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Quality of move-optimal schedules for minimizing total weighted completion time

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Abstract

We study the minimum total weighted completion time problem on identical machines. We analyze a simple local search heuristic, moving jobs from one machine to another. The local optima can be shown to be approximately optimal with approximation ratio $\frac{3}{2}$. In a special case, the approximation ratio is $\frac{3}{2} - 1/\sqrt{6} \approx 1.092$.

MSC: 90B35

Keywords: Parallel machines; Total weighted completion time; Approximation

1. Introduction

We study the strongly $\mathcal{NP}$-hard problem of scheduling n jobs J_j $(j = 1, \dots, n)$ with processing times p_j and weights w_j on m identical parallel machines in order to minimize total weighted completion time $\sum w_j C_j$ without preemption. In the classical scheduling notation this problem is denoted by $P \,||\, \sum w_j C_j$.

For $m = 1$, an optimal assignment is easily obtained by scheduling the jobs in order of non-increasing weight to processing time ratios w_j/p_j (Smith-ratios, cf. Smith [5]). The same argument shows that given any schedule, we may assume w.l.o.g. that the jobs on each machine are scheduled following Smith's rule. Given an assignment A of jobs to machines, we denote by $Z(A)$ the objective function value of the corresponding schedule (obtained by scheduling the jobs assigned to machine i according to Smith's rule, for all $i = 1, \dots, m$).

Smith's rule gives rise to the so-called *LRF-heuristic* ("largest ratio first") for $m \geqslant 2$ machines: An *LRF-assignment* is obtained by first ordering the jobs according to their Smith-ratios and then assigning

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them successively to the first available machine in a greedy manner.

LRF-assignments have been analyzed by Eastman et al. [3] and Kawaguchi and Kyan [4]. Relative to the value $Z(A^*)$ of an optimal assignment A^*, an LRF-assignment A has been shown to satisfy

$$\frac{Z(A)}{Z(A^*)} \leqslant \frac{1}{2}(\sqrt{2}+1) \approx 1.21,$$

and this bound is tight. Indeed, examples approaching the upper bound can be found (with all jobs having equal Smith-ratios, cf. [4]).

Here we study another simple heuristic: a local search which successively modifies a current assignment A by moving a job to another machine. We are interested in the quality of *move-optimal* assignments, i.e., local optima of this local search procedure.

In the general case (arbitrary Smith-ratios) the relation between LRF-assignments and move-optimal schedules is unclear. We can prove certain upper bounds on move-optimal schedules that are identical to corresponding bounds for LRF-schedules from Eastman et al. [3] (although the proofs are completely different). As a consequence of these we obtain our main result:

Theorem 1. *Let A be a move-optimal assignment of jobs to machines and A^* an optimal assignment. Then*

$$\frac{Z(A)}{Z(A^*)} \leqslant \frac{3}{2} - \frac{1}{2m},$$

where $Z(A)$ and $Z(A^*)$ denote the corresponding objective values.

The worst example found so far is a small one for $m = 2$ machines and has a ratio

$$\frac{Z(A)}{Z(A^*)} = \frac{6}{5}.$$

Hence, it is unclear (even in case $m = 2$) whether the bound in Theorem 1 is tight. And it is still not known, whether move-optimal assignments have a better approximation ratio than LRF-assignments.

In a special case the situation looks a bit different. We prove the following:

Theorem 2. *Let all jobs have the same Smith-ratio. Then the objective value $Z(A)$ of a move-optimal assignment A satisfies*

$$\frac{Z(A)}{Z(A^*)} \leqslant \frac{9-\sqrt{6}}{6} \approx 1.092,$$

where $Z(A^*)$ denotes the value of an optimal assignment A^*. Moreover, this bound is tight.

This gives a better approximation ratio than for general LRF-assignments. But, Chandra and Wong [2] study a somehow related problem of minimizing the sum of squared machine load. Their work implies, that in case of all jobs having equal Smith-ratios the approximation guarantee for LRF-assignments can be improved. The jobs have to be ordered by non-increasing processing times before assigning them successively to the first available machine in a greedy manner. Then an LRF-assignment A satisfies $Z(A)/Z(A^*) \leqslant 25/24$.

Recently, some work on the quality of local optima and the efficiency of local search methods for some related scheduling problems has been carried out. Schuurman and Vredeveld [6] and in his PhD thesis, Vredeveld [7] give an overview and present approximation guarantees of local optima for problem $P\,||\,C_{\max}$ as well as $Q\,||C_{\max}$ and $R\,||\,C_{\max}$. Moreover, Brucker et al. [1] have shown that iterative improvement using the move-neighborhood is a polynomial method for problem $P\,||\,C_{\max}$ with complexity $\mathcal{O}(n^2)$. Vredeveld [7] improves this complexity to $\mathcal{O}(nm)$ by using a job selection rule and generalized it for problem $Q\,||\,C_{\max}$ resulting in a complexity of $\mathcal{O}(n^2 m)$.

The remainder of the paper is organized as follows. In Section 2 we prove Theorem 1 and give the best lower bound found so far for the approximation ratio of a move-optimal assignment. Afterwards, in Section 3 we deal with the case of equal Smith-ratios and prove Theorem 2. The paper ends with some open problems.

2. General case

In order to derive an upper bound for the approximation ratio of a move-optimal assignment A, we compare the objective value $Z(A)$ with the optimal objective value Z_1^* of the one machine problem with the same set of jobs.

Let A be an arbitrary assignment of jobs to machines. We obtain an optimal schedule respecting this assignment by scheduling the jobs assigned to the same machine by non-increasing Smith-ratios w_j/p_j. Let M_{ij}^A denote the jth job on machine i in this schedule. If it is clear which assignment A is considered we may write M_{ij} to denote the job and p_{ij} and w_{ij} for the corresponding processing times and weights. For an arbitrary assignment, the objective value calculates as

$$Z(A) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} w_{ij} \left(\sum_{k=1}^{j} p_{ik} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \left(\sum_{k=j}^{n_i} w_{ik} \right), \tag{1}$$

where n_i denotes the number of jobs scheduled on machine i. Now consider an assignment A' arising from A by reassigning the jth job from machine i (which is job M_{ij}) to machine t. Observe, that the job has to be inserted on machine t at the appropriate position (defined by the Smith-ordering). If we denote the insert position of job M_{ij} on machine t with $\tau(i,j,t) = \tau(A,i,j,t)$, the change in the objective value is given by

$$Z(A) - Z(A') = w_{ij} \sum_{k=1}^{j-1} p_{ik} + p_{ij} \sum_{k=j+1}^{n_i} w_{ik} - w_{ij} \sum_{k=1}^{\tau(i,j,t)-1} p_{tk} - p_{ij} \sum_{k=\tau(i,j,t)}^{n_t} w_{tk}. \tag{2}$$

Since in a move-optimal assignment we can find no job and target machine that gives an improvement in the objective value, all differences in (2) have to be non-positive. Thus, if we define

$$\Delta_{ij} := p_{ij} \left(\sum_{k=j+1}^{n_i} w_{ik} \right) + w_{ij} \left(\sum_{k=1}^{j-1} p_{ik} \right) \quad \text{for all } 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n_i,$$

for a move-optimal assignment A the following inequalities hold for all $1 \leqslant i, t \leqslant m$, $i \neq t$, $1 \leqslant j \leqslant n_i$:

$$\Delta_{ij} \leqslant p_{ij} \left(\sum_{k=\tau(i,j,t)}^{n_t} w_{tk} \right) + w_{ij} \left(\sum_{k=1}^{\tau(i,j,t)-1} p_{tk} \right). \tag{3}$$

Furthermore, the values Δ_{ij} and the objective value $Z(A)$ are related. If we sum up all Δ_{ij} for a fixed machine i we get:

$$\begin{aligned} \sum_{j=1}^{n_i} \Delta_{ij} &= \sum_{j=1}^{n_i} w_{ij} \left(\sum_{k=1}^{j-1} p_{ik} \right) + \sum_{j=1}^{n_i} p_{ij} \left(\sum_{k=j+1}^{n_i} w_{ik} \right) \\ &= 2 \sum_{j=1}^{n_i} w_{ij} \left(\sum_{k=1}^{j} p_{ik} \right) - 2 \sum_{j=1}^{n_i} w_{ij} p_{ij}. \end{aligned}$$

Therefore, for any assignment A and the corresponding objective value $Z(A)$ we have

$$2Z(A) = 2 \sum_{i=1}^{m} \sum_{j=1}^{n_i} w_{ij} \left(\sum_{k=1}^{j} p_{ik} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (\Delta_{ij} + 2 w_{ij} p_{ij}). \tag{4}$$

We now consider the single machine problem $1 || \sum w_j C_j$ for the given set of jobs. Here Smith's rule gives an optimal schedule. W.l.o.g. we assume that the jobs are numbered such that $w_1/p_1 \geqslant \cdots \geqslant w_n/p_n$. Then the optimal objective value for the single machine problem calculates as

$$Z_1^* = \sum_{j=1}^{n} w_j \left(\sum_{k=1}^{j} p_k \right) = \sum_{j=1}^{n} p_j \left(\sum_{k=j}^{n} w_k \right).$$

Our goal is to bound the objective value $Z(A)$ of a move-optimal assignment A in terms of Z_1^*. For this we examine the target positions $\tau(i,j,t)$ for a fixed assignment A. All jobs M_{tk} with $t \neq i$ and $1 \leqslant k \leqslant \tau(i,j,t)-1$ have smaller indices and thus a non-smaller Smith-ratio than M_{ij}. To calculate the starting time of a job in an optimal schedule for the single machine problem we have to add the processing times of all jobs with smaller indices. Therefore, for a given assignment A we can expand the sums in the objective value Z_1^* to

$$Z_1^* = \sum_{i=1}^{m}\sum_{j=1}^{n_i} w_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=1}^{\tau(i,j,t)-1} p_{tk} + \sum_{k=1}^{j} p_{ik}\right). \tag{5}$$

By a similar argument, we obtain:

$$Z_1^* = \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=\tau(i,j,t)}^{n_t} w_{tk} + \sum_{k=j}^{n_i} w_{ik}\right). \tag{6}$$

Adding (5) and (6) we arrive at

$$\begin{aligned}
2Z_1^* &= \sum_{i=1}^{m}\sum_{j=1}^{n_i} w_{ij}\left(\sum_{k=1}^{j} p_{ik}\right) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}\left(\sum_{k=j}^{n_i} w_{ik}\right) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n_i} w_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=1}^{\tau(i,j,t)-1} p_{tk}\right) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=\tau(i,j,t)}^{n_t} w_{tk}\right) \\
&= 2Z(A) + \sum_{i=1}^{m}\sum_{j=1}^{n_i} w_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=1}^{\tau(i,j,t)-1} p_{tk}\right) \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n_i} p_{ij}\left(\sum_{\substack{t=1\\t\neq i}}^{m}\sum_{k=\tau(i,j,t)}^{n_t} w_{tk}\right).
\end{aligned}$$

If we now incorporate (3) and use afterwards (4) the following is obtained:

$$\begin{aligned}
2Z_1^* &\geqslant 2Z(A) + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{\substack{t=1\\t\neq i}}^{m} \Delta_{ij} \\
&= 2Z(A) + (m-1)\sum_{i=1}^{m}\sum_{j=1}^{n_i} \Delta_{ij} \\
&= 2Z(A) + 2(m-1)Z(A) \\
&\quad - 2(m-1)\sum_{i=1}^{m}\sum_{j=1}^{n_i} w_{ij}p_{ij} \\
&= 2mZ(A) - 2(m-1)\sum_{j=1}^{n} w_j p_j.
\end{aligned}$$

Thus, a move-optimal assignment A satisfies

$$Z(A) \leqslant \frac{1}{m}Z_1^* + \frac{m-1}{m}\sum_{j=1}^{n} w_j p_j. \tag{7}$$

We get a bound for the quotient of $Z(A)$ and the optimal objective value $Z(A^*)$ by considering a result from Eastman et al. [3]. They give the following lower bound for $Z(A^*)$:

$$Z(A^*) \geqslant \frac{1}{m}Z_1^* + \frac{m-1}{2m}\sum_{j=1}^{n} w_j p_j. \tag{8}$$

Since trivially $Z(A^*) \geqslant \sum_{j=1}^{n} w_j p_j$ holds, we conclude from (8) that

$$\begin{aligned}
Z(A^*) &\geqslant \alpha\left(\frac{1}{m}Z_1^* + \frac{m-1}{2m}\sum_{j=1}^{n} w_j p_j\right) \\
&\quad + (1-\alpha)\sum_{j=1}^{n} w_j p_j
\end{aligned} \tag{9}$$

for every $\alpha \in [0, 1]$.

Comparing (7) with (9) for $\alpha = 2m/(3m-1)$, we find

$$\frac{Z(A)}{Z(A^*)} \leqslant \frac{3}{2} - \frac{1}{2m}, \tag{10}$$

proving Theorem 1.

Consider an example consisting of four jobs and two machines. The following table shows the job data.

job j	1	2	3	4
p_j	1	1	2	2
w_j	1	1	$\frac{1}{2}$	$\frac{1}{2}$

The assignment A in Fig. 1 is move-optimal and has $Z(A) = 6$, whereas the optimum is $Z(A^*) = 5$. So $Z(A)/Z(A^*) = \frac{6}{5}$ which is smaller than $\frac{5}{4}$ (obtained from Eq. (10) with $m = 2$). For any even number of machines we get the ratio $\frac{6}{5}$ by taking multiple copies of the instance for two machines. This leads to the following lemma.

Lemma 3. *The approximation ratio for move-optimal assignments is at least* $\frac{6}{5}$.

3. Instances with equal Smith-ratios

In what follows, we assume that all jobs have equal Smith-ratios and prove an upper bound of $(9-\sqrt{6})/6$ on the approximation ratio of move-optimal assignments. Let A be an assignment of jobs to machines. We denote by M_i^A the set of jobs scheduled on machine i according to assignment A. If it is clear which assignment A is considered, we also write simply M_i. In order to express the objective value we use similar ideas as Kawaguchi and Kyan [4]. Since $w_j/p_j = r$ for all jobs j and some constant r, the objective function value $Z(A)$ corresponding to the assignment A calculates as

$$\begin{aligned} Z(A) &= \sum_{i=1}^{m} \sum_{j \in M_i} w_j \sum_{k \in M_i, k \leqslant j} p_k \\ &= \sum_{i=1}^{m} \sum_{j \in M_i} r p_j \sum_{k \in M_i, k \leqslant j} p_k \\ &= \frac{r}{2} \sum_{i=1}^{m} \left(\sum_{j \in M_i} p_j \sum_{k \in M_i} p_k + \sum_{j \in M_i} p_j^2 \right). \end{aligned}$$

Let L_i^A denote the workload of machine i (we omit the index A if there are no ambiguities), i.e.,

$$L_i = \sum_{j \in M_i} p_j.$$

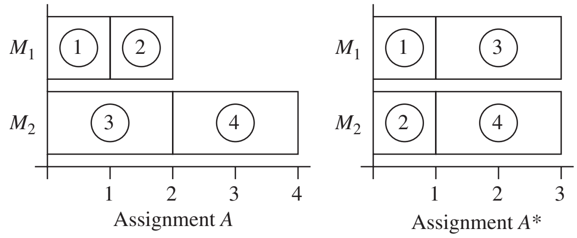

Fig. 1. Gantt-charts for worst-case example found so far.

Then the objective value Z(A) is equal to

$$Z(A) = \frac{r}{2} \left(\sum_{i=1}^{m} L_i^2 + \sum_{j=1}^{n} p_j^2 \right).$$

In the following, let A denote a move-optimal assignment and A^* an optimal assignment. We are interested in an upper bound for the ratio

$$\begin{aligned} \frac{Z(A)}{Z(A^*)} &= \frac{\sum_{i=1}^{m} (L_i^A)^2 + \sum_{j=1}^{n} p_j^2}{\sum_{i=1}^{m} (L_i^{A^*})^2 + \sum_{j=1}^{n} p_j^2} \\ &= 1 + \frac{\sum_{i=1}^{m} (L_i^A)^2 - \sum_{i=1}^{m} (L_i^{A^*})^2}{\sum_{i=1}^{m} (L_i^{A^*})^2 + \sum_{j=1}^{n} p_j^2}. \end{aligned} \tag{11}$$

Therefore, we may scale the processing times and weights such that $r = 1$ and

$$\sum_{j=1}^{n} p_j = m,$$

without changing the value of (11). Moreover, for assignment A we reorder the machines, such that $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_m$ holds. Observe, that for the sum of workloads we have

$$\sum_{i=1}^{m} L_i = \sum_{j=1}^{n} p_j = m. \tag{12}$$

Let $\Delta_i := L_i - L_m$ denote the deviation of the workload of machine i to the minimal workload L_m. From (12) we get:

$$\sum_{i=1}^{m} \Delta_i = m(1 - L_m). \tag{13}$$

With the help of (13) we obtain

$$\sum_{i=1}^{m} \Delta_i L_i = \sum_{i=1}^{m} \Delta_i (L_m + \Delta_i)$$
$$= L_m \sum_{i=1}^{m} \Delta_i + \sum_{i=1}^{m} \Delta_i^2$$
$$= \sum_{i=1}^{m} \Delta_i^2 + m L_m (1 - L_m). \tag{14}$$

We can exploit (14) to rewrite the sum of the squares of the workloads in terms of Δ_i and L_m:

$$\sum_{i=1}^{m} L_i^2 = \sum_{i=1}^{m} L_i (L_m + \Delta_i)$$
$$= L_m \sum_{i=1}^{m} L_i + \sum_{i=1}^{m} \Delta_i L_i$$
$$= \sum_{i=1}^{m} \Delta_i^2 + m L_m (2 - L_m). \tag{15}$$

The quadratic-arithmetic mean inequality together with Eq. (12) yields

$$\sum_{i=1}^{m} L_i^2 \geqslant m. \tag{16}$$

While the above holds for all assignments A, we now use the move-optimality of A to yield a lower bound for the processing times of jobs.

Lemma 4. *Let A be a move-optimal assignment. For all jobs $j \in M_i$ we have $p_j \geqslant \Delta_i$.*

Proof. Assume, for job $j \in M_i$ we have $p_j < \Delta_i$. Because of the equal Smith-ratios we may schedule job j after all other jobs of machine i without changing the objective value $Z(A)$. This yields a completion time $C_j^A = L_i$. Consider now the assignment B arising from A by assigning job j to machine m instead of i. By scheduling job j in this assignment after all jobs of machine m we receive a completion time of $C_j^B = L_m + p_j$. For the corresponding objective values holds $Z(A) - Z(B) = w_j(C_j^A - C_j^B) = w_j(L_i - L_m - p_j) = w_j(\Delta_i - p_j) > 0$. This contradicts the move-optimality of assignment A. □

With the help of Lemma 4 we obtain the following for machine i:

$$\sum_{j \in M_i} p_j^2 \geqslant \Delta_i \sum_{j \in M_i} p_j = \Delta_i L_i.$$

Adding this up for all machines i leads together with (14) to

$$\sum_{j=1}^{n} p_j^2 = \sum_{i=1}^{m} \sum_{j \in M_i} p_j^2 \geqslant \sum_{i=1}^{m} \Delta_i L_i$$
$$= \sum_{i=1}^{m} \Delta_i^2 + m L_m (1 - L_m). \tag{17}$$

Using (15), (16) and (17) we receive for the approximation ratio (11) the following:

$$\frac{Z(A)}{Z(A^*)} \leqslant 2 - \frac{m(2 - L_m)}{\sum_{i=1}^{m} \Delta_i^2 + m(1 + L_m - L_m^2)}. \tag{18}$$

Since we have $L_m \leqslant 1$, the nominator and the denominator are positive. In order to simplify (18) we have to give an upper bound for $\sum_{i=1}^{m} \Delta_i^2$.

We denote with a *worst-case instance* a scaled instance I for the considered scheduling problem for which the ratio $Z(A)/Z(A^*)$ is maximal. The next lemma shows that from a certain workload on there have to be at least two jobs scheduled on a machine.

Lemma 5. *Let I be a worst-case instance. If for a move-optimal assignment A there is a machine i with $L_i > 1$ then there are at least two jobs scheduled on machine i.*

Proof. Let i be a machine with workload $L_i > 1$ and job j be the only job scheduled on this machine, i.e. $p_j > 1$. We prove the lemma by showing that this job j is scheduled alone in any move-optimal assignment B. Since A^* is also move-optimal, this contradicts that I is a worst-case instance. (Removing this job from the instance would yield a higher worst-case ratio.)

Assume to the contrary, that job j is not scheduled alone on machine i in a move-optimal assignment B. Let j_0 be a job also scheduled on machine i. We have $p_{j_0} \leqslant L_i - p_j < L_i - L_m = \Delta_i$ contradicting Lemma 4. □

The preceding lemma also gives an upper bound on the processing times of the jobs in a worst-case instance.

Corollary 6. *Let I be a worst-case instance. Then $p_j \leqslant 1$ for all jobs j.*

The following lemma gives an important upper bound on the deviations Δ_i in a worst-case instance.

Lemma 7. *Let I be a worst-case instance. Then $\Delta_i \leqslant L_m$ for all machines i for a move-optimal assignment A.*

Proof. If $L_1 \leqslant 1$, then $1 = L_1 = \cdots = L_m$, which yields $\Delta_i = 0$ for all machines i.

Consider now the case, that there exists a machine i with $L_i > 1$. Assume $\Delta_i > L_m$, i.e. $L_i > 2L_m$. Thus, due to Lemma 4, each job $j \in M_i$ has $p_j > L_m$. Since we furthermore have $L_i > 1$, we know that $|M_i| \geqslant 2$. Thus, at least one of the jobs on M_i starts later than L_m. Moving this job to machine m reduces the objective value, contradicting the move-optimality of A. $\square$

Using Lemma 7 and (13) we get

$$\sum_{i=1}^{m} \Delta_i^2 \leqslant L_m \sum_{i=1}^{m} \Delta_i = m L_m (1 - L_m).$$

With the help of this we bound the approximation ratio (18) by

$$\frac{Z(A)}{Z(A^*)} \leqslant 2 - \frac{2 - L_m}{1 + 2L_m - 2L_m^2}.$$

The maximum is obtained at $L_m = 2 - \sqrt{6}/2$ yielding

$$\frac{Z(A)}{Z(A^*)} \leqslant \frac{9 - \sqrt{6}}{6}. \tag{19}$$

In case of equal Smith-ratios there is the following worst-case example for m machines. (Thanks to Tjark Vredeveld for providing this example!) For k with $0 < k < \frac{1}{2}$ there are $2km$ jobs with $p = w = 1$ and $(m - km)/\varepsilon$ jobs of size $p = w = \varepsilon$. The move-optimal assignment A shown in Fig. 2 schedules on each of the first km machines two of the jobs with $p = 1$. Moreover, on each of the last $m - km$ machines $1/\varepsilon$ of the jobs with $p = \varepsilon$ are scheduled. The assignment A has

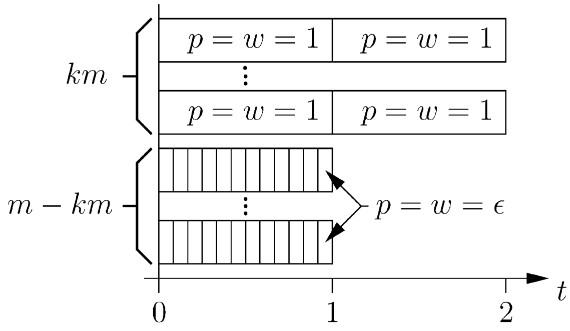

Fig. 2. Move-optimal assignment A for equal Smith-ratios.

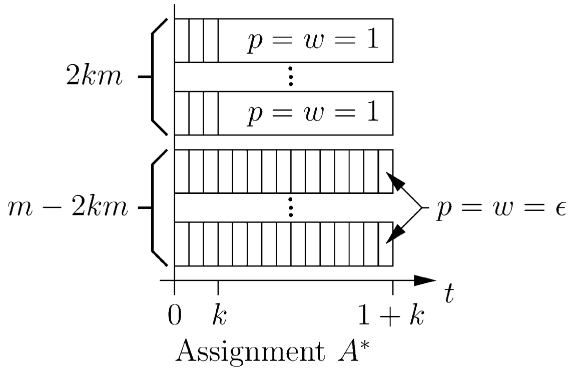

Fig. 3. Optimal assignment A^* for equal Smith-ratios.

an objective value of

$$Z(A) = \tfrac{1}{2} m (5k + 1 + (1 - k)\varepsilon).$$

In an optimal assignment A^* (see Fig. 3) k/ε jobs with $p = \varepsilon$ and 1 job with $p = 1$ are scheduled on each of the first machines. Moreover, on each of the last $m - 2km$ machines $(1 + k)/\varepsilon$ jobs with $p = \varepsilon$ are scheduled, yielding an objective value of

$$Z(A^*) = \tfrac{1}{2} m (k^2 + 4k + 1 + (1 - k)\varepsilon).$$

For $\varepsilon \to 0$, $k \to (\sqrt{6} - 1)/5$ and m sufficiently large such that $km \in \mathbb{N}$, the ratio $Z(A)/Z(A^*)$ approaches the maximum:

$$\frac{Z(A)}{Z(A^*)} \to \frac{9 - \sqrt{6}}{6}.$$

So, the bound (19) is tight. This proves Theorem 2.

4. Open problems

As mentioned already, it would be interesting to also have tight upper bounds in the general case. In particular, it is unclear, whether the approximation ratio indeed grows with the number m of machines (as does the upper bound in Theorem 1).

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