# A regular equilibrium solves the extended HJB system

Kristoffer Lindensjö*

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**Abstract**

Control problems not admitting the dynamic programming principle are known as time-inconsistent. The game-theoretic approach is to interpret such problems as intrapersonal dynamic games and look for subgame perfect Nash equilibria. A fundamental result of time-inconsistent stochastic control is a verification theorem saying that solving the extended HJB system is a sufficient condition for equilibrium. We show that solving the extended HJB system is a necessary condition for equilibrium, under regularity assumptions. The controlled process is a general Itô diffusion.

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## 1 Introduction

Consider a controlled process $X^{\mathbf{u}}$ with initial data $(t,x)$ and the problem of choosing a control $\mathbf{u}$ that maximizes

$$J(t,x,\mathbf{u}) := \mathbb{E}_{t,x}\left[F(x,X_T^{\mathbf{u}})\right] + G(x,\mathbb{E}_{t,x}\left[X_T^{\mathbf{u}}\right]), \tag{1}$$

where $F$ and $G$ are deterministic functions and $t < T < \infty$ for a constant $T$. This problem is inconsistent in the sense that if a control $\mathbf{u}$ is optimal for the initial data $(t,x)$ then $\mathbf{u}$ is generally not optimal for other initial data $(s,y)$, which means that the dynamic programming principle cannot generally hold. This type of inconsistency is known as time-inconsistency.

The game-theoretic approach is to view problem (1) from the perspective of a person who controls the process $X^{\mathbf{u}}$ but whose preferences change when $(t,x)$ changes. Specifically, the problem is viewed as a sequential non-cooperative intrapersonal game regarding how to control $X^{\mathbf{u}}$; where each $(t,x)$ corresponds to one player. See [7, p. 549] for a more comprehensive interpretation along these lines. The approach is formalized by the definition of a subgame perfect Nash equilibrium, which is refinement of the notion of a Nash equilibrium for dynamic games, see Definition 2.4 below. The game-theoretic approach to time-inconsistency was first studied in a seminal paper by Strotz [33] in which utility maximization under non-exponential discounting is studied. Selten [31, 32] gave

*Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden; kristoffer.lindensjo@math.su.se

the first definition of subgame perfect Nash equilibrium, relying on the approach of Strotz.

Time-inconsistent problems were first studied in finance and economics. The time-inconsistency in this literature is typically due to the economic notions of endogenous habit formation, mean-variance utility and non-exponential discounting. These types of problems can be formulated and studied in the framework of the present paper. We formulate simple examples and give references in Section 1.1.

The first general results on the game-theoretic approach to Markovian time-inconsistent stochastic control are due to Björk et al. who around 2010 defined the extended HJB system — which is a system of simultaneously determined PDEs and an extension of the standard HJB equation — and proved a verification theorem in a general Itô diffusion setting, see the recently published [6]. An analogous treatment of discrete time Markovian time-inconsistent stochastic control is presented in [7]. Early papers in mathematical finance to study the game-theoretic approach to time-inconsistent problems are [2, 15, 16, 17] where PDE methods for specific time-inconsistent problems — that are similar to the general method relying on the extended HJB system of [6, 7] — are developed. Recent publications that use different versions of the extended HJB system to study time-inconsistent stochastic control problems include [4, 8, 9, 19, 23, 25, 26, 27, 35]. In [14], the equilibrium of a time-inconsistent control problem is characterized by a stochastic maximum principle. Time-inconsistent stopping problems are studied in e.g. [3, 10, 11, 20]. We refer to [7, 10, 11, 28, 29] for short surveys of the literature on time-inconsistent stochastic control.

Time-inconsistent problems can also be studied using the notion of dynamic optimality defined in [28, 29] and the pre-commitment approach. In the present setting pre-commitment corresponds to finding a control that maximizes (1) for a fixed $(t, x)$. For a definition of dynamic optimality and a comparison of the different approaches to time-inconsistency see [10, 28, 29].

In Section 2 we formulate the time-inconsistent stochastic control problem corresponding to (1) in more detail and give the definition of equilibrium. In Section 3 we define the extended HJB system and prove the main result Theorem 3.8 which says that solving the extended HJB system is a necessary condition for equilibrium, under regularity assumptions. To illustrate the main result we study a simple example in Section 3.1. Section 3.2 contains a more general version of the main result.

## 1.1 Reasons for time-inconsistency

To give an idea of the type of time-inconsistent problems that are typically studied in finance and economics we here formulate three simple examples. We also give references to where problems of these types are studied. For further descriptions of endogenous habit formation, mean-variance utility and non-exponential discounting, and references, see e.g. [7, 10].

**Endogenous habit formation**: Problems of this type are studied in e.g. [1, 6, 10, 13, 18, 30]. As a simple example, consider an investor who controls the evolution of the wealth process $X^{\mathbf{u}}$ by dynamically adjusting the corresponding portfolio weights, see [22] for a standard model. Suppose the terminal time utility of the investor is $F(x, X_T^{\mathbf{u}})$, where $F(x, \cdot)$ is a standard utility function for each fixed current wealth level $x$. In this case (1) becomes

$$J(t, x, \mathbf{u}) := \mathbb{E}_{t,x}\left[(F(x, X_T^{\mathbf{u}})\right].$$

From an economic point of view this may be interpreted as the investor dynamically updating a habitual preference regarding the wealth level.

**Mean-variance utility**: Problems of this type are studied in e.g. [2, 3, 4, 5, 7, 8, 11, 12, 19, 23, 25, 26, 27, 28, 29, 35]. As an example, consider the model above but an investor with mean-variance utility corresponding to

$$J(t, x, \mathbf{u}) = \mathbb{E}_{t,x}\left[X_T^{\mathbf{u}}\right] - \frac{\gamma}{2}\mathrm{Var}_{t,x}\left[X_T^{\mathbf{u}}\right], \quad \text{where } \gamma > 0.$$

The interpretation is that the investor wants a large expected wealth but is averse to risk measured by wealth variance. The parameter $\gamma$ corresponds to risk aversion.

**Non-exponential discounting**: Problems of this type are studied in e.g. [6, 7, 9, 15, 16, 17, 20, 24, 34]. As an example, consider the model above but an investor with a standard utility function $F$ and a deterministic discounting function $\varphi$ which cannot be rewritten as a standard exponential discounting function. Considering the time-space process, (1) becomes in this case

$$J(t, x, \mathbf{u}) = \mathbb{E}_{t,x}\left[\varphi(T - t)F(X_T^{\mathbf{u}})\right],$$

where $\varphi : [0, \infty) \to [0, 1]$ is non-increasing with $\varphi(0) = 1$.

# 2 Problem formulation

Consider a stochastic basis $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ where $\underline{\mathcal{F}}$ is the augmented filtration generated by a $d$-dimensional Wiener process $W$. Consider a constant time horizon $T < \infty$ and an $n$-dimensional controlled SDE

$$dX_s = \mu(s, X_s, \mathbf{u}(s, X_s))ds + \sigma(s, X_s, \mathbf{u}(s, X_s))dW_s, \quad X_t = x, \quad t \leq s \leq T, \quad (2)$$

where $\mathbf{u} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^k$, and $\mu : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to M(n, d)$ are continous and satisfy standard global Lipschitz and linear growth conditions, see e.g. [21, sec 5.2]. $M(n, d)$ denotes the set of $n \times d$ matrices.

We also consider a mapping $U$ that restricts the set of values that controls $\mathbf{u}$ may take, see Definition 2.2. Throughout the present paper we suppose $U$ and the functions $F$ and $G$ in (1) satisfy the following assumption.

**Assumption 2.1** $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $G \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$. The control constraint mapping $U : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^k}$ is such that for each $(t, x) \in [0, T) \times \mathbb{R}^n$ and each $u \in U(t, x)$ there exists a continuous control $\mathbf{u}$ with $\mathbf{u}(t, x) = u$.

Note that constant control constraint mappings, which are used in most applications, trivially satisfy the condition in Assumption 2.1.

**Definition 2.2** The set of admissible controls is denoted by $\mathbf{U}$. A control $\mathbf{u}$ is said to be admissible if: $\mathbf{u}(t, x) \in U(t, x)$ for each $(t, x) \in [0, T] \times \mathbb{R}^n$, and for each $(t, x) \in [0, T) \times \mathbb{R}^n$ the SDE (2) has a unique strong solution $X^{\mathbf{u}}$ with the Markov property satisfying $\mathbb{E}_{t,x}\left[|F(x, X_T^{\mathbf{u}})|\right] < \infty$ and $\mathbb{E}_{t,x}\left[||X_T^{\mathbf{u}}||\right] < \infty$.

**Definition 2.3** For any $\mathbf{u} \in \mathbf{U}$ the auxiliary functions $f_{\mathbf{u}} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g_{\mathbf{u}} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are defined by

$$f_{\mathbf{u}}(t, x, y) = \mathbb{E}_{t,x} \left[ F(y, X_T^{\mathbf{u}}) \right] \text{ and } g_{\mathbf{u}}(t, x) = \mathbb{E}_{t,x} \left[ X_T^{\mathbf{u}} \right].$$

We are now ready to define the subgame perfect Nash equilibrium for the time-inconsistent stochastic control problem (1). Definition 2.4 is in line with the equilibrium definition in e.g. [6, 7] to which we refer for a further motivation.

**Definition 2.4 (Equilibrium)**

- Consider a point $(t, x) \in [0, T) \times \mathbb{R}^n$, two controls $\mathbf{u}, \hat{\mathbf{u}} \in \mathbf{U}$ and a constant $h > 0$. Let

$$\mathbf{u}_h(s, y) := \begin{cases} \mathbf{u}(s, y), & \text{for } t \le s < t + h, \ \ y \in \mathbb{R}^n \\ \hat{\mathbf{u}}(s, y), & \text{for } t + h \le s \le T, \ \ y \in \mathbb{R}^n. \end{cases}$$

- The control $\hat{\mathbf{u}} \in \mathbf{U}$ is said to be an equilibrium control if, for any point $(t, x) \in [0, T) \times \mathbb{R}^n$ and any $\mathbf{u} \in \mathbf{U}$, it satisfies the equilibrium condition

$$\liminf_{h \searrow 0} \frac{J(t, x, \hat{\mathbf{u}}) - J(t, x, \mathbf{u}_h)}{h} \ge 0. \tag{3}$$

- If $\hat{\mathbf{u}}$ is an equilibrium control then $V_{\hat{\mathbf{u}}}$ defined by $V_{\hat{\mathbf{u}}}(t, x) = J(t, x, \hat{\mathbf{u}})$ is said to be the corresponding equilibrium value function and the quadruple $(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}})$ is said to be the corresponding equilibrium.

The following definition will be used throughout the present paper.

**Definition 2.5**

- The differential operator $\mathbf{A}^{\mathbf{u}}$, corresponding to (2), is defined by

$$\mathbf{A}^{\mathbf{u}} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu_i(t, x, \mathbf{u}(t, x)) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma\sigma_{ij}^T(t, x, \mathbf{u}(t, x)) \frac{\partial^2}{\partial x_i x_j}.$$

  Moreover, for any constant $u \in \mathbb{R}^k$ we define

$$\mathbf{A}^{u} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu_i(t, x, u) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma\sigma_{ij}^T(t, x, u) \frac{\partial^2}{\partial x_i x_j}.$$

- Placing the third variable as a superscript for a function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, i.e. $f^y(t, x) = f(t, x, y)$, means that $y$ is to be taken as a constant. For example, $f^y \in C^{1,2}([0, T) \times \mathbb{R}^n)$ means that $f(t, x, y)$ is continuously differentiable with respect to $t$ and twice continuously differentiable with respect to $x$ for a fixed $y$, and $\mathbf{A}^{\mathbf{u}} f^y(t, x)$ involves only derivatives with respect to $t$ and $x$. Moreover, $\mathbf{A}^{\mathbf{u}} f(t, x, x)$ should be interpreted as $\mathbf{A}^{\mathbf{u}} \bar{f}(t, x)$ with $\bar{f}(t, x) := f(t, x, x)$.

- For a function $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ we write $g(t, x) = (g_1(t, x), ..., g_n(t, x))^T$ and let $\mathbf{A}^{\mathbf{u}} g(t, x) := (\mathbf{A}^{\mathbf{u}} g_1(t, x), ..., \mathbf{A}^{\mathbf{u}} g_n(t, x))^T$.

- The operator $\mathbf{H}^{\mathbf{u}}$ is defined by

$$\mathbf{H}^{\mathbf{u}} g(t,x) = G_y(x, g(t,x))\mathbf{A}^{\mathbf{u}} g(t,x), \quad \text{where } G_y(x,y) := \frac{\partial G}{\partial y}(x,y). \tag{4}$$

  $\mathbf{H}^u$ is defined analogously.

- $$G \diamond g(t,x) := G(x, g(t,x)). \tag{5}$$

We will use the observation that (1), Definition 2.3, Definition 2.4 and (5) imply,

$$\begin{aligned} V_{\hat{\mathbf{u}}}(t,x) &= J(t,x,\hat{\mathbf{u}}) \\ &= f_{\hat{\mathbf{u}}}(t,x,x) + G \diamond g_{\hat{\mathbf{u}}}(t,x). \end{aligned} \tag{6}$$

# 3 The main result

The extended HJB system is system a of simultaneously determined PDEs which we here define in line with [6]. Remark 3.2 clarifies what constitutes a solution to the extended HJB system.

**Definition 3.1 (Extended HJB system)** For $(t,x,y) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \mathbf{A}^{\bar{\mathbf{u}}} f^y(t,x) &= 0, \\ f^y(T,x) &= F(y,x), \end{aligned} \tag{7}$$

$$\begin{aligned} \mathbf{A}^{\bar{\mathbf{u}}} g(t,x) &= 0, \\ g(T,x) &= x, \end{aligned} \tag{8}$$

$$\begin{aligned} \sup\nolimits_{u \in U(t,x)} \{ \mathbf{A}^u V(t,x) - \mathbf{A}^u f(t,x,x) + \mathbf{A}^u f^x(t,x) \\ - \mathbf{A}^u G \diamond g(t,x) + \mathbf{H}^u g(t,x) \} = 0, \end{aligned} \tag{9}$$

$$V(T,x) = F(x,x) + G(x,x) \tag{10}$$

where

$$\begin{aligned} \bar{\mathbf{u}}(t,x) \in \arg\max\nolimits_{u \in U(t,x)} \{ \mathbf{A}^u V(t,x) - \mathbf{A}^u f(t,x,x) + \mathbf{A}^u f^x(t,x) \\ - \mathbf{A}^u G \diamond g(t,x) + \mathbf{H}^u g(t,x) \}. \end{aligned} \tag{11}$$

**Remark 3.2** For a fixed function $\bar{\mathbf{u}}$ equations (7) and (8) are Kolmogorov backward equations. For fixed functions $f$ and $g$ equation (9)–(10) is an HJB equation. The non-standard attribute of (7)–(10) is that $\bar{\mathbf{u}}$, $f$ and $g$ are not fixed in this way. Instead, (7)–(10) is a system simultaneously determined through (11). Let us describe what constitutes a solution: If four functions $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{\mathbf{u}} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^k$, where $\bar{\mathbf{u}}(t,x) \in U(t,x)$ for each $(t,x) \in [0,T] \times \mathbb{R}^n$, satisfy the following conditions then $(\bar{\mathbf{u}}, V, f, g)$ is a solution to the extended HJB system:

- $f^y$ and $\bar{\mathbf{u}}$ satisfy (7), for each fixed $y \in \mathbb{R}^n$.
- $g$ and $\bar{\mathbf{u}}$ satisfy (8).
- $V$ satisfies (10).

- $V, f, g$ and $\bar{\mathbf{u}}$ satisfy (9) and (11), i.e for each fixed $(t,x) \in [0,T) \times \mathbb{R}^n$ the inequality $\mathbf{A}^u V(t,x) - \mathbf{A}^u f(t,x,x) + \mathbf{A}^u f^x(t,x) - \mathbf{A}^u G \diamond g(t,x) + \mathbf{H}^u g(t,x) \leq 0$ holds for each constant $u \in U(t,x)$, and it holds with equality for the constant $u := \bar{\mathbf{u}}(t,x)$.

In order to prove the main result, Theorem 3.8, we need Lemma 3.4, Lemma 3.5 and Proposition 3.6 below. We remark that Lemma 3.4 and Lemma 3.5 are versions of the Feynman-Kac formula. A proof is included for the sake of completeness. We will use the following definition.

**Definition 3.3** Consider a control $\mathbf{u} \in \mathbf{U}$. For a function $k : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ we write $k \in L^2_T(X^{\mathbf{u}})$ if, for each $(t,x) \in [0,T) \times \mathbb{R}^n$, there exists a constant $\bar{h} > 0$ satisfying $t + \bar{h} < T$ such that

$$\mathbb{E}_{t,x}\left[\sup_{0\leq h\leq \bar{h}}\left|\int_t^{t+h}\frac{\mathbf{A}^{\mathbf{u}}k(s,X^{\mathbf{u}}_s)}{h}ds\right| + \int_t^{t+\bar{h}}\left|\left|\frac{\partial k}{\partial x}(s,X^{\mathbf{u}}_s)\sigma(s,X^{\mathbf{u}}_s,\mathbf{u}(s,X^{\mathbf{u}}_s))\right|\right|^2 ds\right] < \infty.$$

**Lemma 3.4** Consider a continuous control $\mathbf{u} \in \mathbf{U}$. Suppose the auxiliary function $f_{\mathbf{u}}$ satisfies $f^y_{\mathbf{u}} \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap L^2_T(X^{\mathbf{u}})$, for any fixed $y \in \mathbb{R}^n$. Then, $f^y_{\mathbf{u}}$ is, for any fixed $y \in \mathbb{R}^n$, a solution to the PDE

$$\mathbf{A}^{\mathbf{u}} f^y(t,x) = 0, \;\; f^y(T,x) = F(y,x), \;\; (t,x) \in [0,T) \times \mathbb{R}^n.$$

*Proof.* By definition $f^y_{\mathbf{u}}(t,x) = \mathbb{E}_{t,x}[F(y,X^{\mathbf{u}}_T)]$ and the boundary condition is therefore satisfied. Consider an arbitrary point $(t,x,y) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^n$. Let $X^{\mathbf{u}}$ be the strong solution to the SDE (2) for the initial data $(t,x)$. Consider an arbitrary constant $h$ with $0 < h < \bar{h}$, where $\bar{h}$ is as in Definition 3.3. The Markov property and Itô's formula imply that

$$\begin{aligned} 0 &= \mathbb{E}_{t,x}\left[f^y_{\mathbf{u}}(t+h, X^{\mathbf{u}}_{t+h})\right] - f^y_{\mathbf{u}}(t,x) \\ &= \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{u}} f^y_{\mathbf{u}}(s,X^{\mathbf{u}}_s)ds\right], \end{aligned}$$

where the Itô integral vanished since $f^y_{\mathbf{u}} \in L^2_T(X^{\mathbf{u}})$. Hence,

$$\mathbb{E}_{t,x}\left[\frac{\int_t^{t+h}\mathbf{A}^{\mathbf{u}} f^y_{\mathbf{u}}(s,X^{\mathbf{u}}_s)ds}{h}\right] = 0. \tag{12}$$

The condition $f^y_{\mathbf{u}} \in L^2_T(X^{\mathbf{u}})$ implies that we can use dominated convergence when sending sending $h \searrow 0$ in (12). Moreover, the integrand in (12) is continuous in $s$ for a.e. $\omega$, since $\mu$, $\sigma$ and $\mathbf{u}$ are continuous and $f^y_{\mathbf{u}} \in C^{1,2}([0,T) \times \mathbb{R}^n)$. Hence, $\lim_{h\searrow 0}\mathbb{E}_{t,x}\left[\frac{\int_t^{t+h}\mathbf{A}^{\mathbf{u}} f^y_{\mathbf{u}}(s,X^{\mathbf{u}}_s)ds}{h}\right] = \mathbf{A}^{\mathbf{u}} f^y_{\mathbf{u}}(t,x)$. The result follows. □

**Lemma 3.5** Consider a continuous control $\mathbf{u} \in \mathbf{U}$. Suppose the elements of the auxiliary function $g_{\mathbf{u}}$ satisfy $g_{\mathbf{u},i} \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap L^2_T(X^{\mathbf{u}})$, $i = 1, ..., n$. Then $g_{\mathbf{u}}$ is a solution to the PDE

$$\mathbf{A}^{\mathbf{u}} g(t,x) = 0, \;\; g(T,x) = x, \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^n.$$

*Proof.* The proof is analogous to that of Lemma 3.4 and is omitted. □

**Proposition 3.6** Consider two controls $\mathbf{v}, \tilde{\mathbf{v}} \in \mathbf{U}$ where $\mathbf{v}$ is continuous. Suppose the auxiliary functions $f_{\tilde{\mathbf{v}}}$ and $g_{\tilde{\mathbf{v}}}$ satisfy $f^y_{\tilde{\mathbf{v}}}, g_{\tilde{\mathbf{v}},i} \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap L^2_T(X^{\mathbf{v}})$, for any fixed $y \in \mathbb{R}^n$ and $i = 1, ..., n$. Consider a point $(t,x) \in [0,T) \times \mathbb{R}^n$. Let

$$\mathbf{v}_h(s,y) := \begin{cases} \mathbf{v}(s,y), & \text{for } t \leq s < t+h, \ \ y \in \mathbb{R}^n \\ \tilde{\mathbf{v}}(s,y), & \text{for } t+h \leq s \leq T, \ \ y \in \mathbb{R}^n. \end{cases}$$

Let $v := \mathbf{v}(t,x)$. Then,

$$\lim_{h \searrow 0} \frac{f_{\tilde{\mathbf{v}}}(t,x,x) - f_{\mathbf{v}_h}(t,x,x)}{h} = -\mathbf{A}^v f^x_{\tilde{\mathbf{v}}}(t,x), \tag{13}$$

$$\lim_{h \searrow 0} \frac{G \diamond g_{\tilde{\mathbf{v}}}(t,x) - G \diamond g_{\mathbf{v}_h}(t,x)}{h} = -\mathbf{H}^v g_{\tilde{\mathbf{v}}}(t,x). \tag{14}$$

*Proof.* $\mathbb{E}_{t,x}\left[f^x_{\tilde{\mathbf{v}}}(t+h, X^{\mathbf{v}}_{t+h})\right] - f^x_{\tilde{\mathbf{v}}}(t,x) = \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} f^x_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right]$ is found as in Lemma 3.4. By definition, $\mathbf{v}_h$ and $\mathbf{v}$ coincide on $[t, t+h]$, except at the point $t+h$. By definition, $\mathbf{v}_h(s,y)$ and $\tilde{\mathbf{v}}(s,y)$ coincide on $[t+h, T]$. Thus,

$$\begin{aligned} \mathbb{E}_{t,x}\left[f^x_{\tilde{\mathbf{v}}}(t+h, X^{\mathbf{v}}_{t+h})\right] &= \mathbb{E}_{t,x}\left[\mathbb{E}_{t+h, X^{\mathbf{v}}_{t+h}}[F(x, X^{\tilde{\mathbf{v}}}_T)]\right] \\ &= \mathbb{E}_{t,x}\left[\mathbb{E}_{t+h, X^{\mathbf{v}_h}_{t+h}}[F(x, X^{\mathbf{v}_h}_T)]\right] \\ &= f_{\mathbf{v}_h}(t,x,x). \end{aligned}$$

From the above it follows that $f_{\mathbf{v}_h}(t,x,x) - f^x_{\tilde{\mathbf{v}}}(t,x) = \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} f^x_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right]$. Using arguments analogous to those in the proof of Lemma 3.4 we thus obtain

$$\begin{aligned} \lim_{h \searrow 0} \frac{f_{\tilde{\mathbf{v}}}(t,x,x) - f_{\mathbf{v}_h}(t,x,x)}{h} &= \lim_{h \searrow 0} \frac{-\mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} f^x_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right]}{h} \\ &= -\mathbf{A}^{\mathbf{v}} f^x_{\tilde{\mathbf{v}}}(t,x), \end{aligned}$$

which, since $v := \mathbf{v}(t,x)$, means that (13) holds. Using the same arguments as above we obtain $g_{\mathbf{v}_h}(t,x) = g_{\tilde{\mathbf{v}}}(t,x) + \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} g_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right]$. Standard Taylor expansion gives

$$\begin{aligned} & G\left(x, g_{\tilde{\mathbf{v}}}(t,x) + \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} g_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right]\right) \\ & \quad = G\left(x, g_{\tilde{\mathbf{v}}}(t,x)\right) + G_y\left(x, g_{\tilde{\mathbf{v}}}(t,x)\right) \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} g_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right] + o(h). \end{aligned}$$

Hence, (14) follows from,

$$\begin{aligned} & \lim_{h \searrow 0} \frac{G(x, g_{\tilde{\mathbf{v}}}(t,x)) - G(x, g_{\mathbf{v}_h}(t,x))}{h} \\ & = \lim_{h \searrow 0} \frac{-G_y\left(x, g_{\tilde{\mathbf{v}}}(t,x)\right) \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{v}} g_{\tilde{\mathbf{v}}}(s, X^{\mathbf{v}}_s) ds\right] + o(h)}{h} \\ & = -G_y(x, g_{\tilde{\mathbf{v}}}(t,x)) \mathbf{A}^{\mathbf{v}} g_{\tilde{\mathbf{v}}}(t,x). \end{aligned}$$

□

Let us now define what is meant by a regular equilibrium and present main result Theorem 3.8. An example with a regular equilibrium is studied in Section 3.1.

**Definition 3.7** An equilibrium $(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}})$ is said to be regular if:

(i). The equilibrium control $\hat{\mathbf{u}}$ is continuous.

(ii). $f^y_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}},i} \in L^2_T(X^{\hat{\mathbf{u}}})$ and $f^y_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}},i}, \bar{f} \in C^{1,2}([0,T) \times \mathbb{R}^n)$ for each fixed $y \in \mathbb{R}^n$ and $i = 1, ..., n$, where $\bar{f}(t,x) := f_{\hat{\mathbf{u}}}(t,x,x)$.

(iii). For each $(t,x) \in [0,T) \times \mathbb{R}^n$ and each $u \in U(t,x)$, there exists a continuous control $\mathbf{u} \in \mathbf{U}$ with $\mathbf{u}(t,x) = u$ such that $f^y_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}},i} \in L^2_T(X^{\mathbf{u}})$.

**Theorem 3.8** A regular equilibrium $(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}})$ solves the extended HJB system.

*Proof.* Lemma 3.4 implies that the auxiliary function $f^y_{\hat{\mathbf{u}}}(t,x)$ and the equilibrium control $\hat{\mathbf{u}}$ satisfy (7), for each $y \in \mathbb{R}^n$. Lemma 3.5 implies that the auxiliary function $g_{\hat{\mathbf{u}}}(t,x)$ and $\hat{\mathbf{u}}$ satisfy (8). Sufficient regularity for the use of these lemmas is provided by (i) and (ii) in Definition 3.7. The boundary condition (10) is directly verified using (5), (6) and Definition 2.3. Now consider an arbitrary point $(t,x) \in [0,T) \times \mathbb{R}^n$. In order to show that the equilibrium $(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}})$ is a solution to the extended HJB system we only have left to show that the inequality (15) below holds for any $u \in U(t,x)$ and that it holds with equality for $u := \hat{\mathbf{u}}(t,x)$:

$$\begin{aligned} \mathbf{A}^u V_{\hat{\mathbf{u}}}(t,x) - \mathbf{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathbf{A}^u f^x_{\hat{\mathbf{u}}}(t,x) \\ - \mathbf{A}^u G \diamond g_{\hat{\mathbf{u}}}(t,x) + \mathbf{H}^u g_{\hat{\mathbf{u}}}(t,x) \leq 0. \end{aligned} \tag{15}$$

Consider an arbitrary $u \in U(t,x)$. From (6) it follows that

$$\mathbf{A}^u V_{\hat{\mathbf{u}}}(t,x) = \mathbf{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathbf{A}^u G \diamond g_{\hat{\mathbf{u}}}(t,x), \tag{16}$$

where differentiability is provided by (ii) and Assumption 2.1. Consider a continuous control $\mathbf{u}$ satisfying $\mathbf{u}(t,x) = u$ for which $f^y_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}},i} \in L^2_T(X^{\mathbf{u}})$, cf. (iii). Use Proposition 3.6 and (16) to find

$$\begin{aligned} &\lim_{h \searrow 0} \frac{f_{\hat{\mathbf{u}}}(t,x,x) + G \diamond g_{\hat{\mathbf{u}}}(t,x) - (f_{\mathbf{u}_h}(t,x,x) + G \diamond g_{\mathbf{u}_h}(t,x))}{h} \\ &= -\mathbf{H}^u g_{\hat{\mathbf{u}}}(t,x) - \mathbf{A}^u f^x_{\hat{\mathbf{u}}}(t,x) \\ &= -(\mathbf{A}^u V_{\hat{\mathbf{u}}}(t,x) - \mathbf{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathbf{A}^u f^x_{\hat{\mathbf{u}}}(t,x) \\ &\qquad - \mathbf{A}^u G \diamond g_{\hat{\mathbf{u}}}(t,x) + \mathbf{H}^u g_{\hat{\mathbf{u}}}(t,x)). \end{aligned} \tag{17}$$

Now use the definition of $J(t,x,\mathbf{u})$ and the assumption that $\hat{\mathbf{u}}$ is an equilibrium control, cf. the equilibrium condition (3), to obtain

$$\begin{aligned} &\lim_{h \searrow 0} \frac{f_{\hat{\mathbf{u}}}(t,x,x) + G \diamond g_{\hat{\mathbf{u}}}(t,x) - (f_{\mathbf{u}_h}(t,x,x) + G \diamond g_{\mathbf{u}_h}(t,x))}{h} \\ &= \lim_{h \searrow 0} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_h)}{h} \geq 0. \end{aligned} \tag{18}$$

Recall that $u \in U(t,x)$ was arbitrarily chosen. Hence, (17) and (18) imply that (15) holds for any $u \in U(t,x)$.

Since $f^y_{\hat{\mathbf{u}}}$ and $\hat{\mathbf{u}}$ satisfy (7) for any $y$ it follows that $\mathbf{A}^{\hat{\mathbf{u}}} f^x_{\hat{\mathbf{u}}}(t,x) = 0$. Since $g_{\hat{\mathbf{u}}}$ and $\hat{\mathbf{u}}$ satisfy (8) it follows that $\mathbf{A}^{\hat{\mathbf{u}}} g_{\hat{\mathbf{u}}}(t,x) = 0$ which with (4) gives $\mathbf{H}^{\hat{\mathbf{u}}} g_{\hat{\mathbf{u}}}(t,x) = 0$. From (6) it follows that $\mathbf{A}^{\hat{\mathbf{u}}} V_{\hat{\mathbf{u}}}(t,x) = \mathbf{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}(t,x,x) + \mathbf{A}^{\hat{\mathbf{u}}} G \diamond g_{\hat{\mathbf{u}}}(t,x)$. Hence,

$$\mathbf{A}^{\hat{\mathbf{u}}} V_{\hat{\mathbf{u}}}(t,x) - \mathbf{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}(t,x,x) + \mathbf{A}^{\hat{\mathbf{u}}} f^x_{\hat{\mathbf{u}}}(t,x) - \mathbf{A}^{\hat{\mathbf{u}}} G \diamond g_{\hat{\mathbf{u}}}(t,x) + \mathbf{H}^{\hat{\mathbf{u}}} g_{\hat{\mathbf{u}}}(t,x) = 0.$$

This is equivalent to (15) holding with equality for $u := \hat{\mathbf{u}}(t,x)$. □

## 3.1 An example

Let us study a simple time-inconsistent problem to illustrate Theorem 3.8. Suppose a person controls the evolution of a one-dimensional diffusion process with constant volatility by choosing its drift function. Specifically,

$$dX_t = \mathbf{u}(t, X_t)dt + \sigma dW_t,$$

where admissible controls are restricted to the interval $U = [-a, a]$ for some $a > 0, \sigma > 0$. Suppose the person would like a large difference between the current value of the process and its value at a fixed terminal time $T$. Specifically,

$$J(t,x,\mathbf{u}) = \mathbb{E}_{t,x}\left[(X^{\mathbf{u}}_T - x)^2\right]. \tag{19}$$

This corresponds to $F(x,y) = (y-x)^2$ and $G(x,y) = 0$. We make the ansatz that $\hat{\mathbf{u}} = 0$ is an equilibrium control. Simple calculations give us the corresponding auxiliary functions $g_{\hat{\mathbf{u}}}(t,x) = x$ and $f_{\hat{\mathbf{u}}}(t,x,y) = (x-y)^2 + \sigma^2(T-t)$. Hence, $\frac{\partial f^y_{\hat{\mathbf{u}}}(t,x)}{\partial t} = -\sigma^2$, $\frac{\partial f^y_{\hat{\mathbf{u}}}(t,x)}{\partial x} = 2x - 2y$ and $\frac{\partial^2 f^y_{\hat{\mathbf{u}}}(t,x)}{\partial x^2} = 2$. Let us now show that $\hat{\mathbf{u}} = 0$ does indeed satisfy the equilibrium condition (3). Consider an arbitrary control $\mathbf{u} \in \mathbf{U}$ and an arbitrary point $(t,x)$. From Itô's formula it follows that

$$\begin{aligned}
&\mathbb{E}_{t,x}\left[f^x_{\hat{\mathbf{u}}}(t+h, X^{\mathbf{u}}_{t+h})\right] - f^x_{\hat{\mathbf{u}}}(t,x) \\
&= \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{A}^{\mathbf{u}} f^x_{\hat{\mathbf{u}}}(s, X^{\mathbf{u}}_s)ds + \sigma \int_t^{t+h} \frac{\partial f^x_{\hat{\mathbf{u}}}(s, X^{\mathbf{u}}_s)}{\partial x} dW_s\right] \\
&= \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{u}(s, X^{\mathbf{u}}_s)(2X^{\mathbf{u}}_s - 2x)ds\right].
\end{aligned}$$

Using arguments similar to those in the proof of Proposition 3.6 we find $f_{\mathbf{u}_h}(t,x,x) - f_{\hat{\mathbf{u}}}(t,x,x) = \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{u}(s, X^{\mathbf{u}}_s)(2X^{\mathbf{u}}_s - 2x)ds\right]$. Since $G(x,y) = 0$ it follows that

$$\begin{aligned}
J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_h) &= f_{\hat{\mathbf{u}}}(t,x,x) - f_{\mathbf{u}_h}(t,x,x) \\
&= \mathbb{E}_{t,x}\left[\int_t^{t+h} \mathbf{u}(s, X^{\mathbf{u}}_s)(2x - 2X^{\mathbf{u}}_s)ds\right] \\
&\geq -2a\mathbb{E}_{t,x}\left[\int_t^{t+h} |X^{\mathbf{u}}_s - x| ds\right].
\end{aligned}$$

It follows that the equilibrium condition (3) holds and that $\hat{\mathbf{u}} = 0$ therefore is an equilibrium control. The corresponding equilibrium is

$$(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}}) = (0, \sigma^2(T-t), (x-y)^2 + \sigma^2(T-t), x). \tag{20}$$

From Theorem 3.8 it follows that (20) solves the extended HJB system corresponding to $F(x,y) = (y-x)^2$ and $G(x,y) = 0$, which is also easily verified.

**Remark 3.9** The pre-commitment approach is in this example to maximize (19) over admissible controls, by treating $x$ as an arbitrary but fixed parameter. It is easy to see that the pre-commitment optimal control is, for any fixed $x$,

$$\mathbf{u}(t,y) = \begin{cases} a, & \text{for } y \geq x \\ -a, & \text{for } y < x. \end{cases}$$

## 3.2 A more general problem

In this section we include a running time function $H$ and allow $F$ and $G$ to depend on the initial time $t$. Specifically, we consider

$$\begin{aligned} J(t,x,\mathbf{u}) := \mathbb{E}_{t,x}&\left[\int_t^T H(t,x,r,X_r^{\mathbf{u}},\mathbf{u}(r,X_r^{\mathbf{u}}))dr + F(t,x,X_T^{\mathbf{u}})\right] \\ &+ G(t,x,\mathbb{E}_{t,x}\left[X_T^{\mathbf{u}}\right]) \end{aligned}$$

where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfy conditions analogous to those in Assumption 2.1 and $H : [0,T] \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ is continuous and bounded. The definition of an admissible control is analogous to Definition 2.2. Let

$$\begin{aligned} G \diamond g(t,x) &:= G(t,x,g(t,x)), \\ \mathbf{H}^{\mathbf{u}} g(t,x) &:= G_y(t,x,g(t,x))\mathbf{A}^{\mathbf{u}} g(t,x), \\ f_{\mathbf{u}}(t,x,s,y) &:= \mathbb{E}_{t,x}\left[\int_t^T H(s,y,r,X_r^{\mathbf{u}},\mathbf{u}(r,X_r^{\mathbf{u}}))dr + F(s,y,X_T^{\mathbf{u}})\right]. \end{aligned}$$

The equilibrium definition is analogous to Definition 2.4. Placing the third and fourth variables of a function $f : [0,T] \times \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ as superscripts, i.e. $f^{s,y}(t,x) = f(t,x,s,y)$, means $s$ and $y$ are to be taken as constant.

**Definition 3.10 (Extended HJB system II)** For $(t,x,s,y) \in [0,T) \times \mathbb{R}^n \times [0,T) \times \mathbb{R}^n$,

$$\begin{aligned} \mathbf{A}^{\bar{\mathbf{u}}} f^{s,y}(t,x) + H(s,y,t,x,\bar{\mathbf{u}}(t,x)) &= 0, \\ f^{s,y}(T,x) &= F(s,y,x), \\ \mathbf{A}^{\bar{\mathbf{u}}} g(t,x) &= 0, \\ g(T,x) &= x, \end{aligned}$$

$$\begin{aligned} \sup\nolimits_{u \in U(t,x)} \{ &\mathbf{A}^u V(t,x) - \mathbf{A}^u f(t,x,t,x) + \mathbf{A}^u f^{t,x}(t,x) \\ &-\mathbf{A}^u G \diamond g(t,x) + \mathbf{H}^u g(t,x) + H(t,x,t,x,u)\} = 0, \\ &V(T,x) = F(T,x,x) + G(T,x,x), \end{aligned}$$

where

$$\begin{aligned} \bar{\mathbf{u}}(t,x) \in \arg\max\nolimits_{u \in U(t,x)} \{ &\mathbf{A}^u V(t,x) - \mathbf{A}^u f(t,x,t,x) + \mathbf{A}^u f^{t,x}(t,x) \\ &-\mathbf{A}^u G \diamond g(t,x) + \mathbf{H}^u g(t,x) + H(t,x,t,x,u)\}. \end{aligned}$$

The definition of a regular equilibrium is analogous to that of Definition 3.7. Theorem 3.11 generalizes the main result of this paper to the present setting. The proof is analogous to that of Theorem 3.8 and is omitted.

**Theorem 3.11** A regular equilibrium $(\hat{\mathbf{u}}, V_{\hat{\mathbf{u}}}, f_{\hat{\mathbf{u}}}, g_{\hat{\mathbf{u}}})$ solves the extended HJB system II.

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