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Complements and Substitutes in Generalized Multisided Assignment Economies*

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Abstract

We consider a finitely populated economy in which there are different types of agent, each agent is of exactly one type, and profit is created by coalitions containing at most one agent of each type (or side). The surplus of a so-called generalized multisided assignment economy is defined as the maximum aggregate profit that can be attained by matching agents into pairwise disjoint coalitions of the above kind. We present negative results that establish that when the economy consists of more than two sides *(i)* agents on different sides may not be complements, i.e., they do not necessarily reinforce each other's influence on the surplus and *(ii)* agents on the same side may not be substitutes, i.e., they do not necessarily interfere with each other's influence on the surplus. These findings are in marked contrast with the results for two-sided assignment economies (Shapley, 1962). We propose novel notions for the complementarity and the substitutability of disjoint subsets of agents and we find conditions that ensure that the former are satisfied.

Keywords: multisided assignment economy, complements, substitutes

JEL: C71, D40

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1 Introduction

In a two-sided assignment economy (Shapley and Shubik, 1972),[1] say a real estate market, there are two types of agent, say buyers and sellers. While the former are interested in buying exactly one house, the latter own exactly one house which they want to sell. Thus, profit is created by pairs composed of a seller and a buyer, which as such are the only possible essential coalitions.[2] The two-sided assignment model has been generalized in two directions. On the one hand, Owen (1992) assumes that every agent has an outside option, and therefore singletons might also be essential coalitions.[3] On the other hand, Quint (1991) considers an economy in which there is an arbitrary finite number of types of agents – e.g. buyers, sellers and suppliers – and the only essential coalitions comprise exactly one agent of each type.[4]

In this paper we consider a model that combines both generalizations. A *generalized multisided assignment economy* (henceforth, simply, *assignment economy*) models a finitely populated economy in which there are different types of agent (typically much fewer than the total number of agents), each agent is of exactly one type, and profit is created by coalitions containing at most one agent of each type (or side).[5] Given such an economy, we can attach a non-negative number to any partition of all the agents into coalitions of the above kind by summing the profits associated with the latter. The maximum of any such value is the *surplus of the economy*, which captures the potential profit that all the agents in the economy can jointly produce.[6] When profit is only created by coalitions containing exactly one agent from each side, we speak of a *classical multisided assignment economy.*

Shapley (1962) studies the problem of new entrants in the framework of classical two-sided assignment economies and proves two comparative statics results. He shows that by entering into any such economy *(i)* two agents of different types reinforce each other's influence on the surplus and *(ii)* two agents of the same type interfere with each other's influence on the surplus, so agents of different types are *complements* and agents of the same type are *substitutes.*

We study the robustness of these two results in the framework of generalized multisided as-

[1]Shapley and Shubik (1972) are usually credited with developing the model of these economies, which are typically referred to as *assignment markets* as opposed to assignment economies. It is worth mentioning, however, that assignment models had been studied earlier in similar, related contexts (e.g., Shapley, 1955; Gale, 1960).

[2]An essential coalition creates a profit $x > 0$ and it cannot be properly partitioned so that the sum of the profits associated with all elements of the partition is at least x.

[3]Owen (1992) uses *assignment market with non-null reservation prices* to refer to this model, which is also analyzed in Toda (2005).

[4]Multisided assignment markets (economies) have also been analyzed in Stuart (1997), Sherstyuk (1999) and Tejada and Rafels (2010), for example.

[5]The previous literature includes models in which essential coalitions might be of arbitrary and different sizes (Kaneko and Wooders, 1982; Demange, 2009). Nevertheless, to the best of our knowledge, the model introduced in this paper has never been analyzed before. We note that although our model bears some similarities with partitioning games with replicas as considered by Kaneko and Wooders (1982), we do not consider that different players of the same type can be replicas of each other.

[6]A subsequent problem, not addressed in the present paper, would be that of determining how to divide up the surplus.

signment economies with an arbitrary number of sides. We show that in classical multisided assignment economies with more than two sides agents on different sides may not be complements and agents on the same side may not be substitutes. We propose novel notions that capture the complementarity and the substitutability of disjoint subsets of agents for generalized multisided assignment economies. These definitions are natural generalizations of Shapley's (1962) notions for pairs of agents in classical two-sided assignment economies, i.e., two disjoint subsets of agents that enter an economy are called *subset-complements* (resp. *subset-substitutes*) if they reinforce (resp. interfere with) each other's influence on the surplus. We find sufficient conditions that ensure that, given a status quo assignment economy, the above notions are satisfied for a given pair of subsets of agents that do not belong to the status quo economy.[7] These conditions require the existence of a recursively-defined finite sequence of elements consisting of an assignment economy and two disjoint subsets of agents that do not belong to the economy, and where, for each element of the sequence, the addition of the two subsets of agents into the assignment economy results in the assignment economy of the precedent element of the sequence.[8] Additionally, we show that necessary conditions should involve different kinds of requirement.

The rest of the paper is organized as follows. In Section 2 the formal model and its main concepts are introduced. Section 3 is devoted to a presentation of counterexamples that demonstrate the lack of robustness of Shapley's (1962) results when the assignment economy comprises more than two sides. In Section 4 the notions for the complementarity-substitutability relationships between disjoint subsets of agents are defined and discussed. In Section 5 two different recursively-defined properties for pairs of subsets of agents are also defined. The main results are contained in Section 6. Section 7 concludes.

2 The Generalized Multisided Assignment Economy

A *generalized m-sided assignment economy* $(N^1, ..., N^m; A)$ consists of $m \geq 2$ finite sets (called *types* or *sides*) of agents $N^1, \ldots, N^m$ and a mapping

$$A : \bigcup_{\emptyset \neq K \subseteq M} \left(\prod_{k \in K} N^k \right) \longrightarrow \mathbb{R}_+, \tag{1}$$

where $M = \{1, \ldots, m\}$ denotes the set of types and $\mathbb{R}_+$ is the set of non-negative real numbers. Note that A assigns a non-negative number to any nonempty coalition composed of at most one agent of each type.[9] We assume that $N^1 \subseteq \Omega^1, \ldots, N^m \subseteq \Omega^m$.[10] The nonempty finite sets $\Omega^1, \ldots, \Omega^m$ contain all potential agents of each type and they are pairwise disjoint. An arbitrary agent in N^k is denoted by i_k. Let $N = \cup_{k \in M} N^k$, $\Omega = \cup_{k \in M} \Omega^k$ and

[7] Note that as we are only concerned with an economy's surplus, our approach here in this paper is positive rather than normative. This contrasts with the approach adopted in most papers regarding assignment economies.

[8] See Definitions 3, 4, 5 and 6 in Section 5 for a detailed description of the conditions.

[9] When $m = 2$ the model reduces to the one considered by Owen (1992).

[10] Note that some of the sets $N^1, \ldots, N^m$ might be the empty set.

$\mathcal{T}(N^1, \ldots, N^m) = \cup_{\emptyset \neq K \subseteq M} \left(\prod_{k \in K} N^k \right)$. An arbitrary element in $\mathcal{T}(N^1, \ldots, N^m)$ is denoted by E and called a *tuple*. With some abuse of notation and language, we use indistinctly a tuple and the set composed of all agents in the tuple. We assume that the mapping A is obtained from a (potential) mapping that assigns a non-negative number to each tuple comprising at most one (potential) agent of each type. We refer to the particular case in which $A(E) = 0$ whenever E does not contain exactly one agent of each type as a *classical m-sided assignment economy* (henceforth, simply, *classical assignment economy*).[11] Notice that in the latter case, the mapping A introduced in Eq. (1) can be cast as an array $A = (a_E)_{E \in \prod_{k \in M} N^k}$. A *matching* among $N^1, ..., N^m$ is a finite collection of tuples, $\mu = \{E^r\}_{r=1}^t \subseteq \mathcal{T}(N^1, \ldots, N^m)$, so that any agent belongs to exactly one of the tuples $E^1, ..., E^t$. Different agents are *matched together* under μ if they belong to the same tuple of μ. In this latter case we also say that they are *partners* under μ. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$. A matching μ is *optimal* for an assignment economy $(N^1, ..., N^m; A)$ if it maximizes $\sum_{E \in \mu} A(E)$ within $\mathcal{M}(N^1, ..., N^m)$, where the summation over the empty set is assumed to be zero. We denote by $\mathcal{M}^*(N^1, ..., N^m; A)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$. The *surplus* of an assignment economy $(N^1, ..., N^m; A)$ is

$$\mathcal{S}(N^1, ..., N^m; A) = \sum_{E \in \mu^*} A(E) \text{ for all } \mu^* \in \mathcal{M}^*(N^1, ..., N^m; A).$$

In this paper we are interested in assessing when two disjoint sets of newcomers reinforce or interfere with each other's influence on the surplus of an economy. To analyze the so-called *entry problem*,[12] we consider at all times a *status quo assignment economy* $P = (N^1, ..., N^m; A)$ as the initial situation and denote by $\mathcal{S}$ its surplus. Given a set $T \subseteq \Omega \setminus N$, we denote by $\mathcal{S}^T$ the surplus of the economy that results from the entry of all the agents in T into the status quo economy. Similarly, given a set $T \subseteq N$, we denote by $\mathcal{S}_T$ the surplus of the economy that results from the exit of all the agents in T from the status quo economy. Likewise, we respectively denote by A^T and A_T the mappings that define the corresponding economies, which are, in turn, denoted by P^T and P_T.

For classical two-sided assignment economies we have the following result.

Theorem 1 (Shapley, 1962)
Let $(N^1, N^2; A)$ be a status quo classical two-sided assignment economy. Then,

(a) $\left(\mathcal{S}^{\{i_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{i_2\}} - \mathcal{S}\right) \leq \mathcal{S}^{\{i_1, i_2\}} - \mathcal{S}$, *where* $i_1 \in \Omega^1 \setminus N^1$ *and* $i_2 \in \Omega^2 \setminus N^2$.

(b) $\left(\mathcal{S}^{\{i_k\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{j_k\}} - \mathcal{S}\right) \geq \mathcal{S}^{\{i_k, j_k\}} - \mathcal{S}$, *where* $i_k, j_k \in \Omega^k \setminus N^k$ *and* $k \in \{1, 2\}$.

[11] This case reduces to the model considered by Quint (1991).

[12] Note that the *exit problem*, i.e., the effect that two disjoint subsets of agents leaving the economy have on its surplus, depending on whether they leave simultaneously or separately is symmetric; therefore, we focus only on the entry problem.

That is, agents of different types are *complements*, since the increase in the surplus is (weakly) higher when two agents of different types, $i_1 \in \Omega^1 \setminus N^1$ and $i_2 \in \Omega^2 \setminus N^2$, enter the economy at the same time, $\mathcal{S}^{\{i_1,i_2\}} - \mathcal{S}$, than when they do so separately, $\left(\mathcal{S}^{\{i_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{i_2\}} - \mathcal{S}\right)$. The second statement shows that agents of the same type are *substitutes*.

3 Some Counterexamples

In an arbitrary generalized multisided assignment economy no profit is derived directly from the cooperation of two agents on the same side, whereas agents on different sides can potentially combine with each other and directly create some profit. We might therefore expect agents of different types to be complements and agents of the same type to be substitutes w.r.t. their reinforcement of/interference with the surplus of the economy. Nevertheless, the two examples below show that this expectation is false in assignment economies with at least three sides, even if they are classical.

Example 1
Consider the classical three-sided assignment economy with potential sets of agents $\Omega^1 = \{1_1, 2_1\}$, $\Omega^2 = \{1_2, 2_2\}$, *and* $\Omega^3 = \{1_3, 2_3, 3_3\}$ *defined by the array* A, *which is displayed as follows:*

$$
\begin{array}{c} \\ 1_1 \\ 2_1 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cc} 1_2 & 2_2 \end{array} \\ \left(\begin{array}{cc} 1 & 7 \\ 3 & 1 \end{array}\right) \\ 1_3 \end{array}
\begin{array}{c} \begin{array}{cc} 1_2 & 2_2 \end{array} \\ \left(\begin{array}{cc} 3 & 1 \\ 0 & 1 \end{array}\right) \\ 2_3 \end{array}
\begin{array}{c} \begin{array}{cc} 1_2 & 2_2 \end{array} \\ \left(\begin{array}{cc} 0 & 0 \\ 0 & 10 \end{array}\right) \\ 3_3 \end{array}.
$$

We consider two different status quo economies.

Example 1.A: $N^1 = \{1_1\}, N^2 = \{1_2\}, N^1 = \{1_3\}$

It is an easy exercise to check that $\mathcal{S} = 1, \mathcal{S}^{\{2_1\}} = 3, \mathcal{S}^{\{2_2\}} = 7, \mathcal{S}^{\{2_3\}} = 3, \mathcal{S}^{\{2_1,2_2\}} = 7, \mathcal{S}^{\{2_1,2_3\}} = 3, \mathcal{S}^{\{2_2,2_3\}} = 7$, and $\mathcal{S}^{\{2_1,2_2,2_3\}} = 7$, so

$$
\begin{aligned}
\left(\mathcal{S}^{\{2_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_2\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_2\}} - \mathcal{S}, \\
\left(\mathcal{S}^{\{2_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_3\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_3\}} - \mathcal{S}, \\
\left(\mathcal{S}^{\{2_2\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_3\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_2,2_3\}} - \mathcal{S}, \\
\left(\mathcal{S}^{\{2_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_2,2_3\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_2,2_3\}} - \mathcal{S}, \\
\left(\mathcal{S}^{\{2_2\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_1,2_3\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_2,2_3\}} - \mathcal{S}, \\
\left(\mathcal{S}^{\{2_3\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_1,2_2\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_2,2_3\}} - \mathcal{S}, \text{ and} \\
\left(\mathcal{S}^{\{2_1\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_2\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{2_3\}} - \mathcal{S}\right) &> \mathcal{S}^{\{2_1,2_2,2_3\}} - \mathcal{S}.
\end{aligned}
$$

That is, there is no way in which agents $2_1, 2_2, 2_3$ can combine with each other so as to reinforce each other's influence on the surplus, at least in absolute terms.[13]

Example 1.B: $N^1 = \{1_1, 2_1\}, N^2 = \{1_2, 2_2\}, N^3 = \{1_3\}$

In this case it can be easily checked that $\mathcal{S} = 7, \mathcal{S}^{\{2_3\}} = 7, \mathcal{S}^{\{3_3\}} = 11$, and $\mathcal{S}^{\{2_3,3_3\}} = 13$, so

$$\left(\mathcal{S}^{\{2_3\}} - \mathcal{S}\right) + \left(\mathcal{S}^{\{3_3\}} - \mathcal{S}\right) < \mathcal{S}^{\{2_3,3_3\}} - \mathcal{S}.$$

That is, agents 2_3 and 3_3 are not substitutes although they are agents of the same type.[14]

The failure in general of Theorem 1 when arbitrary multisided assignment economies are considered, as illustrated above by means of Examples 1.A and 1.B, raises two challenges: first, identifying appropriate notions for the complementarity and substitutability of agents – or, more generally, subsets of agents – in a multisided assignment economy; and, second, given the identification of such notions, seeking necessary and sufficient conditions that ensure that the former hold. We address these two issues in the following two sections respectively.

4 Novel Notions of Complements and Substitutes

In this section, novel definitions are proposed for the complementarity and substitutability of two disjoint subsets of agents in the context of arbitrary assignment economies.

Definition 1
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are subset-complements w.r.t. *P if*

$$\left(\mathcal{S}^{T^1} - \mathcal{S}\right) + \left(\mathcal{S}^{T^2} - \mathcal{S}\right) \leq \mathcal{S}^{T^1 \cup T^2} - \mathcal{S}. \tag{2}$$

Definition 2
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are subset-substitutes w.r.t. *P if*

$$\left(\mathcal{S}^{T^1} - \mathcal{S}\right) + \left(\mathcal{S}^{T^2} - \mathcal{S}\right) \geq \mathcal{S}^{T^1 \cup T^2} - \mathcal{S}. \tag{3}$$

The interpretation of Definitions 1 and 2 parallels Shapley's (1962) interpretation of complements and substitutes, with the modification that in our case subsets of agents are involved instead of

[13]Notice that what ensures that agents 2_2 and 2_3 are not complements is that the "complementarity" effect of introducing these agents jointly to the status quo is diminished because of the existence of the first type of agents. This prevents us from simultaneously selecting $a_{(1_1,2_2,1_3)} = 7$ and $a_{(1_1,1_2,2_3)} = 3$.

[14]This is possible since the "substitution effect" of introducing agents 2_3 and 3_3 to the status quo is offset by a complementarity effect between the same agents due to the existence of agents in N^1.

just agents.[15] When there is no possible cause for confusion, we omit the reference to the status quo assignment economy P in both definitions. Theorem 1 in Section 2 shows that, in the case of classical two-sided assignment economies, for the verification as to whether two singletons T^1 and T^2 are subset-complements or subset-substitutes (w.r.t. any P) it is enough to determine whether the two agents are of the same type or not.[16] This simple verification, however, fails to hold in general for assignment economies with at least three sides, even if they are classical.

5 Sufficient Recursively-defined Conditions

When a subset of incoming agents enters an assignment economy, the surplus of the economy weakly increases. There are potentially different optimal matchings that support the new surplus and, for each of them, the incoming agents might be matched with different partners of the status quo economy. When two disjoint subsets of incoming agents enter the economy, however, it is not clear whether they reinforce each other's influence on the surplus or not, i.e., they may be either subset-complements or subset-substitutes. In this section we present certain conditions that restrict the matching possibilities for the incoming agents, but we do not impose bounds on the increase in the surplus due to their entry. Moreover, we consider conditions that impose requirements not only as to whom the incoming agents are matched with, but also as to whom these potential partners (all being agents of the status quo economy) are matched with in the economy obtained from the status quo by removing these latter agents, and so and so forth.

The aforementioned conditions (a fuller understanding of which is provided below) are intended to capture the very essence of the dynamics of a recursive entry problem in an assignment economy[17] as they impose the existence of a certain finite sequence, say $\{T_k^1, T_k^2, P_k\}_{k\geq 1}$, where *(i)* each element of the sequence consists of two disjoint subsets of agents, T_k^1 and T_k^2, together with an assignment economy, P_k, which does not contain any of the former agents such that *(ii)* each assignment economy is obtained from the assignment economy in the antecedent element in the sequence by removing from the latter all the agents of the subsets of the current element in the sequence, i.e., $T_k^1 \cap T_k^2 = \emptyset$ and $P_{k+1} = (P_k)_{T_{k+1}^1 \cup T_{k+1}^2}$.[18]

Before formally presenting the aforementioned conditions, it is convenient at this juncture to

[15] More general definitions in which more than two pairwise disjoint subsets of agents are involved could be introduced and the main results of the paper could likewise be adapted to this more general framework. However, this would mean further complicating the notation. Since the contribution of the paper would be essentially the same, we opted to maintain the definitions for just two disjoint sets.

[16] Note that T^1 and T^2 may be subset-complements and subset-substitutes w.r.t. P at the same time if Eqs. (2) and (3) hold simultaneously. It might also be that T^1 and T^2 are subset-complements (resp. subset-substitutes) w.r.t. P but they are not subset-complements (resp. subset-substitutes) w.r.t. $P' \neq P$.

[17] See Section 7 for an interpretation of the entry problem from the perspective of economics.

[18] The recursive nature of the conditions captures the evolution of a society over time, with new groups of immigrants joining at different junctures. In the light of Theorems 2 and 3 in Section 6, when the "waves of immigrants" satisfy the conditions of the paper regarding who from the "old" society they are matched with, we can properly speak about complements and substitutes in generalized multi-sided assignment economies. Needless to say, such an interpretation requires further development, but this fails outside the scope of the present paper.

introduce further notation so as to facilitate making reference to the subset of agents that are matched with agents of a given subset of agents in a given optimal matching of a certain assignment economy. First, given a subset of incoming agents, $T \subseteq \Omega \setminus N$, we denote by $I^T(\mu) \subseteq N$ the subset of agents in N that are matched under $\mu \in \mathcal{M}^*(P^T)$ together with agents in T, i.e.,

$$I^T(\mu) = \{j \in N \ : \ \exists i \in T \text{ such that } \{i, j\} \subseteq E \text{ for some } E \in \mu\}.$$

Note that $I^T(\mu)$ might be the empty set and that $I^T(\mu)$ might differ from $I^T(\mu')$ for $\mu' \in \mathcal{M}^*(P^T)$, with $\mu' \neq \mu$. Second, given two disjoint subsets of incoming agents, $T, S \subseteq \Omega \setminus N$ such that $T \cap S = \emptyset$, we denote by $I_{T,S}(\mu) \subseteq N \cup S$ the subset of agents in $N \cup S$ that are matched under $\mu \in \mathcal{M}^*(P^{T \cup S})$ together with agents in T, i.e.,

$$I_{T,S}(\mu) = \{j \in N \cup S \ : \ \exists i \in T \text{ such that } \{i, j\} \subseteq E \text{ for some } E \in \mu\}.$$

Note that $I_{T,S}(\mu)$ might be the empty set and that $I_{T,S}(\mu)$ might be different than $I_{T,S}(\mu')$ for $\mu' \in \mathcal{M}^*(P^{T \cup S})$, with $\mu' \neq \mu$. Moreover, $I_{T,S}(\mu) \setminus N \neq \emptyset$ if and only if $I_{S,T}(\mu) \setminus N \neq \emptyset$.

5.1 Complementarity

We present a first recursively-defined condition for a given pair of disjoint subsets of agents that do not belong to the status quo economy. This is subsequently shown to be sufficient for these subsets to be subset-complements – see Theorem 2 in Section 6. On top of the existence of the aforementioned sequence, $\{T_k^1, T_k^2, P_k\}_{k \geq 1}$, this condition requires that, for $l \in \{1, 2\}$, T_{k+1}^l is composed of all the agents that are partners with agents in T_k^l under an optimal matching of the assignment economy $(P_k)^{T_k^l}$. We now formally introduce all the required definitions. We start with the base case.

Definition 3
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are unlinked of degree 0 w.r.t. P if there exist $\mu^1 \in \mathcal{M}^*(P^{T^1})$ and $\mu^2 \in \mathcal{M}^*(P^{T^2})$ such that $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \emptyset$.

We note that T^k, with $k \in \{1, 2\}$, might be the empty set, in which case $I^{T^k}(\mu^k) = \emptyset$ and the above condition is immediately satisfied. When $I^{T^1}(\mu^1) \neq \emptyset$ and $I^{T^2}(\mu^2) \neq \emptyset$, Definition 3 requires that there are at least two optimal matchings of the economies P^{T^1} and P^{T^2}, μ^1 and μ^2 respectively, under which the sets of partners of T^1 and T^2 are disjoint.

Suppose now that, under the same conditions as in Definition 3, we have defined what it means that T^1 and T^2 are *unlinked of degree $t-1$ w.r.t. P*, with t a positive integer.

Definition 4
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are unlinked of degree t w.r.t. P if they are unlinked of degree 0 w.r.t. P

and there exist $\mu^1 \in \mathcal{M}^*(P^{T^1})$ and $\mu^2 \in \mathcal{M}^*(P^{T^2})$ such that $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ are unlinked of degree $t-1$ w.r.t. $P_{I^{T^1}(\mu^1) \cup I^{T^2}(\mu^2)}$.

It is an easy exercise to check that, in a two-sided assignment economy (classical or otherwise), two singletons containing agents from different sides of the economy are always unlinked of any degree with respect to any status quo assignment economy. We stress that Definition 4 requires the existence of a chain of pairs of subsets of agents together with a chain of assignment economies, so that each pair is unlinked of degree 0 w.r.t. the corresponding economy. In particular, notice that as soon as one subset of one of these pairs of subsets is empty, no further checking is required.

5.2 Substitutability

We present a second recursively-defined condition for a given pair of disjoint subsets of agents that do not belong to the status quo economy. This is subsequently shown to be sufficient for these subsets to be subset-substitutes – see Theorem 3 in Section 6. On top of the existence of the sequence $\{T_k^1, T_k^2, P_k\}_{k \geq 1}$, this condition requires now that T_{k+1}^l is composed of all the agents that are partners with agents in T_k^l under an optimal matching of the assignment economy $(P_k)^{T_k^1 \cup T_k^2}$, for $l \in \{1, 2\}$. Notice that the only difference between this and the condition for complementarity is that here we consider the interaction between T_k^1 and T_k^2 regarding the partners with which agents in those two subsets are optimally matched. We now formally introduce all the required definitions. As above, we start with the base case.

Definition 5
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are pairwise unlinked of degree 0 w.r.t. P if there exists $\mu \in \mathcal{M}^*(P^{T^1 \cup T^2})$ such that $I_{T^1,T^2}(\mu) \setminus N = \emptyset$.

Notice that from $I_{T^1,T^2}(\mu) \setminus N = \emptyset$ it follows that $I_{T^1,T^2}(\mu) \cap I_{T^2,T^1}(\mu) = \emptyset$. We also note that T^k, with $k \neq l \in \{1, 2\}$, might be the empty set, in which case $I_{T^k,T^l}(\mu) = \emptyset$ and the above condition is immediately satisfied. When $I_{T^1,T^2}(\mu) \neq \emptyset$ and $I_{T^2,T^1}(\mu) \neq \emptyset$, Definition 5 requires that there is at least one optimal matching of the economy $P^{T^1 \cup T^2}$, μ, under which no two agents, one of T^1 and one of T^2, are matched together.

Suppose that, under the same conditions as in Definition 5, we have defined what it means that T^1 and T^2 are *pairwise unlinked of degree $t-1$ w.r.t. P*, with t a positive integer.

Definition 6
Let P be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. We say that T^1 and T^2 are pairwise unlinked of degree t w.r.t. P if they are pairwise unlinked

of degree 0 w.r.t. P and there exist $\mu \in \mathcal{M}^*(P^{T^1 \cup T^2})$ such that $I_{T^1,T^2}(\mu)$ and $I_{T^2,T^1}(\mu)$ are pairwise unlinked of degree $t-1$ w.r.t. $P_{I_{T^1,T^2}(\mu) \cup I_{T^2,T^1}(\mu)}$.

We stress that, since $I_{T^1,T^2}(\mu) \cap I_{T^2,T^1}(\mu) \neq \emptyset$ implies that $I_{T^1,T^2}(\mu) \setminus N \neq \emptyset$, if T^1 and T^2 are pairwise unlinked of degree $t \geq 0$ w.r.t. P, it is necessarily the case that $I_{T^1,T^2}(\mu)$ and $I_{T^2,T^1}(\mu)$ are disjoint. It is an easy exercise to check that, in a two-sided assignment economy (classical or otherwise), two singletons containing agents from the same side of the economy are always pairwise unlinked of any degree with respect to any status quo assignment economy. We stress that Definition 6 requires the existence of a chain of pairs of subsets of agents together with a chain of assignment economies, so that each pair is pairwise unlinked of degree 0 w.r.t. the corresponding economy. In particular, notice once again that as soon as one subset of one of these pairs of subsets is empty, no further checking is required.

6 Main Results

We are now in a position to present and discuss the two main positive results of the paper, which identify sufficient conditions for the complementarity and substitutability of two disjoint subsets of agents. Moreover, these results are coupled with examples that show that necessary conditions should involve requirements that differ from those presented, even for classical assignment economies.

6.1 Complementarity

We present the main result concerning the complementarity of subsets of agents in an assignment economy, the proof of which can be found in the Appendix.

Theorem 2
Let $P = (N^1, \ldots, N^m; A)$ be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. Then, if T^1 and T^2 are unlinked of any degree w.r.t. P, they are subset-complements w.r.t. P.

We make some remarks regarding the above result. First, Theorem 2 implies Part (a) of Theorem 1. Second, if T^1 and T^2 are not unlinked w.r.t. P, they might not be subset-complements, as Example 1 shows if we consider $N^1 = \{1_1\}, N^2 = \{1_2\}, N^3 = \{1_3\}$ and we take $T^1 = \{2_1\}$ and $T^2 = \{2_2, 2_3\}$. Notice that in this case, $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \{1_3\}$, given $\mu^1 \in \mathcal{M}^*(P^{T^1})$ and $\mu^2 \in \mathcal{M}^*(P^{T^2})$ the unique optimal matchings of the corresponding economies. Third, Eq. (2) might hold even when T^1 and T^2 are not unlinked w.r.t. P. Indeed, consider a classical three-sided assignment economy with $N^1 = \{1_1\}$, $N^2 = \{1_2\}$, $N^3 = \{1_3\}$ and consider a potential array A defined by $a_{(2_1,2_2,2_3)} = 2$, $a_{(2_1,1_2,1_3)} = a_{(1_1,2_2,1_3)} = 1$ and $a_E = 0$ otherwise. Then, Eq. (2) holds if we take $T^1 = \{2_1\}$ and $T^2 = \{2_2, 2_3\}$, although $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \{1_3\}$, given $\mu^1 \in \mathcal{M}^*(P^{T^1})$ and $\mu^2 \in \mathcal{M}^*(P^{T^2})$ the unique optimal matchings of the corresponding economies. Hence the condition stated in Theorem 2 is only sufficient for Eq. (2) to hold. It is therefore natural to inquire about the nature of the conditions that are necessary. We do so in Lemma 1 below, the proof of which can be found in the Appendix.

A property regarding a finite family of sets is said to be *set-like* if it only specifies *(i)* if some of the sets are empty or nonempty, or *(ii)* if some of the sets are related to each other w.r.t. strict inclusion, equality or nonempty intersection. Notice that Definitions 3 and 5 specify set-like properties involving sets such as $T^1, T^2, I^{T^1}(\mu^1), I^{T^2}(\mu^2), I_{T^1,T^2}(\mu) \setminus N$, and $I_{T^2,T^1}(\mu) \setminus N$.

Lemma 1
There is no set-like property involving $T^1, T^2, I^{T^1}(\mu^1), I^{T^2}(\mu^2), I_{T^1,T^2}(\mu) \setminus N, I_{T^2,T^1}(\mu) \setminus N$ that is a necessary condition for Eq. (2) to hold for each status quo assignment economy P and each $T^1, T^2 \subseteq \Omega \setminus N$, where $\mu^1 \in \mathcal{M}^*(P^{T^1})$, $\mu^2 \in \mathcal{M}^*(P^{T^2})$ and $\mu \in \mathcal{M}^*(P^{T^1 \cup T^2})$.

We note that Lemma 1 holds even if we restrict our attention to classical assignment economies.

6.2 Substitutability

The main result concerning the substitutability of agents in an assignment economy is now stated, and proved in the Appendix.

Theorem 3
Let $P = (N^1, \ldots, N^m; A)$ be a status quo assignment economy and let $T^1, T^2 \subseteq \Omega \setminus N$ such that $T^1 \cap T^2 = \emptyset$. Then, if T^1 and T^2 are pairwise unlinked of any degree w.r.t. P, they are subset-substitutes w.r.t. P.

We make some remarks regarding the above theorem. First, Theorem 3 implies Part (b) of Theorem 1. Second, if T^1 and T^2 are not pairwise unlinked w.r.t. P, the inequality in Eq. (3) might not hold, as Example 1 shows by taking $N^1 = N^2 = \emptyset, N^3 = \{1_3\}$, $T^1 = \{1_1, 1_2\}$, and $T^2 = \{2_1, 2_2\}$. Notice that in this case, $I_{T^1,T^2}(\mu) \setminus N = \{2_2\}$, $I_{T^2,T^1}(\mu) \setminus N = \{1_1\}$, where $\mu \in \mathcal{M}^*(P^{T^1 \cup T^2})$ is the only optimal matching of $P^{T^1 \cup T^2}$. Moreover, $\mathcal{S} = 0, \mathcal{S}^{T^1} = \mathcal{S}^{T^2} = 1$, and $\mathcal{S}^{T^1 \cup T^2} = 7$. Third, Eq. (3) might hold even when T^1 and T^2 are not pairwise unlinked w.r.t. P. To verify this, let us consider a classical three-sided assignment economy with $N^1 = \{1_1\}, N^2 = \{2_1, 2_2\}, N^3 = \{1_3\}, T^1 = \{2_1, 2_3\}, T^2 = \{3_3\}$ and potential array A defined by $a_{(1_1,1_2,1_3)} = 1$, $a_{(1_1,2_2,2_3)} = a_{(2_1,1_2,3_3)} = 2$ and $a_E = 0$ otherwise. It can be checked that $I_{T^1,T^2}(\mu) \setminus N = \{3_3\} \neq \emptyset$, but $\mathcal{S}^{T^1 \cup T^2} = 4, \mathcal{S}^{T^1} = 2, \mathcal{S}^{T^2} = 1$, and $\mathcal{S} = 1$, so Eq. (3) holds.

As in the case of complementarity, we investigate how necessary set-like conditions for Eq. (3) to hold should look like by means of the result below, the proof of which is contained in the

Appendix.

Lemma 2
There is no set-like property involving $T^1, T^2, I^{T^1}(\mu^1), I^{T^2}(\mu^2), I_{T^1,T^2}(\mu) \setminus N, I_{T^2,T^1}(\mu) \setminus N$ *that is a necessary condition for Eq. (3) to hold for each status quo assignment economy* P *and each* $T^1, T^2 \subseteq \Omega \setminus N$*, where* $\mu^1 \in \mathcal{M}^*(P^{T^1})$*,* $\mu^2 \in \mathcal{M}^*(P^{T^2})$ *and* $\mu \in \mathcal{M}^*(P^{T^1 \cup T^2})$*.*

We note that Lemma 2 holds even if we restrict our attention to classical assignment economies.

7 Concluding Remarks

To any generalized multisided assignment economy $(N^1, \ldots, N^m; A)$ we can associate a cooperative game (N, ω_A),[19] called a *generalized multisided assignment game*, which is defined as the superadditive cover of A.[20] It is well known that, unlike arbitrary multisided assignment games, two-sided assignment games always have a nonempty core.[21] Mo (1989) considers the core, $C(\omega_A)$, of the assignment game associated with a classical two-sided assignment economy $(N^1, N^2; A)$, and regards it as the set of possible equilibrium payoffs of an economy that comprises two sectors. Mo analyzes the impact on the core due to the entry of new agents. According to his results, the projection of $C(\omega_A)$ into agent i's payoffs, denoted by $C_i(\omega_A)$, *(i)* "weakly increases" when an agent from the other side joins the economy and *(ii)* "weakly decreases" when an agent from the same side joins the economy, reinforcing the notions of complements and substitutes used by Shapley (1962).

If the two-sided assignment game were to be used as a model for immigration within the framework of an assignment economy, as Mo himself proposes, an increase in the supply of one type of agent – those that correspond to one side of the economy – would (weakly) hurt the interests of the agents of the same type that were already in the economy, whereas it would (weakly) benefit the interests of the agents on the other side. A very rough and preliminary implication of this result from the perspective of economics would be that immigration is good for the economy as a whole, but specifically damages those agents that face greater competition as a result of immigration.

Mo's approach is innovative since it does not impose the usual requirement of a single equilibrium. Instead, he compares sets of equilibria. Yet, the fact that there are only two sectors in the economy proves to be crucial to his results. Indeed, consider Example 1 with Mo's interpretation

[19] A *cooperative game* is a pair (N, v), where N is the set of agents and v assigns a number to each coalition of agents, with $v(\emptyset) = 0$.

[20] In the case of a classical assignment economy, the associated game is considered by Quint (1991). We stress that a generalized m-sided assignment game is not in general strategically equivalent to a classical m-sided assignment game, provided that $m > 2$.

[21] The *core*, $C(v)$, of a cooperative game (N, v) is the set of efficient allocations that cannot be improved upon by any coalition on its own, i.e., $C(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subsetneq N\}$.

in mind. It can be seen that[22]

Case 1: $N^1 = \{1_1\}, N^2 = \{1_2, 2_2\}, N^3 = \{1_3\}$ $\quad C_{2_2}(\omega_A) = [0,6] \succ^w [0,4] = C_{2_2}(\omega_{A^{2_1}})$,

Case 2: $N^1 = \{1_1, 2_1\}, N^2 = \{1_2, 2_2\}, N^3 = \{1_3, 2_3\}$ $\quad C_{2_2}(\omega_A) = \{0\} \prec^w [0,3] = C_{2_2}(\omega_{A^{3_3}})$.

In Case 1, agent $2_2 \in N^2$ is weakly worse off after the entry of $2_1 \in \Omega^1 \setminus N^1$, whereas, in Case 2, agent $2_3 \in N^3$ is weakly better off due to the entry of agent $3_3 \in \Omega^3 \setminus N^3$. These two simple examples show that we cannot unambiguously predict the sign of the gain/loss of identifiable participants in a decentralized economy consisting of more than two sectors when new agents enter that economy.

To sum up, this paper calls into question the extension of Shapley's (1962) results by presenting examples that show that in multisided assignment economies agents of the same type may not be substitutes, while agents of different types may not be complements. A fuller understanding of the features of the interaction between agents in economies that are not two-sided would therefore seem necessary. To the best of our knowledge, the present paper is the first to initiate such an attempt with regard to the complementarities and substitutabilities of agents by *(i)* considering a new class of economy in which agents are grouped by type and profit is created by coalitions containing at most one agent of each type, *(ii)* proposing novel definitions of complementarity and substitutability for subsets of agents in the context of this class of economies, *(iii)* identifying certain sufficient conditions that ensure that two disjoint sets are complements or substitutes, and *(iv)* showing that necessary conditions should involve other types of requirement.

Appendix

Proof of Theorem 2.

Let us consider that $\Omega_1, \ldots, \Omega_m$ are given. We note that, when either $I^{T^1}(\mu^1) = \emptyset$ or $I^{T^2}(\mu^2) = \emptyset$, Eq. (2) holds. Indeed, assume w.l.o.g. that $I^{T^1}(\mu^1) = \emptyset$. Then,

$$\mathcal{S}^{T^1} + \mathcal{S}^{T^2} = \mathcal{S} + \sum_l A(E_l) + \mathcal{S}^{T^2} \leq \mathcal{S} + \mathcal{S}^{T^1 \cup T^2},$$

where $\cup_l E_l \subseteq T^1$.[23] Therefore, we henceforth assume that $I^{T^1}(\mu^1) \neq \emptyset$ and $I^{T^2}(\mu^2) \neq \emptyset$. The rest of the proof consists of showing that Eq. (2) holds by a "lexicographic induction"[24] on the non-decreasingly ordered vector containing the number of agents of each side

$$\lambda = \lambda(P) = \left(|N^{k_1}|, \ldots, |N^{k_m}|\right) \in \mathbb{Z}^m_+, \text{ with } \{k_1, \ldots, k_m\} = M \text{ and } |N^{k_1}| \leq \ldots \leq |N^{k_m}|. \quad (4)$$

[22] Let $\geq$ denote the componentwise order $\mathbb{R}^k$. A set $A \subset \mathbb{R}^k$ *weakly dominates* another set $B \subset \mathbb{R}^k$ (we write $A \succ^w B$) if for all $y \in B$ there is $x \in A$ such that $x \geq y$ and $A \neq B$.

[23] Note that it might be that $\cup_l U_l = \emptyset$ and hence $\mathcal{S} = \mathcal{S}^{T^1}$. Analogous remarks are needed in other parts of this proof and the proof of Theorem 3.

[24] We say that $y = (y_i)_{i \in M} \mathbb{Z}^m_+$ is lexicographically larger than $z = (z_i)_{i \in M} \mathbb{Z}^m_+$, and write $z <_{lex} y$, if there exists $k \in \{1, \ldots, m\}$ such that $y_k > z_k$, and $y_t = z_t$ for all $t \in \{1, \ldots, k-1\}$. We write $z \leq_{lex} y$ if either $z <_{lex} y$ or $z = y$.

When $\lambda = \vec{0}$, Eq. (2) trivially holds because T^1 and T^2 are disjoint. Next, let $t \in \mathbb{Z}^m_+$, with $\vec{0} <_{lex} t$, and suppose that

$$\text{Theorem 2 holds if } \vec{0} \leq_{lex} \lambda <_{lex} t. \tag{5}$$

Let P be such that $\lambda = t$ and assume that Eq. (2) does not hold, i.e., there are $T^1, T^2 \subseteq \Omega \setminus N$ such that

$$\left(\mathcal{S}^{T^1} - \mathcal{S}\right) + \left(\mathcal{S}^{T^2} - \mathcal{S}\right) > \mathcal{S}^{T^1 \cup T^2} - \mathcal{S}. \tag{6}$$

Since T^1 and T^2 are unlinked of degree 0 w.r.t. P, there exist $\mu^1 \in \mathcal{M}^*(P^{T^1})$ and $\mu^2 \in \mathcal{M}^*(P^{T^2})$ such that

$$\begin{aligned}
\mathcal{S}_{I^{T^1}(\mu^1)} + \sum_l A\left(H^1_l, \overline{H^1_l}\right) &= \mathcal{S}^{T^1}, \\
\mathcal{S}_{I^{T^2}(\mu^2)} + \sum_l A\left(H^2_l, \overline{H^2_l}\right) &= \mathcal{S}^{T^2}, \\
\mathcal{S}^{T^1 \cup T^2} &\geq \mathcal{S}_{I^{T^1}(\mu^1) \cup I^{T^2}(\mu^2)} + \sum_l A\left(H^1_l, \overline{H^1_l}\right) + \sum_l A\left(H^2_l, \overline{H^2_l}\right),
\end{aligned}$$

where, for $k \in \{1, 2\}$, we have $\cup_l H^k_l \subseteq T^k$, $\cup_l \overline{H^k_l} = I^{T^k}(\mu^k)$, and $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \emptyset$. Let denote $\tilde{P} = P_{I^{T^1}(\mu^1) \cup I^{T^2}(\mu^2)}$ and $\tilde{\mathcal{S}} = \mathcal{S}(\tilde{P})$. Then, by summing the above three inequalities and the inequality in Eq. (6) it follows that

$$\tilde{\mathcal{S}}^{I^{T^1}(\mu^1)} - \tilde{\mathcal{S}} + \tilde{\mathcal{S}}^{I^{T^2}(\mu^2)} - \tilde{\mathcal{S}} > \tilde{\mathcal{S}}^{I^{T^1}(\mu^1) \cup I^{T^2}(\mu^2)} - \tilde{\mathcal{S}},$$

which contradicts the induction hypothesis in (5). Indeed, on the one hand, since $I^{T^1}(\mu^1) \neq \emptyset$ and $I^{T^2}(\mu^2) \neq \emptyset$, it follows that

$$\lambda(\tilde{P}) <_{lex} \lambda(P) = t.$$

On the other hand, since T^1 and T^2 are unlinked of any degree w.r.t. P, $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ are unlinked of any degree w.r.t. $\tilde{P}$. □

Proof of Lemma 1. Since the result of the lemma is negative, it will suffice to focus our attention on classical assignment economies.[25] Note that in general it holds that $I^{T^1}(\mu^1), I^{T^2}(\mu^2) \subseteq N$, on the one hand, and $T^1, T^2 \subseteq \Omega \setminus N$, $I_{T^1,T^2}(\mu) \setminus N \subseteq T^2$, $I_{T^2,T^1}(\mu) \setminus N \subseteq T^1$, and $T^1 \cap T^2 = \emptyset$, on the other hand. Hence, it is sufficient to check set-like properties separately regarding *(i)* $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$, *(ii)* $I_{T^2,T^1}(\mu) \setminus N$ and T^1 and *(iii)* $I_{T^1,T^2}(\mu) \setminus N$ and T^2.[26]

Let $N^1 = \{1_1\}$, $N^2 = \{1_2\}$, $N^3 = \{1_3\}$, $T^1 = \{2_1\}$ and $T^2 = \{2_2, 2_3\}$, and consider a potential array A. The only set-like possibilities regarding T^1, T^2, $I_{T^1,T^2}(\mu) \setminus N$ and $I_{T^2,T^1}(\mu) \setminus N$ are: *(i)* $I_{T^2,T^1}(\mu) \setminus N = T^1$ and $I_{T^1,T^2}(\mu) \setminus N = T^2$, *(ii)* $I_{T^2,T^1}(\mu) \setminus N = T^1$ and $I_{T^1,T^2}(\mu) \setminus N \subsetneq T^2$, and *(iii)* $I_{T^2,T^1}(\mu) \setminus N = \emptyset$ and $I_{T^1,T^2}(\mu) \setminus N = \emptyset$.

[25] To facilitate the reading of the proof we do not specify which economies the different matchings are optimal from. It can be verified that in all cases they are well defined.

[26] The same comment applies for the proof of Lemma 2.

First, by arbitrarily increasingly $a_{(2_1,2_2,2_3)}$ (resp. $a_{(2_1,2_2,1_3)}$), Eq. (2) holds regardless of $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ being empty or not and their relation w.r.t. inclusion. Moreover, it holds that $I_{T^2,T^1}(\mu) \setminus N = T^1$ and $I_{T^1,T^2}(\mu) \setminus N = T^2$ (resp. $I_{T^1,T^2}(\mu) \setminus N \subsetneq T^2$). Second, by arbitrarily and simultaneously increasing $a_{(2_1,1_2,1_3)}$ and $a_{(1_1,2_2,2_3)}$, Eq. (2) holds and $I_{T^2,T^1}(\mu) \setminus N = I_{T^1,T^2}(\mu) \setminus N = \emptyset$. Moreover, by appropriately choosing the remaining entries – but maintaining $\{(2_1, 1_2, 1_3), (1_1, 2_2, 2_3)\}$ as the unique optimal matching of $P^{T^1 \cup T^2}$ – all possible set-like properties regarding $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ can arise. $\square$

Proof of Theorem 3.

Let us consider that $\Omega_1, \dots, \Omega_m$ are given. As in the proof of Theorem 2, we note that, when either $I_{T^1,T^2}(\mu) = \emptyset$ or $I_{T^2,T^1}(\mu) = \emptyset$, Eq. (3) holds. Indeed, assume w.l.o.g. that $I_{T^1,T^2}(\mu) = \emptyset$. Then,

$$\mathcal{S}^{T^1 \cup T^2} + \mathcal{S} = \mathcal{S}^{T^2} + \sum_l A(E_l) + \mathcal{S} \leq \mathcal{S}^{T^1} + \mathcal{S}^{T^2},$$

where $\emptyset \subseteq \cup_l E_l \subseteq T^1$. Therefore, we henceforth assume that $I_{T^1,T^2}(\mu) \neq \emptyset$ and $I_{T^2,T^1}(\mu) \neq \emptyset$. The rest of the proof consists of showing that Eq. (3) holds by a "lexicographic induction" on λ, as defined in Eq. (4). When $\lambda = \vec{0}$, Eq. (3) holds since T^1 and T^2 are pairwise unlinked of degree 0 w.r.t. P, so $I_{T^1,T^2}(\mu) = \emptyset$. Next, let $t \in \mathbb{Z}_+^m$, with $\vec{0} <_{lex} t$, and suppose that

$$\text{Theorem 3 holds if } \vec{0} \leq_{lex} \lambda <_{lex} t. \tag{7}$$

Let P be such that $\lambda = t$ and assume that Eq. (3) does not hold, i.e., there are $T^1, T^2 \subseteq \Omega \setminus N$ such that

$$\left(\mathcal{S}^{T^1} - \mathcal{S}\right) + \left(\mathcal{S}^{T^2} - \mathcal{S}\right) < \mathcal{S}^{T^1 \cup T^2} - \mathcal{S}. \tag{8}$$

Since T^1 and T^2 are pairwise unlinked of degree 0 w.r.t. P, there exists $\mu \in \mathcal{M}^*(P^{T^1} \cup P^{T^2})$ such that

$$\begin{aligned}
\mathcal{S}^{T^1 \cup T^2} &= \mathcal{S}_{I_{T^1,T^2}(\mu) \cup I_{T^2,T^1}(\mu)} + \sum_l A\left(H_l^1, \overline{H_l^1}\right) + \sum_l A\left(H_l^2, \overline{H_l^2}\right), \\
\mathcal{S}_{I_{T^1,T^2}(\mu)} + \sum_l A\left(H_l^1, \overline{H_l^1}\right) &\leq \mathcal{S}^{T^1}, \\
\mathcal{S}_{I_{T^2,T^1}(\mu)} + \sum_l A\left(H_l^2, \overline{H_l^2}\right) &\leq \mathcal{S}^{T^2},
\end{aligned}$$

where, for $k \neq l \in \{1, 2\}$, we have $\cup_l H_l^k \subseteq T^k$, $\cup_l \overline{H_l^k} = I_{T^k,T^l}(\mu)$, and $I_{T^k,T^l}(\mu) \setminus N = \emptyset$. Let denote $\tilde{P} = P_{I_{T^1,T^2}(\mu) \cup I_{T^2,T^1}(\mu)}$ and $\tilde{\mathcal{S}} = \mathcal{S}(\tilde{P})$. Then, by summing the above three inequalities and the inequality in Eq. (8) it follows that

$$\tilde{\mathcal{S}}^{I_{T^1,T^2}(\mu)} - \tilde{\mathcal{S}} + \tilde{\mathcal{S}}^{I_{T^2,T^1}(\mu)} - \tilde{\mathcal{S}} < \tilde{\mathcal{S}}^{I_{T^1,T^2}(\mu) \cup I_{T^2,T^1}(\mu)} - \tilde{\mathcal{S}},$$

which contradicts the induction hypothesis in (7). Indeed, on the one hand, since $I_{T^1,T^2}(\mu) \neq \emptyset$ and $I_{T^2,T^1}(\mu) \neq \emptyset$, it follows that

$$\lambda(\tilde{P}) <_{lex} \lambda(P) = t.$$

On the other hand, since T^1 and T^2 are pairwise unlinked of any degree w.r.t. P, $I_{T^1,T^2}(\mu)$ and $I_{T^2,T^1}(\mu)$ are pairwise unlinked of any degree w.r.t. $\tilde{P}$. $\square$

Proof of Lemma 2.

Since the result of the lemma is negative, it will suffice to focus our attention to classical assignment economies.[27] Let $N^1 = \{1_1, 2_1\}$, $N^2 = N^3 = \emptyset$, $T^1 = \{1_2, 1_3\}$ and $T^2 = \{2_2, 2_3\}$, and consider a potential array A with all entries strictly positive. On the one hand, the only possibilities regarding $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ are: *(i)* $I^{T^1}(\mu^1) = I^{T^2}(\mu^2)$ and *(ii)* $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \emptyset$ and the sets $I^{T^1}(\mu^1)$ and $I^{T^2}(\mu^2)$ are singletons. On the other hand, the only possibilities regarding $I_{T^1,T^2}(\mu) \setminus N$ and $I_{T^2,T^1}(\mu) \setminus N$ are w.l.o.g.: *(i)* $I_{T^2,T^1}(\mu) \setminus N = \emptyset$ and $I_{T^1,T^2}(\mu) \setminus N = \emptyset$ and *(ii)* $I_{T^2,T^1}(\mu) \setminus N = T^1$ and $I_{T^1,T^2}(\mu) \setminus N = T^2$.

Let $\alpha > \beta > 0$. First, assume that $a_{(1_1,1_2,1_3)} = a_{(1_1,2_2,2_3)} = \alpha$ and $a_{(2_1,1_2,1_3)} = a_{(2_1,2_2,2_3)} = \beta$. Notice that $I^{T^1}(\mu^1) = I^{T^2}(\mu^2) = \{1_1\}$. Moreover, by appropriately choosing the remaining entries all possible set-like properties regarding $I_{T^1,T^2}(\mu) \setminus N$ and $I_{T^2,T^1}(\mu) \setminus N$ can arise. Second, let $a_{(1_1,1_2,1_3)} = a_{(2_1,2_2,2_3)} = \alpha$ and $a_{(2_1,1_2,1_3)} = a_{(1_1,2_2,2_3)} = \beta$. Notice that $I^{T^1}(\mu^1) = \{1_1\}$ and $I^{T^2}(\mu^2) = \{2_1\}$, so $I^{T^1}(\mu^1) \cap I^{T^2}(\mu^2) = \emptyset$. As in the previous case, by appropriately choosing the remaining entries all possible set-like properties regarding $I_{T^1,T^2}(\mu) \setminus N$ and $I_{T^2,T^1}(\mu) \setminus N$ can arise.

$\square$

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[27] To facilitate the reading of the proof we do not specify which economies the different matchings are optimal from. It can be verified that in all cases they are well defined.

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