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Extended formulations for radial cones

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ABSTRACT

This paper studies extended formulations for radial cones at vertices of polyhedra, which are the polyhedra defined by the constraints that are active at the vertex. While the perfect-matching polytope cannot be described by subexponential-size extended formulations (Rothvoß 2014), Ventura & Eisenbrand (2003) showed that its radial cones can be described by polynomial-size extended formulations. The authors also asked whether this extends to odd-cut polyhedra, which are related to matching polyhedra by polarity. We answer this question negatively.

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1. Introduction

The concept of extended formulations is an important technique in discrete optimization that allows for replacing the inequality description of some linear program by another inequality description of preferably smaller size using auxiliary variables. Geometrically, given a polyhedron $P \subseteq \mathbb{R}^p$ one searches for a polyhedron $Q \subseteq \mathbb{R}^q$ together with a linear map $\pi : \mathbb{R}^q \rightarrow \mathbb{R}^p$ such that $\pi(Q) = P$. The pair (Q, π) is called a *linear extension* of P whose size is the number of facets of Q .

There are several polyhedra associated to classic combinatorial optimization problems having a large number of facets but admitting linear extensions of small size (polynomial in their dimension). Prominent examples are the spanning tree polytope [11,20], the subtour elimination polytope [20], and the cut dominant [5, §4.2]. The seminal work of Fiorini et al. [9] has shown that such descriptions do not exist for many polytopes associated to hard problems, including the cut polytope or the traveling salesman polytope. Surprisingly, the same is true even for the perfect-matching polytope, a very well-understood polytope over which linear functions can be optimized in polynomial time [7]. In fact, Rothvoß [14] proved that every linear extension of the perfect-matching polytope $P_{\text{pmatch}}(n)$ of the complete graph $K_n = (V_n, E_n)$ on n nodes has size $2^{2^{2(n)}}$.

Thus, in terms of sizes of linear extensions, the perfect matching polytope appears as complicated as certain polytopes associated to hard problems. Ventura & Eisenbrand [19] showed that this situation changes if one aims for *local* descriptions: Given a

vertex v of $P_{\text{pmatch}}(n)$, they showed that the polyhedron defined by *only* those constraints of $P_{\text{pmatch}}(n)$ that are active at v , the *radial cone* at v , has a linear extension of size $O(n^3)$.

Note that such formulations can be used to efficiently test whether a given vertex is optimal with respect to a given linear function. For linear 0/1-optimization problems, efficient routines for such local checks are usually enough to obtain an efficient algorithm for the actual optimization problem, see [17,18]. Thus, the work in [19] yields another proof that the weighted matching problem can be solved in polynomial time. However, this also suggests that such descriptions do not exist for polytopes associated to hard problems, which separates matching from harder optimization problems.

Furthermore, Ventura & Eisenbrand generalized their construction to the V_n -join polyhedron of K_n (which contains $P_{\text{pmatch}}(n)$ as a face), showing that its radial cones also admit linear extensions of size $O(n^3)$. In the same paper, the authors asked whether the same holds for the odd-cut polyhedron, which is the blocker of the V_n -join polyhedron and hence closely related.

Our results.

1. The main purpose of this work is to answer their question negatively by showing the following result.

Theorem 1. *There exists a constant $c > 0$ such that for every even n , the radial cones of the odd-cut polyhedron of K_n cannot be described by linear extensions of size less than 2^{cn} .*

2. To obtain our result, for a polyhedron P of blocking type, we establish a general relationship between its radial cones and certain faces of the blocker of P .

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In the case of the odd-cut polyhedron, we show that its radial cones correspond to certain faces of the V_n -join polyhedron that can be shown to require large linear extensions using Rothvoß' result.

Analogously, it turns out that radial cones of the V_n -join polyhedron correspond to certain faces of the odd-cut polyhedron, which can be easily described by linear extensions of size $O(n^3)$.

This allows us to give an alternative proof of the result by Ventura & Eisenbrand.

3. We complement our results by observing that radial cones of polytopes associated to most classical hard optimization problems indeed do not admit polynomial-size extended formulations in general.

Outline. The paper is structured as follows. In Section 2, we will introduce the relevant concepts and derive straight-forward results on extension complexities of radial cones. Using elementary properties of blocking polyhedra, we will derive a structural relationship between radial cones and certain faces of the blocker in Section 3. Using these insights, our main result is proved in Section 4, where we also provide an alternative proof of the result by Ventura and Eisenbrand. Finally, an upper bound that complements our main result is provided in the appendix.

2. Overview

Recall that, for a polyhedron P and a point $v \in P$, we are interested in describing the *radial cone* $K_P(v)$, which is the polyhedron defined by all inequalities that are valid for P and satisfied with equality by v . We remark that, technically, by our definition, $K_P(v)$ is not necessarily a cone. In fact, standard definitions of the radial cone (or the *cone of feasible directions*) differ from ours in a translation by the vector $-v$, see, e.g., [15, § 2.2]. However, the one given here will be more convenient for us.

Given an inequality description of P , the radial cone is simply defined by dropping some of the inequalities. Note that a polyhedron arising from P by deleting an arbitrary subset of inequalities might require much larger linear extensions than P does. However, radial cones arise in a very structured way, which allows us to carry over linear extensions for P . This is made clear by observing that

$$K_P(v) = \text{cone}(P - v) + v,$$

where $\text{cone}(X) := \{\mu x : \mu \geq 0, x \in X\}$ for convex sets X . Let us formalize the previous claim and other basic observations in the following proposition. To this end, we make use of the (linear) *extension complexity* $\text{xc}(P)$ of a polyhedron P , which is defined as the smallest size of any linear extension of P .

Proposition 2. Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $v \in P$.

- (i) $\text{xc}(K_P(v)) \leq \text{xc}(P)$.
- (ii) Every face F of P satisfies $\text{xc}(F) \leq \text{xc}(P)$.
- (iii) Every face F of P with $v \in F$ satisfies $\text{xc}(K_F(v)) \leq \text{xc}(K_P(v))$.
- (iv) For every linear map $\pi : \mathbb{R}^P \rightarrow \mathbb{R}^d$, we have $\text{xc}(\pi(P)) \leq \text{xc}(P)$ and $\text{xc}(K_{\pi(P)}(\pi(v))) \leq \text{xc}(K_P(v))$.

Proof. To see (i), let Q be a minimum-size extension of P with $P = \pi(Q)$ for some linear map π . Let $w \in Q$ be a preimage of v , i.e., $\pi(w) = v$. By the linearity of π , we have

$$\begin{aligned} K_P(v) &= \text{cone}(P - v) + v = \text{cone}(\pi(Q - w)) + v \\ &= \pi(\text{cone}(Q - w)) + v \\ &= \pi(\text{cone}(Q - w) + w) = \pi(K_Q(w)). \end{aligned}$$

This proves (i) since $K_Q(w)$ is described by a subset of the $\text{xc}(P)$ -many inequalities describing Q .

Let F be a face of P and let H be a corresponding supporting hyperplane, i.e., $F = P \cap H$. Since H is described by an equation, (ii) follows. Moreover, $K_F(v) = K_P(v) \cap H$, i.e., the radial cone of F at v is a face of the radial cone of P at v . An application of (ii) yields (iii).

The first statement of (iv) follows by concatenating the projection map of a minimum-size extension of P with π . To prove the second statement, we will show that $\pi(K_P(v)) = K_{\pi(P)}(\pi(v))$. By translating P to $P - v$ (and by keeping π , also translating $\pi(P)$ to $\pi(P) - \pi(v)$), this is equivalent to showing $\pi(\text{cone}(P)) = \text{cone}(\pi(P))$ for $\mathbb{O} \in P$. Clearly, the last statement holds by linearity of π . \square

On the one hand, Proposition 2 (i) shows that radial cones of polyhedra admitting small extensions, e.g., the ones mentioned in the introduction, also have a small extension complexities. On the other hand, the last two statements of the proposition can be used to derive lower bounds on extension complexities of radial cones of polytopes related to many NP-hard problems.

Radial cones of polytopes associated to hard problems. Consider the cut polytope $P_{\text{CUT}}(n) \in \mathbb{R}^{E_n}$ of the complete graph $K_n = (V_n, E_n)$ defined as the convex hull of characteristic vectors of cuts (in the edge space) in K_n . Braun et al. proved (see Proposition 3 in [3]) that $\text{cone}(P_{\text{CUT}}(n))$ has extension complexity at least $2^{\Omega(n)}$. Note that $\text{cone}(P_{\text{CUT}}(n))$ is the radial cone of $P_{\text{CUT}}(n)$ at the vertex corresponding to the empty cut. Furthermore, it has been shown that several polytopes associated to other NP-hard problems have faces that can be projected onto cut polytopes by (affine) linear maps. Examples are certain stable-set polytopes and traveling-salesman polytopes [9], certain knapsack polytopes [1,13] and 3d-matching polytopes (see [1]).

Consider any such a polytope $P(n)$ and let $F(n)$ be a face that projects to $P_{\text{CUT}}(n)$. Clearly, $F(n)$ must have a vertex v_n whose projection is the vertex \mathbb{O} of $P_{\text{CUT}}(n)$. By Proposition 2 (iii) and (iv), the extension complexity of the radial cone of $P(n)$ at v_n is greater than or equal to the extension complexity of the radial cone of $P_{\text{CUT}}(n)$ at \mathbb{O} . Hence, for such polytopes, we obtain super-polynomial lower bounds on extension complexities of some of their radial cones. Notice that such polytopes may still have radial cones with small extension complexities. For instance, the radial cone of any stable-set polytope at the origin is a nonnegative orthant.

Polyhedra associated to matchings, T-joins, and T-cuts. Throughout the paper, let $T \subseteq V_n$ be a node set of even cardinality. A *T-join* is a subset $J \subseteq E_n$ of edges such that a node $v \in V_n$ has odd degree in the subgraph (V_n, J) if and only if $v \in T$. A *T-cut* is a subset $C \subseteq E_n$ of edges such that $C = \delta(S) := \{\{v, w\} \in E_n : v \in S, w \notin S\}$ holds for some $S \subseteq V_n$ for which $|S \cap T|$ is odd. The V_n -cuts are also known as *odd cuts*. The perfect-matching polytope $P_{\text{pmatch}}(n)$, *T-join-polytope* $P_{T\text{-join}}(n)$ and *T-cut polytope* $P_{T\text{-cut}}(n)$ are defined as the convex hulls of characteristic vectors of all perfect matchings, *T-joins* and *T-cuts* of K_n , respectively. The (weighted) minimization problem for *T-cuts* is NP-hard for arbitrary objective functions, but can be solved in polynomial time for nonnegative ones [12]. For this reason we focus on the *dominant* of the *T-cut polytope*, defined as $P_{T\text{-cut}}(n)^\dagger := P_{T\text{-cut}}(n) + \mathbb{R}_+^{E_n}$, which in turn is related to the dominant of the *T-join polytope* $P_{T\text{-join}}(n)^\dagger := P_{T\text{-join}}(n) + \mathbb{R}_+^{E_n}$. We also refer to $P_{T\text{-cut}}(n)^\dagger$ and $P_{T\text{-join}}(n)^\dagger$ as the *T-cut polyhedron* and the *T-join polyhedron*, respectively. The descriptions of both polyhedra in terms of linear inequalities are well-known [8] (using $\chi(F)$ as a

short-hand notation for $\sum_{e \in F} x_e$:

$$P_{T\text{-join}}(n)^\uparrow = \left\{ x \in \mathbb{R}_+^{E_n} : x(C) \geq 1 \text{ for all } T\text{-cuts } C \right\} \quad (1)$$

$$P_{T\text{-cut}}(n)^\uparrow = \left\{ x \in \mathbb{R}_+^{E_n} : x(J) \geq 1 \text{ for all } T\text{-joins } J \right\}$$

It is worth noting that the vertices of $P_{T\text{-join}}(n)^\uparrow$ are the inclusion-wise minimal T -joins, i.e., those that do not contain cycles [10, §12.2] and hence are edge-disjoint unions of $\frac{1}{2}|T|$ paths whose endnodes are distinct and in T . Setting $n' := |T|$, the perfect-matching polytope $P_{\text{pmatch}}(n')$ is a face of $P_{T\text{-join}}(n)^\uparrow$, induced by $x(\delta(v)) \geq 1$ for all $v \in T$ and $x(\delta(v)) \geq 0$ for all $v \in V_n \setminus T$, where $\delta(v) := \delta(\{v\})$. Thus, from Rothvoß' proof for the exponential lower bound on the extension complexity of the perfect-matching polytope it follows that

$$xc(P_{T\text{-join}}(n)^\uparrow) \geq 2^{\Omega(|T|)}. \quad (2)$$

It turns out that this bound is essentially tight. In fact, in Appendix A we give a linear extension for $P_{T\text{-join}}(n)^\uparrow$ showing

$$xc(P_{T\text{-join}}(n)^\uparrow) \leq O(n^2 \cdot 2^{|T|}). \quad (3)$$

Thus, for case $T = V_n$ with n even we obtain that the extension complexity of the V_n -join polyhedron grows exponentially in n . In the next section we will see that this result carries over to the V_n -cut polyhedron, also known as the *odd-cut polyhedron*.

3. Blocking pairs of polyhedra

The T -cut polyhedron and the T -join polyhedron belong to the class of *blocking polyhedra*. A polyhedron $P \subseteq \mathbb{R}_+^d$ is *blocking* if $x' \geq x$ implies $x' \in P$ for all $x \in P$. Such a polyhedron can be described as $P = \{x \in \mathbb{R}_+^d : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \dots, m\}$ for certain nonnegative vectors $y^{(1)}, \dots, y^{(m)} \in \mathbb{R}_+^d$ or as $P = \text{conv}\{x^{(1)}, \dots, x^{(k)}\} + \mathbb{R}_+^d$ for certain nonnegative vectors $x^{(1)}, \dots, x^{(k)} \in \mathbb{R}_+^d$. The *blocker* of P , defined via

$$B(P) := \{y \in \mathbb{R}_+^d : \langle x, y \rangle \geq 1 \forall x \in P\},$$

is again a blocking polyhedron and satisfies $B(B(P)) = P$. We refer to Section 9.2 in Schrijver's book [16] for the proofs and more properties of blocking polyhedra.

In what follows, we will establish some connections between extension complexities of (certain faces of) blocking polyhedra and (certain faces of) their blockers. We will make use of the following key observation of Martin [11] that relates the extension complexities of certain polyhedra, in particular if they are in a blocking relation.

Proposition 3 ([11], see also [6, Prop. 1]). *Given a non-empty polyhedron Q and $\gamma \in \mathbb{R}$, let*

$$P = \{x : \langle y, x \rangle \geq \gamma \forall y \in Q\}.$$

Then $xc(P) \leq xc(Q) + 1$.

A first consequence of Proposition 3 is that the extension complexities of a blocking polyhedron P and its blocker $B(P)$ differ by at most d (due to the nonnegativity constraints). Thus, the extension complexities of $P_{T\text{-cut}}(n)^\uparrow$ and $P_{T\text{-join}}(n)^\uparrow$ differ by at most $\binom{n}{2}$. In particular, in view of (2) and (3), we obtain

$$2^{\Omega(|T|)} \leq xc(P_{T\text{-cut}}(n)^\uparrow) \leq O(n^2 \cdot 2^{|T|}). \quad (4)$$

Using the same arguments, the above lower bound has been already established in [4, § 5.7].

The main purpose of this section, however, is to show that a radial cone of a blocking polyhedron can be analyzed by considering a certain face of the blocker. To this end, let us now consider a

general pair $(P, B(P))$ of blocking polyhedra in \mathbb{R}_+^d . For every point $v \in P$ we define the set

$$F_{B(P)}(v) := \{y \in B(P) : \langle v, y \rangle = 1\} \\ = \{y \in \mathbb{R}_+^d : \langle v, y \rangle = 1, \langle x, y \rangle \geq 1 \forall x \in P\}, \quad (5)$$

which is a face of $B(P)$. The following lemma establishes structural connections between $K_P(v)$ and $F_{B(P)}(v)$.

Lemma 4. *Let $P \subseteq \mathbb{R}_+^d$ be a blocking polyhedron and let $v \in P$.*

- (i) $F_{B(P)}(v) = \{y \in \mathbb{R}^d : \langle v, y \rangle = 1, \langle x, y \rangle \geq 1 \forall x \in K_P(v)\}$.
- (ii) $K_P(v) = \{x \in \mathbb{R}^d : \langle y, x \rangle \geq 1 \forall y \in F_{B(P)}(v)\}$.

Proof. We first prove “ \subseteq ” of part (i). To this end, let $y \in F_{B(P)}(v)$. We have to show that $\langle x, y \rangle \geq 1$ holds for all $x \in K_P(v)$. Recall that for every $x \in K_P(v)$ there exist $x' \in P$ and $\mu \geq 0$ such that $x = v + \mu(x' - v)$. Since we have $\langle v, y \rangle = 1$ and $\langle x', y \rangle \geq 1$, this implies

$$\langle x, y \rangle = \langle v, y \rangle + \mu(\langle x', y \rangle - \langle v, y \rangle) \\ = 1 + \mu(\langle x', y \rangle - 1) \geq 1,$$

as claimed.

To prove “ \supseteq ” of part (i), we have to show for every $j \in [d] := \{1, \dots, d\}$ that the nonnegativity constraint $y_j \geq 0$ is redundant in the right-hand side of (5). From $v + e_j \in P$ we obtain the valid inequality $\langle v + e_j, y \rangle \geq 1$. Subtracting $\langle v, y \rangle = 1$ implies the desired inequality $y_j \geq 0$.

Before we turn to the proof of part (ii), let us fix some notation. Recall that there exist $y^{(1)}, \dots, y^{(m)} \in \mathbb{R}_+^d$ such that $P = \{x \in \mathbb{R}_+^d : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \dots, m\}$. Denote by $I := \{i \in [m] : \langle v, y^{(i)} \rangle = 1\}$ and $J := \{j \in [d] : v_j = 0\}$ the index sets of the inequalities of P that are tight at v . In other words, $K_P(v) = \{x \in \mathbb{R}^d : \langle y^{(i)}, x \rangle \geq 1 \forall i \in I, x_j \geq 0 \forall j \in J\}$.

To prove “ \subseteq ” of part (ii), we consider vectors $\hat{x} \in K_P(v)$ and $\hat{y} \in F_{B(P)}(v)$ and claim that $\langle \hat{x}, \hat{y} \rangle \geq 1$. In particular, $\hat{y} \in B(P)$, and hence there exists a vector $\bar{y} \leq \hat{y}$ with $\bar{y} \in \text{conv}\{y^{(i)} \mid i \in [m]\}$.

From $\hat{y} \in F_{B(P)}(v)$ and nonnegativity of v we obtain $1 = \langle v, \hat{y} \rangle \geq \langle v, \bar{y} \rangle$. Since $\langle v, y^{(i)} \rangle \geq 1$ holds for all $i \in [m]$, this implies $\langle v, \bar{y} \rangle \geq 1$, and hence $\hat{y}_j = \bar{y}_j$ for all $j \in [d] \setminus J$. Furthermore, only $y^{(i)}$ for $i \in I$ can participate in the convex combination (of \bar{y}) with a strictly positive multiplier. Considering the inequalities that are valid for $K_P(v)$, we observe that $\hat{x}_j \geq 0$ for all $j \in J$ and that $\langle \hat{x}, y^{(i)} \rangle \geq 1$ for all $i \in I$.

This in turn implies $\langle \hat{x}, \bar{y} \rangle \geq 1$. Hence, using $\hat{y}_j = \bar{y}_j$ for all $j \in [d] \setminus J$ and $\hat{x}_j \geq 0$ for all $j \in J$, we obtain $\langle \hat{x}, \hat{y} \rangle \geq \langle \hat{x}, \bar{y} \rangle$ which establishes $\langle \hat{x}, \hat{y} \rangle \geq 1$.

It remains to prove “ \supseteq ” of part (ii). To this end, consider a vector \hat{x} from the set on the right-hand side of the equation. For all $i \in I$, $y^{(i)} \in F_{B(P)}(v)$ implies $\langle y^{(i)}, \hat{x} \rangle \geq 1$. Consider an arbitrary $\bar{y} \in F_{B(P)}(v)$ and some $j \in J$. For all $\mu \geq 0$, we have $(\bar{y} + \mu e_j) \in F_{B(P)}(v)$. To see this, consider (5) and observe that $\langle v, e_j \rangle = 0$ and that $\langle x, e_j \rangle \geq 0$ for all $x \in P$. In particular $1 \leq \langle \hat{x}, \bar{y} + \mu e_j \rangle = \langle \hat{x}, \bar{y} \rangle + \mu \hat{x}_j$, which implies $\hat{x}_j \geq 0$ and concludes the proof. \square

We conclude this section with the following result, which is an immediate consequence of Proposition 3 and parts (i) and (ii) of Lemma 4.

Theorem 5. *Let $P \subseteq \mathbb{R}_+^d$ be a blocking polyhedron and let $v \in P$. Then $xc(K_P(v))$ and $xc(F_{B(P)}(v))$ differ by at most 1.*

4. Radial cones of T -join and T -cut polyhedra

In this section we will apply our structural results from the previous section to the radial cones of T -join and T -cut polyhedra. These results relate the extension complexities of radial cones to the extension complexities of certain faces of the blocker. We start by reproving the result of Ventura and Eisenbrand [19] for which we use the well-known theorem of Balas on unions of polyhedra.

Proposition 6 ([2]). Let $P_1, \dots, P_k \subseteq \mathbb{R}^d$ be non-empty polyhedra, and let P be the closure of $\text{conv}(P_1 \cup \dots \cup P_k)$. Then $\text{xc}(P) \leq \sum_{i=1}^k (\text{xc}(P_i) + 1)$.

Theorem 7 (Ventura & Eisenbrand, 2003 [19]). For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^\uparrow$ corresponding to a T -join $J \subseteq E_n$ in K_n , the extension complexity of the radial cone of $P_{T\text{-join}}(n)^\uparrow$ at v is most $O(|J| \cdot n^2)$.

The crucial observation for (re)proving the result is that the facets of the T -cut polyhedra have small extension complexities.

Proof. By Theorem 5 it suffices to prove that the extension complexity of

$$F := \{x \in P_{T\text{-cut}}(n)^\uparrow : \langle v, x \rangle = 1\}$$

is at most $O(|J| \cdot n^2)$. A vector $y \in \mathbb{R}^{E_n}$ is in the recession cone C of F if and only if it is nonnegative and $\langle v, y \rangle = 0$ holds. Thus, C is generated by all unit vectors corresponding to edges in $E_n \setminus J$. For every edge $e \in J$ we consider the set

$$F_e := \{x \in F : x_{e'} = 0 \forall e' \in J \setminus \{e\}\},$$

which is a face of F . Note that since $\langle v, x \rangle = 1$ is valid for F , so is $x_e = 1$. It is easy to see that F_e also has C as its recession cone. Every vertex w of F satisfies $w_e = 1$ for some edge $e \in J$, and thus $w \in F_e$, which (since F and all faces F_e have the same recession cone) proves

$$F = \text{conv}\left(\bigcup_{e \in J} F_e\right).$$

Hence, by Proposition 6, $\text{xc}(F) \leq |J| \cdot (\text{xc}(F_e) + 1)$ holds, and it remains to prove $\text{xc}(F_e) \leq O(n^2)$ for all $e \in J$. We claim that F_e is equal to

$$G_e := \{x \in P_{T'\text{-cut}}(n)^\uparrow : x_e = 1, x_{e'} = 0 \forall e' \in J \setminus \{e\}\},$$

where $T' := e$ is the set containing the two endnodes of e . Note that G_e is a face of $P_{T'\text{-cut}}(n)^\uparrow$ and hence both polyhedra are integral. Moreover, G_e also has C as its recession cone. To see that also their vertex sets agree, consider a cut $\delta(S)$ for some $S \subseteq V$. If $\delta(S)$ is a T -cut that contains e , then $\delta(S)$ is also a T' -cut. Suppose $\delta(S)$ is a T' -cut with $\delta(S) \cap J = \{e\}$. Since J is the edge-disjoint union of paths whose endnodes are distinct and in T , all such paths, except for the one that contains edge e , have both endnodes either in S or in $V_n \setminus S$. This shows that $|S \cap T|$ is odd and hence that $\delta(S)$ is a T -cut. This concludes the proof of the claim that $F_e = G_e$ holds.

Since T' contains exactly two nodes, Proposition 2 (ii) and the upper bound from (4) already yield $\text{xc}(G_e) \leq \text{xc}(P_{T'\text{-cut}}(n)) \leq O(n^2)$, which concludes the proof. \square

Since T -joins can have at most $O(n^2)$ edges, Lemma 11 establishes an $O(n^4)$ bound for the extension complexities of the radial cones of $P_{T\text{-join}}(n)^\uparrow$ at its vertices. For perfect matchings we obtain a better bound since they have only $O(n)$ edges.

Corollary 8 (Proposition 2.1 in Ventura & Eisenbrand, 2003 [19]). For every n and every vertex v of $P_{\text{pmatch}}(n)$, the extension complexity of the radial cone of $P_{\text{pmatch}}(n)$ at v is most $O(n^3)$.

Proof. The result follows from Theorem 7 and Proposition 2 (iii), using the fact that $P_{\text{pmatch}}(n)$ is a face of $P_{T\text{-join}}(n)^\uparrow$ (see Section 2).

Note that the bound is cubic since v corresponds to a perfect matching, which consists of $n/2$ edges. \square

We now generalize Lemma 11 to radial cones of $P_{T\text{-join}}(n)^\uparrow$ at non-vertices.

Corollary 9. For every n and every $v \in P_{T\text{-join}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)^\uparrow$ at v is most $O(n^4)$.

Proof. Let $P := P_{T\text{-join}}(n)^\uparrow$ and let w be a vertex of P in the smallest face that contains v . Theorem 7 implies that the extension complexity of $K_P(w)$ is at most $O(n^4)$. By definition of the radial cone, $K_P(w) \subseteq K_P(v)$, and thus, by Lemma 4, $F_{B(P)}(v) \subseteq F_{B(P)}(w)$. Using the fact that $F_{B(P)}(v)$ and $F_{B(P)}(w)$ are faces of $B(P)$, this implies that $F_{B(P)}(v)$ is a face of $F_{B(P)}(w)$. Theorem 5 and Proposition 2 (ii) yield

$$\begin{aligned} \text{xc}(K_P(v)) &\leq \text{xc}(F_{B(P)}(v)) + 1 \leq \text{xc}(F_{B(P)}(w)) + 1 \\ &\leq \text{xc}(K_P(w)) + 2 \leq O(n^4), \end{aligned}$$

which concludes the proof. \square

We continue with the main result of this paper. To prove it, we again relate the extension complexity of the radial cones to the extension complexities of certain faces of the blocker, i.e., the T -join polyhedron. In contrast to the situation for Theorem 7, these faces are again very related to T -join polyhedra, and thus have high extension complexities.

Theorem 10. For $T \subseteq V_n$ with $|T|$ even and any vertex v of $P_{T\text{-cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)^\uparrow$ at v is at least $2^{\Omega(|T|)}$.

Proof. By Theorem 5 it suffices to prove that the extension complexity of

$$P := \{x \in P_{T\text{-join}}(n)^\uparrow : \langle v, x \rangle = 1\}$$

is at least $2^{\Omega(|T|)}$. To this end, we will construct a face Q of P that is a Cartesian product of a T_1 -join polyhedron, a single point, and a T_2 -join polyhedron for some $T_1, T_2 \subseteq T$ with $|T_1| + |T_2| + 2 = |T|$. Note that, by Proposition 2 and Inequality (2), this will imply

$$\begin{aligned} \text{xc}(P) &\geq \text{xc}(Q) \\ &\geq \max\{\text{xc}(P_{T_1\text{-join}}(n_1)), \text{xc}(P_{T_2\text{-join}}(n_2))\} \\ &\geq 2^{\Omega(|T|)}. \end{aligned}$$

For subsets $V_1, V_2 \subseteq V$, we will use the notation $V_1 : V_2 := \{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\}$ as well as $E(V_1) := \{v, w\} : v, w \in V_1, v \neq w\}$. Recall that $v \in \mathbb{R}^{E_n}$ is a vertex of $P_{T\text{-cut}}(n)^\uparrow$ and hence we can partition V into sets U_1, U_2 with $|T \cap U_1|$ odd and $|T \cap U_2|$ even, such that v is the characteristic vector of $U_1 : U_2$. With this notation the set P can be rewritten as

$$P = \{x \in P_{T\text{-join}}(n)^\uparrow : x(U_1 : U_2) = 1\}.$$

Fix $t_1 \in T \cap U_1$ and $t_2 \in T \cap U_2$, and define

$$\begin{aligned} V_i &:= U_i \setminus \{t_i\}, \\ T_i &:= (T \cap U_i) \setminus \{t_i\} \quad i = 1, 2. \end{aligned}$$

Let

$$F := (V_1 : V_2) \cup (V_1 : \{t_1, t_2\}) \cup (V_2 : \{t_1, t_2\})$$

denote the set of edges that lie between (any two of) the three sets V_1, V_2 , and $\{t_1, t_2\}$, and consider the set

$$Q := \{x \in P : x_e = 0 \text{ for all } e \in F\},$$

which is a face of P . The support of each point $x \in Q$ is contained in $E(V_1) \cup E(V_2) \cup \{\{t_1, t_2\}\}$. Furthermore, for each $x \in Q$ we have

$$x_{\{t_1, t_2\}} = \underbrace{x(V_1 : V_2)}_{=0} + \underbrace{x(\{t_1\} : V_2)}_{=0} + \underbrace{x(V_1 : \{t_2\})}_{=0} \\ + x_{\{t_1, t_2\}} = x(U_1 : U_2) = 1,$$

and hence $Q = \{x \in P_{T\text{-join}}(n)^\uparrow : x_e = 0 \text{ for all } e \in F, x_{\{t_1, t_2\}} = 1\}$. In particular, we see that the extreme rays of Q are the extreme rays of P whose support is contained in $E(V_1) \cup E(V_2)$; namely, the extreme rays of Q are the characteristic vectors of sets containing a single edge in $E(V_1) \cup E(V_2)$.

Furthermore, a point w is a vertex of Q if and only if w is the characteristic vector of a T -join $H \subseteq E$ satisfying $H \subseteq E(V_1) \cup E(V_2) \cup \{\{t_1, t_2\}\}$ with $\{t_1, t_2\} \in H$. Equivalently, w is the characteristic vector of a set $H = \{t_1, t_2\} \cup H_1 \cup H_2$ where $H_i \subseteq E(V_i)$ and H_i is a T_i -join for $i = 1, 2$.

Thus, Q is the Cartesian product of a T_1 -join polyhedron (with respect to the complete graph formed by the nodes of V_1), a T_2 -join polyhedron (with respect to the complete graph formed by the nodes of V_2), and a set consisting of a single vector in $\mathbb{R}^{F \cup \{t_1, t_2\}}$, which proves the claim. \square

Notice that from Theorem 10 we obtain Theorem 1. by choosing $T := V_n$.

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Appendix A. Upper bound for small cardinalities

In this section we establish an upper bound of $O(n^2 \cdot 2^{|T|})$ on the extension complexities of T -join- and T -cut polyhedra.

Lemma 11. *For every n and every set $T \subseteq V$, the extension complexity of $P_{T\text{-join}}(n)^\uparrow$ is bounded by $O(n^2 \cdot 2^{|T|})$.*

For every node v of a directed graphs we denote by $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$ the sets of arcs leaving (resp. entering) v .

Proof. Trivially, we only have to consider the case of $|T|$ even, since $P_{T\text{-join}}(n) = \emptyset$ otherwise. Let $A := \{(u, v), (v, u) : \{u, v\} \in E_n\}$ denote the set of bidirected edges of K_n . We define, for $S \subseteq T$ with $|S| = |T|/2$ the polyhedron

$$P_S := \{x \in \mathbb{R}^{E_n} : \exists f \in \mathbb{R}_+^A :$$

$$f(\delta^{\text{out}}(v)) - f(\delta^{\text{in}}(v)) = \begin{cases} 1 & \text{if } v \in S \\ -1 & \text{if } v \in T \setminus S \\ 0 & \text{if } v \in V \setminus T \end{cases}$$

for all $v \in V$ and

$$x_{\{u,v\}} \geq f_{(u,v)} + f_{(v,u)} \text{ for all } \{u, v\} \in E_n. \tag{A.1}$$

It is easy to see that the extension of P_S is an integer polyhedron since the first set of constraints defines a totally unimodular system with integral right-hand side, and since every x -variable appears in only one of the further inequalities. Clearly, P_S is an integer polyhedron as well, since the projection on the x -variables maintains integrality.

We claim that $P_S \subseteq P_{T\text{-join}}(n)^\uparrow$ holds. To this end, let $x \in P_S$ and let $f \in \mathbb{R}_+^A$ be such that the constraints in (A.1) are satisfied. For each node $v \in T$, we obtain $x(\delta(v)) \geq \sum_{\{u,v\} \in \delta(v)} (f_{(u,v)} + f_{(v,u)}) \geq 1$. By integrality of P_S , this suffices to prove the claim.

Let now J be a T -join. It is an edge-disjoint union of circuits C_1, \dots, C_k and paths P_1, \dots, P_ℓ for $\ell = \frac{1}{2}|T|$ connecting disjoint pairs of nodes in T . For $i \in [k]$, let $\vec{C}_i \subseteq A$ be a directed version of C_i , that is, a directed cycle whose underlying undirected cycle is C_i . For $j \in [\ell]$, let $\vec{P}_j \subseteq A$ be a directed version of P_j , that is, a directed path whose underlying undirected path is P_j . Let $S \subseteq T$ be the set of starting nodes of the paths \vec{P}_j . Define $x := \chi(J)$ and for all $a \in A$, $f_a := 1$ if $a \in \vec{C}_i$ for some $i \in [k]$ or $a \in \vec{P}_j$ for some $j \in [\ell]$, and $f_a := 0$ otherwise. By construction, (x, f) satisfies the constraints in (A.1), which shows $x \in P_S$.

This proves that the vertex set of $P_{T\text{-join}}(n)^\uparrow$ is covered by the union of the polyhedra P_S for all $S \subseteq T$ with $|S| = \frac{1}{2}|T|$. Proposition 6 yields desired result since there are less than $2^{|T|}$ such sets S and $\text{xc}(P_S) \leq 3|E_n|$ holds. \square

Corollary 12. *For every n and every set $T \subseteq V$, the extension complexity of $P_{T\text{-cut}}(n)^\uparrow$ is bounded by $O(n^2 \cdot 2^{|T|})$.*

Proof. Apply Proposition 3 to Lemma 11. \square

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