The Undirected Two Disjoint Shortest Paths Problem

Marinus Gottschau¹, Marcus Kaiser¹, Clara Waldmann¹

Abstract

The k disjoint shortest paths problem (k-DSPP) on a graph with k source-sink pairs (s_i, t_i) asks for the existence of k pairwise edge- or vertex-disjoint shortest $s_i - t_i$ -paths. It is known to be NP-complete if k is part of the input. Restricting to 2-DSPP with strictly positive lengths, it becomes solvable in polynomial time. We extended edge lengths and give a polynomial time algorithm based on dynamic programming for with non-negative edge lengths. Keywords: disjoint paths, disjoint shortest paths, dynamic programming, mixed graphs 1. Introduction up with an $\mathcal{O}(|V|^3)$ algorithe In contrast to that, Fortune to 2-DSPP with strictly positive lengths, it becomes solvable in polynomial time. We extend this result by allowing zero edge lengths and give a polynomial time algorithm based on dynamic programming for 2-DSPP on undirected graphs

1. Introduction Due to many practical applications, e.g., in communication networks, the k disjoint paths problem (k-DPP) is a well studied problem in the literature. The input of the prob-lem is an undirected graph G = (V, E) as well as k pairs of vertices $(s_i, t_i) \in V^2$ for $i \in [k] := \{1, \ldots, k\}$ and the task is to decide whether there exist k paths P_1, \ldots, P_k such that P_i is an s_i - t_i -path and all paths are pairwise disjoint. Here, disjoint can either mean vertex-disjoint or edge-disjoint. The k disjoint shortest path problem (k-DSPP) is a gen-eralization of the k disjoint paths problem. The input of the problem is an undirected graph G = (V, E) with edge lengths $\ell : E \to \mathbb{R}$ and k pairs of vertices $(s_i, t_i) \in V^2$ for $i \in [k]$. But here, all paths P_i for $i \in [k]$ are addition-ally required to be shortest s_i - t_i -paths. Note, if $\ell \equiv 0$, this agrees with k-DPP. We shall refer to the versions of the problems in directed graphs by k-dDPP and k-dDSPP. 1.1. Related Work Probably most famously, Menger's theorem [9] deals with disjoint paths which gave rise to one of the most funda-

disjoint paths which gave rise to one of the most fundamental results for network flows: the max-flow-min-cut theorem [4, 6]. Using these results, an application of any flow algorithm solves the k-dDPP if $s_i = s_j$ for all $i, j \in [k]$ or $t_i = t_j$ for all $i, j \in [k]$. Without restrictions on the input instances, all variants of the discussed problems are NP-complete if k is considered part of the input [5, 8].

Due to this, a lot of research focuses on the setting where k is considered fixed. Robertson and Seymour [10] came

marcus.kaiser@tum.de, clara.waldmann@tum.de

up with an $\mathcal{O}(|V|^3)$ algorithm for k-DPP.

In contrast to that, Fortune et al. [7] prove that k-dDPP is still NP-hard, even if k = 2. They give an algorithm that solves k-dDPP for any fixed k on directed acyclic graphs in polynomial time. Zhang and Nagamochi [12] then extended the work of Fortune et al. [7] to solve the problem on acyclic mixed graphs, which are graphs that contain arcs and edges where directing any set of edges does not close a directed cycle.

Since k-dDSPP and k-dDPP agree for $\ell \equiv 0$, all hardness results carry over. However, if all edge lengths are strictly positive Bérczi and Kobayashi [1] give a polynomial time algorithm for 2-dDSPP. Also, for 2-DSPP with strictly positive edge lengths a polynomial time algorithm is due to Eilam-Tzoreff [3]. However, the complexity of k-DSPP on undirected graphs with non-negative edge lengths and constant $k \ge 2$ is unknown. We settle the case k = 2 in this paper.

Other than restricting the paths to be shortest s_i - t_i -paths, e.g., Suurballe [11] gave a polynomial time algorithm minimizing the total length, if all arc lengths are non-negative and $s_i = s_j$, $t_i = t_j$ for all $i, j \in [k]$. Björklund and Husfeldt [2] came up with a polynomial time algebraic Monte Carlo algorithm for solving 2-DPP with unit lengths where the total length of the paths is minimized.

	$\ell \equiv 0$		ℓ non-negative	
k	$k ext{-}\mathrm{DPP}$	k-dDPP	k-DSPP	k-dDSPP
arb.	NP-hard [5, 8]	NP-hard [5]	NP-hard [3]	NP-hard [3]
fixed	P [10]	NP-hard [7]	open $(\ell > 0)$	open $(\ell > 0)$
			- (- /	NP-hard $(\ell \ge 0)$ [7]
2	P [10]	NP-hard [7]	$P(\ell > 0)$ [3]	$P(\ell > 0)[1]$
			P ($\ell \ge 0$) *	NP-hard $(\ell \ge 0)$ [7]

Table 1: Complexity of the disjoint paths problem and its variants. * A polynomial time algorithm for the 2-DSPP on undirected graphs with non-negative edge lengths is the main result of this paper.

 $Email \ addresses: \verb"marinus.gottschau@tum.de", \\$

¹Department of Mathematics, Technische Universität München.

1.2. Our Results

We give a polynomial time algorithm for 2-DSPP on undirected graphs with non-negative edge lengths. Combining techniques from [7] and [1] enables us to deal with edges of length zero. We consider the following problem.

Problem 1 (Undirected Two Edge-Disjoint Shortest Paths Problem)

Input: An undirected graph G = (V, E) with non-negative edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$, a tuple of sources $s \in V^2$, and a tuple of sinks $t \in V^2$

Task: Decide whether there exist two edge-disjoint paths P_1 and P_2 in G such that P_1 is a shortest s_1 - t_1 -path and P_2 is a shortest s_2 - t_2 -path w.r.t. the edge lengths ℓ .

Our paper is organized as follows. In Section 2, based on the ideas of [7], we give a dynamic algorithm that solves the k-DPP in polynomial time on weakly acyclic mixed graphs, which are a generalization of directed acyclic graphs.

These results are then used in Section 3 together with a similar approach as in [1] to solve the undirected 2-DSPP with non-negative edge lengths in polynomial time.

The results of this paper have been obtained independently by Kobayashi and Sako.

2. Disjoint Paths in Weakly Acyclic Mixed Graphs

In this section, we give an algorithm that solves *k*-DPP in a generalization of directed acyclic graphs. We first define mixed graphs, introduce some notations, and state the problem.

A graph $G = (V, A \cup E)$ is a *mixed graph* on the vertex set V with arc set $A \subseteq V^2$ and edge set $E \subseteq {V \choose 2}$. We define $\mathscr{E}(G) := A \cup E$. The set of ingoing (outgoing) arcs of a set of vertices $W \subseteq V$ is denoted by $\delta_A^-(W)$ ($\delta_A^+(W)$).

For pairwise disjoint vertex sets W_1, \ldots, W_h , we denote by $G/\{W_1, \ldots, W_h\}$ the graph that results from G by contracting W_1, \ldots, W_h into h vertices.

A (directed) u-w-path P in G is a sequence of h arcs and edges $(\mathfrak{a}_1, \ldots, \mathfrak{a}_h) \in \mathbb{E}^h$ such that there exists a sequence of vertices $(u = v_1, \ldots, v_{h+1} = w) \in V^{h+1}$ satisfying either $\mathfrak{a}_i = (v_i, v_{i+1})$ or $\mathfrak{a}_i = \{v_i, v_{i+1}\}$ for all $i \in [h]$. Two paths are arc/edge-disjoint (vertex-disjoint) if they do not have a common arc or edge (vertex).

Note that a directed acyclic graph induces natural orderings of its vertices. A linear ordering of the vertices is called a *topological ordering* if, for every arc (v, w), the tail v precedes the head w in the ordering. An ordering is called a *reverse topological ordering* if its reverse ordering is a topological ordering.

On a ground set U, a binary relation R is a subset of U^2 . For $(u, v) \in R$, we write u R v. A relation R is called reflexive, if $u \ R \ u$ holds for all $u \in U$. For two binary relations $R, S \subseteq U^2$, the composition $S \circ R$ is defined by $\{(u, w) \in U^2 \mid \exists v \in U : u \ R \ v \land v \ S \ w\}$. Note that \circ is an associative operator.

We consider the following problem for fixed k.

Problem 2 (Mixed k Arc/Edge-Disjoint Paths Problem) **Input:** A mixed graph $G = (V, \mathcal{E})$, a k-tuple of sources $s \in V^k$, and a k-tuple of sinks $t \in V^k$

Task: Decide whether there exist k pairwise $arc/edge-disjoint paths P_1, \ldots, P_k$ in G such that P_i is an s_i-t_i -path, for all $i \in [k]$.

We give an algorithm that solves this problem on a class of mixed graphs, that generalize directed acyclic graphs:

Definition 1 (Weakly Acyclic Mixed Graphs) We call a mixed graph $G = (V, A \cup E)$ weakly acyclic if the contraction of all edges E yields a directed acyclic graph without loops.

Note that a weakly acyclic mixed graph can contain (undirected) cycles in its edge set.

For a mixed graph $G = (V, \mathbb{E})$, we use the following notation in order to discuss the existence of disjoint paths.

Definition 2 (Arc/Edge-Disjoint Paths Relation)

For $k \in \mathbb{N}$, we define the binary relation $\rightrightarrows_{\mathscr{H}}$ on the set V^k as follows. For $v, w \in V^k$, we have $v \rightrightarrows_{\mathscr{H}} w$ if there exist pairwise arc/edge-disjoint v_i -w_i-paths for all $i \in [k]$ in \mathscr{H} . We will also write $\rightrightarrows_{\mathscr{G}}$ short for $\rightrightarrows_{\mathscr{H}(G)}$.

Since paths of length zero are allowed, the relation $\rightrightarrows_{\mathcal{E}}$ is reflexive. In general, it is not transitive. When considering two relations based on two disjoint sets of arcs and edges, however, these two act in a transitive manner. In that case, the respective underlying arc/edge-disjoint paths from both relations can be concatenated. The resulting arc/edge-disjoint paths correspond to an element in the composition of the two relations.

Observation 3 (Partial Transitivity)

For disjoint arc/edge sets $\mathscr{E}_1, \mathscr{E}_2 \subseteq \mathscr{E}$ and vectors of vertices $u, v, w \in V^k$, it holds

$$u \rightrightarrows_{\mathscr{E}_1} v \wedge v \rightrightarrows_{\mathscr{E}_2} w \implies u \rightrightarrows_{\mathscr{E}_1 \cup \mathscr{E}_2} w.$$

This observation is exploited in Algorithm 1 in order to solve Problem 2 for fixed k for weakly acyclic mixed graphs. It computes the relation \rightrightarrows_G in polynomial time by dealing with the edges and arcs in G separately.

For the undirected components, i.e., the connected components of the subgraph (V, E), it uses an algorithm for edge-disjoint paths in undirected graphs (e.g., [10]) to find the relation \Rightarrow on each component. **Algorithm 1:** Dynamic Program for *k*-DPP in Weakly Acyclic Mixed Graphs

Input: weakly acyclic mixed graph $G = (V, A \cup E)$ **Output:** \rightrightarrows_G on V^k

- Find connected components V₁,..., V_h of the subgraph (V, E) sorted according to a topological ordering of G/{V₁,..., V_h};
- 2 for j = 1, ..., h do

Compute $\rightrightarrows_{G[V_i]}$ using an algorithm for k-DPP;

- **3** Initialize \Rightarrow to the relation $\{(v, v) | v \in V^k\};$
- 4 for j = 1, ..., h do

Update
$$\Rightarrow$$
 to $\Rightarrow_{G[V_i]} \circ \Rightarrow_{\delta^-(V_i)} \circ \Rightarrow;$

5 return \Rightarrow

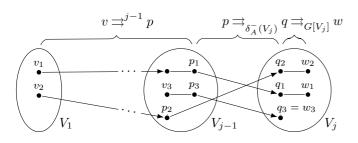


Figure 1: In iteration j of Algorithm 1, relation \Rightarrow^j is built by concatenating previously computed paths (\Rightarrow^{j-1}) , pairwise different arcs to the next component $(\Rightarrow_{\delta_A^-(V_j)})$, and undirected edge-disjoint paths in the next component $(\Rightarrow_{G[V_i]})$.

Afterwards, dynamic programming is used to compute \Rightarrow on successively larger parts of the mixed graph. As G is weakly acyclic, contracting all undirected components results in an acyclic graph. The algorithm iterates over the components in a topological ordering. Based on Observation 3, previously found arc/edge-disjoint paths are extended alternately by arcs between components and edge-disjoint paths within one component. This approach is a generalization of the methods presented in [7].

Theorem 4 (Algorithm 1: Correctness and Running Time) Let $k \in \mathbb{N}$ be fixed. Given a weakly acyclic mixed graph $G = (V, A \cup E)$, Algorithm 1 computes the relation \rightrightarrows_G on V^k in polynomial time.

Proof. Let $V = \bigcup_{j=1}^{h} V_j$ be the partition of V into the vertex sets of the h connected components of (V, E) as computed by the algorithm.

For all $j \in \{0, \ldots, h\}$, let \mathbb{E}_j be the arc and edge set of $G[\bigcup_{l=1}^j V_l]$. In particular, $\mathbb{E}_0 = \emptyset$ holds true. For each $j \in \{0, \ldots, h\}$, let \rightrightarrows^j be the relation \rightrightarrows as computed by Algorithm 1 after the *j*-th iteration of Line 4. In particular, \rightrightarrows^0 is the relation after Line 3. In the following, we proof by induction on j that \rightrightarrows^j is equal to $\rightrightarrows_{\mathcal{H}_i}$.

After the initialization, this is true for j = 0, as \mathcal{E}_0 contains no arcs or edges. Consider an iteration $j \in [h]$ and assume that the claim was true after the previous iteration.

"⊆": Let $v, w \in V^k$ such that $v \rightrightarrows^j w$. There exist $p, q \in V^k$ such that $v \rightrightarrows^{j-1} p \rightrightarrows_{\delta_A^-(V_j)} q \rightrightarrows_{G[V_j]} w$. Using the induction hypothesis, we know $v \rightrightarrows_{\mathcal{E}_{j-1}} p$. Since the arc and edge sets in the three relations are pairwise disjoint, Observation 3 yields $v \rightrightarrows_{\mathcal{E}_i} w$.

" \supseteq ": Let $v, w \in V^k$ with $v \rightrightarrows_{E_j} w$, and $P_i, i \in [k]$ be arc/edge-disjoint $v_i \neg w_i$ -paths in \mathcal{E}_j . Let $q_i \in V$ be the first vertex on P_i in V_i and p_i be its predecessor if they exist, otherwise set them to w_i and q_i , respectively. As G is weakly acyclic, we have $v \rightrightarrows_{E_{j-1}} p, p \rightrightarrows_{\delta_A^-(V_j)} q$, as well as $q \rightrightarrows_{G[V_j]} w$. It follows from the induction hypothesis that $v \rightrightarrows^j w$.

The connected components of (G, E) and their topological ordering in $G/\{V_1, \ldots, V_h\}$ can be computed in polynomial time. Finding edge-disjoint paths in the undirected components can also be done efficiently (e.g., [10]). A binary relation on V^k contains at most $|V|^{2k}$ elements and composing two of them can be done in time polynomial in their sizes. Hence, Algorithm 1 runs in time polynomial in the size of the input if k is fixed.

In many settings, the problem of finding arc/edge-disjoint paths can be reduced to finding vertex-disjoint paths. Observe that arc/edge-disjoint paths in a graph correspond to the vertex-disjoint paths in its line graph and an appropriate notion of a line graph can be defined for mixed graphs as well.

For directed graphs, there is a generic reduction from vertexdisjoint to arc-disjoint instances based on splitting vertices. This reduction, however, cannot be applied to undirected or mixed graphs. Yet, Algorithm 1 can be modified slightly as follows to compute vertex-disjoint paths. An algorithm for the undirected vertex-disjoint path problem is used in Line 2. Only vectors with pairwise different elements are included in the initial relation in Line 3. Finally in Line 4, tuples $v, w \in V^k$ are related only if their sets of endpoints $\{v_i, w_i\}, i \in [k]$ are pairwise disjoint.

3. Undirected Disjoint Shortest Paths

In this section, we study Problem 1 on undirected graphs with non-negative edge lengths. We first transform the undirected graph G into a mixed graph and then use the results of the previous section to solve the transformed instance.

Let an instance of Problem 1 be given by an undirected graph G = (V, E), non-negative edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$, and $s, t \in V^2$. We are going to transform the graph G into a mixed graph such that the shortest source-sink-paths in G correspond to directed source-sink-paths in the resulting mixed graph.

Since we are interested in shortest s_1-t_1 - and s_2-t_2 -paths, we consider the shortest path networks rooted at s_1 and s_2 . For $i \in [2]$, we define the distance function $d_i : V \to \mathbb{R}_{\geq 0}$ induced by ℓ w.r.t. s_i by $d_i(v) := \min_{s_i-v-\text{path }P} \sum_{e \in P} \ell(e)$. The shortest path network rooted at s_i is given by the set

$$E_i := \{\{v, w\} \in E \mid \ell(\{v, w\}) = |d_i(v) - d_i(w)|\}.$$

See Figure 3a for an example of the sets E_i .

The distances d_i induce an orientation for all edges in E_i which have a strictly positive length. We would like to replace an edge $\{v, w\} \in E$ with $d_i(v) < d_i(w)$ by the arc (v, w) (with the same length). The orientations induced by d_1 and d_2 , however, do not have to agree on the set $E_1 \cap E_2$. Introducing both arcs would neglect the fact that only one of them can be included in any set of arc/edge-disjoint paths. We will overcome this by replacing such edges by a standard gadget of directed arcs as depicted in Figure 2.

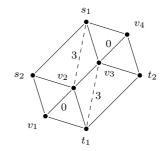


Figure 2: Gadget for resolving conflicts during the orientation of an edge $\{v, w\}$ induced by d_1 and d_2 .

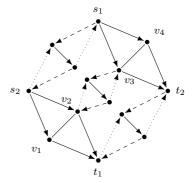
Consider the gadget for an edge $\{v, w\} \in E$. It contains exactly one v-w-path and one w-v-path corresponding to the two possible orientations of $\{v, w\}$. Since both share an arc, only one of two arc/edge-disjoint paths in the transformed graph can use the gadget. As further both paths consist of three arcs, setting the length of all the arcs in the gadget to $\frac{1}{3}\ell(\{v, w\})$ preserves the distances in the graph. That way, the distance functions d_i can be extended to the new vertices introduced with gadgets.

For $i \in [2]$, A_i denotes the set of arcs that result from orienting E_i w.r.t. d_i . More precisely, for $\{v, w\} \in E_i$ with $d_i(v) < d_i(w)$ the arc (v, w) is included into the set A_i if $\{v, w\} \in E_1 \triangle E_2$ or the orientation induced by d_1 and d_2 agree. Otherwise, the arcs of the *v*-*w*-path in the gadget replacing $\{v, w\}$ are added to A_i .

The induced orientation is only well-defined for edges with strictly positive lengths. Therefore, the set of edges with length zero $E_0 := \{e \in E | \ell(e) = 0\}$ are left undirected and have to be treated in a different manner.



(a) Example: edges without label have length 1, solid edges are in $E_1 \cup E_2$



(b) Partially oriented expansion of (a): solid arcs are in A₁ ∩ A₂, dashed arcs are in A₁ \ A₂, dotted arcs are in A₂ \ A₁

Figure 3: Exemplary construction of partially oriented expansion

Definition 5 (Partially Oriented Expansion)

Let G = (V, E) be an undirected graph with non-negative edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and $s \in V^2$.

The partially oriented expansion of G w.r.t. ℓ and s is the graph $\vec{G} := (W, E_0 \cup A_1 \cup A_2)$ where W is the set of vertices V augmented with additional vertices introduced with gadgets, and E_0 , A_1 , and A_2 are as defined above.

The partially oriented expansion of the example from Figure 3a is depicted in Figure 3b. As we are going to discuss the existence of shortest edge-disjoint paths in G and the existence of arc/edge-disjoint paths restricted to different arc and edge sets in \vec{G} , the following notation will be useful.

Definition 6 (Two Disjoint Paths Relations)

i) Let G = (V, E) be an undirected graph with non-negative edge lengths $\ell : E \to \mathbb{R}_{>0}$.

For $v, w \in V^2$, we write $v \stackrel{\ell}{\Longrightarrow}_E w$ if there exist edge-disjoint shortest $v_i - w_i$ -paths w.r.t. ℓ for $i \in [2]$ in E.

ii) Let $G = (V, A \cup E)$ be a mixed graph and let $\mathcal{E}_1, \mathcal{E}_2$ be two subsets of arcs and edges of $A \cup E$.

For $v, w \in V^2$, we write $v \rightleftharpoons_{\mathcal{E}_2}^{\mathcal{E}_1} w$ if there exist a v_1 - w_1 -path in \mathcal{E}_1 and a w_2 - v_2 -path in \mathcal{E}_2 which are arc/edge-disjoint.

As described above, the distance functions of the original graph G extend to the vertices of \vec{G} . For $i \in [2]$ and $v \in W$, $d_i(v)$ is the length of a shortest s_i -v-path in \vec{G} .

Lemma 7 (Paths in the Partially Oriented Expansion) Let G = (V, E) be an undirected graph with non-negative edge lengths $\ell: E \to \mathbb{R}_{\geq 0}$ and $s \in V^2$. Furthermore, let $\vec{G} = (W, E_0 \cup A_1 \cup A_2)$ be the partially oriented expansion of G w.r.t. ℓ and s.

Then for every $t \in V^2$, we have $s \stackrel{\ell}{\Rightarrow}_E t$ in G if and only if $\binom{s_1}{t_2} \rightleftharpoons_{E_0 \cup A_2}^{E_0 \cup A_1} \binom{t_1}{s_2}$ in \overrightarrow{G} .

Proof. " \Rightarrow ": Assume there exist two edge-disjoint shortest s_i - t_i -paths P_i in E_i for $i \in [2]$. Replace each edge with non-zero length in P_i by the respective oriented arc or path in the respective gadget to obtain \overrightarrow{P}_i in $E_0 \cup A_i$. \overline{P}_1 and \overline{P}_2 are arc/edge-disjoint as different edges are replaced by disjoint (sets of) arcs.

" \Leftarrow ": Assume there are arc/edge-disjoint $s_i - t_i$ -paths \vec{P}_i in $E_0 \cup A_i$ for $i \in [2]$. Replace the subpath of P_i within one gadget with the corresponding edge in E_i . The remaining arcs are translated directly to the respective edges in E_i . Due to the mentioned equality of distances in G and Gand the fact that d_i is non-decreasing along arcs in G, P_i is a shortest path in G. Any path that uses a gadget in G, uses its inner arc. Therefore, P_1 and P_2 inherit being edge-disjoint from \overline{P}_1 and \overline{P}_2 .

3.2. Disjoint Paths in the Partially Oriented Expansion

Lemma 7 shows that \overline{G} captures the shortest paths in G by using orientation. We will use the distances, however, to prove the main structural result. It concern the subgraph of \overline{G} potentially used by both paths and its weakly connected components, which are its connected components when ignoring the arcs' directions.

Lemma 8 (Structure of Partially Oriented Expansion) Let G = (V, E) be an undirected graph with non-negative edge lengths $\ell: E \to \mathbb{R}_{\geq 0}$ and $s \in V^2$. Furthermore, let $\overline{G} = (W, E_0 \cup A_1 \cup A_2)$ be the partially oriented expansion of G w.r.t. ℓ and s. Let $W = \bigcup_{j=1}^{h} W_j$ be the partition of W into the vertex sets of the h weakly connected components of the subgraph $(W, E_0 \cup (A_1 \cap A_2))$.

Then

i) $\vec{G}[W_i]$ is weakly acyclic for all $j \in [h]$,

ii) sorting the components $W_j, j \in [h]$ in non-decreasing order w.r.t. the function $d_1 - d_2$ is a topological ordering of $(W, A_1)/\{W_1, \ldots, W_h\}$ and a reverse topological ordering of $(W, A_2)/\{W_1, \ldots, W_h\}$, and

iii) $\overline{G}[W_j]$ contains arcs only from $A_1 \cap A_2$ and edges only from E_0 for all $j \in [h]$.

Proof. i) By definition of A_1 , we know that d_1 increases strictly along arcs in $A_1 \cap A_2$. Further, d_1 is constant on **6 return** $\left\{ t \in V^2 \mid {s_1 \choose t_2} \rightleftharpoons {t_1 \choose s_2} \right\}$ edges in E_0 . Assume there is $j \in [h]$ and a (directed) cycle C in $\overline{G}[W_i]$ such that there exists $a \in C \cap A_1 \cap A_2$.

Along of a the distance d_1 strictly increases. However, d_1 cannot decrease along C, which yields a contradiction.

ii) Consider the function on the vertex set of \overline{G} . Based on the common underlying lengths in G and the definitions of A_1 and A_2 , it is strictly increasing along arcs in $A_1 \setminus A_2$ and strictly decreasing along arcs in $A_2 \setminus A_1$. Opposed to that, it is constant on edges in E_0 as well as along of arcs in $A_1 \cap A_2$.

iii) The function $d_1 - d_2$ is constant along all arcs $A_1 \cap A_2$ and edges in E_0 . Hence, it is constant on each weakly connected component w.r.t. those arcs and edges. At the same time, the function is not constant along arcs in $A_1 \triangle A_2$.

This structural result allows to use dynamic programming for solving Problem 2 on the partially oriented expansion. Similar to Section 2, the problem is split into two parts. First, the two arc/edge-disjoint paths problem on the weakly connected components W_1, \ldots, W_h of the subgraph $(W, E_0 \cup (A_1 \cap A_2))$ is solved by Algorithm 1. Afterwards, a dynamic program is used to incorporate the results into arc-disjoint paths in $G/\{W_1,\ldots,W_h\}$ to get $\operatorname{arc/edge-disjoint}$ paths in G.

We know that the two arc/edge-disjoint paths that we are looking for, if they exist, pass through $\overline{G}/\{W_1,\ldots,W_h\}$ in opposite directions. In order to accomplish simultaneous construction of both, one of the paths is created backwards. Apart from that, Algorithm 2 resembles Algorithm 1.

Algorithm 2: Dynamic Program for 2-DSPP with nonnegative edge lengths

Input: undirected graph G = (V, E), non-negative edge lengths $\ell: E \to \mathbb{R}_{>0}, s \in V^2$

Output: set of pairs in V^2 that succeed s w.r.t. $\stackrel{\ell}{\Rightarrow}_E$

- 1 Construct $\vec{G} = (W, E_0 \cup A_1 \cup A_2)$ for G w.r.t. ℓ and s;
- **2** Find weakly connected components W_1, \ldots, W_h of the subgraph $(W, E_0 \cup (A_1 \cap A_2))$ sorted non-decreasingly w.r.t. $d_1 - d_2;$

B for
$$j = 1, ..., h$$
 do
Compute $\rightleftharpoons_{\vec{G}[W_j]}^{\vec{G}[W_j]}$ using Algorithm 1;

4 Initialize
$$\rightleftharpoons$$
 to the relation $\{(v, v) | v \in W^2\};$

5 for
$$j = 1, ..., h$$
 do
Update \rightleftharpoons to $\rightleftharpoons \overrightarrow{G}[W_j] \circ \rightleftharpoons \overrightarrow{\delta}_{A_1}^{-}(W_j) \circ \rightleftharpoons$

Theorem 9 (Algorithm 2: Correctness and Running Time) Given an undirected graph G = (V, E) with non-negative edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and $s \in V^2$, Algorithm 2 computes all successors of $s w.r.t. \xrightarrow{\ell} in polynomial time.$

Proof. Let $W = \bigcup_{j=1}^{h} W_j$ be the partition of W into the vertex sets of the h weakly connected components of the subgraph $(W, E_0 \cup (A_1 \cap A_2))$ as computed by the algorithm. Lemma 8 ii) shows that the W_j 's are sorted in a topological ordering of $(W, A_1)/\{W_1, \ldots, W_h\}$ and in a reverse topological ordering of $(W, A_2)/\{W_1, \ldots, W_h\}$.

For $i \in [2]$ and $j \in \{0, \ldots, h\}$, set \mathbb{H}_i^j to be the arcs of A_i and edges of E_0 in the induced subgraph $\vec{G}[\bigcup_{l=1}^j W_l]$. In particular, we have $\mathbb{H}_i^0 = \emptyset$. For $j \in [h]$, let \rightleftharpoons^j denote the relation \rightleftharpoons computed by Algorithm 2 after the *j*-th iteration. In particular, \rightleftharpoons^0 is as defined in Line 4. We will prove by induction on $j = 0, \ldots, h$ that \rightleftharpoons^j is equal to $\rightleftharpoons_{\mathbb{H}_2^j}^{\mathbb{H}_1^j}$. The correctness of the algorithm then follows from Lemma 7.

The claim holds for j = 0, since $\mathbb{A}_1^0 = \mathbb{A}_2^0 = \emptyset$ by definition. Consider iteration $j \in [h]$ and assume that the claim holds for the preceding iteration.

" \subseteq ": Let $v, w \in W^2$ such that $v \rightleftharpoons^j w$. Considering Line 5 and using induction hypothesis, there exist $p, q \in W^2$ with

$$v \rightleftharpoons_{\mathbb{H}_{2}^{j-1}}^{\mathbb{H}_{1}^{j-1}} p \rightleftharpoons_{\delta_{A_{2}}^{+}(W_{j})}^{\delta_{A_{1}}^{-}(W_{j})} q \rightleftharpoons_{\overline{G}[W_{j}]}^{\overline{G}[W_{j}]} w$$

Lemma 8 iii) guarantees that the arc and edge sets of the three relations are pairwise disjoint. As a result, $v \rightleftharpoons_{E_2^j}^{E_1^j} w$ follows from Observation 3.

" \supseteq ": Let $v, w \in W^2$ such that $v \rightleftharpoons_{\mathbb{H}_2^j}^{\mathbb{H}_1^j} w$. Thus, there have to be a simple $v_1 - w_1$ -path P_1 in \mathbb{H}_1^j and a simple $w_2 - v_2$ -path P_2 in \mathbb{H}_2^j that are arc/edge-disjoint. Define $q_1 \in W$ to be the first vertex on P_1 in W_j , if it exists, or w_1 . Let p_1 be the predecessor of q_1 on P_1 or q_1 if it is the first vertex of P_1 . Similarly, let $q_2 \in W$ be the last vertex on P_2 in W_j or w_2 if it does not exist, and let p_2 be the successor of q_2 or q_2 if q_2 does not have a successor. The topological ordering of the W_j 's implies that the subpaths of P_1 and P_2 prove

$$v \rightleftharpoons_{\mathbb{H}_{2}^{j-1}}^{\mathbb{H}_{1}^{j-1}} p \rightleftharpoons_{\delta_{A_{2}}^{+}(W_{j})}^{\delta_{A_{1}}^{-}(W_{j})} q \rightleftharpoons_{\widehat{G}[W_{j}]}^{\widehat{G}[W_{j}]} w.$$

Finally, $v \rightleftharpoons^{j} w$ follows by induction hypothesis.

As for the running time, finding the weakly connected components and sorting them in a topological ordering can be done in polynomial time. Computing the relations $\rightleftharpoons_{\overline{G}[W_j]}^{\overrightarrow{G}[W_j]}$ also can be done efficiently by virtue of Algorithm 1. Finally, relations on V^2 have at most $|V|^4$ elements and can be composed efficiently. Therefore, the total running time of the algorithm is polynomial in the input size.

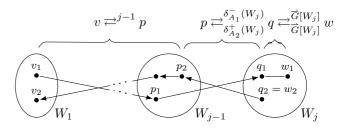


Figure 4: Iteration j of Algorithm 2: relation \rightleftharpoons^j is built by concatenating already computed paths, pairwise different arcs to the next component, and arc/edge-disjoint paths in the next mixed component

Similar to Section 2, Algorithm 2 can be adapted to check for the existence of two vertex-disjoint shortest paths. In that case, the gadget from Figure 2 is not needed anymore, but can be replaced by two opposite arcs.

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