# The Undirected Two Disjoint Shortest Paths Problem 

Marinus Gottschau ${ }^{1}$, Marcus Kaiser ${ }^{1}$, Clara Waldmann ${ }^{1}$


#### Abstract

The $k$ disjoint shortest paths problem ( $k$-DSPP) on a graph with $k$ source-sink pairs $\left(s_{i}, t_{i}\right)$ asks for the existence of $k$ pairwise edge- or vertex-disjoint shortest $s_{i}-t_{i}$-paths. It is known to be NP-complete if $k$ is part of the input. Restricting to 2-DSPP with strictly positive lengths, it becomes solvable in polynomial time. We extend this result by allowing zero edge lengths and give a polynomial time algorithm based on dynamic programming for 2-DSPP on undirected graphs ,with non-negative edge lengths.


Keywords: disjoint paths, disjoint shortest paths, dynamic programming, mixed graphs

## 1. Introduction

Due to many practical applications, e.g., in communication networks, the $k$ disjoint paths problem ( $k$-DPP) is a well 'studied problem in the literature. The input of the problem is an undirected graph $G=(V, E)$ as well as $k$ pairs 'of vertices $\left(s_{i}, t_{i}\right) \in V^{2}$ for $i \in[k]:=\{1, \ldots, k\}$ and the task is to decide whether there exist $k$ paths $P_{1}, \ldots, P_{k}$ such that $P_{i}$ is an $s_{i}-t_{i}$-path and all paths are pairwise disjoint. Here, disjoint can either mean vertex-disjoint or 'edge-disjoint.
The $k$ disjoint shortest path problem ( $k$-DSPP) is a gen'eralization of the $k$ disjoint paths problem. The input of the problem is an undirected graph $G=(V, E)$ with edge lengths $\ell: E \rightarrow \mathbb{R}$ and $k$ pairs of vertices $\left(s_{i}, t_{i}\right) \in V^{2}$ for $i \in[k]$. But here, all paths $P_{i}$ for $i \in[k]$ are additionally required to be shortest $s_{i}-t_{i}$-paths. Note, if $\ell \equiv 0$, this agrees with $k$-DPP.
We shall refer to the versions of the problems in directed graphs by $k$-dDPP and $k$-dDSPP.

### 1.1. Related Work

'Probably most famously, Menger's theorem [9] deals with disjoint paths which gave rise to one of the most fundamental results for network flows: the max-flow-min-cut theorem [4, 6]. Using these results, an application of any flow algorithm solves the $k$-dDPP if $s_{i}=s_{j}$ for all $i, j \in[k]$ or $t_{i}=t_{j}$ for all $i, j \in[k]$. Without restrictions on the input instances, all variants of the discussed problems are NP-complete if $k$ is considered part of the input [5, 8].
Due to this, a lot of research focuses on the setting where $k$ is considered fixed. Robertson and Seymour [10] came

[^0]up with an $\mathcal{O}\left(|V|^{3}\right)$ algorithm for $k$-DPP.
In contrast to that, Fortune et al. [7] prove that $k$-dDPP is still NP-hard, even if $k=2$. They give an algorithm that solves $k$-dDPP for any fixed $k$ on directed acyclic graphs in polynomial time. Zhang and Nagamochi [12] then extended the work of Fortune et al. [7] to solve the problem on acyclic mixed graphs, which are graphs that contain arcs and edges where directing any set of edges does not close a directed cycle.
Since $k$-dDSPP and $k$-dDPP agree for $\ell \equiv 0$, all hardness results carry over. However, if all edge lengths are strictly positive Bérczi and Kobayashi [1] give a polynomial time algorithm for 2-dDSPP. Also, for 2-DSPP with strictly positive edge lengths a polynomial time algorithm is due to Eilam-Tzoreff [3]. However, the complexity of $k$-DSPP on undirected graphs with non-negative edge lengths and constant $k \geq 2$ is unknown. We settle the case $k=2$ in this paper.
Other than restricting the paths to be shortest $s_{i}-t_{i}$-paths, e.g., Suurballe [11] gave a polynomial time algorithm minimizing the total length, if all arc lengths are non-negative and $s_{i}=s_{j}, t_{i}=t_{j}$ for all $i, j \in[k]$. Björklund and Husfeldt [2] came up with a polynomial time algebraic Monte Carlo algorithm for solving 2-DPP with unit lengths where the total length of the paths is minimized.

| $k$ | $\ell \equiv 0$ |  | $\ell$ non-negative |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $k$-DPP | $k$-dDPP | $k$-DSPP | $k$-dDSPP |
| arb. | NP-hard [5, 8] | NP-hard [5] | NP-hard [3] | NP-hard [3] |
| fixed | P [10] | NP-hard [7] | open ( $\ell>0$ ) | open ( $\ell>0$ ) |
|  |  |  | open ( $\ell \geq 0$ ) | NP-hard ( $\ell \geq 0$ ) [7] |
| 2 | P [10] | NP-hard [7] | $\mathrm{P}(\ell>0)[3]$ | $\mathrm{P}(\ell>0)[1]$ |
|  |  |  | $\mathrm{P}(\ell \geq 0)$ * | NP-hard ( $\ell \geq 0$ ) [7] |

Table 1: Complexity of the disjoint paths problem and its variants.

* A polynomial time algorithm for the 2-DSPP on undirected graphs with non-negative edge lengths is the main result of this paper.


### 1.2. Our Results

We give a polynomial time algorithm for 2-DSPP on undirected graphs with non-negative edge lengths. Combining techniques from [7] and [1] enables us to deal with edges of length zero. We consider the following problem.

Problem 1 (Undirected Two Edge-Disjoint Shortest Paths Problem)
Input: An undirected graph $G=(V, E)$ with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$, a tuple of sources $s \in V^{2}$, and a tuple of sinks $t \in V^{2}$

Task: Decide whether there exist two edge-disjoint paths $P_{1}$ and $P_{2}$ in $G$ such that $P_{1}$ is a shortest $s_{1}-t_{1}$-path and $P_{2}$ is a shortest $s_{2}-t_{2}$-path w.r.t. the edge lengths $\ell$.

Our paper is organized as follows. In Section 2, based on the ideas of [7], we give a dynamic algorithm that solves the $k$-DPP in polynomial time on weakly acyclic mixed graphs, which are a generalization of directed acyclic graphs.
These results are then used in Section 3 together with a similar approach as in [1] to solve the undirected 2-DSPP with non-negative edge lengths in polynomial time.

The results of this paper have been obtained independently by Kobayashi and Sako.

## 2. Disjoint Paths in Weakly Acyclic Mixed Graphs

In this section, we give an algorithm that solves $k$-DPP in a generalization of directed acyclic graphs. We first define mixed graphs, introduce some notations, and state the problem.

A graph $G=(V, A \cup E)$ is a mixed graph on the vertex set $V$ with arc set $A \subseteq V^{2}$ and edge set $E \subseteq\binom{V}{2}$. We define $\nVdash(G):=A \cup E$. The set of ingoing (outgoing) arcs of a set of vertices $W \subseteq V$ is denoted by $\delta_{A}^{-}(W)\left(\delta_{A}^{+}(W)\right)$.
For pairwise disjoint vertex sets $W_{1}, \ldots, W_{h}$, we denote by $G /\left\{W_{1}, \ldots, W_{h}\right\}$ the graph that results from $G$ by contracting $W_{1}, \ldots, W_{h}$ into $h$ vertices.

A (directed) $u-w$-path $P$ in $G$ is a sequence of $h \operatorname{arcs}$ and edges $\left(æ_{1}, \ldots, æ_{h}\right) \in \mathbb{E}^{h}$ such that there exists a sequence of vertices $\left(u=v_{1}, \ldots, v_{h+1}=w\right) \in V^{h+1}$ satisfying either $æ_{i}=\left(v_{i}, v_{i+1}\right)$ or $æ_{i}=\left\{v_{i}, v_{i+1}\right\}$ for all $i \in[h]$. Two paths are arc/edge-disjoint (vertex-disjoint) if they do not have a common arc or edge (vertex).
Note that a directed acyclic graph induces natural orderings of its vertices. A linear ordering of the vertices is called a topological ordering if, for every $\operatorname{arc}(v, w)$, the tail $v$ precedes the head $w$ in the ordering. An ordering is called a reverse topological ordering if its reverse ordering is a topological ordering.
On a ground set $U$, a binary relation $R$ is a subset of $U^{2}$. For $(u, v) \in R$, we write $u R v$. A relation $R$ is called
reflexive, if $u R u$ holds for all $u \in U$. For two binary relations $R, S \subseteq U^{2}$, the composition $S \circ R$ is defined by $\left\{(u, w) \in U^{2} \mid \exists v \in U: u R v \wedge v S w\right\}$. Note that $\circ$ is an associative operator.

We consider the following problem for fixed $k$.
Problem 2 (Mixed $k$ Arc/Edge-Disjoint Paths Problem) Input: A mixed graph $G=(V, E)$, a $k$-tuple of sources $s \in V^{k}$, and a $k$-tuple of sinks $t \in V^{k}$

Task: Decide whether there exist $k$ pairwise arc/edgedisjoint paths $P_{1}, \ldots, P_{k}$ in $G$ such that $P_{i}$ is an $s_{i}-t_{i}$-path, for all $i \in[k]$.

We give an algorithm that solves this problem on a class of mixed graphs, that generalize directed acyclic graphs:

Definition 1 (Weakly Acyclic Mixed Graphs)
We call a mixed graph $G=(V, A \cup E)$ weakly acyclic if the contraction of all edges $E$ yields a directed acyclic graph without loops.

Note that a weakly acyclic mixed graph can contain (undirected) cycles in its edge set.
For a mixed graph $G=(V, \nVdash)$, we use the following notation in order to discuss the existence of disjoint paths.

Definition 2 (Arc/Edge-Disjoint Paths Relation)
For $k \in \mathbb{N}$, we define the binary relation $\exists_{E}$ on the set $V^{k}$ as follows. For $v, w \in V^{k}$, we have $v \rightrightarrows_{A} w$ if there exist pairwise arc/edge-disjoint $v_{i}$ - $w_{i}$-paths for all $i \in[k]$ in $\notin$. We will also write $\rightrightarrows_{G}$ short for $\rightrightarrows_{E(G)}$.

Since paths of length zero are allowed, the relation $\rightrightarrows_{\notin}$ is reflexive. In general, it is not transitive. When considering two relations based on two disjoint sets of arcs and edges, however, these two act in a transitive manner. In that case, the respective underlying arc/edge-disjoint paths from both relations can be concatenated. The resulting arc/edge-disjoint paths correspond to an element in the composition of the two relations.

Observation 3 (Partial Transitivity)
For disjoint arc/edge sets $\mathscr{E}_{1}, E_{2} \subseteq \notin$ and vectors of vertices $u, v, w \in V^{k}$, it holds

$$
u \rightrightarrows_{E_{1}} v \wedge v \rightrightarrows_{E_{2}} w \Longrightarrow u \rightrightarrows_{E_{1} \cup E_{2}} w
$$

This observation is exploited in Algorithm 1 in order to solve Problem 2 for fixed $k$ for weakly acyclic mixed graphs. It computes the relation $\rightrightarrows_{G}$ in polynomial time by dealing with the edges and arcs in $G$ separately.
For the undirected components, i.e., the connected components of the subgraph $(V, E)$, it uses an algorithm for edge-disjoint paths in undirected graphs (e.g., [10]) to find the relation $\rightrightarrows$ on each component.

## Algorithm 1: Dynamic Program for $k$-DPP in Weakly Acyclic Mixed Graphs

Input: weakly acyclic mixed graph $G=(V, A \cup E)$
Output: $\rightrightarrows_{G}$ on $V^{k}$
1 Find connected components $V_{1}, \ldots, V_{h}$ of the subgraph $(V, E)$ sorted according to a topological ordering of $G /\left\{V_{1}, \ldots, V_{h}\right\}$;
2 for $j=1, \ldots, h$ do
Compute $\rightrightarrows_{G\left[V_{j}\right]}$ using an algorithm for $k$-DPP;
3 Initialize $\rightrightarrows$ to the relation $\left\{(v, v) \mid v \in V^{k}\right\}$;
4 for $j=1, \ldots, h$ do

$$
\text { Update } \rightrightarrows \text { to } \rightrightarrows_{G\left[V_{j}\right]}^{\circ} \rightrightarrows_{\delta_{A}^{-}\left(V_{j}\right)} \circ \rightrightarrows ;
$$

5 return $\rightrightarrows$


Figure 1: In iteration $j$ of Algorithm 1, relation $\rightrightarrows^{j}$ is built by concatenating previously computed paths $\left(\rightrightarrows^{j-1}\right)$, pairwise different arcs to the next component $\left(\rightrightarrows_{\delta_{A}^{-}\left(V_{j}\right)}\right)$, and undirected edge-disjoint paths in the next component $\left(\rightrightarrows_{G\left[V_{j}\right]}\right)$.

Afterwards, dynamic programming is used to compute $\rightrightarrows$ on successively larger parts of the mixed graph. As $G$ is weakly acyclic, contracting all undirected components results in an acyclic graph. The algorithm iterates over the components in a topological ordering. Based on Observation 3, previously found arc/edge-disjoint paths are extended alternately by arcs between components and edgedisjoint paths within one component. This approach is a generalization of the methods presented in [7].

Theorem 4 (Algorithm 1: Correctness and Running Time) Let $k \in \mathbb{N}$ be fixed. Given a weakly acyclic mixed graph $G=(V, A \cup E)$, Algorithm 1 computes the relation $\rightrightarrows_{G}$ on $V^{k}$ in polynomial time.

Proof. Let $V=\bigcup_{j=1}^{h} V_{j}$ be the partition of $V$ into the vertex sets of the $h$ connected components of $(V, E)$ as computed by the algorithm.
For all $j \in\{0, \ldots, h\}$, let $Æ_{j}$ be the arc and edge set of $G\left[\bigcup_{l=1}^{j} V_{l}\right]$. In particular, $Æ_{0}=\emptyset$ holds true. For each $j \in\{0, \ldots, h\}$, let $\rightrightarrows^{j}$ be the relation $\rightrightarrows$ as computed by Algorithm 1 after the $j$-th iteration of Line 4 . In
particular, $\rightrightarrows^{0}$ is the relation after Line 3. In the following, we proof by induction on $j$ that $\rightrightarrows^{j}$ is equal to $\rightrightarrows_{\Phi_{j}}$.
After the initialization, this is true for $j=0$, as $Æ_{0}$ contains no arcs or edges. Consider an iteration $j \in[h]$ and assume that the claim was true after the previous iteration.
$" \subseteq$ ": Let $v, w \in V^{k}$ such that $v \rightrightarrows{ }^{j} w$. There exist $p, q \in V^{k}$ such that $v \rightrightarrows^{j-1} p \not \rightrightarrows_{\delta_{A}^{-}\left(V_{j}\right)} q \rightrightarrows_{G\left[V_{j}\right]} w$. Using the induction hypothesis, we know $v \rightrightarrows_{\Phi_{j-1}} p$. Since the arc and edge sets in the three relations are pairwise disjoint, Observation 3 yields $v \rightrightarrows_{Æ_{j}} w$.
$" \supseteq ":$ Let $v, w \in V^{k}$ with $v \rightrightarrows_{\oiint_{j}} w$, and $P_{i}, i \in[k]$ be arc/edge-disjoint $v_{i}$ - $w_{i}$-paths in $Æ_{j}$. Let $q_{i} \in V$ be the first vertex on $P_{i}$ in $V_{i}$ and $p_{i}$ be its predecessor if they exist, otherwise set them to $w_{i}$ and $q_{i}$, respectively. As $G$ is weakly acyclic, we have $v \not \rightrightarrows_{\mathbb{E}_{j-1}} p, p \not \rightrightarrows_{\delta_{A}^{-}\left(V_{j}\right)} q$, as well as $q \not \rightrightarrows_{G\left[V_{j}\right]} w$. It follows from the induction hypothesis that $v \not \rightrightarrows^{j} w$.

The connected components of $(G, E)$ and their topological ordering in $G /\left\{V_{1}, \ldots, V_{h}\right\}$ can be computed in polynomial time. Finding edge-disjoint paths in the undirected components can also be done efficiently (e.g., [10]). A binary relation on $V^{k}$ contains at most $|V|^{2 k}$ elements and composing two of them can be done in time polynomial in their sizes. Hence, Algorithm 1 runs in time polynomial in the size of the input if $k$ is fixed.

In many settings, the problem of finding arc/edge-disjoint paths can be reduced to finding vertex-disjoint paths. Observe that arc/edge-disjoint paths in a graph correspond to the vertex-disjoint paths in its line graph and an appropriate notion of a line graph can be defined for mixed graphs as well.
For directed graphs, there is a generic reduction from vertexdisjoint to arc-disjoint instances based on splitting vertices. This reduction, however, cannot be applied to undirected or mixed graphs. Yet, Algorithm 1 can be modified slightly as follows to compute vertex-disjoint paths. An algorithm for the undirected vertex-disjoint path problem is used in Line 2. Only vectors with pairwise different elements are included in the initial relation in Line 3. Finally in Line 4, tuples $v, w \in V^{k}$ are related only if their sets of endpoints $\left\{v_{i}, w_{i}\right\}, i \in[k]$ are pairwise disjoint.

## 3. Undirected Disjoint Shortest Paths

In this section, we study Problem 1 on undirected graphs with non-negative edge lengths. We first transform the undirected graph $G$ into a mixed graph and then use the results of the previous section to solve the transformed instance.

### 3.1. From Shortest to Directed Paths

Let an instance of Problem 1 be given by an undirected graph $G=(V, E)$, non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$, and $s, t \in V^{2}$. We are going to transform the graph $G$ into a mixed graph such that the shortest source-sink-paths in $G$ correspond to directed source-sink-paths in the resulting mixed graph.

Since we are interested in shortest $s_{1}-t_{1}$ - and $s_{2}-t_{2}$-paths, we consider the shortest path networks rooted at $s_{1}$ and $s_{2}$. For $i \in[2]$, we define the distance function $d_{i}: V \rightarrow \mathbb{R}_{\geq 0}$ induced by $\ell$ w.r.t. $s_{i}$ by $d_{i}(v):=\min _{s_{i}-v \text {-path } P} \sum_{e \in P} \ell(e)$. The shortest path network rooted at $s_{i}$ is given by the set

$$
E_{i}:=\left\{\{v, w\} \in E\left|\ell(\{v, w\})=\left|d_{i}(v)-d_{i}(w)\right|\right\} .\right.
$$

See Figure 3a for an example of the sets $E_{i}$.
The distances $d_{i}$ induce an orientation for all edges in $E_{i}$ which have a strictly positive length. We would like to replace an edge $\{v, w\} \in E$ with $d_{i}(v)<d_{i}(w)$ by the $\operatorname{arc}(v, w)$ (with the same length). The orientations induced by $d_{1}$ and $d_{2}$, however, do not have to agree on the set $E_{1} \cap E_{2}$. Introducing both arcs would neglect the fact that only one of them can be included in any set of arc/edge-disjoint paths. We will overcome this by replacing such edges by a standard gadget of directed arcs as depicted in Figure 2.


Figure 2: Gadget for resolving conflicts during the orientation of an edge $\{v, w\}$ induced by $d_{1}$ and $d_{2}$.

Consider the gadget for an edge $\{v, w\} \in E$. It contains exactly one $v-w$-path and one $w-v$-path corresponding to the two possible orientations of $\{v, w\}$. Since both share an arc, only one of two arc/edge-disjoint paths in the transformed graph can use the gadget. As further both paths consist of three arcs, setting the length of all the arcs in the gadget to $\frac{1}{3} \ell(\{v, w\})$ preserves the distances in the graph. That way, the distance functions $d_{i}$ can be extended to the new vertices introduced with gadgets.
For $i \in[2], A_{i}$ denotes the set of arcs that result from orienting $E_{i}$ w.r.t. $d_{i}$. More precisely, for $\{v, w\} \in E_{i}$ with $d_{i}(v)<d_{i}(w)$ the $\operatorname{arc}(v, w)$ is included into the set $A_{i}$ if $\{v, w\} \in E_{1} \triangle E_{2}$ or the orientation induced by $d_{1}$ and $d_{2}$ agree. Otherwise, the arcs of the $v-w$-path in the gadget replacing $\{v, w\}$ are added to $A_{i}$.
The induced orientation is only well-defined for edges with strictly positive lengths. Therefore, the set of edges with length zero $E_{0}:=\{e \in E \mid \ell(e)=0\}$ are left undirected and have to be treated in a different manner.

(a) Example: edges without label have length 1, solid edges are in $E_{1} \cup E_{2}$

(b) Partially oriented expansion of (a): solid arcs are in $A_{1} \cap A_{2}$, dashed arcs are in $A_{1} \backslash A_{2}$, dotted arcs are in $A_{2} \backslash A_{1}$

Figure 3: Exemplary construction of partially oriented expansion

Definition 5 (Partially Oriented Expansion)
Let $G=(V, E)$ be an undirected graph with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$ and $s \in V^{2}$.

The partially oriented expansion of $G$ w.r.t. $\ell$ and $s$ is the graph $\vec{G}:=\left(W, E_{0} \cup A_{1} \cup A_{2}\right)$ where $W$ is the set of vertices $V$ augmented with additional vertices introduced with gadgets, and $E_{0}, A_{1}$, and $A_{2}$ are as defined above.

The partially oriented expansion of the example from Figure 3a is depicted in Figure 3b. As we are going to discuss the existence of shortest edge-disjoint paths in $G$ and the existence of arc/edge-disjoint paths restricted to different arc and edge sets in $\vec{G}$, the following notation will be useful.

## Definition 6 (Two Disjoint Paths Relations)

i) $\operatorname{Let} G=(V, E)$ be an undirected graph with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$.
For $v, w \in V^{2}$, we write $v \stackrel{\ell}{\rightrightarrows}_{E} w$ if there exist edge-disjoint shortest $v_{i}-w_{i}$-paths w.r.t. $\ell$ for $i \in[2]$ in $E$.
ii) Let $G=(V, A \cup E)$ be a mixed graph and let $E_{1}, E_{2}$ be two subsets of arcs and edges of $A \cup E$.
For $v, w \in V^{2}$, we write $v \rightleftarrows{ }_{A_{2}}^{\mathscr{H}_{1}} w$ if there exist a $v_{1}-w_{1}-p a t h$ in $\mathscr{E}_{1}$ and a $w_{2}-v_{2}$-path in $E_{2}^{2}$ which are arc/edge-disjoint.

As described above, the distance functions of the original graph $G$ extend to the vertices of $\vec{G}$. For $i \in[2]$ and $v \in W$, $d_{i}(v)$ is the length of a shortest $s_{i}-v$-path in $\vec{G}$.

Lemma 7 (Paths in the Partially Oriented Expansion) Let $G=(V, E)$ be an undirected graph with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$ and $s \in V^{2}$. Furthermore, let $\vec{G}=\left(W, E_{0} \cup A_{1} \cup A_{2}\right)$ be the partially oriented expansion of $G$ w.r.t. $\ell$ and s.
Then for every $t \in V^{2}$, we have $s \stackrel{\ell}{\rightrightarrows}_{E} t$ in $G$ if and only if $\binom{s_{1}}{t_{2}} \rightleftarrows{ }_{E_{0} \cup A_{2}}^{E_{0} \cup A_{1}}\binom{t_{1}}{s_{2}}$ in $\stackrel{\rightharpoonup}{G}$.

Proof. " $\Rightarrow$ ": Assume there exist two edge-disjoint shortest $s_{i}-t_{i}$-paths $P_{i}$ in $E_{i}$ for $i \in[2]$. Replace each edge with non-zero length in $P_{i}$ by the respective oriented arc or path in the respective gadget to obtain $\vec{P}_{i}$ in $E_{0} \cup A_{i}$. $\vec{P}_{1}$ and $\vec{P}_{2}$ are arc/edge-disjoint as different edges are replaced by disjoint (sets of) arcs.
" $\Leftarrow$ ": Assume there are arc/edge-disjoint $s_{i}-t_{i}$-paths $\vec{P}_{i}$ in $E_{0} \cup A_{i}$ for $i \in[2]$. Replace the subpath of $P_{i}$ within one gadget with the corresponding edge in $E_{i}$. The remaining arcs are translated directly to the respective edges in $E_{i}$. Due to the mentioned equality of distances in $G$ and $\vec{G}$ and the fact that $d_{i}$ is non-decreasing along arcs in $\vec{G}$, $P_{i}$ is a shortest path in $G$. Any path that uses a gadget in $\vec{G}$, uses its inner arc. Therefore, $P_{1}$ and $P_{2}$ inherit being edge-disjoint from $\vec{P}_{1}$ and $\vec{P}_{2}$.

### 3.2. Disjoint Paths in the Partially Oriented Expansion

Lemma 7 shows that $\vec{G}$ captures the shortest paths in $G$ by using orientation. We will use the distances, however, to prove the main structural result. It concern the subgraph of $\vec{G}$ potentially used by both paths and its weakly connected components, which are its connected components when ignoring the arcs' directions.

Lemma 8 (Structure of Partially Oriented Expansion) Let $G=(V, E)$ be an undirected graph with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$ and $s \in V^{2}$. Furthermore, let $\vec{G}=\left(W, E_{0} \cup A_{1} \cup A_{2}\right)$ be the partially oriented expansion of $G$ w.r.t. $\ell$ and s. Let $W=\bigcup_{j=1}^{h} W_{j}$ be the partition of $W$ into the vertex sets of the $h$ weakly connected components of the subgraph $\left(W, E_{0} \cup\left(A_{1} \cap A_{2}\right)\right)$.

## Then

i) $\stackrel{\rightharpoonup}{G}\left[W_{j}\right]$ is weakly acyclic for all $j \in[h]$,
ii) sorting the components $W_{j}, j \in[h]$ in non-decreasing order w.r.t. the function $d_{1}-d_{2}$ is a topological ordering of $\left(W, A_{1}\right) /\left\{W_{1}, \ldots, W_{h}\right\}$ and a reverse topological ordering of $\left(W, A_{2}\right) /\left\{W_{1}, \ldots, W_{h}\right\}$, and
iii) $\vec{G}\left[W_{j}\right]$ contains arcs only from $A_{1} \cap A_{2}$ and edges only from $E_{0}$ for all $j \in[h]$.

Proof. i) By definition of $A_{1}$, we know that $d_{1}$ increases strictly along arcs in $A_{1} \cap A_{2}$. Further, $d_{1}$ is constant on edges in $E_{0}$. Assume there is $j \in[h]$ and a (directed) cycle $C$ in $\stackrel{\rightharpoonup}{G}\left[W_{j}\right]$ such that there exists $a \in C \cap A_{1} \cap A_{2}$.

Along of $a$ the distance $d_{1}$ strictly increases. However, $d_{1}$ cannot decrease along $C$, which yields a contradiction.
ii) Consider the function on the vertex set of $\vec{G}$. Based on the common underlying lengths in $G$ and the definitions of $A_{1}$ and $A_{2}$, it is strictly increasing along arcs in $A_{1} \backslash A_{2}$ and strictly decreasing along arcs in $A_{2} \backslash A_{1}$. Opposed to that, it is constant on edges in $E_{0}$ as well as along of arcs in $A_{1} \cap A_{2}$.
iii) The function $d_{1}-d_{2}$ is constant along all $\operatorname{arcs} A_{1} \cap A_{2}$ and edges in $E_{0}$. Hence, it is constant on each weakly connected component w.r.t. those arcs and edges. At the same time, the function is not constant along arcs in $A_{1} \triangle A_{2}$.

This structural result allows to use dynamic programming for solving Problem 2 on the partially oriented expansion. Similar to Section 2, the problem is split into two parts. First, the two arc/edge-disjoint paths problem on the weakly connected components $W_{1}, \ldots, W_{h}$ of the subgraph $\left(W, E_{0} \cup\left(A_{1} \cap A_{2}\right)\right)$ is solved by Algorithm 1. Afterwards, a dynamic program is used to incorporate the results into arc-disjoint paths in $\vec{G} /\left\{W_{1}, \ldots, W_{h}\right\}$ to get arc/edge-disjoint paths in $\vec{G}$.

We know that the two arc/edge-disjoint paths that we are looking for, if they exist, pass through $\vec{G} /\left\{W_{1}, \ldots, W_{h}\right\}$ in opposite directions. In order to accomplish simultaneous construction of both, one of the paths is created backwards. Apart from that, Algorithm 2 resembles Algorithm 1.

## Algorithm 2: Dynamic Program for 2-DSPP with nonnegative edge lengths

Input: undirected graph $G=(V, E)$, non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}, s \in V^{2}$
Output: set of pairs in $V^{2}$ that succeed $s$ w.r.t. $\stackrel{\ell}{\rightrightarrows}_{E}$
${ }_{1}$ Construct $\stackrel{\rightharpoonup}{G}=\left(W, E_{0} \cup A_{1} \cup A_{2}\right)$ for $G$ w.r.t. $\ell$ and $s$;
2 Find weakly connected components $W_{1}, \ldots, W_{h}$ of the subgraph $\left(W, E_{0} \cup\left(A_{1} \cap A_{2}\right)\right)$ sorted non-decreasingly w.r.t. $d_{1}-d_{2}$;

3 for $j=1, \ldots, h$ do
Compute $\rightleftarrows \underset{\vec{G}\left[W_{j}\right]}{\stackrel{\rightharpoonup}{G}\left[W_{j}\right]}$ using Algorithm 1;
4 Initialize $\rightleftarrows$ to the relation $\left\{(v, v) \mid v \in W^{2}\right\}$;
5 for $j=1, \ldots, h$ do
Update $\rightleftarrows$ to $\rightleftarrows \stackrel{\stackrel{\rightharpoonup}{G}\left[W_{j}\right]}{\vec{G}\left[W_{j}\right]} \circ \stackrel{\underbrace{+}_{\delta_{A_{2}}}\left(W_{j}\right)}{\delta_{A_{1}}^{-}\left(W_{j}\right)} \circ \rightleftarrows ;$
6 return $\left\{t \in V^{2} \left\lvert\,\binom{ s_{1}}{t_{2}} \rightleftarrows\binom{t_{1}}{s_{2}}\right.\right\}$

Theorem 9 (Algorithm 2: Correctness and Running Time) Given an undirected graph $G=(V, E)$ with non-negative edge lengths $\ell: E \rightarrow \mathbb{R}_{\geq 0}$ and $s \in V^{2}$, Algorithm 2 computes all successors of $s$ w.r.t. $\stackrel{\ell}{\rightrightarrows}_{E}$ in polynomial time.

Proof. Let $W=\bigcup_{j=1}^{h} W_{j}$ be the partition of $W$ into the vertex sets of the $h$ weakly connected components of the subgraph ( $W, E_{0} \cup\left(A_{1} \cap A_{2}\right)$ ) as computed by the algorithm. Lemma 8 ii) shows that the $W_{j}$ 's are sorted in a topological ordering of $\left(W, A_{1}\right) /\left\{W_{1}, \ldots, W_{h}\right\}$ and in a reverse topological ordering of $\left(W, A_{2}\right) /\left\{W_{1}, \ldots, W_{h}\right\}$.

For $i \in[2]$ and $j \in\{0, \ldots, h\}$, set $\mathbb{Æ}_{i}^{j}$ to be the $\operatorname{arcs}$ of $A_{i}$ and edges of $E_{0}$ in the induced subgraph $\vec{G}\left[\bigcup_{l=1}^{j} W_{l}\right]$. In particular, we have $Æ_{i}^{0}=\emptyset$. For $j \in[h]$, let $\rightleftarrows^{j}$ denote the relation $\rightleftarrows$ computed by Algorithm 2 after the $j$-th iteration. In particular, $\rightleftarrows^{0}$ is as defined in Line 4. We will prove by induction on $j=0, \ldots, h$ that $\rightleftarrows^{j}$ is equal to $\underset{\Phi_{2}^{j}}{\rightleftarrows \Phi_{1}^{j}}$. The correctness of the algorithm then follows from Lemma 7.
The claim holds for $j=0$, since $\mathbb{E}_{1}^{0}=Æ_{2}^{0}=\emptyset$ by definition. Consider iteration $j \in[h]$ and assume that the claim holds for the preceding iteration.
$" \subseteq "$ Let $v, w \in W^{2}$ such that $v \rightleftarrows^{j} w$. Considering Line 5 and using induction hypothesis, there exist $p, q \in W^{2}$ with

$$
v \rightleftarrows \underset{E_{2}^{j-1}}{\mathbb{E}_{1}^{j-1}} p \rightleftarrows{ }_{\delta_{A_{2}}^{+}\left(W_{j}\right)}^{\delta_{-}^{-}\left(W_{j}\right)} q \underset{\vec{G}\left[W_{j}\right]}{\stackrel{\rightharpoonup}{G}\left[W_{j}\right]} w .
$$

Lemma 8 iii) guarantees that the arc and edge sets of the three relations are pairwise disjoint. As a result, $v \not \rightleftarrows_{\Phi_{2}^{j}}^{\Phi_{1}^{j}} w$ follows from Observation 3.
$" \supseteq ":$ Let $v, w \in W^{2}$ such that $v \rightleftarrows{ }_{\AA_{2}^{j}}^{\AA_{1}^{j}} w$. Thus, there have to be a simple $v_{1}-w_{1}$-path $P_{1}$ in $\mathbb{E}_{1}^{j}$ and a simple $w_{2}-v_{2}$-path $P_{2}$ in $\mathbb{E}_{2}^{j}$ that are arc/edge-disjoint. Define $q_{1} \in W$ to be the first vertex on $P_{1}$ in $W_{j}$, if it exists, or $w_{1}$. Let $p_{1}$ be the predecessor of $q_{1}$ on $P_{1}$ or $q_{1}$ if it is the first vertex of $P_{1}$. Similarly, let $q_{2} \in W$ be the last vertex on $P_{2}$ in $W_{j}$ or $w_{2}$ if it does not exist, and let $p_{2}$ be the successor of $q_{2}$ or $q_{2}$ if $q_{2}$ does not have a successor. The topological ordering of the $W_{j}$ 's implies that the subpaths of $P_{1}$ and $P_{2}$ prove

$$
v \rightleftarrows \underset{\mathbb{E}_{2}^{j-1}}{\mathbb{E}_{1}^{j-1}} p \rightleftarrows{ }_{\delta_{A_{2}}^{+}\left(W_{j}\right)}^{\delta_{-}^{-}\left(W_{j}\right)} q \rightleftarrows \frac{\vec{G}\left[W_{j}\right]}{\vec{G}\left[W_{j}\right]} w .
$$

Finally, $v \rightleftarrows^{j} w$ follows by induction hypothesis.
As for the running time, finding the weakly connected components and sorting them in a topological ordering can be done in polynomial time. Computing the relations $\rightleftarrows \underset{\vec{G}\left[W_{j}\right]}{\vec{G}\left[W_{j}\right]}$ also can be done efficiently by virtue of Algorithm 1. Finally, relations on $V^{2}$ have at most $|V|^{4}$ elements and can be composed efficiently. Therefore, the total running time of the algorithm is polynomial in the input size.


Figure 4: Iteration $j$ of Algorithm 2: relation $\rightleftarrows^{j}$ is built by concatenating already computed paths, pairwise different arcs to the next component, and arc/edge-disjoint paths in the next mixed component

Similar to Section 2, Algorithm 2 can be adapted to check for the existence of two vertex-disjoint shortest paths. In that case, the gadget from Figure 2 is not needed anymore, but can be replaced by two opposite arcs.

## Acknowledgments

This work has been supported by the Alexander von Humboldt Foundation with funds from the German Federal Ministry of Education and Research (BMBF). Additionally, we want to thank Jannik Matuschke for his valuable comments and helpful discussions.

## References

[1] Bérczi, K., Kobayashi, Y., 2017. The directed disjoint shortest paths problem, in: Pruhs, K., Sohler, C. (Eds.), 25th Annual European Symposium on Algorithms (ESA 2017), Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Dagstuhl, Germany. pp. 13:1-13:13.
[2] Björklund, A., Husfeldt, T., 2014. Shortest two disjoint paths in polynomial time, in: Esparza, J., Fraigniaud, P., Husfeldt, T., Koutsoupias, E. (Eds.), Automata, Languages, and Programming, Springer Berlin Heidelberg, Berlin, Heidelberg. pp. 211-222.
[3] Eilam-Tzoreff, T., 1998. The disjoint shortest paths problem. Discrete Applied Mathematics 85, 113-138.
[4] Elias, P., Feinstein, A., Shannon, C., 1956. A note on the maximum flow through a network. IRE Transactions on Information Theory 2, 117-119.
[5] Even, S., Itai, A., Shamir, A., 1976. On the complexity of timetable and multicommodity flow problems. SIAM Journal on Computing 5, 691-703.
[6] Ford, Jr., L.R., Fulkerson, D.R., 1956. Maximal flow through a network. Canadian Journal of Mathematics 8, 399-404.
[7] Fortune, S., Hopcroft, J., Wyllie, J., 1980. The directed subgraph homeomorphism problem. Theoretical Computer Science 10, 111-121.
[8] Karp, R.M., 1975. On the computational complexity of combinatorial problems. Networks 5, 45-68.
[9] Menger, K., 1927. Zur allgemeinen Kurventheorie. Fundamenta Mathematicae 10, 96-115.
[10] Robertson, N., Seymour, P.D., 1995. Graph minors .XIII. The disjoint paths problem. Journal of Combinatorial Theory, Series B 63, 65-110.
[11] Suurballe, J.W., 1974. Disjoint paths in a network. Networks 4, 125-145.
[12] Zhang, C., Nagamochi, H., 2012. The next-to-shortest path in undirected graphs with nonnegative weights, in: Mestre, J. (Ed.), Computing: The Australasian Theory Symposium (CATS 2012), Australian Computer Society, Melbourne, Australia. pp. 13-20.


[^0]:    Email addresses: marinus.gottschau@tum.de,
    marcus.kaiser@tum.de, clara.waldmann@tum.de
    ${ }^{1}$ Department of Mathematics, Technische Universität München.

