

Solving the continuous nonlinear resource allocation problem with an interior point method

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Abstract

Resource allocation problems are usually solved with specialized methods exploiting their general sparsity and problem-specific algebraic structure. We show that the sparsity structure alone yields a closed-form Newton search direction for the generic primal-dual interior point method. Computational tests show that the interior point method consistently outperforms the best specialized methods when no additional algebraic structure is available.

1 Introduction

We consider the resource allocation problem in the form

$$\text{minimize } f(x) := \sum_{i=1}^n f_i(x_i) \text{ over all } x \quad (1)$$

$$\text{subject to } g(x) := \sum_{i=1}^n g_i(x_i) = b, \quad (2)$$

$$l \leq x \leq u. \quad (3)$$

Here x , l , and u are n -vectors of real numbers, b is a real scalar, and the functions f_i and g_i are convex and twice differentiable on an open set containing the interval $[l_i, u_i]$. Inequalities of vectors are interpreted coordinate-wise.

The recent survey paper of Patriksson [5] shows that such problems have a long history and diverse applications. The contexts in which the problem appears often demand that it be solved very quickly, even in high dimensions. Consequently, researchers long ago moved beyond general-purpose nonlinear programming procedures and focused on exploiting the special structure of the optimality conditions for

the problem. As noted by Patriksson, two frameworks have emerged as the most competitive for solving resource allocation problems: the *pegging* or *variable-fixing* methods and the *breakpoint-search* methods. Patriksson also observes that computational studies in the literature have generally indicated that pegging is superior to breakpoint search when certain subproblems (see §2) common to both methods are easily solved, whereas breakpoint search is faster otherwise. Moreover, numerical comparisons of either method with general-purpose solvers are essentially nonexistent in the literature.

Here we present evidence that a primal-dual interior point method outperforms breakpoint search on problems for which the latter is traditionally considered the best possible choice, namely, when its subproblems do not admit closed-form solutions and must be solved numerically. We show that the special structure of (1)–(3) allows for a closed-form solution of the linear system defining the search directions and we present computational results showing the method’s superiority. This addresses two questions posed by Patriksson [5]. First, it shows that the sparsity can be exploited within the setting of a general-purpose optimizer. Second, it provides an efficient method that also avoids the usual assumptions (see §2) imposed by pegging or breakpoint search methods on the domain, monotonicity or strict convexity of f_i and g_i .

In the next section, we review the optimality conditions for (1)–(3). In §3 we describe the breakpoint search and interior point methods, along with details of their implementation. Section 4 lays out the problem instances used for the computational tests, and the results are discussed in §5.

2 Optimality conditions

In this study we make the following assumptions:

- A1. The *relaxed* problem, in which (2) is replaced by $g(x) \leq b$, has no optimum with $g(x) < b$.

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A2. The function f_i is decreasing on $[l_i, u_i]$ and g_i is increasing on $[l_i, u_i]$ with $g(l) < b < g(u)$.

The randomly generated test instances of §4 all satisfy these assumptions, which are needed for breakpoint search but not for the interior point method.

In practice, we are more interested in the relaxed problem mentioned in Assumption A1. However, we can easily determine whether either assumption holds if we know the intervals of monotonicity for each f_i and g_i . Indeed, many treatments of resource allocation problems include one or both of these assumptions because they can be inexpensively enforced through some combination of initialization, preprocessing, and data generation.

Assumptions A1–A2 imply that (1)–(3) and the relaxed problem are equivalent and admit an optimal solution; they also guarantee that the Slater constraint qualification holds for the relaxed problem. By Lagrangian duality, necessary and sufficient optimality conditions for (1)–(3) can therefore be expressed as follows: $g(x) = b$ and, for some real number ρ , x is a solution to the separable optimization subproblem

$$\text{minimize } f(x) + \rho g(x) \text{ subject to } l \leq x \leq u. \quad (4)$$

The dual objective is

$$\rho \mapsto -b\rho + \sum_{i=1}^n \min_{x_i \in [l_i, u_i]} [f_i(x_i) + \rho g_i(x_i)], \quad (5)$$

which attains its maximum; moreover, any maximizer ρ is necessarily nonnegative. The subproblem (4) has coordinate-wise optimality conditions given by

$$\begin{aligned} f'_i(x_i) + \rho g'_i(x_i) &= 0, & \text{if } l_i < x_i < u_i, \\ f'_i(x_i) + \rho g'_i(x_i) &\geq 0, & \text{if } x_i = l_i, \\ f'_i(x_i) + \rho g'_i(x_i) &\leq 0, & \text{if } x_i = u_i. \end{aligned}$$

The left-hand sides give the Karush-Kuhn-Tucker multipliers for the bounds $l_i \leq x_i$ and $x_i \leq u_i$, respectively, as

$$\begin{aligned} \lambda_i &:= \max\{0, -[f'_i(x_i) + \rho g'_i(x_i)]\}, \\ \mu_i &:= \max\{0, f'_i(x_i) + \rho g'_i(x_i)\}. \end{aligned}$$

Letting $s := u - x$ denote the vector of slack variables for the upper bounds on x , we express the Karush-Kuhn-Tucker (KKT) conditions for (1)–(3) as

$$\nabla f(x) + \rho \nabla g(x) - \lambda + \mu = 0, \quad (6)$$

$$x + s = u, \quad (7)$$

$$x \geq l, \lambda \geq 0, s \geq 0, \mu \geq 0, \quad (8)$$

$$\text{diag}(x - l)\lambda = 0, \text{diag}(s)\mu = 0, \quad (9)$$

$$g(x) = b. \quad (10)$$

Here $\text{diag}(z)$ denotes the diagonal matrix whose diagonal entries are the entries of the vector z .

The three solution frameworks discussed in §1 utilize the optimality conditions in different ways:

- Pegging methods solve subproblems of the form (1)–(2), but for which some variables are held fixed while the bounds (3) for all remaining variables are omitted.
- Breakpoint search methods maximize the dual objective (5) by solving a sequence of subproblems of the form (4) at various values of ρ .
- Primal-dual interior point methods apply Newton's method to perturbations of the KKT system (6)–(10).

The pegging and breakpoint search methods both benefit considerably when minimization of $x_i \mapsto f_i(x_i) + \rho g_i(x_i)$ can be handled efficiently. Because we focus on problems for which breakpoint search dominates pegging, we do not include pegging methods in this study. In fact, the pegging approach is not even well-defined for some of the problems we consider, because the pegging subproblems do not admit optimal solutions.

3 Methods and implementation

In this section, we describe the two main approaches considered in our computational study.

3.1 Breakpoint search

Breakpoint search is based on the observation that the dual objective (5) is concave and defined piecewise with a finite number of easily calculated breakpoints. The derivative, or subdifferential, of this objective is nonincreasing. A binary search of the breakpoints therefore identifies either one that is a root or a pair that most closely bracket a root.

There are at most $2n$ breakpoints, occurring at ρ -values where some $x_i \mapsto f_i(x_i) + \rho g_i(x_i)$ attains its minimum over $[l_i, u_i]$ at an endpoint l_i or u_i . Equivalently, a breakpoint makes the derivative $x_i \mapsto f'_i(x_i) + \rho g'_i(x_i)$ nonnegative at l_i or nonpositive at u_i . Consequently, all breakpoints have the form $\rho_i^+ := -f'_i(l_i)/g'_i(l_i)$ or $\rho_i^- := -f'_i(u_i)/g'_i(u_i)$. The monotonicity of f_i and g_i allow us to define $\rho_i^+ = \infty$ when $g'_i(l_i) = 0$ and to guarantee that $g'_i(u_i) > 0$ in the definition of ρ_i^- . The convexity and monotonicity of f_i and g_i also guarantee that $0 \leq \rho_i^- \leq \rho_i^+$.

The binary search sequentially refines a bracketing $\rho^- < \rho^* < \rho^+$ until the true root ρ^* lies between

two consecutive breakpoints. The bracket is adjusted inward by finding a breakpoint ρ within it and testing the sign of the derivative of the dual objective (5). To evaluate that derivative at ρ , we first fix

$$x_i := \begin{cases} l_i, & \text{if } \rho \geq \rho_i^+, \\ u_i, & \text{if } \rho \leq \rho_i^-. \end{cases} \quad (11)$$

The remaining minimizers are critical points: $f'_i(x_i) + \rho g'_i(x_i) = 0$ and $l_i < x_i < u_i$. Depending on the problem data, these critical points might be found (a) in closed form, (b) by using a problem-specific implementation of Newton's method, or (c) by means of a general-purpose Newton's method with Armijo linesearch for sufficient decrease and damping (as needed) to maintain $l_i < x_i < u_i$. The derivative value at ρ is then given by $-b + \sum_i g_i(x_i)$, the sign of which determines whether ρ becomes the new ρ^- or ρ^+ . This in turn determines, through (11), that some values of x_i shall remain fixed and can therefore be removed from further consideration.

The final bracket, if nontrivial, consists of two closest breakpoints with the optimal value of ρ lying somewhere between them. To interpolate between them, our implementation finds ρ and the unfixed x_i -coordinates (denoted by $i \in I$) simultaneously by applying a multi-dimensional Newton's method with Armijo linesearch to the corresponding Lagrange multiplier conditions $\sum_{i \in I} g_i(x_i) = \hat{b}$ and $f'_i(x_i) + \rho g'_i(x_i) = 0$ for $i \in I$.

Throughout the procedure, the subproblem optimizations are initialized using the corresponding solutions from prior iterations. Also, we extract the required median values without sorting the list of breakpoints in advance, which can yield significant computational savings if each subproblem solution requires only a few operations per index i [1, 2, 4, 6].

3.2 Interior point method

The primal-dual interior point method solves the KKT optimality conditions (6)–(10) for the variables $(x, \lambda, s, \mu, \rho)$. Its operation preserves strict inequality for the simple bounds (8), only allowing them to become active in the limit. The method is based on the perturbed KKT system

$$\nabla f(x) + \rho \nabla g(x) - \lambda + \mu = 0, \quad (12)$$

$$x + s = u, \quad (13)$$

$$\text{diag}(x - l)\lambda = \tau e, \text{diag}(s)\mu = \tau e, \quad (14)$$

$$g(x) = b, \quad (15)$$

where e denotes the n -vector of all ones and the inequalities $x > l$, $\lambda > 0$, $s > 0$, $\mu > 0$ are enforced sep-

arately. The system (12)–(15) is algebraically equivalent to the Lagrange multiplier equations for a related optimization problem involving barrier functions for the bounds (3):

$$\text{minimize } f(x) - \tau \sum_{i=1}^n [\ln(x_i - l_i) + \ln(s_i)]$$

$$\text{over all } x > l, s > 0$$

$$\text{subject to } g(x) = b, x + s = u.$$

As the barrier parameter $\tau > 0$ is driven to zero, we expect the (unique) solution $(x, \lambda, s, \mu, \rho)$ of (12)–(15) to tend toward the solution set of the original KKT system (6)–(10).

In each iteration of the interior point method, we calculate a Newton search direction for the perturbed system (12)–(15) and then take a step along that direction, damped so as to preserve $x > l$, $\lambda > 0$, $s > 0$, $\mu > 0$. Next, the value of τ is adjusted and the iteration repeats. The method stops when the residuals $r_d := \nabla f(x) + \rho \nabla g(x) - \lambda + \mu$, $r_l := \text{diag}(x - l)\lambda$, $r_u := \text{diag}(s)\mu$, and $r_g := g(x) - b$ are small enough.

Ours is a rudimentary implementation aimed at any nonlinear programming formulation involving simple bounds and an equality constraint. The algorithmic parameters were assigned values that gave reliable performance in preliminary testing: all relative tolerances for residuals were set to 10^{-10} , τ was set to 0.25 of the current duality gap $(x \cdot \lambda + s \cdot \mu)/2n$, and the step size was taken to be the smaller of unity or 0.8 of the feasible step. No attempt was made to provide theoretical guarantees of global convergence, superlinear convergence, or complexity. However, the implementation correctly solved all the instances described in §4 and easily outperformed breakpoint search on challenging problems of moderate to very large dimension. It therefore met the needs of the present study. The key to making it competitive is the fact that the linear system defining the Newton search direction can be solved in Cn arithmetic operations for a small fixed value of C , as we show next.

To simplify the notation, we introduce a vector h with entries $h_i := f''_i(x_i) + \rho g''_i(x_i)$ and let ξ denote $x - l$. We also use uppercase letters to denote these diagonal matrices: $\Xi := \text{diag}(\xi)$, $\Lambda := \text{diag}(\lambda)$, $S := \text{diag}(s)$, $M := \text{diag}(\mu)$, $H := \text{diag}(h)$. The linear system satisfied by the search direction is then

$$\begin{bmatrix} H & -I & 0 & I & \nabla g \\ \Lambda & \Xi & 0 & 0 & 0 \\ 0 & 0 & M & S & 0 \\ I & 0 & I & 0 & 0 \\ \nabla g^T & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \\ \Delta \mu \\ \Delta \rho \end{bmatrix} = \begin{bmatrix} r_d \\ r_l \\ r_u \\ 0 \\ r_g \end{bmatrix}. \quad (16)$$

To solve (16), first calculate the vectors

$$\begin{aligned} w &:= h + \Xi^{-1}\lambda + S^{-1}\mu, \\ y &:= r_d + \Xi^{-1}r_l - S^{-1}r_u, \\ z &:= W^{-1}\nabla g \end{aligned}$$

and let $\eta := -1/\nabla g^T z$. Multiplying (16) from the left by

$$\begin{bmatrix} W^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ -\eta z^T & 0 & 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} I & \Xi^{-1} & S^{-1} & S^{-1}M & 0 \\ 0 & \Xi^{-1} & 0 & 0 & 0 \\ 0 & 0 & S^{-1} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} I & 0 & 0 & 0 & z \\ \Xi^{-1}\Lambda & I & 0 & 0 & 0 \\ 0 & 0 & S^{-1}M & I & 0 \\ I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \\ \Delta \mu \\ \Delta \rho \end{bmatrix} = \begin{bmatrix} W^{-1}y \\ \Xi^{-1}r_l \\ S^{-1}r_u \\ 0 \\ \eta(r_g - z^T y) \end{bmatrix}$$

from which we can read off the solution to (16) as:

$$\begin{aligned} \Delta \rho &= \eta(r_g - z^T y), \\ \Delta x &= W^{-1}y - (\Delta \rho)z, \\ \Delta s &= -\Delta x, \\ \Delta \lambda &= \Xi^{-1}r_l - \Xi^{-1}\Lambda \Delta x, \\ \Delta \mu &= S^{-1}r_u - S^{-1}M \Delta s. \end{aligned}$$

The saved values of $\Xi^{-1}r_l$, $\Xi^{-1}\Lambda$, $S^{-1}r_u$, $S^{-1}M$ from the calculation of w and y can be reused here. The solution of (16) requires $9n - 1$ additions/subtractions, $5n + 1$ multiplications, and $6n + 1$ divisions.

Table 1 shows that the proposed method for solving (16) is much faster than a standard linear solver, namely, the MATLAB sparse LU factorization with approximate minimum-degree reordering of columns. The proportionate speed-up seen here completely accounts for the superiority of our interior point method over the general breakpoint search (see §5).

Table 1: Solution time in ms for linear system (16).

solver	dimension				
	10 ²	10 ³	10 ⁴	10 ⁵	10 ⁶
proposed	0.03	0.08	0.50	5.53	69
LU colamd	0.85	3.25	43.16	526.98	5627

4 Test instances

For our computational tests, we selected five problem classes involving mathematical forms of potential

interest in operations research. None admits closed-form solutions for its separable breakpoint subproblems. Instances were generated so that assumptions A1–A2 of §2 were satisfied, after possible reorientation of intervals. In the following, the notation $z \sim U(a, b)$ indicates that the value z was selected according to a continuous uniform distribution on the open interval (a, b) , whereas $z \sim N(\mu, \sigma)$ indicates that z was selected according to a normal distribution with mean μ and standard deviation σ .

4.1 Resource renewal

Problems in this class have $f_i(x_i) = a_i x_i (e^{-1/x_i} - 1)$ and $g_i(x_i) = c_i x_i$ for $x_i > 0$, as studied by Melman and Rabinowitz [3]. For convenience, we extend f_i to a C^∞ convex function on the real line by defining $f_i(x_i) = -x_i$ for $x_i \leq 0$. Instances were generated as follows:

- $a_i, c_i \sim U(0.001, 1000)$;
- $b = 1.1 \sum_i c_i \xi_i$, where $\gamma = \min_j \{a_j/c_j\}$ and

$$\xi_i = \begin{cases} 0, & \text{if } a_i/c_i > \gamma, \\ \operatorname{argmin}_{x_i} f_i(x_i) + \gamma g_i(x_i), & \text{if } a_i/c_i \leq \gamma; \end{cases}$$

- $l_i = 0$ and $u_i = b/c_i$.

4.2 Weighted p -norm over a ball

Problems in this class have $f_i(x_i) = a_i |x_i - y_i|^p$ and $g_i(x_i) = |x_i|^r$. Note that f_i and g_i are everywhere twice differentiable when $p, r \geq 2$. Instances with $p, r \in \{2, 2.5, 3, 4\}$ and $p \neq r$ were generated as follows:

- $a_i \sim U(1, 10)$;
- $l_i \sim U(0, 5)$, $u_i \sim U(l_i, l_i + 5)$;
- $y_i \sim U(u_i, u_i + 5)$, $b \sim U(g(l), g(u))$.

4.3 Sums of powers

Problems in this class have $f_i(x_i) = a_i |x_i - y_i|^{p_i}$ and $g_i(x_i) = |x_i|^{r_i}$. Instances were generated as follows:

- $a_i \sim U(1, 10)$;
- $p_i, r_i \sim U(2, 4)$;
- $l_i \sim U(0, 5)$, $u_i \sim U(l_i, l_i + 5)$;
- $y_i \sim U(u_i, u_i + 5)$, $b \sim U(g(l), g(u))$.

4.4 Convex quartic over a simplex

This class of problems has $f_i(x_i) = a_i x_i^4 + b_i x_i^3 + c_i x_i^2 + d_i x_i$ and $g_i(x_i) = x_i$. Instances of these problems were generated as follows:

- $a_i = (\xi_i^2 + \eta_i^2)/\sqrt{8}$, $b_i = (\xi_i \zeta_i + \eta_i \chi_i)/\sqrt{3}$ and $c_i = (\zeta_i^2 + \chi_i^2)/\sqrt{8}$, with $\xi_i, \eta_i, \zeta_i, \chi_i \sim N(0, 1)$;
- $d_i = -f'_i(\tau_i)$, with $\tau_i \sim U(0, 10)$;
- $u_i = \min(\tau_i, \lambda_i)$, with $\lambda_i \sim U(0, \tau_i)$;
- $l_i \sim U(0, u_i)$;
- $b \sim U(g(l), g(u))$.

The choice of coefficients for f_i guarantees that f_i is strictly convex on the real line, which is true if and only if $8a_i c_i > 3b_i^2$, $a_i > 0$, and $c_i > 0$. Equivalently, the matrix

$$\begin{bmatrix} \sqrt{8}a_i & \sqrt{3}b_i \\ \sqrt{3}b_i & \sqrt{8}c_i \end{bmatrix}$$

must be positive definite. This can be ensured by selecting a_i, b_i, c_i to be the rescaled entries of a matrix formed as $A^T A$, where the entries of A are given by $\xi_i, \eta_i, \zeta_i, \chi_i$. The choice of d_i guarantees that the critical point of f_i is positive, after which the bounds are chosen so that each f_i has the same monotonicity.

4.5 Log-exponential

Problems in this class have $g_i(x_i) = c_i x_i$ and

$$f_i(x_i) = \ln \left[\sum_{j=1}^5 \exp(a_{ij} x_i + d_{ij}) \right].$$

Instances were generated as follows:

- $d_{ij} \sim N(0, 1)$ and $c_i \sim U(0, 10)$;
- $\xi_{ij} \sim N(0, 1)$ and $\zeta_i \sim \begin{cases} U(0, 1), & \text{if } i \in I, \\ N(0, 1), & \text{if } i \notin I; \end{cases}$
- $a_{ij} = \begin{cases} |\xi_{ij}|, & \text{if } \xi_{ij} > 0 \text{ for all } j, \\ \xi_{ij}, & \text{otherwise;} \end{cases}$
- $\chi_i = \operatorname{argmin} f_i$;
- $u_i = \begin{cases} \min(\chi_i, 1.2\zeta_i \chi_i), & \text{if } i \in I, \\ 5\zeta_i, & \text{if } i \notin I; \end{cases}$
- $l_i = u_i - 0.05|u_i| - 5|\eta_i|$, with $\eta_i \sim N(0, 1)$;
- $b \sim U(g(l), g(u))$.

4.6 A note on easier problems

In addition to the problems described above, we ran similar tests on randomly generated instances from several classes of problems admitting closed-form algebraic solutions to the either pegging or breakpoint subproblems. We don't report the results on these easier problems beyond the following brief summary. Unsurprisingly, problem-specific pegging methods (when available) handily outperformed all other approaches. However, the results were mixed concerning problem-specific breakpoint search versus the interior point method: the former was faster for low- to medium-dimensional instances but lost its edge as the dimension increased, so that the interior point method was generally faster for $n \geq 10^4$ or $n \geq 10^5$. Regardless of the problem class, the performance of the interior point method relative to the general breakpoint search of §3.1 was similar to the results described in §5 below.

5 Computational results

The procedures of §3 were coded in MATLAB 7.12 and their performance compared on randomly generated instances as described in §4. All tests were performed on a dedicated 2×quad-core Intel 64-bit (2.26 GHz) platform with 24GB RAM running CentOS Linux.

We attempted to solve instances of dimension 10^k for $k \in \{2, 3, 4, 5, 6\}$ with both methods for each problem class. One hundred random instances were generated at each dimension for four of the five problem classes. The exception was the class described in §4.2, for which we generated 100 instances at each dimension for each value of $p, r \in \{2, 2.5, 3, 4\}$. Performance differences among these combinations of p and r were detectable, but too small to warrant separating out the results for discussion. We therefore report them in aggregate over all p and r , and remark that the higher values of p or r tended to require slightly longer running times than did the smaller values.

Based on preliminary testing, an *a priori* time limit of 10^{k-2} seconds was imposed on each attempt at solution. The interior point method was run first and never exceeded this time limit. Consequently, for higher dimensional instances on some problem types, the breakpoint searches were limited to at most a factor of ten over the worst runtime for the interior point method on problems of the same type and size.

The results of the tests are presented by problem class in Figure 1. Because the breakpoint search often exceeded the given time limits, the graphs consist mainly of the median runtimes. Mean runtimes are

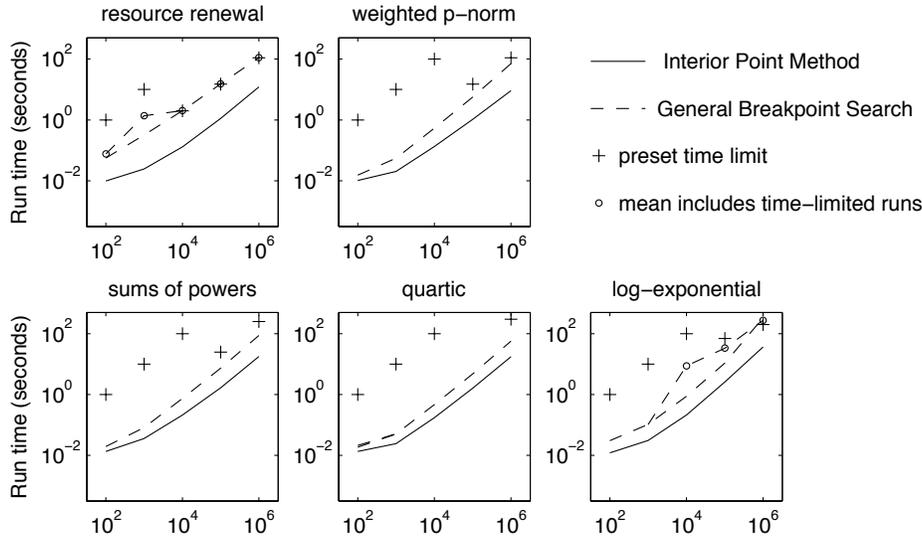


Figure 1: Mean and median running times (seconds) for three methods on ten problem classes.

drawn separately whenever they can be visually distinguished from the medians on the scale shown; the mean curves are always the upper branches when two curves of the same line type are shown. Means that include runtimes at their limits are specially marked.

The interior point method clearly dominates the general breakpoint search, often by an order of magnitude. Table 2 shows the frequency with which such dominance occurs. The common scale and position on the five graphs in Figure 1 suggest that running times for the interior point method do not depend greatly on the specific type problem (aside from the expense of function evaluations). On the other hand, the performance gap between the two optimization methods is smaller than the gap between the two linear-system solvers considered in §3.2. We conclude that when algebraic simplifications due to the form of f_i and g_i are unavailable, an interior point method is a strong option for solving problems of the form (1)–(3). However, its competitiveness relies even more heavily than usual on the efficiency of the underlying linear-system solver.

Table 2: Win percentage of IPM over general breakpoint method

problem class	dimension				
	10^2	10^3	10^4	10^5	10^6
resource renewal	0	73	86	100	100
weighted p -norm	4	95	99	100	100
sums of powers	4	97	100	100	100
quartic	86	100	100	100	97
log-exponential	100	100	100	100	100

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