]$, and $S_{-}(0,+\infty)=\cup_{k=0}^{+\infty}[2 k+1,2 k+1+0.002]$. Let $t_{0}=0$, and $N=1$, then $\mu_{-}\left(t_{k}, t_{k+1}\right)=0.004$, $\mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)=0.5$, and we have

$$
\frac{\left[e^{M_{b} \mu_{-}\left(t_{k}, t_{k+1}\right)}-1\right] e^{M_{a} \mu_{+}(N+1) \tau_{\max }}}{\min \left\{\frac{1}{M_{a}}, \mu_{\eta}\left(t_{k}, t_{k+1}^{-}\right)\right\}}=\frac{\left(e^{1.2 \times 0.004}-1\right) e^{2}}{\min \{1,0.5\}} \simeq 0.0711<0.1=\frac{\eta}{2} .
$$



Figure 1: Asymptotic stability of the zero solution of Eq. (34)

Then from Theorem 1, the zero solution of (34) is asymptotically stable. The simulation results are provided in Fig. [1, where the initial value are chosen randomly.

### 4.2 Periodic neural network with delays

In this simulation, we consider the following delay periodic neural network with 3 neurons:

$$
\begin{aligned}
& \frac{d x_{i}(t)}{d t}=-\left(2+\sin ^{2}(\pi t)\right) x_{i}(t)+\left|\sin ^{3}(\pi t)\right| \tanh \left(x_{i}(t)\right)+\sin ^{2}(2 \pi t) \tanh \left(x_{i+1}(t)\right) \\
& +\cos ^{2}(2 \pi t) \tanh \left(x_{i+2}(t)\right)+\sin ^{2}(4 \pi t) \arctan \left(x_{i+1}(t-|\sin (2 \pi t)|)\right) \\
& +\cos ^{2}(4 \pi t) \arctan \left(x_{i+2}(t-|\cos (2 \pi t)|)\right)+\sin (i \pi t), \quad i=1,2,3 .
\end{aligned}
$$

Here, $i+1$ and $i+2$ are understood as $i+1 \bmod 3, i+2 \bmod 3$ if they exceed 3 . Now, we verify that the conditions in Theorem 3 can be satisfied. In accordance to model (29), $d_{i}(t)=2+\sin ^{2}(\pi t)$,

$$
\left[a_{i j}(t)\right]=\left[\begin{array}{ccc}
\left|\sin ^{3}(\pi t)\right| & \sin ^{2}(2 \pi t) & \cos ^{2}(2 \pi t) \\
\cos ^{2}(2 \pi t) & \left|\sin ^{3}(\pi t)\right| & \sin ^{2}(2 \pi t) \\
\sin ^{2}(2 \pi t) & \cos ^{2}(2 \pi t) & \left|\sin ^{3}(\pi t)\right|
\end{array}\right],\left[b_{i j}(t)\right]=\left[\begin{array}{ccc}
0 & \sin ^{2}(4 \pi t) & \cos ^{2}(4 \pi t) \\
\cos ^{2}(4 \pi t) & 0 & \sin ^{2}(4 \pi t) \\
\sin ^{2}(4 \pi t) & \cos ^{2}(4 \pi t) & 0
\end{array}\right] .
$$

And we can choose $\tau_{\max }=1, F_{i}=G_{i}=1$. Thus,

$$
\begin{gathered}
0<d_{i}(t)-\sum_{j=1}^{3} G_{j}\left|a_{i j}(t)\right|=1+\sin ^{2}(\pi t)-\left|\sin ^{3}(\pi t)\right| \leq \frac{29}{27}, \\
\sum_{j=1}^{3} F_{j}\left|b_{i j}(t)\right|=\sum_{j=1}^{3}\left|b_{i j}(t)\right|=1 .
\end{gathered}
$$



Figure 2: Asymptotic stability of periodic solutions in a delayed periodic neural networks.

This implies that we can set $M_{a}=29 / 27, M_{b}=1$. On the other hand, $d_{i}(t)-\sum_{j=1}^{3}\left[G_{j}\left|a_{i j}(t)\right|+\right.$ $\left.F_{j}\left|b_{i j}(t)\right|\right]=\sin ^{2}(\pi t)-\left|\sin ^{3}(\pi t)\right| \geq 0$. This means that $\bar{\mu}_{-}=0$, so the left-hand term in Ineq. (31) is 0 and Ineq. (31) holds for any $\eta>0$. Since the maximum of $d_{i}(t)-\sum_{j=1}^{3}\left[G_{j}\left|a_{i j}(t)\right|+F_{j}\left|b_{i j}(t)\right|\right]$ is $2 / 27$, we can choose $\eta=1 / 27$, and from the continuity of $d_{i}(t)-\sum_{j=1}^{3}\left[G_{j}\left|a_{i j}(t)\right|+F_{j}\left|b_{i j}(t)\right|\right]$, we have $\mu\left(\bar{S}_{\eta}\right)>0$. So the requirements in Theorem 3 are satisfied and this network has a periodic solution which is asymptotically stable. This is verified by the simulation results in Fig. 2.

## 5 Conclusions

In this paper, we discuss generalized Halanay inequality and its applications. First, we prove a new generalized Halanay inequalities under less restricted conditions, which are useful for the asymptotic stability of the zeros solution of a delayed differential equation. To our knowledge, these conditions are the least restricted ones known. We also give two applications of the theoretical results. First, we provide more general sufficient conditions for the self synchronization of the neural networks with time varying delays. Then, under more relaxed requirements, we prove a sufficient condition for the existence and exponential stability periodical solutions for a class of neural networks with periodic coefficients and time varying delays. Yet, we only consider bounded time varying delays. The case of unbounded time-varying delays is also very important and will be our next research topic.

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