Certificates of linear mixed integer infeasibility

Kent Andersen [a], Quentin Louveaux [b], Robert Weismantel [c,*]

[a] *Department of Mathematical Sciences, University of Copenhagen, Denmark.*

[b] *Montefiore Institute, University of Liège, Belgium.*

[c] *Department of Mathematics/IMO, Otto-von-Guericke-University of Magdeburg, Germany.*

Abstract

We derive a certificate of integral infeasibility for linear systems with equations and inequalities by generating algebraically an outer description of a lattice point free polyhedron that contains the given integer infeasible system. The extension to the mixed integer setting is also derived.

Key words: Mixed integer programming, disjunction, split cuts

1 Introduction

It is a fundamental result in the theory of integer optimization that one can give a certificate for a vector not being a member of a lattice. This result can be viewed as an integer version of the Farkas Lemma.

Theorem 1.1 *[3] Let $A \in \mathbb{Z}^{m\times n}$ be of full row rank and let $b \in \mathbb{Z}^m$. The system $Ax = b$ has no integral solution iff the system $y^T A$ integer, $y^T b$ fractional is solvable over the rational numbers.*

Among other applications, this result is important in developing the theory of totally dual integral systems and for proving finiteness of cutting plane algorithms in the pure integer case, see [5]. Its applicablility, however, is limited to systems of equations and unbounded variables. Indeed, if inequalities are present, it is easy to design examples even in three variables for which a certificate of this kind cannot be given.

It is the purpose of this paper to generalize Theorem 1.1 to a mixed system of inequalities and equations,

$$\begin{aligned} A\,x + G\,z &= b \\ C\,x + H\,z &\le d \end{aligned}, \quad x \in \mathbb{Z}^n,\ z \in \mathbb{R}^q. \qquad (1)$$

* Corresponding author. Address: Otto-von-Guericke-Universität Magdeburg, Department of Mathematics/IMO, Universitätsplatz 2, 39106 Magdeburg, Germany

Email addresses: kha@math.ku.dk (Kent Andersen), q.louveaux@ulg.ac.be (Quentin Louveaux), weismantel@imo.math.uni-magdeburg.de (Robert Weismantel).

The paper is organized as follows. In Section 2 we deal with the pure integer version of system (1) (i.e., $q = 0$) and treat the case when the rank of C is equal to one. We generalize this result to higher order ranks of the inequality system in Section 3. In Section 4 we develop an algebraic certificate for the infeasibility of system (1) in the presence of both integer and continuous variables. We also discuss the following application of our result. The certificate of infeasibility of system (1) will be used for deriving a multiterm disjunction from a face of a polyhedron not containing a mixed integer point that only depends on the dimension of the face.

In this paper we use the $+$-operator to denote the Minkowski-sum of two sets in $\mathbb{R}^n$. The linear (conic) space generated by the vectors $w^1, \ldots, w^d$ is denoted $\text{lin}(w^1, \ldots, w^d)$ ($\text{cone}(w^1, \ldots, w^d)$), while the null space of a matrix B is denoted by $\ker(B)$. For a set $S \subseteq \mathbb{R}^n$, the symbol $S^\perp$ denotes the orthogonal complement of S. The notation $\text{int}(S)$ denotes the relative interior of a set S.

2 The integer case: rank of $C = 1$

Theorem 1.1 can be interpreted geometrically. To this end, let A be of full row rank and let b be an integral vector. Then, $Ax = b$ defines an affine space that we can represent in the form $\{v^*\} + \text{lin}\,(W)$, where $v^* \in \mathbb{Q}^n$ and $W = \{w^1, \ldots, w^d\} \subseteq \mathbb{Z}^n$ is a set of linearly independent vectors. Then it follows that the set $\{y^T A \mid y \in \mathbb{Q}^m\}$ is a subset of $\text{lin}\,(W)^\perp$. Hence, Theorem 1.1 is equivalent to the following result.

Theorem 2.1

$$\Big(\{v^*\} + \text{lin}\,(W)\Big) \cap \mathbb{Z}^n = \emptyset$$

iff there exists $\pi \in \text{lin}\,(W)^\perp \cap \mathbb{Z}^n$ *such that* $\pi^\top v^* \notin \mathbb{Z}$.

As a first step we generalize this result to polyhedra that one can represent as the Minkowski sum of an edge plus a linear span. We obtain

Theorem 2.2 *Let* $v_1^*, v_2^* \in \mathbb{Q}^n$ *and let* $E^* = \text{conv}(v_1^*, v_2^*)$ *denote an edge.*

$$(E^* + \text{lin}\,(W)) \cap \mathbb{Z}^n = \emptyset$$

iff there exists $\pi \in \text{lin}\,(W)^\perp \cap \mathbb{Z}^n$ *such that*

$$\pi^\top v \notin \mathbb{Z} \textit{ for all } v \in E^*.$$

PROOF. We begin to show that both systems cannot have a solution simultaneously. Suppose that $(E^* + \text{lin}\,(W)) \cap \mathbb{Z}^n \neq \emptyset$. Then it follows that there exists an $x^* \in \mathbb{Z}^n$ and multipliers $0 \leq \lambda \leq 1$, $\mu_1, \ldots, \mu_d \in \mathbb{Q}$ such that $x^* = \lambda v_1^* + (1-\lambda) v_2^* + \sum \mu_i w^i$. This implies that for all $\pi \in \text{lin}\,(W)^\perp \cap \mathbb{Z}^n$ we have that

$$\begin{aligned} &\pi^\top (\lambda v_1^* + (1-\lambda) v_2^*) = \\ &\pi^\top \left(\lambda v_1^* + (1-\lambda) v_2^* + \sum \mu_i w^i\right) = \\ &0 + \pi^\top x^* \in \mathbb{Z}, \end{aligned}$$

i.e., the system

$$\pi \in \text{lin}\,(W)^\perp \cap \mathbb{Z}^n,\ \pi^\top v \notin \mathbb{Z} \text{ for all } v \in E^*$$

is infeasible.

As a next step we assume that $(E^* + \text{lin}\,(W)) \cap \mathbb{Z}^n = \emptyset$. The following two cases may be distinguished. In the first case, the set $(v_1^* + \text{lin}\,(W, v_2^* - v_1^*)) \cap \mathbb{Z}^n = \emptyset$. Then the result follows from the Farkas Lemma using v_1^* in place of v^*. Otherwise, there exist smallest positive rational numbers $\lambda_1,\ \lambda_2 \in \mathbb{Q}$ such that

$$(v_2^* + \lambda_2(v_2^* - v_1^*) + \text{lin}\,(W)) \cap \mathbb{Z}^n \neq \emptyset,$$
$$(v_1^* + \lambda_1(v_1^* - v_2^*) + \text{lin}\,(W)) \cap \mathbb{Z}^n \neq \emptyset.$$

Let us denote by z^1 and z^2 the corresponding integer points, respectively, i.e., there exist $\mu_{i,1}, \mu_{i,2} \in \mathbb{Q}$, $i = 1, \ldots, d$ such that

$$z_1 = v_1^* + \lambda_1(v_1^* - v_2^*) + \textstyle\sum_{i=1}^d \mu_{i,1} w^i,$$
$$z_2 = v_2^* + \lambda_2(v_2^* - v_1^*) + \textstyle\sum_{i=1}^d \mu_{i,2} w^i.$$

Noting that $\lambda_1 > 0$ and $\lambda_2 > 0$, it follows that for all $0 < \sigma < 1$ we have that

$$\Big(\{z_1 + \sigma(z_2 - z_1)\} + \operatorname{lin}(W)\Big) \cap \mathbb{Z}^n = \emptyset. \quad (2)$$

As a next step we consider the following system of equations in integer variables $\pi_1, \ldots, \pi_n$:

$$\begin{array}{ll} (z_2 - z_1)^\top & \pi = 1 \\ w_1^\top & \pi = 0 \\ \vdots & \\ w_d^\top & \pi = 0 \\ & \pi \in \mathbb{Z}^n. \end{array}$$

If this system is inconsistent, then by invoking Theorem 1.1 we may conclude that the following dual system is solvable:

There exists $y \in \mathbb{Q}^{d+1}$ such that

$$(z_2 - z_1)y_1 + \sum_{i=1}^d w^i y_i \in \mathbb{Z}^n, \text{ but } y_1 \notin \mathbb{Z}.$$

Since $z_2 - z_1 \in \mathbb{Z}^n$, we can assume without loss of generality that $0 < y_1 < 1$. Then setting $\sigma := y_1$, $z_1 + \sigma(z_2 - z_1) + \sum w^i y^i \in \mathbb{Z}^n$ contradicts Equation (2). Hence, the primal integral system is feasible and determines the desired split with normal vector π. This completes the proof. $\Box$

This geometric statement can be directly turned into a certificate for the infeasibility of an integral system of equations and an inequality system of row rank equal to one.

Corollary 2.3 *The set $X = \{x \in \mathbb{R}^n | Ax = b, l \le c^T x \le u\}$ has no integral solution if and only if there exist $y \in \mathbb{Q}^m$ and $z \in \mathbb{Q}_+$ such that $y^T A + zc \in \mathbb{Z}^n$ and the interval $[y^T b + zl, y^T b + zu]$ contains no integer point.*

PROOF.

Case 1: If X is empty, then the result can be derived from the Farkas lemma.
Case 2: Suppose that for all x such that $Ax = b$, we have $l \le c^T x \le u$. Then we can apply Theorem 1.1 to the system $Ax = b$ and obtain a vector y such that $y^T A$ is integral and $y^T b$ is fractional. Then, $(y, 0)$ yields the desired result.
Case 3: In this case we have that $\operatorname{rank}(A) \le n - 1$, otherwise we are in one of the two previous cases. Notice also that if c is in the subspace spanned by the rows of A, we are in one of the two previous cases. We can therefore express the set X as $X = \{x \in \mathbb{R}^n | x = \lambda x^0 + (1-\lambda)x^1 + \sum_{i=1}^{n-m} \mu^i y^i, \lambda \in [0,1]\}$, where x^0 satisfies $Ax = b, c^T x = l$ and x^1 satisfies $Ax = b, c^T x = u$ and (y^i) are a basis of $Ax = 0, c^T x = 0$. We now obtain the result from Theorem 2.1. $\Box$

Example 1. Let $X \subseteq \mathbb{R}^4$ be given by

$$2x_1 + x_2 + 3x_3 - x_4 = 3 \quad (3)$$
$$6x_1 - x_2 - 2x_3 + x_4 = 5 \quad (4)$$
$$5 \le 4x_2 + x_3 - 4x_4 \le 8. \quad (5)$$

A short proof of the fact that X has no integral solution is to compute $\frac{2}{5}(3) + \frac{1}{5}(4) + \frac{1}{5}(5)$. It follows that $2x_1 + x_2 + x_3 - x_4$ is (i) integral and (ii) must be included in the interval $[\frac{16}{5}, \frac{19}{5}]$. Since this is not possible, $X \cap \mathbb{Z}^4 = \emptyset$.

3 From edges to higher dimensional polyhedra

This section deals with an extension of Corollary 2.3 to general systems $X \cap \mathbb{Z}^n$, where

$$X = \{x \in \mathbb{R}^n | Ax = b, Cx \leq d\}. \qquad (6)$$

There is a way to prove that $X \cap \mathbb{Z}^n$ is empty by establishing a system with rank(C) variables which has no integral solution. First we outline this construction. Assuming that A has full row rank equal to m, the subsystem $Ax = b,\ x \in \mathbb{Z}^n$ can be rewritten as $x = x^0 + L\lambda,\ \lambda \in \mathbb{Z}^{n-m}$ where $L \in \mathbb{Z}^{n\times(n-m)}$. With this transformation (6) becomes

$$C\,L\,\lambda \quad \leq \quad d - Cx^0, \quad \lambda \in \mathbb{Z}^{n-m}. \qquad (7)$$

From the equivalence of the two systems it follows that (6) is infeasible if and only if (7) is infeasible. From a theorem of Doignon [2] it follows that if (6) is infeasible, then at most 2^{n-m} inequalities of CL suffice to determine an infeasible integral system. This certificate is however large if m is small.

As a next step one applies the techniques outlined in the proof of Corollary 18.7c in [5] with the goal to reduce the number of variables.

Let l denote the rank of C. Then one may observe that $\hat{C} = CL$ has row rank at most equal to l. Let

$$\hat{C} \;=\; \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix},$$

where $\hat{C}_1$ contains the l linearly independent rows of $\hat{C}$. Hence, by applying a Hermite normal form computation (multiplication by a unimodular matrix U) to $\hat{C}_1$, we arrive at system

$$C\,L\,U\,\mu \quad \leq \quad d - Cx^0, \quad \mu \in \mathbb{Z}^{n-m},$$

such that CLU has zero-entries in all columns with indices larger than l. It then follows from the theorem of Doignon that if (6) is infeasible, then at most 2^l inequalities of CLU suffice to determine an infeasible integral system.

It is, however, geometrically not so clear what this certificate means in terms of the given constraints encoded in A and C. This fact motivated us to derive an infeasibility statement that directly involves A and C. In other words, we search for a small number of multipliers $y \in Y$ for the rows of A and C in order to obtain a certificate of infeasibility of the form

$$y^T \begin{bmatrix} A \\ C \end{bmatrix} x \leq y^T \begin{bmatrix} b \\ d \end{bmatrix} \quad \text{for all } y \in Y.$$

If rank$(C) \leq 1$, then Y is a singleton and delivers a "split vector" $\pi = y^T[A^T\, C^T]^T$. In fact, for rank$(C) \leq 1$, the product $y^T A$ determines the normal vector of the outer description of a lattice point free split that contains the given infeasible system. As a next step we will generalize this construction to general systems X. For higher rank of C, however, we certainly need to account for more complicated lattice point free bodies that contain our infeasible system, i.e., more and more constraints come into play and are needed to give the outer description of the lattice point free body that constitutes the infeasibility certificate. This is precisely the role of the multipliers y_i in our next result.

Theorem 3.1 *Let $A \in \mathbb{Z}^{m\times n}$, $C \in \mathbb{Z}^{p\times n}$ and let $l =$ rank (C). For integer vectors b and d, either (6) contains an integer point or there exist at most l linearly independent vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$, $t \leq 2^l$ vectors $y_1, \ldots, y_t \in$*

$\mathbb{Q}^m \times \mathbb{Q}^p_+$ *and integer coefficients* λ_i^k *satisfying* $\left[A^T\ C^T\right] y_k = \sum_{i=1}^l \lambda_i^k v^i$, *and such that*

$$\sum_{j=1}^{l} \lambda_j^k z_j \leq y_k^T \begin{bmatrix} b \\ d \end{bmatrix},\ k = 1, \ldots, t, \quad (8)$$

has no solution in integer variables z.

PROOF. Let X be as in (6). If $x \in X \cap \mathbb{Z}^n \neq \emptyset$, then for all l linearly independent vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$ and $t \leq 2^l$ vectors $y_1, \ldots, y_t \in \mathbb{Q}^m \times \mathbb{Q}^p_+$ and coefficients λ satisfying $\left[A^T\ C^T\right] y_k = \sum_{i=1}^l \lambda_i^k v^i,\ \lambda_i^k \in \mathbb{Z}\ \forall i, k$ we may define $z_i := (v^i)^T x \in \mathbb{Z}$ for all $i \in \{1, \ldots, l\}$ so as to construct an integer solution to (8).

It remains to show that if $X \cap \mathbb{Z}^n = \emptyset$, then there exist l linearly independent vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$, $t \leq 2^l$ vectors $y_1, \ldots, y_t \in \mathbb{Q}^m \times \mathbb{Q}^p_+$ and coefficients λ satisfying $\left[A^T\ C^T\right] y_k = \sum_{i=1}^l \lambda_i^k v^i,\ \lambda_i^k \in \mathbb{Z}\ \forall i, k$ so that system (8) has no integral solution. Since X is lattice point free, there exists a maximal lattice point free body L strictly containing X. From [4] it then follows that

$$X \subset L = L^* + \text{lin}\,(W),$$

where L^* is a polytope of dimension $l' \leq l$ and $W = \{w^1 \ldots w^{n-l'}\} \subseteq \mathbb{Z}^n$ consists of linearly independent vectors. W.l.o.g. we may assume that $l' = l$. Next, we complete $w^1, \ldots w^{n-l}$ to a basis of $\mathbb{R}^n$ by adding some vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$ such that (1) the Smith normal form (SNF) of the matrix $V = (v^1 \cdots v^l)$ is $SNF(V) = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and (2) $(w^j)^T v^k = 0$ for all $j \in \{1, \ldots, n-l\}$ and $k \in \{1, \ldots, l\}$. It is an exercise to prove that such vectors v^i exist.

L can be described by linear inequalities,

$$L = \left\{x \in \mathbb{R}^n \mid \pi_1^T x \leq \pi_1^0, \ldots, \pi_t^T x \leq \pi_t^0\right\}$$

with integral normal vectors $\pi_1, \ldots, \pi_t$ and integral right-hand-side vector π^0. In fact, since $L = L^* + \text{lin}\,(W)$, we can conclude that $\pi_k^T w^j = 0$ for all k and $j \leq n - l$, i.e.,

$$\pi_k = \sum_{i=1}^{l} \lambda_i^k v^i,\ \lambda_i^k \in \mathbb{Z} \ \text{ for all } k.$$

On the other hand, since X is fully contained in the interior of L, we have that $\max\,\{\pi_k^T x \mid x \in X\ \} < \pi_k^0$ for all $k = 1, \ldots, t$. Therefore, this maximum value exists. From linear programming duality we obtain that

$$\begin{array}{llcll} & \max \pi_k^T x & = & & \min[b^T, d^T]\ y_k \\ \text{s.t.} & Ax = b, & & \text{s.t.} & [A^T, C^T] y_k = \pi_k \\ & Cx \leq d & & & y_{k,m+1}, \ldots, y_{k,m+l} \geq 0 \end{array}$$

Hence, the minimum-value in the LP-duality relation satisfies $[b^T, d^T]\ y_k < \pi_k^0$. We next claim that

$$\sum_{i=1}^{l} \lambda_i^k (v^i)^T x \leq [b^T\ d^T] y_k \quad (9)$$

has no integer solution in variables x if and only if the system

$$\sum_{i=1}^{l} \lambda_i^k z_i \leq [b^T\ d^T] y_k \quad (10)$$

has no integer solution in variables z. In order to verify this claim, let U, W be unimodular matrices so that $U^T V W^T = \begin{pmatrix} I_l \\ 0 \end{pmatrix}$. We decompose $U = (U_1\ U_2)$ such that $W V^T U_1 = I_l$. Then for every $z \in \mathbb{Z}^l$, there exists $x \in \mathbb{Z}^n$ such that $z = V^T x$, since $w := Wz \in \mathbb{Z}^l$, $U_1 w \in \mathbb{Z}^l$ and $W V^T (U_1 w) = w \iff$

$V^T(U_1 w) = W^{-1}w \iff V^T x = z$. Hence, it has been established that system (10) is integer infeasible. From the theorem of Doignon [2] it follows that if system (10) in rank(C) variables has no integral solution, then at most $2^{\text{rank}(C)}$ of its inequalities suffice to determine an infeasible integral system. This shows that $t \leq 2^l$. $\square$

Let us return to the geometric picture underlying Theorem 3.1. In the course of the proof we construct the maximal lattice point free body L^* such that $X \subset L = L^* + \text{lin}\,(W)$. In fact, the system (10) gives precisely the outer description of L^* – and L^* is living in a subspace of the original space – whereas system (9) is an outer description of a lattice point free body in the original space that contains our integer infeasible set X. We next illustrate Theorem 3.1 on an example.

Example 2. Consider the set X of solutions to

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 0 && (11)\\
-3x_1 + 4x_2 &\leq 0 && (12)\\
-x_1 - 2x_2 &\leq -3 && (13)\\
2x_1 - x_2 &\leq 5. && (14)
\end{aligned}
$$

Notice that one can represent X in the form $X = \{x \in \mathbb{R}^n | Ax = b, Cx \leq d\}$ with rank(C) $= l = 2$. It is readily checked that although both $Ax = b$ and $Cx \leq d$ have integral solutions, their intersection $X \cap \mathbb{Z}^3$ is empty. We next establish a certificate in the spirit of Theorem 3.1.

To this end, define $v^1 = (1,0,0)$, $v^2 = (0,1,1)$ which are linearly independent integral vectors. Next define multipliers $y_1, y_2, y_3 \in \mathbb{Q}^1 \times \mathbb{Q}^3_+$ as follows:

$$
\begin{aligned}
y_1 &= (-2,0,1,0)\\
y_2 &= (-1,0,0,1)\\
y_3 &= (4,1,0,0).
\end{aligned}
$$

Then we obtain for $\begin{bmatrix} A^T \; C^T \end{bmatrix} y_k$ the following relations:

$$
\begin{aligned}
\begin{bmatrix} A^T \; C^T \end{bmatrix} y_1 &= -3v^1 - 6v^2\\
\begin{bmatrix} A^T \; C^T \end{bmatrix} y_2 &= v^1 - 3v^2\\
\begin{bmatrix} A^T \; C^T \end{bmatrix} y_3 &= v^1 - 12v^2,
\end{aligned}
$$

and hence, all coefficients λ_i^k are integral. Then, system (8) becomes

$$
\begin{aligned}
-2(11) + (13) &: -3z_1 - 6z_2 \leq -3\\
-(11) + (14) &: \quad z_1 - 3z_2 \leq 5\\
4(11) + (12) &: \quad z_1 + 12z_2 \leq 0.
\end{aligned}
$$

The corresponding feasible region is a two-dimensional triangle that contains no integer points. Hence the initial feasible region X does not contain any integral point.

4 A certificate for mixed integer sets

This section is devoted to the extension of Theorem 3.1 from the pure integer setting to the mixed integer scenario. We have

Theorem 4.1 *Let $A \in \mathbb{Z}^{m \times n}$, $G \in \mathbb{Z}^{m \times q}$, $C \in \mathbb{Z}^{p \times n}$, $H \in \mathbb{Z}^{p \times q}$ and let $1 \leq l =$ rank $([C, H])$. For integer vectors b and d, either System (1) is feasible or there exist at most l linearly independent vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$, $t \leq 2^l$ vectors $y_1, \ldots, y_t \in \mathbb{Q}^m \times \mathbb{Q}^p_+$ and integer coefficients λ_i^k satisfying*

$$
(y_k)^T \begin{bmatrix} G \\ H \end{bmatrix} = 0, \; (y_k)^T \begin{bmatrix} A \\ C \end{bmatrix} = \sum_{i=1}^{l} \lambda_i^k (v^i)^T,
$$

and such that system (15) in variables z_j, $j \in$

$\{1, \ldots, l\}$ *has no integral solution:*

$$\sum_{j=1}^{l} \lambda_j^k z_j \le y_k^T \begin{bmatrix} b \\ d \end{bmatrix} \; \forall k. \qquad (15)$$

We will prove Theorem 4.1 by projecting the mixed integer set first to the space of discrete variables. Then one may apply Theorem 3.1 to the projected set from which the result follows. The key observation why this approach works is that the projection operation does not increase the rank of the inequality subsystem. Indeed, we have

Lemma 4.2 *Let* $A \in \mathbb{Q}^{m \times n}$, $Q \in \mathbb{Q}^{p \times n}$ *and* $g \in \{0,1\}^m$, $h \in \{0, \pm 1\}^p$ *with* $1 \le l =$ *rank* $([C,\, h])$. *The projection of the set*

$$\begin{array}{l} A\,x + g\,z = b \\ C\,x + h\,z \le d \end{array}, \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}. \qquad (16)$$

to the space of x*-variables is a system of equations and inequalities where the rank of the inequality subsystem is at most* l.

PROOF. Letting $M = \{1, \ldots, m\}$ and $P = \{1, \ldots, p\}$, we define

$$M_k = \{i \in M \mid g_i = k\}; \;\text{ for } k = 0, 1;$$
$$P_k = \{j \in P \mid h_j = k\}; \;\text{ for } k = -1, 0, 1.$$

W.l.o.g., we assume that $g_1 = 1$, i.e., $1 \in M_1$. In order to compute an outer description for the projection of system (16) to the space of x-variables, we first determine the generators of the polyhedral cone $\mathcal{C} = \Big\{(u, v) \in \mathbb{R}^{m+p} \mid \sum_{i \in M_1} u_i + \sum_{j \in P_{-1} \cup P_1} h_j v_j = 0,\ u_i \in \mathbb{R},\ v_j \ge 0\Big\}$. We denote the first m unit vectors in $\mathbb{R}^{m+p}$ by $e^i : \; i = 1, \ldots, m$. The remaining p unit vectors are denoted by e^{m+j}, $j = 1, \ldots, p$. The cone $\mathcal{C}$ is the sum of a pointed cone C^0 and a linear space,

$$\mathrm{lin}(e^i : \; i \in M_0) \,+\, \mathrm{lin}(e^1 - e^i : \; i \in M_1 \setminus \{1\}).$$

The extreme rays of C^0 can be grouped into:

Type (A) e^j, for all $j \in P_0$;

Type (B) $-h_j e^1 + e^j$, for all $j \in P_1 \cup P_{-1}$;

Type (C) $e^k + e^j$, for all $k \in P_1$ and $j \in P_{-1}$.

From this representation of the extreme rays of C^0 and the basis of the linear space we obtain a description of the projected set.

$$A_{i,\cdot}^T x = b_i, \; i \in M_0;$$
$$(A_{1,\cdot} - A_{i,\cdot})^T x = b_1 - b_i, \; i \in M_1 \setminus \{1\};$$
$$C_{j,\cdot}^T x \le d_j, \; j \in P_0;$$
$$(-A_{1,\cdot} + C_{j,\cdot})^T x \le -b_1 + d_j, \; j \in P_1;$$
$$(A_{1,\cdot} + C_{j,\cdot})^T x \le b_1 + d_j, \; j \in P_{-1};$$
$$(C_{k,\cdot} + C_{j,\cdot})^T x \le d_k + d_j, \; k \in P_1, \; j \in P_{-1}.$$

We next observe that every row vector $(C_{k,\cdot} + C_{j,\cdot})$ for $k \in P_1$, $j \in P_{-1}$ is the sum of the two row vectors $(-A_{1,\cdot} + C_{k,\cdot})$ and $(A_{1,\cdot} + C_{j,\cdot})$. Indeed, it is now a minor exercise to show that the rank of the row vectors $C_{j,\cdot}$, $j \in P_0$ together with the row vectors $(-h_j A_{1,\cdot} + C_{j,\cdot})$, $j \in P_1 \cup P_{-1}$ is equal to the rank of the row vectors of $[C,\, h]$. $\square$

We are now prepared to finalize the proof of Theorem 4.1.

PROOF of Theorem 4.1. (1) is mixed integer infeasible if and only if its projection on the space of integer variables is lattice point free. The projection S_x of the System (1) to the space of x-variables can be accomplished by iteratively removing one continuous variable. By scaling each row of the matrix we can transform it in a way that the column of the variable that is to be eliminated next is of the type as stated in Lemma 4.2. It then follows that the operation of projection does not change the rank of $[C, H]$. Indeed, by inductively applying Lemma 4.2, we end up with a description for the projected system S_x of the form,

$$S_x = \left\{ \begin{array}{ll} UA & x = Ub \\ V \begin{bmatrix} A \\ C \end{bmatrix} & x \leq V \begin{bmatrix} b \\ d \end{bmatrix} \end{array} \right\}$$

where U and V are rational matrices of appropriate dimension and where the rank of $V \begin{bmatrix} A \\ C \end{bmatrix}$ is bounded by the rank of the submatrix $[C,\ H]$. As a next step we apply Theorem 3.1 to the description of S_x. The certificate is given by multipliers $y_1, \ldots, y_t$ which, in turn, yield multipliers for the original mixed integer system, if $[U^T,\ V^T]y_i$ is put in place of y_i, for $i = 1, \ldots, t$. $\square$

Let us finally discuss an application of this result to derive multiterm disjunctions. A multiterm disjunction D is a polyhedron $D = \{(x, z) \mid C(x, z) \leq \Gamma\} \subseteq \mathbb{R}^{n+q}$ not containing mixed integer points in its relative interior. I.e., letting $M = \{(x, z) \in \mathbb{R}^{n+q} \mid x \in \mathbb{Z}^n\}$, we have that $\text{int}(D) \cap M = \emptyset$. This is equivalent to saying that every $(x, z) \in M$ satisfies at least one of the inequalities $C_i^T(x, z) \geq \gamma_i$. Of interest are disjunctions that cannot be further enlarged. From [4] it then follows that $D = \text{conv}(V) + \text{lin}(W)$. Theorem 4.1 can be used so as to derive a multiterm disjunction from a face of a polyhedron not containing a mixed integer point that only depends on the dimension of the face. The key statement here is not the fact that such a disjunction always exists. Rather, the "complexity" of the multiterm disjunction is nicely controllable. In fact, since $D = \text{conv}(V) + \text{lin}(W)$, we can measure the "complexity" of the disjunction by means of the dimension of $\text{conv}(V)$. In light of this Theorem 4.1 ensures that there exists a multiterm disjunction of complexity no more than the dimension of the face to certify that the face is mixed integer free. This is a higher dimensional mixed integer version of the fact that a vertex of a polyhedron is either integral or there exists a split fully containing the vertex.

Theorem 4.3 *Let F be a face of a polyhedron $P = \{(x, z) \in \mathbb{R}^{n+q} \mid Ax + Bz \leq b\}$. Either F contains a mixed integer point or there exists a multiterm disjunction $D = \{(x, z) \mid G(x, z) \leq g\} = \text{conv}(V) + \text{lin}(W)$, with $G \in \mathbb{Z}^{t\times(n+q)}$, $g \in \mathbb{Z}^t$, $V \subseteq Q^{n+q}$, $W \subseteq Q^{n+q}$ such that*

- *$F \subset \text{int}(D)$,*
- *$t \leq 2^{\dim(F)}$ and*
- *$\dim(\text{conv}(V)) \leq \max\{1, \dim(F)\}$.*

PROOF. Let F be a face of P. It follows that there exists subsystems $Cx + Hz \leq \gamma$ of $Ax + Bz \leq b\}$ and $A_I x + B_I z \leq b_I$ that lead to a minimal description of F in the form $F = \{(x, z) \in \mathbb{R}^{n+q} \mid A_I x + B_I z = b_I,\ Cx + Hz \leq \gamma\}$ such that the rank of the matrix $[C\ H]$ is less or equal to $l := \dim(F)$, see (1) in page 103 of [5], for example. By m and p we denote the number of equalities and inequalities of this minimal description of F. From Theorem 4.1 it follows that F contains no mixed integer point iff there exist at most l linearly independent vectors $v^1, \ldots, v^l \in \mathbb{Z}^n$, $t \leq 2^l$ vectors $y_1, \ldots, y_t \in \mathbb{Q}^m \times \mathbb{Q}^p_+$ and integer coefficients λ_i^k satisfying

$$(y_k)^T \begin{bmatrix} B \\ H \end{bmatrix} = 0,\ (y_k)^T \begin{bmatrix} A \\ C \end{bmatrix} = \sum_{i=1}^{l} \lambda_i^k (v^i)^T,$$

and such that system (15) in variables $z_j,\ j \in \{1,\dots,l\}$ has no integral solution. This implies that also the set

$$D = \{(x,z)\ |\forall k = 1,\dots,t \quad (y_k)^T \begin{bmatrix} A \\ C \end{bmatrix} (x,z) \le (y_k)^T \begin{bmatrix} b \\ d \end{bmatrix}\}$$

contains no integer point. Hence, $D = \mathrm{conv}(V) + \mathrm{lin}(W)$ defines precisely the desired multiterm disjunction. $\square$

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