# A short proof for the polyhedrality of the Chvátal–Gomory closure of a compact convex set<sup>\*†</sup>

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#### Abstract

Recently Schrijver's open problem, whether the Chvátal-Gomory closure of an irrational polytope is polyhedral was answered independently in the seminal works of Dadush et al. [2011] and Dunkel and Schulz [2010]; the former even applies to general compact convex sets. We present a very short, easily accessible proof.

# 1 Introduction

The Chvátal-Gomory procedure was the first cutting-plane procedure introduced (in Gomory [1958, 1960, 1963], Chvátal [1973]) and has been studied thoroughly from a theoretical as well as a practical point of view (see e.g., Chvátal et al. [1989], Caprara and Fischetti [1996], Bockmayr et al. [1999], Letchford and Lodi [2002], Eisenbrand and Schulz [2003], Fischetti and Lodi [2007], Cornuéjols [2008], Bonami et al. [2008]). Recall that the Chvátal-Gomory closure K' of a polyhedron or a compact convex set  $K \subseteq \mathbb{R}^n$  is defined as

$$K' := \bigcap_{\substack{(c,\delta) \in \mathbb{Z}^n \times \mathbb{R}, \\ K \subseteq \{cx \le \delta\}}} \{cx \le \lfloor \delta \rfloor\},$$

where we use  $\{ax \le b\}$  as a shorthand for  $\{x \mid ax \le b\}$ ; for brevity we refer to it as CG closure and to the defining inequalities as CG cuts. One of the fundamental questions in cutting-plane theory is whether the closure arising from a cutting-plane procedure (i.e., adding all potential cuts that can be derived from valid inequalities) is a polyhedron. Clearly, we add an infinite number of cuts here and thus it is not clear *a priori* whether *K'* is a polyhedron. However, for the case where *K* is a rational polyhedron it is well-known that the CG closure is a rational polyhedron again (see Schrijver [1980], Chvátal [1973]). As a natural consequence, in Schrijver [1980] the question was raised whether the CG closure of an irrational polytope *P* is a polytope. This important question was answered in the affirmative independently in the works by Dunkel and Schulz [2010] and Dadush et al. [2011] (the latter established the even more general case of arbitrarily compact convex set). The relevance of this result is many-fold, from the convergence of adding cutting planes of the CG type to Mixed-Integer Nonlinear Programming over compact convex sets to the theory of proof systems where we consider proofs of assertions with infinitely many defining sentences.

\*Keywords and phrases: Chvátal–Gomory closure, polyhedrality, compactness, convex body

<sup>&</sup>lt;sup>+</sup>2010 Mathematics Subject Classification: Primary 90C25; Secondary 90C05, 90C10

# Contribution

We provide a short proof of the more general result Dadush et al. [2011]. In contrast to Dadush et al. [2011], we do not rely so much on convex analysis but take a rather direct topological approach while maintaining the overall high-level strategy. Key is here a strengthened, quantitative homogeneity lemma (see Lemma 3) from which many required properties follow immediately. Before, these had been established separately using different strategies. We believe that the proposed proof lends itself to potential applications to many more classes of cutting planes.

The proof consists of three steps:

- 1. *Continuity:* Faces and implied cuts deform smoothly when perturbing the coefficients. This is crucial for the actual finiteness argument via compactness, and is valid for inequalities in general. See Lemma 1.
- 2. *Homogeneity* for a procedure *M*: The cutting-plane closure *M* commutes with intersections with faces, i.e.,

$$M(K \cap F) = M(K) \cap F,$$

where *F* is a face-defining hyperplane of the convex body *K*. See Lemma 3.

Not only does homogeneity hold for many cutting-plane procedures *M* (such as the split closure, the Lovász-Schrijver closures, Sherali-Adams hierarchies, and the Lift-and-Project closure, see Pokutta and Schulz [2010]) in the case of rational polyhedra, but homogeneity also allows a very clean, inductive approach to polyhedrality. See Theorem 7.

Moreover, it is also homogeneity that ensures that finitely many CG cuts suffice to restrict the CG closure to a rational subspace of the affine space spanned by the convex set (which is necessary for polyhedrality here). See Corollary 5.

3. *Locality:* Informally, every point *x* in the relative interior of a *polytope P* can be cut out by a finite number of CG cuts.

This step is contained in the proof of Theorem 6.

From all those properties the hardest one to establish and the cornerstone of our proof is the quantitative version of the homogeneity lemma in Section 3. In fact, for the proof we need a generalization of a famous theorem due to Kronecker and Weyl provided in Lemma 2. Once homogeneity is established, the conclusion of polyhedrality follows naturally in our framework and it is actually very similar to the proof for rational polyhedra given in Schrijver [1980].

# 2 Preliminaries

In the following, we only consider exposed faces of convex sets and for the sake of brevity we refer to them as *faces*. In other words, a face *F* of a compact convex set *K* is a subset of the form  $F = K \cap \{\pi x = \pi_0\}$  for some supporting hyperplane  $\pi x = \pi_0$ , i.e.,  $K \subseteq \{\pi x \le \pi_0\}$ , and there exists  $x_0 \in K$  with  $\pi x_0 = \pi_0$ ; we will call the face *F* the  $\pi$ -*face of K*. In particular, F = K is allowed if *K* is lower dimensional. Recall that a compact convex set is uniquely determined by its exposed proper faces (i.e., exposed faces different from *K*). All facts that we mention without pointers to the literature can be found in Schrijver [1998] and Barvinok [2002].

To formalize continuity of directions, we identify a direction with the unit vector pointing to that direction, therefore for any non-zero vector  $\pi$ , let  $\hat{\pi} \coloneqq \pi / \|\pi\|$ . The next lemma shows that compact  $\pi$ -faces change "upper semi-continuously" in the direction of  $\pi$ . In the following we denote the dimension of the ambient space  $\mathbb{R}^n$  by n and we use the shorthand  $[k] \coloneqq \{1, \ldots, k\}$ . Let  $\partial K$  denote the relative boundary of a closed convex set K.

**Lemma 1** (Continuity). Let *K* be a closed convex set in  $\mathbb{R}^n$ . Let  $\pi$  be a non-zero vector, and let the  $\pi$ -face *F* of *K* be compact. Then for every neighbourhood *U* of *F* there exists an  $\varepsilon > 0$  such that whenever

 $\|\hat{\pi}' - \hat{\pi}\| < \varepsilon$ , the  $\pi'$ -face of K is a subset of U, i.e., all the maximizers of the function  $x \mapsto \pi' x$  on K lie in U.

*Proof.* Without loss of generality, we may assume that U is compact and convex. By assumption, for some constant c, we have  $\pi x = c$  for all  $x \in F$ , and  $\pi x < c$  for all  $x \in K \setminus F$ , in particular for all  $x \in \partial U \cap K$ . As  $\partial U \cap K$  is compact, actually  $\pi x < c_1$  for some  $c_1 < c$  and all  $x \in \partial U \cap K$ .

As  $\pi' x$  is continuous in  $\pi'$  and x, and F and  $\partial U \cap K$  are compact, for all  $\pi'$  in a neighbourhood of  $\pi$ , we have  $\pi' x > c_1$  for all  $x \in F$ , but  $\pi' x < c_1$  for all  $x \in \partial U \cap K$ . In particular, all maximizers of  $x \mapsto \pi' x$  on K lie in U. This is obvious if  $K \subseteq U$ . If  $K \nsubseteq U$ , then  $\pi' x$  is everywhere smaller on  $K \setminus U$  than on F: take arbitrary points  $x_0 \in K \setminus U$  and  $x_1 \in F$ . There is an  $x_2 \in \partial U \cap K$  in the line segment  $[x_0, x_1]$ . As  $\pi' x_2 < \pi' x_1$ , we obtain  $\pi' x_0 < \pi' x_1$ . This finishes the proof of the lemma.

*Remark* 1. If *K* is a polyhedron one can show more: the  $\pi'$ -face is contained in *F*. In particular, there is no need for *U*. To see this we choose *U* to be a polytope in the proof. Then  $U \cap K$  is also a polytope, and  $x \mapsto \pi' x$  is everywhere larger on the vertices of the  $\pi$ -face than on the other vertices, when the direction of  $\pi'$  is close to that of  $\pi$ . Hence the  $\pi'$ -face is contained in the  $\pi$ -face, as claimed.

We will use a well-known approximation theorem due to Kronecker. We state a version suitable for our needs, which we derive from Weyl's criterion.

**Lemma 2** ([Kronecker, 1884],[Weyl, 1916, Satz 3]). Let  $n, N_0 \in \mathbb{N}$  and  $\pi \in \mathbb{R}^n$  with  $\pi \neq 0$ . Then  $\mathbb{Z}^n + \pi \mathbb{Z}_{>N_0}$  contains a dense subset of a linear subspace V of  $\mathbb{R}^n$ . In particular,  $\mathbb{Z}^n + \pi \mathbb{Z}_{>N_0}$  contains points arbitrarily close to 0, i.e., for every  $\epsilon > 0$  there exists  $N > N_0$  and  $a \in \mathbb{Z}^n$  with  $||a - N\pi|| < \epsilon$ .

*Proof.* When the components of  $\pi$  together with 1 are linearly independent over  $\mathbb{Q}$ , this is a special case of Weyl's criterion with  $V = \mathbb{R}^n$ . We reduce the general case to this one.

First we define *V*. Let  $\pi_1, ..., \pi_n$  denote the components of  $\pi$ . We can assume without loss of generality that a linear basis of 1,  $\pi_1, ..., \pi_n$  over  $\mathbb{Q}$  is 1,  $\pi_1, \pi_2, ..., \pi_k$ .

Thus for j > k there are integers  $n_{j,i}$  and  $n_j$  together with a positive integer *m* such that

$$m\pi_j = n_j + \sum_{i=1}^k n_{j,i}\pi_i, \qquad j > k.$$

We use these as the defining equations of *V*, i.e., *V* is defined by

$$mx_j = \sum_{i=1}^k n_{j,i} x_i, \qquad j > k.$$

Let  $e_1, \ldots, e_n$  denote the canonical basis of  $\mathbb{Z}^n$ . The following elements lie in *V*:

$$\widetilde{e_i} \coloneqq me_i + \sum_{j=k+1}^n n_{j,i}e_j, \qquad i < k,$$
  
$$\widetilde{\pi} \coloneqq m\pi - \sum_{j=k+1}^n n_je_j.$$

By Weyl's criterion,  $\mathbb{Z}^k + (\pi_1, \ldots, \pi_k)\mathbb{Z}_{>N_0}$  is dense in  $\mathbb{R}^k$ , and hence also  $m\mathbb{Z}^k + m(\pi_1, \ldots, \pi_k)\mathbb{Z}_{>N_0}$  is dense in  $\mathbb{R}^k$ . We reformulate this for V via the projection to the first k coordinates, which is obviously an isomorphism between V and  $\mathbb{R}^k$ : a dense subset of V is  $\sum_{i=1}^k \mathbb{Z}\tilde{e}_i + \tilde{\pi}\mathbb{Z}_{>N_0}$ , which is a subset of  $\mathbb{Z}^n + \pi\mathbb{Z}_{>N_0}$ . This finishes the proof. (For k = 0 the argument above is overkill, as  $\pi$  is rational and hence  $\mathbb{Z}^n + \pi\mathbb{Z}_{>N_0}$  contains 0.)

# 3 Homogeneity

In this section we compare K' with the CG closure F' of a face F.

**Lemma 3** (Homogeneity for compact faces). Let  $K \subseteq \mathbb{R}^n$  be a closed convex set and let

$$F := K \cap \{\pi x = \pi_0\}$$

be a compact  $\pi$ -face of K for some  $\pi \in \mathbb{R}^n$  and  $\pi_0 \in \mathbb{R}$  with  $K \subseteq \{\pi x \leq \pi_0\}$ . Further, assume that F satisfies  $cx \leq \delta$  with  $c \in \mathbb{Z}^n$  (and hence F' satisfies  $cx \leq \lfloor \delta \rfloor$ ). Then there are finitely many CG cuts of K defining a polyhedron P satisfying  $(c + \alpha \pi)x \leq \lfloor \delta \rfloor + \alpha \pi_0$  for some  $\alpha > 0$ .

*Proof.* Rescaling  $(\pi_0, \pi)$  we may assume, without loss of generality, that  $\pi_0 \in \mathbb{Z}$ . By increasing  $\delta$  a little if necessary, we may assume that  $\delta$  is not an integer. Note that a small increase of  $\delta$  does not change  $\lfloor \delta \rfloor$ . Let  $0 < \varepsilon < \delta - \lfloor \delta \rfloor$  be a small positive number. Choose a small compact neighbourhood U of F such that  $cx \leq \delta + \varepsilon$  for  $x \in U$ .

There is clearly an  $\varepsilon_1 > 0$  such that for all  $y \in \mathbb{R}^n$  with  $||y|| < \varepsilon_1$ , we have  $|yx| < \varepsilon$  for all  $x \in U$ . By Lemma 1, there exists an  $\varepsilon_2 > 0$  such that whenever  $\left\| \widehat{\pi'} - \widehat{\pi} \right\| < \varepsilon_2$ , all maximizers of the function  $x \mapsto \pi' x$  on K lie in U. There is a large positive integer N such that for all positive integer  $m \ge N$  and  $a \in \mathbb{Z}^n$  with  $||a - m\pi|| < \varepsilon_1$  we have  $\left\| \widehat{c+a} - \widehat{\pi} \right\| = \left\| \frac{\widehat{c+a}}{m} - \widehat{\pi} \right\| < \varepsilon_2$ . In particular, all maximizers of (c + a)x on K lie in U for all such m and a. All in all, for all positive integer  $m \ge N$  and  $a \in \mathbb{Z}^n$  with  $||a - m\pi|| < \varepsilon_1$ , all maximizers of the function  $x \mapsto (c + a)x$  on K lie in U, and we have  $||(a - m\pi)x|| < \varepsilon$  for all  $x \in U$ .

Now we choose a finite collection of such pairs (m, a). By Lemma 2 the collection  $\mathbb{Z}^n - \mathbb{Z}_{\geq N}\pi$  contains a dense subset of a linear subspace V of  $\mathbb{R}^n$ . Let  $v_1, \ldots, v_k$  be the vertices of a small simplex in V containing 0 in its relative interior, with  $||v_i|| < \varepsilon_1$  for all i. We choose the simplex to be full dimensional in V, i.e., here k - 1 is the dimension of V. By slightly perturbing the  $v_i$  in V, the vertices remain in the  $\varepsilon_1$ -ball, and 0 in the interior of the simplex. As  $\mathbb{Z}^n - \mathbb{Z}_{\geq N}\pi$  contain a dense subset of V, by a slight perturbation we can even move the vertices inside  $\mathbb{Z}^n - \mathbb{Z}_{\geq N}\pi$ , obtaining a new simplex with vertices  $a_i - m_i\pi$ , with 0 still contained in the relative interior of the new simplex: i.e., there exist coefficients  $\lambda_i$  satisfying

$$\sum_{i \in [k]} \lambda_i (a_i - m_i \pi) = 0, \quad \text{where } \lambda_1, \dots, \lambda_k > 0 \text{ and } \sum_{i \in [k]} \lambda_i = 1$$
(1)

with some  $a_i \in \mathbb{Z}^n$ ,  $m_i \in \mathbb{N}$ ,  $m_i \ge N$  satisfying  $||a_i - m_i \pi|| < \varepsilon_1$ .

As a consequence, for all  $x \in U \cap K$ , one has

$$(c+a_i)x = cx + m_i\pi x + (a_i - m_i\pi)x \le (\delta + \varepsilon) + m_i\pi_0 + \varepsilon,$$
(2)

which is also valid for all  $x \in K$  as  $(c + a_i)x$  attains its maximum in U. Hence

$$(c+a_i)x \leq \lfloor \delta + m_i\pi_0 + 2\varepsilon \rfloor = \lfloor \delta \rfloor + m_i\pi_0$$

is a CG cut for *K* for  $i \in [k]$ .

Let *P* be the polyhedron defined by these CG cuts. The convex combination of the CG cuts with coefficients  $\lambda_i$  is valid for *P*, which is exactly the claimed inequality

$$(c+\alpha\pi)x\leq \lfloor\delta\rfloor+\alpha\pi_0,$$

for *P* with  $\alpha := \sum_{i \in [k]} \lambda_i m_i > 0$  as (1) can be rewritten to

$$\sum_{i \in [k]} \lambda_i(c+a_i) = c + \underbrace{\sum_{i \in [k]} \lambda_i m_i \, \pi}_{\alpha}.$$

Apart from establishing the basis for the later induction, the main advantage of Lemma 3 is that many important properties of the CG closure follow as simple corollaries.

**Corollary 4.** Let  $K \subseteq \mathbb{R}^n$  be a compact convex set. Then  $K' \subseteq K$  and we have  $K' \cap F = F'$  for every *face F* of *K*.

*Proof.* Applying Lemma 3 to c = 0 and  $\delta = 0$ , we obtain that K' satisfies every inequality  $\pi x \le \pi_0$  satisfied by K.

For a face *F*, Lemma 3 implies that  $K' \cap F$  satisfies the CG cuts defining *F'*, hence  $K' \cap F \subseteq F'$ . The inclusion in the other direction  $F' \subseteq K' \cap F$  is obvious.

*Remark* 2. If one merely wants to establish  $K' \cap F \subseteq (K \cap F)'$  and one is not interested in the finiteness statement of Lemma 3 then it suffices to consider a single vector  $c + a_1$  in the proof of Lemma 3 instead of a finite family. From (2) the proof can then be concluded as follows: For every  $x \in K' \cap F$ 

$$cx = (c+a_1)x + (m_1\pi - a_1)x - m_1\pi_0 \le \lfloor \delta \rfloor + \varepsilon$$

for every  $\varepsilon > 0$  small enough. Thus  $cx \leq |\delta|$  is valid for  $K' \cap F$ .

Also note that  $K' \subseteq K$  alternatively follows with [Dey and Pokutta, 2011, Lemma 2] (see Dadush et al. [2010] for a similar result).

We further obtain

**Corollary 5.** Let *K* be a compact convex set. Then finitely many CG cuts of *K* define a polyhedron in a rational affine subspace *V* with  $V \subseteq aff(K)$ .

*Proof.* The affine subspace aff(K) is defined by finitely many inequalities  $a_i x \leq b_i$  with  $i \in [\ell]$  for some  $\ell \in \mathbb{N}$ . These are consequences of finitely many CG cuts via Lemma 3 with  $\pi = a_i$ ,  $\pi_0 = b_i$  and c = 0,  $\delta = 0$ . Therefore the polyhedron defined by these CG cuts spans a rational affine subspace *V* of aff(K).

# 4 The CG closure of a compact convex set

We will now prove the main theorem:

**Theorem 6.** *Let K be a compact convex set. Then K' is a rational polytope defined by finitely many CG cuts of K.* 

The proof will proceed via induction on the dimension of *K* using the following step lemma.

**Lemma 7.** Let *K* be a compact convex set. Let us assume that for every proper face *F* of *K*, the CG closure *F'* is defined by finitely many CG cuts of *F* (i.e., Theorem 6 holds for *F*). Then there is a polytope *P* defined by finitely many CG cuts of *K*, which is contained in *K* and  $P \cap \partial K = K' \cap \partial K$ .

*Proof.* For all unit vectors  $\pi$  in the lineality space of aff(K), let  $\pi x \leq \pi_0$  define the associated supporting hyperplane of K. Now  $F_{\pi} = K \cap \{\pi x = \pi_0\}$  is a proper face, the  $\pi$ -face of K, and hence  $F'_{\pi}$  is defined by finitely many CG cuts of  $F_{\pi}$  by our assumption. By Lemma 3 there are finitely many CG cuts of K defining a polyhedron  $P_{\pi}$  satisfying  $\pi x \leq \pi_0$  and  $P_{\pi} \cap \{\pi x = \pi_0\} = F'_{\pi}$ . For  $F'_{\pi} \neq \emptyset$  this means exactly that the  $\pi$ -face of  $P_{\pi}$  is  $F'_{\pi}$ . By adding finitely many CG cuts, we may assume that  $P_{\pi}$  is a polytope.

We claim that for vectors  $\pi'$  in a neighbourhood  $U_{\pi}$  of  $\pi$ , the polytope  $P_{\pi}$  still satisfies  $\pi' x \leq \pi'_0$  and  $P_{\pi} \cap \{\pi' x = \pi'_0\} = F'_{\pi'}$ . This is immediate if  $F'_{\pi} = \emptyset$ , as  $F_{\pi'}$  is disjoint from  $P_{\pi}$  by Lemma 1. If  $F'_{\pi} \neq \emptyset$ , let  $F_{P_{\pi},\pi'}$  denote the  $\pi'$ -face of  $P_{\pi}$ . The inequality  $\pi' x \leq \pi'_0$  is satisfied by  $P_{\pi}$  as  $F_{P_{\pi},\pi'}$  is also the  $\pi'$ -face of  $F'_{\pi}$ . If  $F_{P_{\pi},\pi'}$  is not contained in the  $\pi'$ -face  $F_{\pi'}$  of K, then  $P_{\pi}$  satisfies  $\pi' x < \pi'_0$  and  $F'_{\pi'} = \emptyset$ . However, if  $F_{P_{\pi},\pi'}$  is contained in the  $\pi'$ -face  $F_{\pi'}$ , then clearly  $F'_{\pi'} = F_{P_{\pi},\pi'} = P_{\pi} \cap \{\pi' x = \pi'_0\}$ , proving the claim.

We obtain an open cover of the unit sphere of  $\operatorname{aff}(K)$  with neighborhoods  $U_{\pi}$  such that for each  $\pi' \in U_{\pi}$  we have  $P_{\pi} \cap \{\pi' x = \pi'_0\} = F'_{\pi'}$  and  $\pi' x \leq \pi'_0$  for all  $x \in P_{\pi}$ . Since the unit sphere is compact, it follows by choosing a finite subcover that finitely many CG cuts define a polytope Q with  $Q \cap \{\pi x = \pi_0\} = F'_{\pi}$  and  $\pi x \leq \pi_0$  for all  $x \in Q$ .

By Corollary 5, by adding finitely many cuts we obtain a polytope *P* in a rational affine subspace of aff(*K*), which is contained in *Q*. In particular, it lies in *K* and  $P \cap \partial K = K' \cap \partial K$ .

Finally, we are ready to prove the main theorem.

*Proof of Theorem 6.* The proof proceeds via induction on the dimension of *K*. By the induction hypothesis, the Theorem holds for proper faces of *K*. From Lemma 7 we know that finitely many CG cuts define a polytope *P* in *K* with  $P \cap \partial K = K' \cap \partial K$ . The polytope *P* spans a rational affine subspace *V* of aff(*K*).

Let *D* denote the orthogonal projection of  $\mathbb{Z}^n$  onto the lineality space *W* of *V*. As *W* is rational, the orthogonal projection *D* is a lattice. We claim that there are only finitely many  $d \in D$  with a preimage  $c \in \mathbb{Z}^n$ , for which a CG cut  $cx \leq \lfloor \delta \rfloor$  cuts out something from *P* i.e., at least one vertex *v* of *P*.

As vertices on the boundary of *K* belong to *K'*, these cannot be cut out. Therefore *v* has to be contained in the relative interior of *K* and so does a small ball *U* in *V* around *v*. Let *r* denote the radius of *U*. Now whenever  $d \in D$  is too long, i.e.,  $||d|| \ge 1/r$ , we have  $\max_{x \in K} cx \ge \max_{x \in U} cx \ge cv + 1$  as  $x - v \in W$  and so cx - cv = dx - dv. Hence  $cx \le \lfloor \max_{x \in K} cx \rfloor$  cannot cut off *v*. As there are only a finite number of vertices *v* of *P*, there is a global upper bound on the length of the *d* which could cut out a vertex in the relative interior of *K*. As *D* is discrete, there are only finitely many such vectors *d*, and adding these CG cuts to *P* we obtain *K'*.

Actually, for every *d* we need to add only one cut, thus defining *K'* by finitely many CG cuts, as claimed. To prove this, we consider all the CG cuts  $cx \le \lfloor \delta \rfloor$  where *c* is a preimage of a fixed  $d \in D$ . We claim that unless  $P = \emptyset$  (when P = K' and the theorem holds), there is a deepest cut among these. This will be the only cut we need to add to *P* for the vector *d*.

To prove the last claim, let  $x_0$  be a rational point of V. Restricted to V, every CG cut  $cx \le \lfloor \delta \rfloor$  can be rewritten to  $dx \le \lfloor \delta \rfloor - (c - d)x_0$ . As  $\lfloor \delta \rfloor$  is an integer, c - d is rational with bounded denominator (as D is discrete), and  $x_0$  is rational, therefore the right-hand side can take only a discrete set of values, and dx is a lower bound on the set of values for every  $x \in P$ . Therefore there is a cut with  $\lfloor \delta \rfloor - (c - d)x_0$  minimal, which is obviously a deepest cut.

# Acknowledgements

The authors are grateful to Daniel Dadush and Santanu Dey for helpful comments and discussions and to the anonymous reviewers whose comments improved the presentation significantly.

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