Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Operations Research Letters 34 (2006) 219–227
Operations Research Letters
www.elsevier.com/locate/orl

Optimal picking of large orders in carousel systems

Nelly Litvak

Faculty of EEMCS, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

Received 21 December 2004; accepted 2 May 2005
Available online 14 June 2005

Abstract

A carousel is an automated ring-shaped warehousing system that rotates either direction bringing items to a picker. We obtain the limiting behavior of the shortest rotation time and the number of steps before a turn, as well as approximate mean rotation time, for one large order with non-uniform items locations.

MSC: 90B06; 90B80; 62E15; 60F05; 60J20

Keywords: Carousel systems; Travel times; Spacings; Weak convergence; Allocation policy

1. Introduction

A carousel is an automated storage and retrieval system which is widely used in modern warehouses as one of major technologies for small parts' storage. The system consists of a circular disk with a large number of shelves and drawers along its circumference. The disk rotates either direction bringing the items to a picker who has a stationary position in front of the carousel. This takes away the walking time and thus enhances efficiency of the picker. Other important benefits include a better control over materials and a greater utilization of available space.

A natural model of a carousel is a circle of length 1. An order consists of n items whose locations are modeled as points on a circumference. The time needed to fulfill an order consists of a pick time (collecting the items from drawers), plus the travel (rotation) time of the carousel. For a given carousel unit with a human picker, the pick times are essentially pre-defined and not subject to control. The travel time however can be optimized by adopting an efficient allocation policy and choosing a picking sequence that provides a reasonably short route along the circle. Thus, one of relevant research problems in carousel systems is to characterize the travel time for all kinds of lay-outs and various picking sequences.

The analysis of the travel time under various strategies is in general a non-trivial problem. This problem however has been resolved for independent uniformly distributed items locations. Refs. [9,7] provide a complete analysis of the commonly used nearest item heuristic, where the next item to be collected is always the nearest one. The shortest route has been studied in [10]. As the optimal route allows at most one turn

E-mail address: n.litvak@math.utwente.nl.

0167-6377/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2005.05.001

(see e.g. [1]), it can be successfully approximated by so-called m-step strategies proposed in [14] and analyzed in detail in [8]. These strategies allow at most one turn after collecting at most $m \geqslant 0$ items. In the recent paper [11], a no-reversal strategy is adopted, which is in fact an m-step strategy with $m = 0$, i.e., the carousel rotates either only clockwise or only counterclockwise choosing the shortest of the two possible routes.

The results on the order picking time in one carousel can be further applied in performance analysis for more complex warehousing systems. For instance, in [11] the expected travel time in one carousel is needed in order to evaluate the throughput for a carousel pod. An interesting analysis of two carousels operated by one picker was presented by Park et al. [12] and considerably extended by Vlasiou et al. [16,15]. For such analysis, the full knowledge of the travel time distribution is required.

The model with non-uniform items locations reflects a relevant situation when some of the drawers are required more frequently than others. One interesting special case of non-uniform items locations was studied by Wan and Wolff [17] who analyzed the problem of picking clumpy orders, i.e. the orders concentrated on a relatively small segment of a circle. To the best of our knowledge, there is no other paper regarding travel times in carousel systems with non-uniform items locations.

In this paper, we focus on the length of the shortest rotation time needed to collect one order when the order size n is large and the items locations have a non-uniform continuous distribution with a positive density f on [0, 1]. In the next section, we formally describe the problem and provide the background. In Section 3, we obtain the limiting behavior of the travel time when the order size goes to infinity. In Section 4, we use these limiting results to derive a simple approximation formula for the mean travel time and verify this formula numerically. We also show that if the picker's starting position before collecting each order is fixed, then the optimal allocation rule depends on the order size. Finally, Section 5 presents asymptotic results on the number of items collected before a turn.

2. Problem description

We model a carousel as a circle of length 1. The picker has a position at point zero, and he has to collect one order of n items by moving along the circle at unit speed in either direction. The locations of the items are independent and identically distributed continuous random variables with probability density function $f(\cdot)$ which is positive and bounded on [0, 1]. For $i = 1, \ldots, n$, let Y_i denote a location of the ith item. Set $Y_0 = 0$, $Y_{n+1} = 1$. Further, let $0 = Y_{0:n} < Y_{1:n} < \cdots < Y_{n:n} < Y_{n+1:n} = 1$ denote the order statistics of $Y_0, Y_1, \ldots, Y_{n+1}$. Then the picker's starting point and the positions of the n items partition the circle into $n + 1$ spacings

$$D_{i,n} = Y_{i:n} - Y_{i-1:n}, \quad 1 \leqslant i \leqslant n + 1.$$

We assume that the optimal picking strategy is adopted, that is, the picker chooses the shortest possible route. Let T_n be the minimal (optimal) travel time of the picker. Since the optimal route admits at most one turn, we may write T_n as

$$T_n = 1 - \max\left\{ \max_{1 \leqslant j \leqslant n} \{D_{j,n} - Y_{j-1:n}\}, \max_{1 \leqslant j \leqslant n} \{D_{n+2-j,n} - (1 - Y_{n+2-j:n})\} \right\}. \tag{1}$$

Indeed, for $j = 1, 2, \ldots, n$, the quantity $D_{j,n} - Y_{j-1:n}$ is the gain in travel time (compared to one full rotation) obtained by skipping the spacing $D_{j,n}$ and going back instead. The same applies to $D_{n+2-j,n} - (1 - Y_{n+2-j:n})$ except here the picker makes his last move in the opposite direction. Optimally, the picker chooses the shortest route, i.e., largest possible gain.

It is convenient to write the travel time via spacings because various helpful properties of the spacings have been extensively studied in literature. The key reference is the classical paper of Pyke [13]. If the Y_i's have a uniform distribution, then the spacings satisfy the following distributional identity:

$$(D_{1,n}, D_{2,n}, \ldots, D_{n+1,n}) \stackrel{d}{=} (X_1/S_{n+1}, X_2/S_{n+1}, \ldots, X_{n+1}/S_{n+1}), \tag{2}$$

where $X_1, X_2, \ldots$ are independent exponentially distributed random variables with mean 1 and $S_{n+1} = X_1 + \cdots + X_{n+1}$. For non-uniform spacings, one can apply a well-known *asymptotic* independence and exponentiality. In Pyke [13], this result is written as follows. Let F be a continuous distribution function of Y_i, $i=1,\dots,n$, and f be the corresponding density function. For $0<u<v<1$, suppose that $s=F^{-1}(u)$ and $t=F^{-1}(v)$. Denote $D_{ni}=(n+1)D_{i,n}$, $i=1,\dots,n+1$. Then, if $i/n\to u$ and $j/n\to v$,

$$\lim_{n\to\infty} F_{D_{ni},D_{nj}}(x,y)=(1-\mathrm{e}^{-f(s)x})(1-\mathrm{e}^{-f(t)y}). \tag{3}$$

If f is positive and bounded on a finite closed interval and zero elsewhere, then (3) can be extended to any finite set of spacings.

In [7,8,10] we analyzed various order picking strategies and, in particular, the optimal strategy, exploiting (2). The following lemma from [8] was particularly useful in analyzing the shortest path.

Lemma 2.1. *Let $X_1, X_2, \dots$ be i.i.d. exponential random variables with mean 1. Define $S_0=0$ and $S_j=X_1+\cdots+X_j$, $j\geqslant 1$. Then for any $m=0,1,\dots,$*

$$\max_{1\leqslant j\leqslant m+1}\{X_j-S_{j-1}\}\overset{d}{=}\sum_{j=1}^{m+1}(2^j-1)^{-1}X_j. \tag{4}$$

In this note, we shall combine Lemma 2.1 and (3) to obtain a limiting behavior of T_n for non-uniform items locations as n goes to infinity.

3. The main result

Let $X_1, X_2, \dots, X_1', X_2', \dots$ be independent exponential random variables with mean 1, $S_0=S_0'=0$, $S_j=X_1+\cdots+X_j$, and $S_j'=X_1'+\cdots+X_j'$, for all $j\geqslant 1$. For any fixed m, when n goes to infinity, result (3) extended to a finite set of spacings implies that

$$\begin{aligned}&(n+1)(D_{1,n},\dots,D_{m+1,n},D_{n+1-m,n},\dots,D_{n+1,n})\\ &\quad\overset{d}{\to}\left(\frac{1}{f(0)}X_1,\dots,\frac{1}{f(0)}X_{m+1},\frac{1}{f(1)}X_{m+1}',\dots,\right.\\ &\qquad\left.\frac{1}{f(1)}X_1'\right),\end{aligned} \tag{5}$$

where for $i=1,\dots,m+1$, the expressions $(1/f(0))X_i$ and $(1/f(1))X_i'$ stand for exponential random variables with parameters $f(0)$ and $f(1)$, respectively. Furthermore, for all $j=1,\dots,m+1$, the quantities $(n+1)Y_{j-1:n}$ and $(n+1)(1-Y_{n+2-j:n})$ converge, respectively, to $(1/f(0))S_{j-1}$ and $(1/f(1))S_{j-1}'$. Then from Lemma 2.1 and (1) it is natural to foretell the following limiting result.

Theorem 3.1. *Let f be a density function of Y_i, $i=1,\dots,n$, and assume that f is positive and bounded on $[0,1]$. Then*

$$\begin{aligned}&(n+1)(1-T_n)\\ &\quad\overset{d}{\longrightarrow}\max\left\{\frac{1}{f(0)}\sum_{j=1}^{\infty}\frac{1}{2^j-1}X_j,\right.\\ &\qquad\left.\frac{1}{f(1)}\sum_{j=1}^{\infty}\frac{1}{2^j-1}X_j'\right\}\quad \textit{as } n\to\infty.\end{aligned}$$

The idea of the proof is as follows. First, we show that the probability of making a turn after collecting m items decreases exponentially with m. Thus, the travel time is determined essentially by a finite number of spacings close to the picker's starting position. For these spacings, we shall apply (5) and then (4).

Proof of Theorem 3.1. Note that the picker makes a turn after collecting the $(k-1)$th item only if $[D_{k,n}>Y_{k-1:n}]$ or $[D_{n+2-k,n}>1-Y_{n+2-k:n}]$. The probability of the former event is given by

$$\begin{aligned}&\mathbb{P}(D_{k,n}>Y_{k-1:n})\\ &\quad=\int_0^1\mathbb{P}(D_{k,n}>Y_{k-1:n}|Y_{k:n}=u)f_{Y_{k:n}}(u)\,\mathrm{d}u.\end{aligned} \tag{6}$$

For the conditional probability under the integral, we obtain

$$\begin{aligned}&\mathbb{P}(D_{k,n}>Y_{k-1:n}|Y_{k:n}=u)\\ &\quad=\mathbb{P}(Y_{k-1:n}\leqslant u/2|Y_{k:n}=u)\\ &\quad=\mathbb{P}(Y_1,\dots,Y_{k-1}\leqslant u/2|Y_1,\dots,Y_{k-1}<u)\\ &\quad=\frac{[F(u/2)]^{k-1}}{[F(u)]^{k-1}}.\end{aligned} \tag{7}$$

Define $g_1(u)=F(u/2)/F(u)$, $u\in(0,1]$, and put $g_1(0)=\lim_{u\to 0}F(u/2)/F(u)=1/2$. Note that $0<g_1(u)<1$ for all $u\in[0,1]$, since F is a continuous distribution function and the density f is strictly positive on $[0,1]$. Furthermore, g_1 is defined on a compact set, and thus there exists $u_1^*\in[0,1]$ such that $\gamma_1=g_1(u_1^*)=\max_{0\leqslant u\leqslant 1}g_1(u)$. Hence,

$g_1(u_1) \leqslant \gamma_1 < 1$ for all $u_1 \in [0, 1]$. Thus, substituting (7) in (6), we obtain

$$\mathbb{P}(D_{k,n} > Y_{k-1:n}) = \int_0^1 (g_1(u))^{k-1} f_{Y_{k:n}}(u)\,\mathrm{d}u \leqslant \gamma_1^{k-1}. \tag{8}$$

Analogously, we show that there exists $\gamma_2 < 1$ such that $\mathbb{P}(D_{n+2-k,n} > 1 - Y_{n+2-k:n}) \leqslant \gamma_2^{k-1}$.

Next, for some fixed $m \geqslant 0$, we approximate the minimal travel time by the travel time $T_n^{(m)}$ under the m-step strategy, where the picker may turn after collecting at most m items, and, exactly as under the optimal strategy, at most one turn is allowed. We write $T_n^{(m)}$ as

$$T_n^{(m)} = 1 - \max\left\{ \max_{1 \leqslant j \leqslant m+1} \{D_{j,n} - Y_{j-1:n}\}, \max_{1 \leqslant j \leqslant m+1} \{D_{n+2-j,n} - (1 - Y_{n+2-j:n})\} \right\}. \tag{9}$$

Since the right-hand side of (9) is a continuous function of the spacings, we consequently apply (5), the continuous mapping theorem (see e.g. [4]) and Lemma 2.1 getting

$$\begin{aligned}
&(n+1)(1 - T_n^{(m)}) \\
&\xrightarrow{d} \max\left\{ \frac{1}{f(0)} \max_{1 \leqslant j \leqslant m+1} \{X_j - S_{j-1}\}, \frac{1}{f(1)} \max_{1 \leqslant j \leqslant m+1} \{X'_j - S'_{j-1}\} \right\}, \\
&\stackrel{d}{=} \max\left\{ \frac{1}{f(0)} \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X_j, \frac{1}{f(1)} \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X'_j \right\}.
\end{aligned} \tag{10}$$

Now, consider the distribution functions

$$P_n(t) = \mathbb{P}((n+1)(1 - T_n) \leqslant t),$$
$$P_n^{(m)}(t) = \mathbb{P}((n+1)(1 - T_n^{(m)}) \leqslant t), \quad t \geqslant 0.$$

Note that $T_n \leqslant T_n^{(m)}$, and the strict inequality holds only if it is optimal to turn after collecting more than m items, which is possible only if either $[D_{k,n} > Y_{k-1:n}]$ or $[D_{n+2-k,n} > 1 - Y_{n+2-k:n}]$ occurs for some $k > m + 1$. Thus, for any $t \geqslant 0$, we have

$$\begin{aligned}
0 &\leqslant P_n^{(m)}(t) - P_n(t) \\
&= \mathbb{P}((n+1)(1 - T_n^{(m)}) \leqslant t, (n+1)(1 - T_n) > t) \\
&\leqslant \mathbb{P}(T_n < T_n^{(m)}) \\
&\leqslant \mathbb{P}\left(\left[\bigcup_{k=m+2}^{n} [D_{k,n} > Y_{k-1:n}] \right] \cup \left[\bigcup_{k=m+2}^{n} [D_{n+2-k,n} > 1 - Y_{n+2-k:n}] \right] \right) \\
&\leqslant \sum_{k=m+2}^{n} \mathbb{P}(D_{k,n} > Y_{k-1:n}) + \sum_{k=m+2}^{n} \mathbb{P}(D_{n+2-k,n} > 1 - Y_{n+2-k:n}) \\
&\leqslant \frac{\gamma_1^{m+1}}{1 - \gamma_1} + \frac{\gamma_2^{m+1}}{1 - \gamma_2},
\end{aligned} \tag{11}$$

where the last inequality follows from (8).

Now we can prove the limiting result. Denote

$$J = \sum_{j=1}^{\infty} \frac{1}{2^j - 1} X_j, \quad J' = \sum_{j=1}^{\infty} \frac{1}{2^j - 1} X'_j,$$
$$J_m = \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X_j, \quad J'_m = \sum_{j=1}^{m+1} \frac{1}{2^j - 1} X'_j.$$

Further, for $t \geqslant 0$, define

$$P(t) = \mathbb{P}\left(\max\left\{ \frac{1}{f(0)} J, \frac{1}{f(1)} J' \right\} \leqslant t \right),$$
$$P^{(m)}(t) = \mathbb{P}\left(\max\left\{ \frac{1}{f(0)} J_m, \frac{1}{f(1)} J'_m \right\} \leqslant t \right).$$

To show that the limit $\lim_{n\to\infty} P_n(t)$ exists and equals $P(t)$, first choose any $\varepsilon > 0$ and fix m large enough so that $\gamma_1^{m+1}/(1 - \gamma_1) + \gamma_2^{m+1}/(1 - \gamma_2) < \varepsilon/3$. Next, writing (10) as $\lim_{n\to\infty} P_n^{(m)}(t) = P^{(m)}(t)$ we see that there exists some $N > m$ such that for any $n_1, n_2 > N$ holds $|P_{n_1}^{(m)}(t) - P_{n_2}^{(m)}(t)| < \varepsilon/3$. Then, for such n_1, n_2, using (11), we get

$$\begin{aligned}|P_{n_1}(t)-P_{n_2}(t)| &\leqslant |P_{n_1}(t)-P_{n_1}^{(m)}(t)| \\ &\quad + |P_{n_1}^{(m)}(t)-P_{n_2}^{(m)}(t)| \\ &\quad + |P_{n_2}^{(m)}(t)-P_{n_2}(t)| \\ &\leqslant \gamma_1^{m+1}/(1-\gamma_1)+\gamma_2^{m+1}/(1-\gamma_2) \\ &\quad + \varepsilon/3+\gamma_1^{m+1}/(1-\gamma_1) \\ &\quad + \gamma_2^{m+1}/(1-\gamma_2) < \varepsilon.\end{aligned}$$

Hence, according to the Cauchy criterion, $P_n(t)$ converges to a limit for any $t>0$. Letting n go to infinity in (11), we obtain

$$P^{(m)}(t) \leqslant \lim_{n\to\infty} P_n(t) \leqslant P^{(m)}(t) + \frac{\gamma_1^{m+1}}{1-\gamma_1} + \frac{\gamma_2^{m+1}}{1-\gamma_2}$$

for an arbitrarily large m. Thus, for any $t>0$, $\lim_{n\to\infty} P_n(t) = \lim_{m\to\infty} P^{(m)}(t) = P(t)$. $\square$

Define $Q(t)=\mathbb{P}(J\leqslant t)$, $t\geqslant 0$. Then it follows from the definition of P that

$$\begin{aligned}P(t) &= \mathbb{P}\left(\frac{1}{f(0)}J\leqslant t,\ \frac{1}{f(1)}J'\leqslant t\right) \\ &= Q(f(0)t)Q(f(1)t), \quad t\geqslant 0.\end{aligned} \tag{12}$$

The properties of Q have been studied in detail in [10]. In particular, we show that

$$\begin{aligned}Q(t) &= 1-\sum_{j=1}^{\infty}(-1)^{j-1}2^j\exp\{-(2^j-1)t\} \\ &\quad \times \prod_{l=1}^{j}\frac{1}{2^l-1}, \quad t\geqslant 0.\end{aligned} \tag{13}$$

This equation together with (12) provides the explicit expression for P. We note that the exponential functionals similar to J appear in various contexts and have received a considerable attention in literature (see e.g. [3,5]).

The convergence in distribution in Theorem 3.1 also implies the convergence of moments. Writing $\bar{P}=1-P$ and $\bar{Q}=1-Q$, it is easy to see from (12) that for all $t\geqslant 0$,

$$\begin{aligned}\bar{P}(t) &= 1-(1-\bar{Q}(f(0)t))(1-\bar{Q}(f(1)t)) \\ &= \bar{Q}(f(0)t)+\bar{Q}(f(1)t)-\bar{Q}(f(0)t)\bar{Q}(f(1)t).\end{aligned} \tag{14}$$

Substituting (13) in (14), we find the kth moment from $k\int_0^\infty t^{k-1}\bar{P}(t)\,\mathrm{d}t$ as follows:

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}[(n+1)(1-T_n)]^k \\ &\quad = \mathbb{E}\left[\frac{1}{f(0)}J\right]^k + \mathbb{E}\left[\frac{1}{f(1)}J\right]^k \\ &\qquad - k\int_0^\infty t^{k-1}\bar{Q}(f(0)t)\bar{Q}(f(1)t)\,\mathrm{d}t \\ &\quad = k!\sum_{\alpha=0,1}\frac{1}{[f(\alpha)]^k}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{2^j}{(2^j-1)^k} \\ &\qquad \times \prod_{l=1}^{j}\frac{1}{2^l-1} - k!\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{i+j} \\ &\qquad \times \frac{2^{i+j}}{[f(0)(2^i-1)+f(1)(2^j-1)]^k} \\ &\qquad \times \prod_{l=1}^{j}\frac{1}{2^l-1}\prod_{r=1}^{i}\frac{1}{2^r-1}.\end{aligned}$$

The last expression can be written in many equivalent forms. For instance, for the expectation, we obtain the first term directly using the definition of J:

$$\begin{aligned}\mu &= \lim_{n\to\infty}\mathbb{E}[(n+1)(1-T_n)] \\ &= \mathbb{E}\left[\frac{1}{f(0)}J\right] + \mathbb{E}\left[\frac{1}{f(1)}J\right] \\ &\quad - \int_0^\infty \bar{Q}(f(0)t)\bar{Q}(f(1)t)\,\mathrm{d}t \\ &= \sum_{\alpha=0,1}\frac{1}{f(\alpha)}\sum_{j=1}^{\infty}\frac{1}{2^j-1} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j} \\ &\quad \times \frac{2^{i+j}}{(2^i-1)f(0)+(2^j-1)f(1)}\prod_{l=1}^{j}\frac{1}{2^l-1} \\ &\quad \times \prod_{r=1}^{i}\frac{1}{2^r-1}.\end{aligned} \tag{15}$$

4. Approximation for the mean travel time

For given $f(0)$, $f(1)$, the value of μ in (15) can be easily computed, and it follows from the convergence of moments that $(n+1)\mathbb{E}[1-T_n]\approx\mu$ for large enough n. Hence, we find a simple approximation for the average minimal travel time:

$$\mathbb{E}[T_n] \approx 1 - \frac{\mu}{n+1}. \tag{16}$$

In this section, we compare the approximation (16) with simulation results obtained for a carousel with 100 pick faces (shelves or drawers). The frequency f_i with which an item $i=1, \ldots, 100$ is demanded is modeled according to the truncated normal distribution

$$f_i = \frac{\int_{i-1}^{i} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x}{\int_{0}^{100} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x},$$

where we chose $\mu=50$ and $\sigma=30$ which implies that the maximal frequency is approximately four times larger than the minimal frequency. The position $j = 1, \ldots, n$ contains an item $i = v(j)$ where $v(\cdot)$ models the allocation rule. The picker's starting position is zero, and the numbers $f(0)$ and $f(1)$ in (15) are defined, respectively, as $100 \cdot f_{v(1)}$ and $100 \cdot f_{v(100)}$, where the scaling factor 100 appears since the carousel is modeled as a circle of a unit length.

In this research we focus on the travel time needed to collect a single order. The results apply directly to sequential orders if so-called *fixed dwell point strategy* is adopted, that is, the carousel returns to its starting position every time after collecting an order. Below we describe several allocation rules used in simulations.

If an order consists of one item then the following allocation policy is clearly optimal under the fixed dwell point strategy:

Organ-pipe (OP) allocation policy: Place the *most* frequently asked item at the bin located at the starting point of the picker. Repetitively place next *most* frequently asked item alternating between the positions to the left and to the right of the already placed item.

When the order size equals one, it was shown in [2] that the OP allocation rule is also optimal when collecting sequential orders under the more practical *floating dwell point strategy*, where the endpoint of picking an order becomes a starting point of collecting the next order.

Although the OP allocation is perfectly reasonable for small orders, our asymptotic results suggest that for large order sizes, it may perform quite poorly if the fixed dwell point strategy is applied. Indeed, it follows from Theorem 3.1 that for large enough n, the random variable T_n stochastically *increases* with $f(0)$ and $f(1)$. Hence, if orders are large, then *less* frequently asked items have to be stored *close* to the picker's starting point. Intuitively, if an order is small, then one stores most frequently asked items close by, hoping that the whole order can be collected by traveling only a small part of the circle. However, if an order is large, then, most probably, the picker has to cover the major part of the circle anyway. Hence, in this case, the travel time can be reduced only by skipping a large spacing close to the picker's starting point. Such large spacing is more likely to occur if the items that are stored close to the picker's starting point, are not been demanded frequently. This calls for the next two allocation strategies that may suite for collecting large orders under the fixed dwell point strategy.

Reversed organ-pipe (ROP) allocation policy: Place the *least* frequently asked item at the bin located at the starting point of the picker. Repetitively place next *least* frequently asked item alternating between the positions to the left and to the right of the already placed item.

Monotone (Mo) allocation policy: Place the *least* frequently asked item at the bin located at the starting point of the picker. Repetitively place next *least* frequently asked item to the right of the already placed item.

We shall also consider a random allocation policy that models, for instance, the allocation in alphabetic order.

Random (Ra) allocation policy: Repetitively place an item in a randomly chosen available bin.

Finally, we will compare the quality of our approximation with the well-studied case of the uniform items locations. The results are presented in Table 1. Although the approximation for non-uniform items locations is not as accurate as in the uniform case and is clearly unsuitable for small orders, it gives a correct indication of the magnitude of the mean travel time if an order is large enough.

Looking at the numerical results, one may also discuss a proper choice of an allocation policy. For instance, it is clear that the OP policy is only suitable for very small orders. Already for $n = 5$, the monotone policy performs as good as the OP policy, and for large orders, the monotone and the ROP policy perform the best as expected. Also note that for very large orders, the OP policy performs even worse than

Table 1
Approximation of the mean travel time

Allocation rule		$n=3$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$
OP	$\mathbb{E}[T_n]$	.4793	.6307	.8083	.9186	.9492	.9700
$f(0)=f(1)=1.3296$	Approx. (16)	.6331	.7554	.8665	.9301	.9527	.9712
ROP	$\mathbb{E}[T_n]$	.5564	.6617	.7712	.8523	.8874	.9220
$f(0)=f(1)=0.3409$	Approx. (16)	−.4312	.0459	.4796	.7274	.8153	.8876
Mo							
$f(0)=0.3409$	$\mathbb{E}[T_n]$	.5157	.6366	.7704	.8588	.8967	.9310
$f(1)=1.3296$	Approx. (16)	−.0776	.2816	.6081	.7947	.8610	.9155
Ra							
$f(0)=1.2356$	$\mathbb{E}[T_n]$	.5229	.6561	.8038	.8960	.9289	.9554
$f(1)=0.9785$	Approx. (16)	.5496	.6998	.8362	.9142	.9419	.9647
Uniform	$\mathbb{E}[T_n]$	.5262	.6588	.8049	.8977	.9299	.9568
$f(0)=f(1)=1$	Approx. (16)	.4605	.6404	.8038	.8972	.9304	.9577

the random allocation. The choice between the ROP and the monotone policy depends on given demand frequencies. For instance, in experiments with $\sigma=20$, the monotone allocation appears to be the best for any order size larger than 5.

One can also see that the average travel time for uniformly distributed items is larger than for non-uniformly distributed items with a reasonable allocation policy. This suggests that in a carousel pod, one should make sure that each carousel accommodates items of diverse demand frequencies. This is in lines with results of Hassini and Vickson [6] on optimal storing of products in carousels grouped in pods of two. For one-item orders, their nearly-optimal solutions are characterized by variability of demand frequencies in each of the two carousels.

5. Number of steps before a turn

As we discussed earlier, the optimal route implies at most one turn. In the remainder of the paper we shall present several results on the asymptotic behavior of the number of items collected before a turn as the order size n goes to infinity. In [8] we proved that in case of independent uniform items locations, the limiting distribution is geometric with parameter $\frac{1}{2}$. Below we show that this surprising result also holds for non-uniform items locations. Moreover, the number of items collected before a turn and the travel time turn out to be asymptotically independent.

Denote by K_n and $K_n^{(m)}$ the number of items collected before a turn when collecting an order of n items under the optimal strategy and the m-step strategy, respectively. If there is no turn, these numbers are set equal to zero. Observe that the number $K_n^{(m)}+1$ equals one of the values j_1 or j_2 where either the first or the second internal maximum in (9) is achieved. The choice of j_1 or j_2 depends on whether the first or the second internal maximum is larger. Let C denote an event that the first maximum is larger, and let $\bar{C}$ be the event complementary to C. Then $K_n^{(m)}$ can be formally written as

$$\begin{aligned}&\arg\max_{1\leqslant j\leqslant m+1}\{D_{j,n}-Y_{j-1:n}\}\mathbf{1}_{\{C\}}\\&\quad+\arg\max_{1\leqslant j\leqslant m+1}\{D_{n+2-j,n}-(1-Y_{n+2-j:n})\}\mathbf{1}_{\{\bar{C}\}}\\&\quad-1,\end{aligned}\tag{17}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. We can now prove the following lemma.

Lemma 5.1. *For any $m\geqslant 0$,*

(i) *if $n\to\infty$, then $K_n^{(m)}\xrightarrow{d}K^{(m)}$ where*

$$\begin{aligned}\mathbb{P}(K^{(m)}=k)=\frac{1}{2^{k+1}-2^{k-m}}&=\frac{1}{2^{k+1}}\\&+\frac{1}{2^{m+1}(2^{k+1}-2^{k-m})},\\&k=0,\ldots,m,\end{aligned}$$

(ii) $(n+1)(1-T_n^{(m)})$ *and* $K_n^{(m)}$ *are asymptotically independent as* $n \to \infty$.

Proof. In (17), multiply the expressions in argmax by $(n+1)$ and let $n \to \infty$. Then it follows from (5) that $K_n^{(m)}$ converges in distribution to $K^{(m)}$ given by

$$\arg\max_{1\leqslant j\leqslant m+1}\left\{\frac{1}{f(0)}[X_j - S_{j-1}]\right\}\mathbf{1}_{\{C\}} + \arg\max_{1\leqslant j\leqslant m+1}\left\{\frac{1}{f(1)}[X'_j - S'_{j-1}]\right\}\mathbf{1}_{\{\bar{C}\}} - 1,$$

where C reduces to the event

$$\left[\frac{1}{f(0)}\max_{1\leqslant j\leqslant m+1}\{X_j - S_{j-1}\} \geqslant \frac{1}{f(1)}\max_{1\leqslant j\leqslant m+1}\{X'_j - S'_{j-1}\}\right].$$

It follows from Corollary 3.2 in [8] that the random variables

$$\arg\max_{1\leqslant j\leqslant m+1}\{X_j - S_{j-1}\} \tag{18}$$

and

$$\max_{1\leqslant j\leqslant m+1}\{X_j - S_{j-1}\} \tag{19}$$

are independent, and hence, the distribution of $K^{(m)}$ does not depend on the maxima involved in the event C. Thus, for $0\leqslant k\leqslant m$, we subsequently use the total probability formula and the independence of (18) and (19) to obtain

$$\begin{aligned}\mathbb{P}(K^{(m)}=k) &= \mathbb{P}(C)\mathbb{P}\left(\arg\max_{1\leqslant j\leqslant m+1}\{X_j - S_{j-1}\}=k+1|C\right) \\ &\quad + \mathbb{P}(\bar{C})\mathbb{P}\left(\arg\max_{1\leqslant j\leqslant m+1}\{X'_j - S'_{j-1}\}=k+1|\bar{C}\right) \\ &= \mathbb{P}\left(\arg\max_{1\leqslant j\leqslant m+1}\{X_j - S_{j-1}\}=k+1\right) \\ &= \frac{1}{2^{k+1}-2^{k-m}},\end{aligned}$$

where the last equality is from Corollary 3.2 in [8]. Note that by (10), the asymptotic behavior of $(n+1)(1-T_n^{(m)})$ is determined by the maxima involved in C and $\bar{C}$, whereas the asymptotic behavior of $K_n^{(m)}$ does not depend on these maxima. Thus, $K_n^{(m)}$ and $(n+1)(1-T_n^{(m)})$ are asymptotically independent. □

We are now ready to prove the following theorem.

Theorem 5.2. (i) *For any fixed* $k\geqslant 0$,

$$\lim_{n\to\infty}\mathbb{P}(K_n=k)=2^{-(k+1)}, \quad k=0,1,\ldots .$$

(ii) $(n+1)(1-T_n)$ *and* K_n *are asymptotically independent when* n *goes to infinity.*

Proof. (i) Observe that for $0\leqslant k\leqslant m$,

$$\mathbb{P}(K_n^{(m)}=k)-\mathbb{P}(K_n>m) \leqslant \mathbb{P}(K_n=k)\leqslant \mathbb{P}(K_n^{(m)}=k).$$

It follows from (11) that $\mathbb{P}(K_n>m)=\mathbb{P}(T_n<T_n^{(m)})\leqslant \gamma_1^{m+1}/(1-\gamma_1)+\gamma_2^{m+1}/(1-\gamma_2)$. Further, Lemma 5.1 implies that $K_n^{(m)} \xrightarrow{d} K^{(m)}$ as $n\to\infty$. Now, for any fixed $k\geqslant 0$, we can apply the Cauchy criterion for $\mathbb{P}(K_n=k)$ exactly as it was done for $P_n(t)$ in the last part of the proof of Theorem 3.1, herewith showing that $\lim_{n\to\infty}\mathbb{P}(K_n=k)$ exists and equals $\lim_{m\to\infty}\mathbb{P}(K^{(m)}=k)=2^{-(k+1)}$, where the last equality is by Lemma 5.1(i).

The proof of (ii) is along the same lines. We first write

$$\begin{aligned}0 &\leqslant \mathbb{P}(K_n^{(m)}=k,\,(n+1)(1-T_n^{(m)})\leqslant t) - \mathbb{P}(K_n=k,\,(n+1)(1-T_n)\leqslant t) \\ &\leqslant \mathbb{P}(T_n<T_n^{(m)})\leqslant \frac{\gamma_1^{m+1}}{1-\gamma_1}+\frac{\gamma_2^{m+1}}{1-\gamma_2}.\end{aligned}$$

Applying again the Cauchy criterion and using the asymptotic independence of $K_n^{(m)}$ and $(n+1)(1-T_n^{(m)})$ as $n\to\infty$, we prove that $\lim_{n\to\infty}\mathbb{P}(K_n=k,\,T_n<1-t/(n+1))$ exists and equals

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(K_n^{(m)}=k,\,(n+1)(1-T_n^{(m)})\leqslant t) \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(K_n^{(m)}=k) \times \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}((n+1)(1-T_n^{(m)})\leqslant t) \\ &= 2^{-(k+1)}P(t).\end{aligned}$$

The asymptotic independence of K_n and $(n+1)(1-T_n)$ now follows from (i) and Theorem 3.1. □

Acknowledgements

The author is grateful to anonymous referees whose constructive comments helped to considerably improve the presentation of the paper.

References

[1] J.J. Bartholdi III, L.K. Platzman, Retrieval strategies for a carousel conveyor, IIE Trans. 18 (1986) 166–173.
[2] J. Bengu, An optimal storage assignment for automating rotating carousels, IIE Trans. 27 (1995) 105–107.
[3] J. Bertoin, Ph. Biane, M. Yor, Poissonian exponential functionals, q-series, q-integrals, and the moment problem for log-normal distributions, Seminar on Stochastic Analysis, Random Fields and Applications IV, Progr. Probab., 58, Birkhäuser, Basel, 2004, pp. 45–56.
[4] P. Billingsley, Convergence of Probability Measures, second ed., Wiley, New York, 1999.
[5] F. Guillemin, Ph. Robert, B. Zwart, AIMD algorithms and exponential functionals, Ann. Appl. Probab. 14 (2004) 90–117.
[6] E. Hassini, R. Vickson, A two-carousel storage location problem, Comput. Oper. Res. 30 (4) (2003) 527–539.
[7] N. Litvak, I. Adan, The travel time in carousel systems under the nearest item heuristic, J. Appl. Probab. 38 (1) (2001) 45–54.
[8] N. Litvak, I. Adan, On a class of order pick strategies in paternosters, Oper. Res. Lett. 30 (2002) 377–386.
[9] N. Litvak, I.J.B.F. Adan, J. Wessels, W.H.M. Zijm, Order picking in carousel systems under the nearest item heuristic, Probab. Engrg. Inform. Sci. 15 (2) (2001) 135–164.
[10] N. Litvak, W.R. van Zwet, On the minimal travel time needed to collect n items on a circle, Ann. Appl. Probab. 14 (2004) 881–902.
[11] R.D. Meller, J.F. Klote, A throughput model for carousel/VLM pods, IIE Trans. 36 (8) (2004) 725–741.
[12] B.C. Park, J.Y. Park, R.D. Foley, Carousel system performance, J. Appl. Probab. 40 (3) (2003) 602–612.
[13] R. Pyke, Spacings, J. Roy. Statist. Soc. Ser. B 27 (1965) 395–449.
[14] B. Rouwenhorst, J.P. van den Berg, G.J. van Houtum, W.H.M. Zijm, Performance analysis of a carousel system, in: Progress in Material Handling Research: 1996, The Material Handling Industry of America, Charlotte, NC, 1996, pp. 495–511.
[15] M. Vlasiou, I.J.B.F. Adan, An alternating service problem, Probab. Engrg. Inform. Sci. 19 (2005), no. 4. (to appear).
[16] M. Vlasiou, I.J.B.F. Adan, J. Wessels, A Lindley-type equation arising from a carousel problem, J. Appl. Probab. 41 (2004) 1171–1181.
[17] Y.-W. Wan, R.W. Wolff, Picking clumpy orders on a carousel, Probab. Engrg. Inform. Sci. 18 (1) (2004) 1–11.