Product Sequencing and Pricing under Cascade Browse Model

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Abstract

In this paper, we study the joint product sequencing and pricing problem faced by many online retailers such as Amazon. We assume that a consumer's purchasing behavior can be explained by a "consider-then-choose" model: she first forms a consideration set by screening a subset of products sequentially, and then decides which product to purchase from her consideration set. We propose a cascade browse model to capture the consumer's browsing behavior, and use the Multinomial Logit (MNL) model as our choice model. We study two problems in this paper: in the first problem, we assume that each product has a fixed revenue and preference weight, the goal is to identify the best sequence of products to offer so as to maximize the expected revenue subject to a cardinality constraint. We propose a constant approximate solution to this problem. As a byproduct, we propose the first fully polynomial-time approximation scheme (FPTAS) for the classic assortment optimization problem subject to one capacity constraint and one cardinality constraint. In the second problem, we treat the price of each product as a decision variable and our objective is to jointly decide a sequence of product and their prices to maximize the expected revenue. We propose a constant approximate solution to this problem.

Preprint submitted to Journal of LTFX Templates

1. Introduction

In this paper, we consider the setting where the platform has a set of products and a limited number of vertically differentiated display positions. Our objective is to select a sequence of products, as well as their prices, so as to maximize the expected revenue. We assume that the consumer's purchasing behavior is governed by a "consider-then-choose" model. Once a sequence of products is displayed to a consumer, she first forms her consideration set by browsing the products sequentially, then chooses one product (or nothing) from her consideration set based on MNL model. This modeling approach is well established in the literature in quantitative marketing and operations [1].

We depart from existing literature in assuming that the browsing behavior of a consumer is governed by a cascade browse model. Our model is inspired by the cascade click model proposed in [2], however, their purpose is different from ours, e.g., their goal is to find a good model that captures the click behavior of a consumer in the context of online advertising. We next give a brief introduction to our cascade browse model. Under the cascade browse model, each product is associated with a continuation probability which represents the probability that a consumer continues to browse the next product after browsing the current product. The purchasing decision process of a customer can be roughly described as follows. Upon viewing a sequence of products, she adds the product displayed in the first position to her consideration set. Then she will decide whether to make a purchasing decision, including non-purchase option, or to continue adding the product in the next position to her consideration set. Once a consumer decides to make a purchase from among her current consideration set, we assume that her purchasing decision is governed by MNL model, and she will leave the system after the purchase. Otherwise, if she decides to continue adding the next product to her consideration set, the whole process continues with the next product until she makes a purchasing decision at some point.

Our main contributions are summarized as follows.

 We introduce a cascade browse model to capture the consumer's browsing behavior. Our model not only captures the position-bias effect but also considers the externality among displayed products.

- 2. We propose constant approximate solutions to the corresponding assortment optimization and pricing problems.
- As a byproduct, we propose the first fully polynomial-time approximation scheme (FPTAS) for the classic assortment optimization problem subject to one capacity constraint and one cardinality constraint.

Literature Review: Our work is closely related to the assortment optimization problems [3, 4, 5, 6, 7], which have been extensively studied in the revenue management literature. We have limited our review to studies under the MNL model with position bias. [8] and [9] were the first to study the assortment optimization problem under MNL model with position bias. However, they assume that consumers browse all displayed products, and the location of a product only affects its MNL-based preference weight. In our study, we assume that the consumer only browses a subset of displayed products, and then makes a purchase from among all browsed products. Recently, [10, 11] considers the assortment optimization problem with vertically differentiated locations under MNL model. Similar to our work, they adopt the consider-then-choose model, e.g., the consumer first browses a random number of products and then makes a choice within these products according to the MNL model. Different from our cascade browse model, their model does not capture the externality among displayed products, e.g., they assume that the probability of a displayed product being browsed is solely dependent on its location, which is not affected by other displayed products. [1] also study the consider-then-choose model, however, they assume that each product is considered by a customer with a fixed probability. Another stream of literature introduces the search cost in the consumer choice model. For example, [12] assume that a consumer selects a group of products as her consideration set such that the expected utility net of search cost is maximized. Our study is also related to assortment pricing. In addition to the literature on assortment pricing under standard MNL model [13, 14, 15], [16] studied a pricing problem based on a cascade click model. Our model differs from theirs in that we use a combination of cascade browse model and MNL model to capture the consumer's purchasing behavior. Moreover, they assume that the sequence of displayed products is fixed.

2. Preliminaries and Problem Formulation

We consider the setting where the platform has N products $\Omega = [N]$ and B vertically differentiated display positions. Our objective is to allocate B products to B display locations so as to maximize the expected revenue of the platform. We use a "consider-then-choose" model to capture the consumers' purchasing behavior. Once products are allocated to display locations, a consumer chooses among products in two phases: she first forms a *consideration set* by sequentially examining the products according to the linear order of locations, then decides which product to purchase from among her consideration set. In this paper, we propose a cascade browse model to model the consumer's browsing behavior in the first phase and use the MNL model to capture the consumer's purchase behavior in the second phase. We next introduce our "consider-then-choose" model in more details.

Some notations. Throughout this paper, we use capital letter to denote a sorted sequence of products, and use corresponding calligraphy letter to denote a set of products. For example, given a sequence of products S, we use S denote the set of products involved in S. Moreover, we use $S_{\leq i}$ (resp. $S_{<i}, S_{>i}, S_{\geq i}$) to denote the longest subsequence of S which is placed no later than (resp. before, after, no earlier than) product i. Correspondingly, we use $S_{\leq i}$ (resp. $S_{<i}, S_{>i}, S_{\geq i}$) to denote the set of products in $S_{\leq i}$ (resp. $S_{<i}, Q_{>i}, S_{\geq i}$). For example, given a sequence of prodicts $S = \{3, 4, 1, 2, 5\}$, $S_{<1} = \{3, 4\}$, we have $S_{>1} = \{2, 5\}, S_{\leq 1} = \{3, 4, 1\}$, and $S_{\geq 1} = \{1, 2, 5\}$. Given two sequences S_1 and S_2 , we define $S_1 \oplus S_2$ as a new sequence by first displaying S_1 and then displaying S_2 .

2.1. Phase 1: Forming a consideration set

Under our cascade browse model, each product $i \in \Omega$ has a continuation probability θ_i which represents the probability that a consumer continues to browse the next product (if any) after browsing *i*.

Given a sequence of products S and a product $i \in S$, we define the *reachability* Θ_i^S

of i given S as the probability that a consumer browses i given S:

$$\Theta_i^S = \prod_{j \in \mathcal{S}_{< i}} \theta_j$$

It follows that for any S and $i \in S$, the probability $\Pr[S_{\leq i} | S]$ that a consumer browses all and only products in $S_{\leq i}$ is

$$\Pr[\mathcal{S}_{\leq i} \mid S] = \begin{cases} \Theta_i^S, \text{ if } i \text{ is the last product in } S\\ \Theta_i^S(1-\theta_i), \text{ otherwise} \end{cases}$$

We refer to $S_{\leq i}$ as the random consideration set induced by S.

2.2. Phase 2: Making a purchase

In the second phase, the consumer follows MNL model to make a purchase from among her consideration set. In the MNL model, each product $i \in \Omega$ is associated with a revenue α_i , and a MNL-based preference weight β_i . For any given realized consideration set S, the probability $P_i(S)$ that $i \in S$ is purchased by a single representative consumer is

$$P_i(\mathcal{S}) = \frac{\beta_i}{\sum_{i \in \mathcal{S}} \beta_i + 1} \tag{1}$$

The expected revenue g(S) of the consideration set S is

$$g(\mathcal{S}) = \sum_{i \in \mathcal{S}} \alpha_i P_i(\mathcal{S}) \tag{2}$$

The expected revenue f(S) of a sequence of products S can be calculated as

$$f(S) = \sum_{i \in S} \Pr[\mathcal{S}_{\leq i} \mid S] g(\mathcal{S}_{\leq i})$$

2.3. Problem Formulation

We study two problems in this paper. In the first problem, given that α_i and β_i are constants for each $i \in \Omega$, we aim to determine the best sequence of products that maximizes the expected revenue. In the second problem, we relax the assumption that α_i and β_i are pre-fixed and assume that α_i and β_i are functions of *i*'s price. By treating the price of each product as a decision variable, we study the joint product selection, sequencing, and pricing problem.

2.3.1. Revenue maximizing product selection and sequencing

We first introduce the revenue maximizing product selection and sequencing problem. Assume α_i and β_i are given for each $i \in \Omega$, our objective is to identify the best subset of *B* products, as well as their sequence, so as to maximize the expected revenue. We present the formal definition of our problem in **P.A**.

P.A max $f(S)$	
subject to: $ S \leq B$;	

2.3.2. Joint product selection, sequencing, and pricing

In addition to optimizing the positioning of the products, the platform could also adjust the price of each product to improve the expected revenue. We next study the case when the price of a product is also a decision variable. To capture the impact of price on a consumer's purchase decision, it is common practise to interpret α_i , as well as β_i , as a function of *i*'s quality, price, and cost for any $i \in \Omega$. Assume each product $i \in \Omega$ is associated with a *fixed* quality q_i , an *adjustable* price p_i , and a *fixed* cost c_i . If we set $\alpha_i = p_i - c_i$ and $\beta_i = e^{q_i - p_i}$ and plug into (1) and (2), we get

$$P_i(\mathcal{S}, \mathbf{p}) = \frac{e^{q_i - p_i}}{\sum_{i \in \mathcal{S}} e^{q_i - p_i} + 1}$$
(3)

where $\mathbf{p} = \{p_i \mid i \in \Omega\} \in \mathbb{R}^{|\Omega|}$ is a price vector. The expected revenue g(S) of S is

$$g(\mathcal{S}, \mathbf{p}) = \sum_{i \in \mathcal{S}} (p_i - c_i) P_i(\mathcal{S})$$
(4)

The expected revenue of S and price \mathbf{p} can be calculated as

$$f(S, \mathbf{p}) = \sum_{i \in \mathcal{S}} \Pr[\mathcal{S}_{\leq i} \mid S] g(\mathcal{S}_{\leq i}, \mathbf{p})$$

Our objective is to jointly decide a sequence of products and their prices to maximize the expected revenue subject to a cardinality constraint. We next present the formal definition of the second problem **P.B**.

P.B max $f(S, \mathbf{p})$	
subject to: $ S \leq B$;	

3. Revenue maximizing product selection and sequencing

We first study the case when α_i and β_i are fixed for all products $i \in \Omega$. Our objective is to identify the best sequence of products that maximizes the expected revenue subject to a cardinality constraint. Before presenting our solution to **P.A**, we show that given any optimal solution O to **P.A**, we can safely remove those products whose reachability is sufficiently small from O such that it does not affect the expected revenue of O much.

Lemma 1. For any $\rho \in [0, 1]$, there is a solution Q of expected revenue at least

$$f(Q) \ge (1-\rho)f(O)$$

such that $|Q| \leq B$ and $\forall i \in Q : \Theta_i^Q \geq \rho$.

Proof: Let O[t] denote the *t*-th product in *O*. Assume O[k] is the last question in *O* whose reachability is no smaller than ρ , e.g., $k = \arg \max_t(\Theta_{O[t]}^O \ge \rho)$. Recall that we use $O_{>k}$ (resp. $O_{\le k}$) to denote the sequence of questions scheduled after (resp. no later than) slot *k*. Therefore the reachability of every question in $O_{\le k}$ is no smaller than ρ .

We first show that $\rho f(O_{>k}) \ge f(O) - f(O_{\le k})$. Let e_S denote the event that a consumer browses all and only products in S. We define $\Lambda_1 = \{O_{\le t} \mid t \le k\}$ and $\Lambda_2 = \{O_{\le t} \mid t > k\}.$

$$f(O) = \sum_{A \in \Lambda_1} \Pr[e_A]g(\mathcal{A}) + \Pr[e_{O_{\leq k}}]g(\mathcal{O}_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A]g(\mathcal{A})$$

$$\leq \sum_{A \in \Lambda_1} \Pr[e_A]g(\mathcal{A}) + \Pr[e_{O_{\leq k}}]g(\mathcal{O}_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A](g(\mathcal{A}) - g(\mathcal{O}_{\leq k}) + g(\mathcal{O}_{\leq k}))$$
(5)

$$= \sum_{A \in \Lambda_1} \Pr[e_A]g(\mathcal{A}) + \Pr[e_{\mathcal{O}_{\leq k}}]g(\mathcal{O}_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A]g(\mathcal{O}_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A](g(\mathcal{A}) - g(\mathcal{O}_{\leq k}))$$

$$\leq f(O_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A](g(\mathcal{A}) - g(\mathcal{O}_{\leq k}))$$
(6)

$$\leq f(O_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A] g(\mathcal{A} \setminus \mathcal{O}_{\leq k})$$
(7)

$$= f(O_{\leq k}) + \Theta^{O}_{O[k]} \theta_{O[k]} f(O_{>k})$$
(8)

$$\leq f(O_{\leq k}) + \rho f(O_{>k}) \tag{9}$$

Inequality (6) is due to $\sum_{A \in \Lambda_1} \Pr[e_A]g(\mathcal{A}) + \Pr[e_{O_{\leq k}}]g(\mathcal{O}_{\leq k}) + \sum_{A \in \Lambda_2} \Pr[e_A]g(\mathcal{O}_{\leq k}) = f(O_{\leq k})$. Inequality (9) is due to $k = \arg\max_t(\Theta_{O[t]}^O \geq \rho)$. It remained to prove inequality (7), e.g., $g(\mathcal{A}) - g(\mathcal{O}_{\leq k}) \leq g(\mathcal{A} \setminus \mathcal{O}_{\leq k})$ for any $\mathcal{A} \in \Lambda_2$.

$$g(\mathcal{A}) - g(\mathcal{O}_{\leq k}) = \sum_{i \in \mathcal{A}} \frac{\alpha_i \beta_i}{\sum_{i \in \mathcal{A}} \beta_i + 1} - \sum_{i \in \mathcal{O}_{\leq k}} \frac{\alpha_i \beta_i}{\sum_{i \in \mathcal{O}_{\leq k}} \beta_i + 1}$$
(10)

$$= \sum_{i \in \mathcal{A} \cap \mathcal{O}_{\leq k}} \alpha_i \left(\frac{\beta_i}{\sum_{i \in \mathcal{A}} \beta_i + 1} - \frac{\beta_i}{\sum_{i \in \mathcal{O}_{\leq k}} \beta_i + 1} \right) \quad (11)$$

$$+\sum_{i\in\mathcal{A}\setminus\mathcal{O}_{\leq k}}\frac{\alpha_i\beta_i}{\sum_{i\in\mathcal{A}}\beta_i+1}\tag{12}$$

$$\leq \sum_{i \in \mathcal{A} \setminus \mathcal{O}_{\leq k}} \frac{\alpha_i \beta_i}{\sum_{i \in \mathcal{A}} \beta_i + 1}$$
(13)

$$\leq \sum_{i \in \mathcal{A} \setminus \mathcal{O}_{\leq k}} \frac{\alpha_i \beta_i}{\sum_{i \in \mathcal{A} \setminus \mathcal{O}_{\leq k}} \beta_i + 1} = g(\mathcal{A} \setminus \mathcal{O}_{\leq k})$$
(14)

This finishes the proof of $\rho f(O_{>k}) \ge f(O) - f(O_{\le k})$. Due to O is the optimal solution to **P.A**, we have $f(O_{>k}) \le f(O)$. It follows that $(1 - \rho)f(O) \le f(O_{\le k})$. Because $|O_{\le k}| \le B$ and all products in $O_{\le k}$ can be browsed with probability at least $\rho, O_{\le k}$ is one such sequence that satisfies all conditions in Lemma 1. \Box

Lemma 2. For any $\rho \in [0, 1]$, there is a sequence R with $|R| \leq b$ and $\forall i \in R : \Theta_i^R \geq \rho$ such that

$$g(\mathcal{R}) \ge (1-\rho)f(O)$$

Proof: Based on Lemma 1, for any $\rho \in [0, 1]$, there is a solution Q of expected profit at least $f(Q) \ge (1 - \rho)f(Q)$. Let Q[t] denote the *t*-th product in Q. Assume that |Q| = k, we have

$$f(Q) = \sum_{t \in [1,k-1]} \Theta_{Q[t]}^Q (1 - c_{Q[t]}) g(\mathcal{Q}_{\leq t}) + \Theta_{Q[k]}^Q g(\mathcal{Q}_{\leq k}) \ge (1 - \rho) f(O)$$

The equality is due to the definition of f(Q). Because $\sum_{t \in [1,k-1]} \Theta_{Q[t]}^Q (1 - c_{Q[t]}) + \Theta_{Q[k]}^Q = 1$, we have

$$\max_{t \in [1,k]} g(\mathcal{Q}_{\le t}) \ge (1-\rho)f(O)$$

Because $\max_{t \in [1,k]} g(\mathcal{Q}_{\leq t})$ is a subsequence of Q, we have $|\max_{t \in [1,k]} g(\mathcal{Q}_{\leq t})| \leq B$ and every product in $\max_{t \in [1,k]} g(\mathcal{Q}_{\leq t})$ has readability at least ρ . Therefore, $\max_{t \in [1,k]} g(\mathcal{Q}_{\leq t})$ is one such sequence that satisfies all conditions in Lemma 2. \Box

Based on Lemma 2, in order to obtain a near-optimal solution, it suffice to consider those products whose reachability is sufficiently high. This motivates us to introduce a new problem **P.A.1**. The goal of **P.A.1** is to find a sequence of products that maximizes the expected revenue while ignoring those products whose reachability is sufficiently small. The solution to **P.A.1** is composed of two parts: a set of products S and a single product $y \in \Omega$, where y is displayed after S. The reason we separate y from other products in S is that y is scheduled at the last slot, thus there is no restriction on y's continuation probability. Constraint (C1) ensures that every product can be viewed with probability at least ρ , and constraint (C2) ensures that the size of our solution is upper bounded by B.

> **P.A.1** *Maximize*_{$y,S \subseteq \Omega \setminus \{y\}$} $g(S \cup y)$ **subject to:** $\begin{cases}
> -\sum_{i \in S} \log(\theta_i) \leq -\log \rho \quad (C1) \\
> |S| < B \quad (C2)
> \end{cases}$

We next present our algorithm Algorithm 1.

Description of Algorithm 1.

- 1. We first propose a fully polynomial-time approximation scheme (FPTAS) for **P.A.1**.
- 2. After solving **P.A.1** (approximately) and obtaining a solution $(S^{\text{alg}_1}, y^{\text{alg}_1})$, we build the final solution by first displaying S^{alg_1} (an arbitrary sequence of S^{alg_1}) and then displaying y^{alg_1} .

In the rest of this section, we first present a FPTAS for **P.A.1** and then analyze the performance of Algorithm 1.

3.1. FPTAS for P.A.1

Our basic idea, inspired by [17], is to enumerate the last product $y \in \Omega$ in the optimal solution to **P.A.1**, for each fixed y, we present a FPTAS for **P.A.1**. At last, we choose the solution with the largest expected revenue as the final solution to **P.A.1**. Note that given a fixed y, **P.A.1** reduces to an assortment optimization problem subject to one capacity constraint and one cardinality constraint. [18] develops a FPTAS for the assortment optimization problem subject to one capacity constraint, we extend their solution and provide a FPTAS, inspired by [19], to **P.A.1** when y is fixed, e.g., we provide the first FPTAS for the assortment optimization problem subject to one capacity constraint and one cardinality constraint.

Note that when y is fixed, we only consider those products in $\Omega \setminus \{y\}$. For ease of presentation, we relabel all products in $\Omega \setminus \{y\}$ such that $\Omega \setminus \{y\} = [N - 1]$. We first introduce some notations. Let $\alpha_{\min} = \min_{i \in \Omega} \alpha_i$ be the minimum revenue of a single product and $\alpha_{\max} = \max_{i \in \Omega} \alpha_i$ be the maximum revenue of a single product. Let $\beta_{\min} = \min_{i \in \Omega} \beta_i$ and $\beta_{\max} = \max_{i \in \Omega} \beta_i$. Let $\gamma_i = \alpha_i \beta_i$, $\gamma_{\min} = \min_{i \in \Omega} \gamma_i$, and $\gamma_{\max} = \max_{i \in \Omega} \gamma_i$.

For a given $\epsilon > 0$, we build the following group of guesses.

$$I = \{\gamma_{\min}(1+\epsilon)^a \mid a \le \ln \frac{N\gamma_{\max}}{\epsilon\gamma_{\min}}\}, J = \{\beta_{\min}(1+\epsilon)^b \mid b \le \ln \frac{N\beta_{\max}}{\epsilon\beta_{\min}}\}$$

For a given guess $\gamma_{\min}(1+\epsilon)^a \in I$ and $\beta_{\min}(1+\epsilon)^b \in J$, we discretize the values of γ_i and β_i as follows,

$$\tilde{\gamma_i} = \lceil \frac{\gamma_i}{\gamma_{\min}(1+\epsilon)^a \epsilon/B} \rceil, \tilde{\beta_i} = \lfloor \frac{\beta_i}{\beta_{\min}(1+\epsilon)^b \epsilon/B} \rfloor$$

We use ω_i to denote $-\log(\theta_i)$ for all $i \in \Omega \setminus \{y\}$. Denote by function h(j, u, v, l)for $j \in [N], u \in [\lceil N^2/\epsilon \rceil], v \in [\lceil N^2/\epsilon \rceil], l \in [B]$ the optimal solution value of the following problem:

$$h(j, u, v, l) := \min\{\sum_{i=1}^{j} \omega_i x_i : \sum_{i=1}^{j} \tilde{\gamma}_i x_i = u, \sum_{i=1}^{j} \tilde{\beta}_i x_i = v, \sum_{i=1}^{j} x_i = l, x_i \in \{0, 1\}, l \in [B]\}$$

We set the initial values as follows: we first set $h(j, u, v, l) = +\infty$ for $i = 0, u \in [\lceil N^2/\epsilon \rceil], v \in [\lceil N^2/\epsilon \rceil], l \in [B]$, and then set h(0, 0, 0, 0) = 0.

Then we fill up the dynamic program table using the following recurrence function.

$$h(j, u, v, l) = \begin{cases} h(j - 1, u, v, l) & \text{if } u < \gamma_j \text{ or } v < \beta_j; \\\\ \min \begin{cases} h(j - 1, u, v, l) & \\ h(j - 1, u - \gamma_j, v - \beta_j, l - 1) + \omega_j \end{cases} & \text{otherwise.} \end{cases}$$

After filling up the dynamic programming table, we go through all entries with $h(j, u, v, l) \leq -\log \rho$, and return the solution with the largest expected revenue. We repeat the above process for every guess in $I \times J$, then return the one with the largest expected revenue as the final solution to **P.A.1**. Denote the returned solution by $(S^{\text{alg}_1}, y^{\text{alg}_1})$. We next prove that the above solution achieves $\frac{1-\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}$ approximation ratio for **P.A.1**.

Lemma 3. Let (S^*, y^*) denote the optimal solution to **P.A.1**. For any $\epsilon > 0$,

$$g(\mathcal{S}^{\mathrm{alg}_1} \cup y^{\mathrm{alg}_1}) \ge \frac{1 - \epsilon(1 + \epsilon)}{1 + \epsilon(1 + \epsilon)} g(\mathcal{S}^* \cup y^*)$$

Proof: Assume $\gamma_{\min}(1+\epsilon)^a \leq \sum_{i\in\mathcal{S}^*} \gamma_i \leq \gamma_{\min}(1+\epsilon)^{a+1}$ and $\beta_{\min}(1+\epsilon)^b \leq \sum_{i\in\mathcal{S}^*} \beta_i \leq \beta_{\min}(1+\epsilon)^{b+1}$. Recall that to obtain a FPTAS for **P.A.1**, we enumerate the last product $o \in \Omega$ and solve the dynamic program for each y and each guess in $I \times J$. Consider the case when $(y^*, \gamma_{\min}(1+\epsilon)^{a+1} \in I, \beta_{\min}(1+\epsilon)^{b+1} \in J)$ is enumerated, let $u^* = \sum_{i\in\mathcal{S}^*} \tilde{\gamma}_i$ and $v^* = \sum_{i\in\mathcal{S}^*} \tilde{\beta}_i$ denote the summation of the scaled values. It is clear that $h(N, u^*, v^*, |\mathcal{S}^*|) \leq -\log \rho$, let S' denote the solution stored in $h(N, u^*, v^*, |\mathcal{S}^*|)$. We first give a lower bound on $\sum_{z\in S'} \gamma_z$,

$$\sum_{z \in S'} \gamma_z \geq \sum_{z \in S'} \tilde{\gamma_z} \epsilon \gamma_{\min} (1+\epsilon)^{a+1} / B - \epsilon \gamma_{\min} (1+\epsilon)^{a+1}$$
(15)

$$= u^* \epsilon \gamma_{\min}(1+\epsilon)^{a+1} / B - \epsilon \gamma_{\min}(1+\epsilon)^{a+1}$$
(16)

$$\geq u^* \epsilon \gamma_{\min}(1+\epsilon)^{a+1} / B - \epsilon (1+\epsilon) \sum_{i \in \mathcal{S}^*} \gamma_i \tag{17}$$

$$\geq \sum_{i \in \mathcal{S}^*} \gamma_i - \epsilon (1+\epsilon) \sum_{i \in \mathcal{S}^*} \gamma_i \tag{18}$$

$$= (1 - \epsilon(1 + \epsilon)) \sum_{i \in \mathcal{S}^*} \gamma_i$$
(19)

where the last inequality is due to $\tilde{\gamma_i} \ge \frac{\gamma_i}{\gamma_{\min}(1+\epsilon)^{a+1}\epsilon/n}$ for all $i \in \Omega$.

Then we give an upper bound on $\sum_{z \in S'} \beta_z$,

$$\sum_{z \in S'} \beta_z \leq \sum_{z \in S'} \tilde{\beta}_z \epsilon \beta_{\min} (1+\epsilon)^{b+1} / B + \epsilon \beta_{\min} (1+\epsilon)^{b+1}$$
(20)

$$= v^* \epsilon \beta_{\min} (1+\epsilon)^{b+1} / B + \epsilon \beta_{\min} (1+\epsilon)^{b+1}$$
(21)

$$\leq v^* \epsilon \beta_{\min} (1+\epsilon)^{b+1} / B + \epsilon (1+\epsilon) \sum_{i \in \mathcal{S}^*} \beta_i$$
(22)

$$\leq \sum_{i \in \mathcal{S}^*} \beta_i + \epsilon (1+\epsilon) \sum_{i \in \mathcal{S}^*} \beta_i$$
(23)

$$= (1 + \epsilon(1 + \epsilon)) \sum_{i \in S^*} \beta_i$$
(24)

where the last inequality is due to $\tilde{\beta}_i \leq \frac{\beta_i}{\beta_{\min}(1+\epsilon)^{a+1}\epsilon/n}$ for all $i \in \Omega$.

It follows that

$$g(\mathcal{S}^{\mathrm{alg}_1} \cup y^{\mathrm{alg}_1}) \geq g(\mathcal{S}' \cup y^*)$$
(25)

$$= \frac{\sum_{z \in S'} \gamma_z + \gamma_{y^*}}{\sum_{z \in S'} \beta_z + \beta_{y^*} + 1}$$
(26)

$$\geq \frac{(1-\epsilon(1+\epsilon))\sum_{i\in\mathcal{S}^*}\gamma_i+\gamma_{y^*}}{(1+\epsilon(1+\epsilon))\sum_{i\in\mathcal{S}^*}\beta_i+\beta_{y^*}+1}$$
(27)

$$\geq \frac{1-\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}g(\mathcal{S}^* \cup y^*)$$
(28)

3.2. Performance analysis of Algorithm 1

We next analyze the performance bound of Algorithm 1. Recall that after obtaining $(S^{\text{alg}_1}, y^{\text{alg}_1})$ from the previous stage, Algorithm 1 returns $S^{\text{alg}_1} \oplus y^{\text{alg}_1}$ as the final solution where S^{alg_1} is an arbitrary sequence of S^{alg_1} . We next prove that for any $\epsilon > 0$ and $\rho \in [0, 1]$, Algorithm 1 achieves $\frac{1-\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}\rho(1-\rho)$ approximation ratio.

Theorem 1. For any $\epsilon > 0$ and $\rho \in [0, 1]$,

$$f(S^{\text{alg}_1} \oplus y^{\text{alg}_1}) \ge \frac{1 - \epsilon(1 + \epsilon)}{1 + \epsilon(1 + \epsilon)}\rho(1 - \rho)f(O)$$

Proof: Due to constraint (C2), a consumer views all product in $S^{alg_1} \oplus y^{alg_1}$ with probability at least ρ , then we have

$$f(S^{\operatorname{alg}_1} \oplus y^{\operatorname{alg}_1}) \ge \rho g(\mathcal{S}^{\operatorname{alg}_1} \cup y^{\operatorname{alg}_1})$$
⁽²⁹⁾

Based on Lemma 3, we have

$$g(\mathcal{S}^{\mathrm{alg}_1} \cup y^{\mathrm{alg}_1}) \ge \frac{1 - \epsilon(1 + \epsilon)}{1 + \epsilon(1 + \epsilon)} g(\mathcal{S}^* \cup y^*)$$
(30)

(29) and (30) imply that

$$f(S^{\text{alg}_1} \oplus y^{\text{alg}_1}) \ge \rho \frac{1 - \epsilon(1 + \epsilon)}{1 + \epsilon(1 + \epsilon)} g(\mathcal{S}^* \cup y^*)$$
(31)

In Lemma 4 (whose proof we defer until later), we prove that

$$g(\mathcal{S}^* \cup y^*) \ge (1 - \rho)f(O) \tag{32}$$

(31) and (32) imply that $f(S^{\text{alg}_1} \oplus y^{\text{alg}_1}) \ge \frac{1-\epsilon(1+\epsilon)}{1+\epsilon(1+\epsilon)}\rho(1-\rho)f(O)$. \Box

We next focus on proving Lemma 4.

Lemma 4. *For any* $\rho \in [0, 1]$ *,*

$$g(\mathcal{S}^* \cup y^*) \ge (1-\rho)f(O)$$

Proof: Recall that in Lemma 2, we prove that there exists a sequence R with $|R| \leq B$ and $\forall i \in R : \Theta_i^R \geq \rho$ such that $g(\mathcal{R}) \geq (1 - \rho)f(O)$. Let s denote the last product in R, it is easy to verify that $(\mathcal{R} \setminus s, s)$ is a feasible solution to **P.A.1**. As a result, $g(\mathcal{S}^* \cup y^*) \geq g(\mathcal{R}) \geq (1 - \rho)f(O)$. \Box

4. Joint product selection, sequencing, and pricing

We next study the case when the price of each product is also a decision variable. As described in Section 2.3.2, assume each product $i \in \Omega$ is associated with a fixed quality q_i , an adjustable price p_i , and a fixed cost c_i , we set $\alpha_i = p_i - c_i$ and $\beta_i = e^{q_i - p_i}$. Our objective is to find a solution (S, \mathbf{p}) to **P.B**, where S is a sequence of products and $\mathbf{p} = \{p_i \mid i \in \Omega\}$ is the corresponding pricing vector. Let (O, \mathbf{p}^{OPT}) denote the optimal solution to **P.B**. Based on similar proofs of Lemma 1 and 2, we have the following two lemmas.

Lemma 5. For any $\rho \in [0,1]$, there is a sequence Q with $|Q| \leq B$ and $\forall i \in Q : \Theta_i^Q \geq \rho$ such that

$$f(Q, \mathbf{p}^{OPT}) \ge (1 - \rho)f(O, \mathbf{p}^{OPT})$$

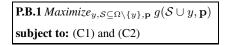
Proof Sketch: We fix the price \mathbf{p}^{OPT} , then apply the proof of Lemma 1 to complete the proof.

Lemma 6. For any $\rho \in [0, 1]$, there is a sequence R with $|R| \leq B$ and $\forall i \in R : \Theta_i^R \geq \rho$ and price **p** such that

$$g(\mathcal{R}, \mathbf{p}^{OPT}) \ge (1 - \rho)f(O, \mathbf{p}^{OPT})$$

Proof Sketch: We fix the price \mathbf{p}^{OPT} , then apply the proof of Lemma 2 to complete the proof.

We next introduce a new problem **P.B.1** whose goal is to jointly decide a consideration set and a pricing to maximize the expected revenue, while ignoring those products whose reachability is sufficiently small. Similar to **P.A.1**, we still use y to denote the last product in a sequence, (C1) ensures that all products in a feasible solution can be viewed with probability at least ρ , and (C2) ensures that the cardinality of the solution is bounded by B. We next describe the design of our algorithm (Algorithm 2).



Description of Algorithm 2.

- 1. We first propose a FPTAS for **P.B.1**.
- After solving **P.B.1** (approximately) and obtaining a (1 − ε)-approximate solution (S^{alg2}, y^{alg2}, p^{alg2}), we build the final solution by first displaying S^{alg2} (an arbitrary sequence of S^{alg1}) and then displaying y^{alg2} using price p^{alg2}.

In the rest of this section, we first present a FPTAS for **P.B.1** and then analyze the performance bound of Algorithm 2.

4.1. A FPTAS for P.B.1

To facilitate our study, we first introduce a well-known result in the field of assortment optimization.

Lemma 7. [15] Given any consideration set S, the maximum revenue $\max_{\mathbf{p}} g(S, \mathbf{p})$ is achieved at $p_i = W(\sum_{i \in S} e^{q_i - c_i - 1}) + c_i + 1$ for all products $i \in \Omega$ where W(z) is the

solution to $xe^x = z$. Moreover, the value of the maximum revenue is $\max_{\mathbf{p}} g(\mathcal{S}, \mathbf{p}) = W(\sum_{i \in \mathcal{S}} e^{q_i - c_i - 1}).$

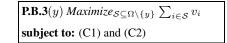
Based on Lemma 7, we obtain a closed form solution of the maximum revenue for any consideration set. This enables us to remove the decision variable **p** from **P.B.1** to obtain an equivalent, but simplified, formulation in **P.B.2**. In particular, we replace the objective function $g(S \cup y, \mathbf{p})$ in **P.B.1** by $W(\sum_{i \in S} v_i + v_y)$ in **P.B.2**.

P.B.2 $Maximize_{y,S\subseteq\Omega\setminus\{y\}} W(\sum_{i\in S} v_i + v_y)$
subject to: (C1) and (C2)

Note that maximizing $W(\sum_{i \in S} v_i + v_y)$ is equivalent to maximizing $\sum_{i \in S} v_i + v_y$. We can further simplify **P.B.2** by replacing the objective function $W(\sum_{i \in S} v_i + v_y)$ by $\sum_{i \in S} v_i + v_y$. We denote by **P3.2** the new formulation of **P.B.2**.

P.B.3 $Maximize_{y,S \subseteq \Omega \setminus \{y\}} \sum_{i \in S} v_i + v_y$
subject to: (C1) and (C2)

Now we are ready to present the solution to **P.B.1**. We first enumerate the last product $o \in \Omega$ and solve **P.B.3** for each y. The solution with the largest expected revenue is returned as the final solution to **P.B.1**. Let **P.B.3**(y) denote **P.B.3** when the last product y is given. It is easy to verify that for a fixed y, **P.B.3**(y) is a classic knapsack problem subject to a capacity constraint and a cardinality constraint. Given that **P.B.3**(y) admits an FPTAS [19], **P.B.3** also admits an FPTAS. A detailed proof is provided in Lemma 8.



Lemma 8. If **P.B.3**(y) admits an FPTAS for any fixed y, **P.B.1** also admits an FPTAS.

Proof: Recall that to solve **P.B.1**, we solve **P.B.3**(y) approximately for all $y \in \Omega$ and choose the best one as the final solution. Because **P.B.1** and **P.B.2** are equivalent, to prove this lemma, it is equivalent to proving that any $(1 - \epsilon)$ -approximate solution to **P.B.3**(y) is also a $(1 - \epsilon)$ -approximate solution to **P.B.2** for a fixed y and any $\epsilon < 1$. Given any y, assume there exists a $(1 - \epsilon)$ -approximate solution \mathcal{D} to **P.B.3**(y), i.e.,

$$\begin{split} \sum_{i\in\mathcal{D}} v_i &\geq (1-\epsilon) \max_{\mathcal{S}} (\sum_{i\in\mathcal{S}} v_i) \text{ subject to all constraints in } \textbf{P.B.3}(y). \text{ It follows} \\ \text{that } \sum_{i\in\mathcal{D}} v_i + v_y &\geq (1-\epsilon) (\max_{\mathcal{S}} \sum_{i\in\mathcal{S}} v_i + v_y). \text{ It implies that } W(\sum_{i\in\mathcal{D}} v_i + v_y) &\geq \\ W((1-\epsilon) (\max_{\mathcal{S}} \sum_{i\in\mathcal{S}} v_i + v_y)). \text{ Because } W \text{ is concave, we have } W(\sum_{i\in\mathcal{D}} v_i + v_y) &\geq \\ (1-\epsilon) W(\max_{\mathcal{S}} (\sum_{i\in\mathcal{S}} v_i + v_y)) = (1-\epsilon) \max_{\mathcal{S}} W(\sum_{i\in\mathcal{S}} v_i + v_y). \Box \end{split}$$

4.2. Performance analysis of Algorithm 2

We next analyze the performance bound of Algorithm 2. We use (S^*, y^*, \mathbf{p}^*) to denote the optimal solution to **P.B.1**.

Theorem 2. For any $\epsilon < 1$, we have

$$f(\mathcal{S}^{\mathrm{alg}_2} \oplus y^{\mathrm{alg}_2}, \mathbf{p}^{\mathrm{alg}_2}) \ge (1 - \epsilon)\rho(1 - \rho)f(O, \mathbf{p}^{OPT})$$

Proof: Recall that in Lemma 6, we prove that there exists a sequence R with $|R| \leq B$ and $\forall i \in R : \Theta_i^R \geq \rho$ such that $g(\mathcal{R}, \mathbf{p}^{OPT}) \geq (1 - \rho)f(O, \mathbf{p}^{OPT})$. Let s denote the last product in R, it is easy to verify that $(\mathcal{R} \setminus s, s, \mathbf{p}^{OPT})$ is a feasible solution to **P.B.1**. As a result,

$$g(\mathcal{S}^* \cup y^*, \mathbf{p}^*) \geq g(\mathcal{R}, \mathbf{p}^{OPT})$$
 (33)

$$\geq (1-\rho)f(O, \mathbf{p}^{OPT}) \tag{34}$$

The first inequality is due to (S^*, y^*, \mathbf{p}^*) is the optimal solution to **P.B.1**. It follows that

$$f(\mathcal{S}^{\text{alg}_2} \oplus y^{\text{alg}_2}, \mathbf{p}^{\text{alg}_2}) \geq \rho g(\mathcal{S}^{\text{alg}_2} \cup y, \mathbf{p}^{\text{alg}_2})$$
(35)

$$\geq (1-\epsilon)\rho(1-\rho)f(O,\mathbf{p}^{OPT})$$
(36)

The first inequality is due to the following observation: Because $(S^{\text{alg}_2}, y^{\text{alg}_2})$ is a feasible solution to **P.A.1**, constraint (C2) ensures that all products in $S^{\text{alg}_2} \oplus y^{\text{alg}_2}$ can be browsed with probability at least ρ . Thus, $f(S^{\text{alg}_2} \oplus y^{\text{alg}_2}, \mathbf{p}^{\text{alg}_2}) \ge \rho g(S^{\text{alg}_2} \cup y, \mathbf{p}^{\text{alg}_2})$. The second inequality is due to the assumption that **P.B.1** admits an FPTAS, i.e., $g(S^{\text{alg}_2} \cup y, \mathbf{p}^{\text{alg}_2}) \ge (1 - \epsilon)g(S^* \cup y^*, \mathbf{p}^*)$ for any $\epsilon < 1$, and (34). \Box

5. Conclusion

We study the product sequencing and pricing problem under the cascade browse model. In the first setting, we assume that both the revenue and MNL-based preference weight are fixed for all products, and focus on finding the best sequence of products subject to a cardinality constraint. In the second setting, we consider the joint sequencing and pricing problem. We develop approximate solutions to both settings. As a by product, we propose the first FPTAS for the assortment optimization problem subject to one capacity constraint and one cardinality constraint.

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