# The Hierarchical Chinese Postman Problem: the slightest disorder makes it hard, yet disconnectedness is manageable 

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#### Abstract

The Hierarchical Chinese Postman Problem is finding a shortest traversal of all edges of a graph respecting precedence constraints given by a partial order on classes of edges. We show that the special case with connected classes is NP-hard even on orders decomposable into a chain and an incomparable class. For the case with linearly ordered (possibly disconnected) classes, we get 5/3-approximations and fixed-parameter algorithms by transferring results from the Rural Postman Problem.


Keywords: approximation algorithm, fixed-parameter algorithm, NP-hardness, arc routing, rural postman problem, temporal graphs

## 1. Introduction

The following NP-hard arc routing problem arises in snow plowing, garbage collection, flame and laser cutting [8, 23].

Problem 1.1 (Hierarchical Chinese Postman Problem, HCPP). Input: An undirected graph $G=(V, E)$, edge weights $\omega: E \rightarrow$ $\mathbb{N}$, a partition $\mathcal{P}$ of $E$ into $k$ classes, a partial order $<$ on $\mathcal{P}$.
Find: A least-weight closed walk traversing each edge in $E$ at least once such that each edge $e$ in a class $E^{\prime}$ is traversed only after all edges in all classes $E^{\prime \prime}<E^{\prime}$ are traversed.

The case $k=1$ is the Chinese Postman Problem (CPP), which reduces to a minimum-weight perfect matching problem $[4,6$, $9,28]$. We study the following special cases of HCPP:
$\operatorname{HCPP}(1)$ : the order $<$ is linear,
HCPP(c): each edge class induces a connected subgraph,
$\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ : both of the above restrictions.
$\operatorname{HCPP}(1)$ and $\operatorname{HCPP}(\mathrm{c}, 1)$ can also be understood as variants of the Travelling Salesman Problem (TSP) in temporal graphs [26], with the difference that it is required to explore all edges instead of all vertices and that edges never disappear from the graph. $\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ is polynomial-time solvable $[8,14,21]$. This naturally raises two questions about $\mathrm{HCPP}(\mathrm{c})$ and $\operatorname{HCPP}(\mathrm{l})$ :
(a) Is HCPP(c) effectively solvable on other order types, like several scheduling problems on, for example, tree orders [18], interval orders [27], and bounded-width orders [1, 30]?
(b) Is $\operatorname{HCPP}(1)$ effectively solvable when the number of connected components in each class is sufficiently small? If the number of connected components is unbounded, $\operatorname{HCPP}(1)$ is NP-hard already for $k=2$ [3].

[^0]Our contributions. In Section 3, we show that HCPP(c) is NPhard even on partial orders that are decomposable into a linear order and a class that is incomparable to all others classes, thus negatively answering (a) for all order types mentioned there. The remaining sections are dedicated to question (b).

In Section 4, we revisit a construction that reduces $\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ to the $s-t$-Rural Postman Path Problem ( $s-t$-RPP, Problem 4.1) [8]. We show that, when applied to $\operatorname{HCPP}(1)$, the construction transfers performance guarantees of approximation and randomized algorithms from $s-t$-RPP to $\mathrm{HCPP}(1)$.

In Section 5, we show a 5/3-approximation algorithm for HCPP(1). This contrasts TSP in temporal graphs, which is not better than 2-approximable unless $\mathrm{P}=\mathrm{NP}$ [26]. To get $5 / 3$-approximations for $\operatorname{HCPP}(1)$, we use the construction from Section 4 and show a 5/3-approximation algorithm for $s-t$-RPP analogously to that for $s$ - $t$-TSP [17]. Any better approximation factor for $s$ - $t$-RPP will directly carry over to HCPP(1).

In Section 6, we use the construction from Section 4 and known algorithms for the Rural Postman Problem [3, 10, 15] to show that $\operatorname{HCPP}(1)$ is solvable in polynomial-time if each class induces a constant number $c$ of connected components. When the edge weights are polynomially bounded, one can even obtain randomized fixed-parameter algorithms with respect to $c$.

## 2. Preliminaries

By $\mathbb{N}$, we denote the set of natural numbers, including zero. For two multisets $A$ and $B, A \uplus B$ is the multiset obtained by adding the multiplicities of elements in $A$ and $B$. By $A \backslash B$ we denote the multiset obtained by subtracting the multiplicities of elements in $B$ from the multiplicities of elements in $A$. Finally, given some weight function $\omega: A \rightarrow \mathbb{N}$, the weight of a multiset $A$ is $\omega(A):=\sum_{e \in A} v(e) \omega(e)$, where $v(e)$ is the multiplicity of $e$ in $A$.

We mostly consider simple undirected graphs $G=(V, E)$ with a set $V(G):=V$ of vertices and a set $E(G):=E \subseteq\{\{u, v\} \mid$ $u, v \in V, u \neq v\}$ of edges. Unless stated otherwise, $n$ denotes the
number of vertices and $m$ denotes the number of edges. Within proofs, there may occur multigraphs, where $E$ is a multiset, and directed graphs $G=(V, A)$ with a set of $\operatorname{arcs} A \subseteq V^{2}$. The degree of a vertex in an undirected multigraph is its number of incident edges. We call a vertex balanced if it has even degree. We call a graph balanced if all its vertices are balanced. For a multiset $R$ of edges, we denote by $V(R)$ the set of their incident vertices. For a multiset $R$ of edges of $G, G\langle R\rangle:=(V(R), R)$ is the (multi)graph induced by the edges in $R$.

A walk from $v_{0}$ to $v_{\ell}$ in a graph $G$ is a sequence $w=\left(v_{0}, e_{1}\right.$, $v_{1}, e_{2}, v_{2}, \ldots, e_{\ell}, v_{\ell}$ ) such that $e_{i}=\left\{v_{i-1}, v_{i}\right\}$ (if $G$ is undirected) or $e_{i}=\left(v_{i-1}, v_{i}\right)$ (if $G$ is directed) for each $i \in\{1, \ldots, \ell\}$. When there is no ambiguity (like in simple graphs), we will also specify walks simply as a list of vertices. If $v_{0}=v_{\ell}$, then we call $w$ a closed walk. If all vertices on $w$ are pairwise distinct, then $w$ is a path. If only its first and last vertex coincide, then $w$ is a cycle. A subwalk $w^{\prime}$ of $w$ is any subsequence $w^{\prime}$ of $w$ that is itself a walk. By $E(w)$, we denote the multiset of edges on $w$, that is, each edge appears on $w$ and in $E(w)$ equally often. The weight of walk $w$ is $\omega(w):=\sum_{i=1}^{\ell} \omega\left(e_{\ell}\right)$. For a walk $w$, we denote $G\langle w\rangle:=G\langle E(w)\rangle$. Note that $G\langle R\rangle$ and $G\langle w\rangle$ do not contain isolated vertices yet might contain edges with a higher multiplicity than $G$ and, therefore, are not necessarily sub(multi)graphs of $G$. An Euler walk for $G$ is a walk that traverses each edge or arc of $G$ exactly as often as it is present in G. An Euler tour is a closed Euler walk. A graph is Eulerian if it allows for an Euler tour. A connected undirected graph is Eulerian if and only if all its vertices are balanced.

For any $\alpha \geq 1$, an $\alpha$-approximate solution for a minimization problem is a feasible solution whose weight does not exceed the weight of an optimal solution by more than a factor of $\alpha$, called the approximation factor [13]. The Exponential Time Hypothesis (ETH) is that 3-SAT (Problem 3.2) with $n$ variables is not solvable in $2^{o(n)}$ time [19].

## 3. NP-hardness for $\operatorname{HCPP}(\mathrm{c})$ with one incomparable class

$\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ is polynomial-time solvable [8, 14, 21]. We prove that adding a single incomparable class makes the problem NP-hard.

Theorem 3.1. Even on edges with weight one and orders that are decomposable into a linear order and a class that is incomparable to all others classes,
(i) $\mathrm{HCPP}(\mathrm{c})$ is NP-hard and
(ii) not solvable in $2^{o(n+m+k)}$ time unless ETH fails.

To prove Theorem 3.1, we use a polynomial-time many-one reduction from 3-SAT, which is NP-hard [20] and, unless the ETH fails, is not solvable in $2^{o(n+m)}$ time [19].

Problem 3.2 (3-SAT).
Input: A formula $\varphi$ in conjunctive normal form with $n$ variables and $m$ clauses, each containing at most three literals.
Question: Is there a assignment to the variables satisfying $\varphi$ ?
The reduction is carried out by the following construction, which is illustrated in Figure 1.

Construction 3.3. Let $\varphi$ be an instance of 3-SAT. First, delete each clause containing both $x$ and $\bar{x}$ : they are always satisfied. Consider now the variables $x_{1}, \ldots, x_{n}$ and clauses $C_{1}, \ldots, C_{m}$. For each $i \in\{1, \ldots, n\}$, let $\mu_{i}$ denote the number of clauses containing either $x_{i}$ or $\bar{x}_{i}$. We describe an instance $I_{\varphi}=(G, \omega, \mathcal{P},<)$ of $\operatorname{HCPP}(\mathrm{c})$. Each edge will have weight one.

Graph $G$ contains a path $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$ for each $j \in\{1, \ldots, m\}$ and, for each $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, 6 \mu_{i}\right\}$, a path $P_{i}^{\ell}:=$ $\left(t_{i}^{\ell}, z_{i}^{\ell}, f_{i}^{\ell}\right)$. For each $i \in\{1, \ldots, n\}$, it contains a cycle

$$
X_{i}:=\left(t_{i}^{1}, t_{i}^{2}, \ldots, t_{i}^{6 \mu_{i}-1}, t_{i}^{6 \mu_{i}}, f_{i}^{6 \mu_{i}}, f_{i}^{6 \mu_{i}-1}, \ldots, f_{i}^{2}, f_{i}^{1}, t_{i}^{1}\right)
$$

For each $i \in\{1, \ldots, n\}$ and $i^{\prime}=i \bmod n+1$, cycles $X_{i}$ and $X_{i^{\prime}}$ are connected to each other via a cycle

$$
Y_{i i^{\prime}}=\left(t_{i}^{6 \mu_{i}}, f_{i^{\prime}}^{1}, f_{i}^{6 \mu_{i}}, t_{i^{\prime}}^{1}, t_{i}^{6 \mu_{i}}\right)
$$

For each literal $x_{i}$ in a clause $C_{j}$, graph $G$ contains a cycle

$$
Z_{i j}:=\left(t_{i}^{6 \ell-3}, c_{j}^{1}, a_{i j}, c_{j}^{2}, t_{i}^{6 \ell-2}, b_{i j}, t_{i}^{6 \ell-3}\right),
$$

where $\ell \leq \mu_{i}$ is such that $C_{j}$ is the $\ell$-th clause containing $x_{i}$ or $\bar{x}_{i}$. For each literal $\bar{x}_{i}$ in a clause $C_{j}$, graph $G$ contains a cycle

$$
\bar{Z}_{i j}:=\left(f_{i}^{6 \ell-3}, c_{j}^{1}, a_{i j}, c_{j}^{2}, f_{i}^{6 \ell-2}, b_{i j}, f_{i}^{6 \ell-3}\right)
$$

where $\ell \leq \mu_{i}$ is such that $C_{j}$ is the $\ell$-th clause containing $x_{i}$ or $\bar{x}_{i}$.
The edges are ordered as follows. For each $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, 6 \mu_{i}\right\}, E_{i}^{\ell}=E\left(P_{i}^{\ell}\right)$ is a connected class. They are lexicographically ordered, that is,

$$
E_{i}^{\ell}<E_{i^{\prime}}^{\ell^{\prime}} \Longleftrightarrow\left(i<i^{\prime}\right) \vee\left(i=i^{\prime} \wedge \ell \leq \ell^{\prime}\right) .
$$

They are preceded by the connected class

$$
E_{0}=\bigcup_{i=1}^{n} E\left(X_{i}\right) \cup \bigcup_{i=1}^{n} E\left(Y_{i, i \bmod n+1}\right) \cup \bigcup_{x_{i} \in C_{j}} E\left(Z_{i j}\right) \cup \bigcup_{\bar{x}_{i} \in C_{j}} E\left(\bar{Z}_{i j}\right),
$$

Finally, the edge set $E^{*}$ consisting of all edges incident to $c^{*}$ forms a connected class incomparable to all other classes.

For convenience, we collect the vertices of $I_{\varphi}$ of the form $t_{i}^{\ell}$ and $f_{i}^{\ell}$ and of the form $c_{j}^{1}$ and $c_{j}^{2}$ into sets

$$
V_{F T}:=\bigcup_{i=1}^{n} \bigcup_{\ell=1}^{6 \mu_{i}}\left\{t_{i}^{\ell}, f_{i}^{\ell}\right\} \quad \text { and } \quad V_{C}:=\bigcup_{j=1}^{m}\left\{c_{j}^{1}, c_{j}^{2}\right\} .
$$

Observation 3.4. Let $I_{\varphi}=(G, \omega, \mathcal{P},<)$ be the $\operatorname{HCPP}(\mathrm{c})$ instance constructed by Construction 3.3 from a 3-SAT instance $\varphi$ with $n$ variables and $m$ clauses. Then,
(i) the subgraph $G\left\langle E_{0}\right\rangle$ is Eulerian: it is connected and the union of pairwise edge-disjoint cycles, that is, balanced,
(ii) the imbalanced vertices of $G$ are therefore $V_{F T} \cup V_{C}$, and
(iii) the number of vertices, edges, and classes is $O(n+m)$, since $\sum_{i=1}^{n} \mu_{i} \leq 3 m$ in any 3 -SAT formula.

In the following, we will show that the $\operatorname{HCPP}(\mathrm{c})$ instance $I_{\varphi}$ allows for a tight tour if and only if $\varphi$ is satisfiable:


Figure 1: Illustration of Construction 3.3. The graph is generated from the formula $\varphi=\left(\bar{x}_{1} \vee x_{4}\right) \wedge\left(\bar{x}_{2} \vee x_{3}\right)$, that is, $C_{1}=\left(\bar{x}_{1} \vee x_{4}\right)$ and $C_{2}=\left(\bar{x}_{2} \vee x_{3}\right)$. The dotted edges form the edge set $E_{0}$. The solid edges form the paths $P_{i}^{\ell}$ and $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$. The gray areas illustrate the types of cycles introduced in the construction: they consist of the dashed edges enclosed in the gray areas. Note that $E_{0}$ (the dotted edges) forms an Eulerian subgraph: it is the union of cycles and thus each vertex has an even number of incident edges in $E_{0}$. Thus, each vertex of the form $t_{i}^{\ell}, f_{i}^{\ell}, c_{j}^{1}$ and $c_{j}^{2}$ is imbalanced and they are the only imbalanced vertices.


Figure 2: A minimum-weight closed walk for the graph in Figure 1 first visits each edge in $E_{0}$ exactly once (the dotted edges in Figure 1), and then follows the arrows as shown in this figure. The walk corresponds to $x_{1}=0$ and $x_{2}=x_{3}=x_{4}=1$.

Definition 3.5. A feasible solution for an $\operatorname{HCPP}(\mathrm{c})$ instance ( $G, \omega$, $\mathcal{P}, \prec)$ with $G=(V, E)$ is a tight tour if it has weight at most $|E|+b / 2$, where $b$ is the number of imbalanced vertices in $G$.

Proposition 3.6. The $\operatorname{HCPP}(\mathrm{c})$ instance $I_{\varphi}=(G, \omega, \mathcal{P}, \prec)$ created from a 3-SAT instance $\varphi$ by Construction 3.3 allows for a tight tour if and only if $\varphi$ is satisfiable.

In the rest of this section, it remains to prove Proposition 3.6, which, together with Observation 3.4(iii), yields Theorem 3.1.

Satisfiability of $\varphi$ implies a tight tour in $I_{\varphi}$. Assume that $\varphi$ is satisfiable. We show a tight tour $T$ for $I_{\varphi}$. Without loss of generality, assume that $x_{1}$ is "true": otherwise, we can replace $x_{1}$ by $\bar{x}_{1}$ throughout the formula $\varphi$.

The tight tour $T$ for $I_{\varphi}$ then looks as follows (an example is shown in Figure 2). It starts in $f_{1}^{1}$, first visits each edge of $E_{0}$ exactly once and returns to $f_{1}^{1}$. This is possible by Observation 3.4(i). Then, it remains to traverse the paths $\left(t_{i}^{\ell}, z_{i}^{\ell}, f_{i}^{\ell}\right)$ for each $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, 6 \mu_{i}\right\}$ and the paths $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$ for each $j \in\{1, \ldots, m\}$. This is done as follows. For $i$ from 1 to $n$, if $x_{i}$ is "true", $T$ visits the vertices

$$
f_{i}^{1}, z_{i}^{1}, t_{i}^{1}, \quad t_{i}^{2}, z_{i}^{2}, f_{i}^{2}, \quad f_{i}^{3}, z_{i}^{3}, t_{i}^{3}, \quad t_{i}^{4}, z_{i}^{4}, f_{i}^{4} \ldots, t_{i}^{6 \mu_{i}}, z_{i}^{6 \mu_{i}}, f_{i}^{6 \mu_{i}}
$$

for some $\ell \in\left\{1, \ldots, \mu_{i}\right\}$ taking a detour through the vertices $t_{i}^{6 \ell-3}, c_{j}^{1}, c^{*}, c_{j}^{2}, t_{i}^{t \ell-2}$ if clause $C_{j}$ contains $x_{i}$ and $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$ has not been traversed before. If $x_{i}$ is "false", then $T$ visits

$$
t_{i}^{1}, z_{i}^{1}, f_{i}^{1}, f_{i}^{2}, z_{i}^{2}, t_{i}^{2}, t_{i}^{3}, z_{i}^{3}, f_{i}^{3}, f_{i}^{4}, z_{i}^{4}, t_{i}^{4}, \ldots, f_{i}^{6 \mu_{i}}, z_{i}^{6 \mu_{i}}, t_{i}^{6 \mu_{i}}
$$

for some $\ell \in\left\{1, \ldots, \mu_{i}\right\}$ taking a detour through the vertices $f_{i}^{6 \ell-3}, c_{j}^{1}, c^{*}, c_{j}^{2}, f_{i}^{6 \ell-2}$ if clause $C_{j}$ contains $\bar{x}_{i}$ and $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$ has not been traversed before. Finally, after $f_{n}^{6 \mu_{n}}$ or $t_{n}^{6 \mu_{n}}$, the walk $T$ returns to $f_{1}^{1}$. Note that this traversal is possible due to the cycle $Y_{i, i \bmod n+1}$ for each $i \in\{1, \ldots, n\}$.

Observe that the closed walk $T$ contains all edges and respects precedence constraints: For the edges in $E_{0}$ and all paths $\left(t_{i}^{\ell}, z_{i}^{\ell}, f_{i}^{\ell}\right)$ for $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, 6 \mu_{i}\right\}$, this is obvious. To see that the path $\left(c_{j}^{1}, c^{*}, c_{j}^{2}\right)$ has been traversed for each $j \in\{1, \ldots, m\}$, observe that each clause $C_{j}$ contains a true literal, so that a detour via $c_{j}^{1}, c^{*}, c_{j}^{2}$ is taken.

To see that $T$ is tight, we check which edges are traversed a second time. When $x_{i}$ is "true", the edges $\left\{f_{i}^{2 \ell}, f_{i}^{2 \ell+1}\right\} \in E_{0}$ for $\ell \in\left\{1, \ldots, \mu_{i}-1\right\}$ are visited a second time, whereas each edge $\left\{t_{i}^{2 \ell-1}, t_{i}^{2 \ell}\right\} \in E_{0}$ for $\ell \in\left\{1, \ldots, \mu_{i}\right\}$ is traversed a second time or skipped by a detour that traverses the edges $\left\{t_{i}^{2 \ell-1}, c_{j}^{1}\right\}$ and $\left\{t_{i}^{2 \ell}, c_{j}^{2}\right\}$ a second time. Analogously when $x_{i}$ is "false". Moreover, for each $i \in\{1, \ldots, n\}$, one edge of the cycle $Y_{i, i} \bmod n+1$ is visited a second time: it joins the last vertex visited by $T$ in $X_{i}$ to the first vertex visited by $T$ in $X_{i \bmod n+1}$. We thus see that the edges visited a second time form a matching. Their endpoints are the $b$ imbalanced vertices $V_{F T} \cup V_{C}$. Thus, $T$ traverses not more than $|E|+b / 2$ edges.

Tight tour for $I_{\varphi}$ implies satisfiability of $\varphi$. Assume that $I_{\varphi}$ allows for a tight tour $T$. We show that $\varphi$ is satisfiable.

Lemma 3.7. Let $M=E(T) \backslash E$, that is, $M$ is the multiset of the edges that the tight tour $T$ traverses additionally to $E$ (taking into account the multiplicity of additional visits). Then,
(i) $M \subseteq E_{0}$ is a perfect matching on the vertices $V_{F T} \cup V_{C}$, in particular, $M$ contains each edge at most once,
(ii) each edge in $M$ has an endpoint in $V_{F T}$,
(iii) $G\left\langle\left(E \backslash E_{0}\right) \cup M\right\rangle$ is connected.

Proof. (i) Since $T$ is a closed walk, all vertices in $G\langle T\rangle$ are balanced, whereas its subgraph $G\langle E\rangle=G$ has $b$ imbalanced vertices. Since $T$ contains at most $|E|+b / 2$ edges, the graph $G\langle T\rangle$ contains at most $b / 2$ edges additionally to those in $G\langle E\rangle$. Thus, $G\langle T\rangle$ contains a set $M$ of at most $b / 2$ edges whose endpoints are the $b$ imbalanced vertices of $G\langle E\rangle$. By Observation 3.4(ii), these are exactly the vertices $V_{F T} \cup V_{C}$. This is only possible if $M$ is a perfect matching on $V_{F T} \cup V_{C}$. Since each edge of $G$ that is not in $E_{0}$ has at least one balanced endpoint (namely, $c^{*}$ or one of the $z_{i}^{\ell}$, we easily get $M \subseteq E_{0}$.
(ii) The only edges in $E_{0}$ that have no endpoints in $V_{F T}$ have one of the vertices of the form $a_{i j}$ or $b_{i j}$ as endpoints. Since these are only on the cycle $Z_{i j}$ or $\bar{Z}_{i j}$, they are balanced. Thus, $M$ cannot contain such edges.
(iii) Let $T^{*}$ be a tight tour for $(G, \omega, \mathcal{P}, \prec)$ such that $T^{*}$ visits exactly the edges in $M$ twice and the minimal prefix $T_{1}$ of $T^{*}$ traversing all edges in $E_{0}$ has minimum length (the tight tour $T$ and (i) witness the existence of $T^{*}$ ). Let $T_{2}$ be the rest of $T^{*}$. We will prove that $T_{1}$ is an Euler tour for $G\left\langle E_{0}\right\rangle$. Then (iii) follows since $G\left\langle\left(E \backslash E_{0}\right) \cup M\right\rangle$ is even Eulerian: $T_{2}$ visits all edges in $\left(E \backslash E_{0}\right) \cup M$, since they are not visited by $T_{1} ; T_{2}$ is closed since $T^{*}$ and its prefix $T_{1}$ are; and, by choice of $T^{*}, T_{2}$ does not visit any edge in $\left(E \backslash E_{0}\right) \cup M$ more than once. It remains to prove that $T_{1}$ is indeed an Euler tour for $G\left\langle E_{0}\right\rangle$.

We first prove that $T_{1}$ is a closed walk. By the minimality of $T_{1}$ and choice of $\prec, T_{1}$ does not end in $c^{*}$ and does not contain edges from any class $E_{i}^{\ell}$. It might contain edges from $E^{*}$. Assume, for the sake of a contradiction, that $T_{1}$ starts at some vertex $s$ and ends at some vertex $t \neq s$. Then, $t$ is not balanced in $G\left\langle T_{1}\right\rangle$ but balanced in $G\left\langle E_{0}\right\rangle$ by Observation 3.4(i). Thus, there is an edge $e=\left\{t^{\prime}, t\right\} \in E^{*} \cup M$ on $T_{1}$. The graph $G\left\langle E\left(T_{1}\right) \backslash\{e\}\right\rangle$ contains $E_{0}$, is connected, all its vertices except for $s$ and $t^{\prime}$ are balanced, and it therefore has an Euler walk $T_{1}^{\prime}$. It follows that ( $T_{1}^{\prime}, e, T_{2}$ ) is another tight tour for $G$ visiting all edges with the same multiplicity as $T^{*}$, yet its prefix $T_{1}^{\prime}$ containing $E_{0}$ satisfies $\left|T_{1}^{\prime}\right|<\left|T_{1}\right|$. This contradicts the choice of $T^{*}$.

We now show that $T_{1}$ traverses each edge $e \in E_{0} \cup E^{*}$ at most once. Towards a contradiction, assume that it traverses $e=\{u, v\}$ twice. Then, $v \in V_{F T}$ by (ii). Thus, $v$ is not incident to any edges in $E^{*}$ and, because $v$ is balanced in $G\left\langle E_{0}\right\rangle$ by Observation 3.4(i), it is also balanced in $G\left\langle E_{0} \cup E^{*}\right\rangle$. Since $v$ is balanced in $G\left\langle E_{0} \cup\right.$ $\left.E^{*}\right\rangle$, balanced in $G\left\langle T_{1}\right\rangle$, and $T_{1}$ traverses $e$ twice, $T_{1}$ also traverses another edge incident to $v$ twice, contradicting (i).

Finally, we prove that $T_{1}$ contains only edges of $E_{0}$. Towards a contradiction, assume that $T_{1}$ contains an edge $\left\{c^{*}, c\right\} \in E^{*}$. Vertex $c \in V_{C}$ is balanced in $G\left\langle T_{1}\right\rangle$, yet not balanced in its subgraph $G\left\langle E_{0} \cup\left\{c^{*}, c\right\}\right\rangle$ by Observation 3.4(i). Thus, $G\left\langle T_{1}\right\rangle$ contains some edge $e \in E_{0} \cup E^{*}$ twice, which is impossible.


(b) Case 2: $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ is not covered.

Figure 3: The two cases in the proof of Lemma 3.9. Dotted edges are all the possibly present edges in $E_{0}$ available for inclusion in $M$ by Lemma 3.7(i) (the edge $\left\{t_{i}^{6 \ell-5}, f_{i}^{6 \ell-5}\right\}$ is present if $\ell=1$, the edge $\left\{t_{i}^{6 \ell}, f_{i}^{6 \ell}\right\}$ is present if $\left.\ell=\mu_{i}\right)$. Including $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ in $M$ or excluding it from $M$ force all the bold edges into $M$ due to the fact that all vertices must be contained in some edge of $M$ and that edges drawn above each other cannot both be part of $M$.

We now show that the matching $M$ from Lemma 3.7 takes one of two possible forms in each variable cycle $X_{i}$. This will correspond to setting a variable to "true" or "false".

Definition 3.8. Let $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, \mu_{i}\right\}$. We call an edge $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\}$ covered if $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\} \in M$ or if there is a $j \in\{1, \ldots, m\}$ such that both $\left\{t_{1}^{6 \ell-3}, c_{j}^{1}\right\}$ and $\left\{t_{1}^{6 \ell-2}, c_{j}^{2}\right\}$ are in $M$.

We call an edge $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ covered if $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\} \in M$ or if there is a $j \in\{1, \ldots, m\}$ such that both $\left\{f_{1}^{6 \ell-3}, c_{j}^{1}\right\}$ and $\left\{f_{1}^{6 \ell-2}, c_{j}^{2}\right\}$ are in $M$.

Lemma 3.9. For each $i \in\{1, \ldots, n\}$, either all $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\}$ are covered or all $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ are covered for $\ell \in\left\{1, \ldots, \mu_{i}\right\}$.

Proof. For any $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, \mu_{i}\right\}$, we first show that exactly one of $\left\{t_{i}^{6 \ell-2}, t_{i}^{6 \ell-3}\right\}$ and $\left\{f_{i}^{6 \ell-2}, f_{i}^{6 \ell-3}\right\}$ is covered. Note that, by Construction 3.3, at most one of these pairs of vertices is attached to $\left\{c_{j}^{1}, c_{j}^{2}\right\}$ for any $j \in\{1, \ldots, m\}$. Without loss of generality, let this be $\left\{t_{i}^{6 \ell-2}, t_{i}^{6 \ell-3}\right\}$. The other case is symmetric.

Denote $R:=E \backslash E_{0}$. By Lemma 3.7(iii), $G\langle R \cup M\rangle$ is connected. Thus, there is at least one edge of $M$ leaving any subset of connected components of $G\langle R\rangle$ and, for each $h \in$ $\{6 \ell-5, \ldots, 6 \ell-1\}$, only one of $\left\{t_{i}^{h}, t_{i}^{h+1}\right\}$ and $\left\{f_{i}^{h}, f_{i}^{h+1}\right\}$ is in $M$ : otherwise, the matching $M$ could not contain any edge leaving the set of connected components $\left\{\left\{t_{i}^{h}, z_{i}^{h}, f_{i}^{h}\right\},\left\{t_{i}^{h+1}, z_{i}^{h+1}, f_{i}^{h+1}\right\}\right\}$. We also exploit that, by Lemma 3.7(i), all vertices in $V_{F T}$ must be incident to an edge of $M$.

We now distinguish two cases, illustrated in Figure 3. First, assume that $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ is covered, that is, in $M$. Then all bold edges shown in Figure 3a are in $M$. Thus, the edge $\left\{t_{i}^{6 \ell-2}, t_{i}^{6 \ell-3}\right\}$ is not covered. If the edge $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ is not covered, that is, not in $M$, then all bold edges shown in Figure 3 b are in $M$. To match the vertices $t_{i}^{6 \ell-3}$ and $t_{i}^{6 \ell-2}$, one either has $\left\{t_{i}^{6 l-3}, t_{i}^{6 \ell-2}\right\} \in M$ or $\left\{\left\{t_{i}^{6 \ell-3}, c_{j}^{1}\right\},\left\{t_{i}^{6 \ell-2}, c_{j}^{2}\right\}\right\} \subseteq M$. That is, $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\}$ is covered.

Finally, towards a contradiction, assume that there are $\ell, \ell^{\prime}$ such that $\left\{f_{i}^{6 \ell-2}, f_{i}^{6 \ell-3}\right\}$ and $\left\{t_{i}^{6 \ell^{\prime}-2}, t_{i}^{6 \ell^{\prime}-3}\right\}$ are covered. Then we can choose $\ell, \ell^{\prime}$ so that $\left|\ell-\ell^{\prime}\right|=1$. Assume $\ell^{\prime}=\ell+1$, the other case is symmetric. Then, as illustrated in Figure 3a, vertex $t_{i}^{6 \ell}$ has to be matched to $t_{i}^{6 \ell^{\prime}-5}$ (there is no edge $\left\{t_{i}^{6 \ell}, f_{i}^{6 \ell}\right\}$ in this case by Construction 3.3, since $\ell<\mu_{i}$ ). However, vertex $t_{i}^{6 \ell^{\prime}-5}$ is already matched to $t_{i}^{6 \ell^{\prime}-4}$, so that this is impossible.

We can now easily prove that, since $I_{\varphi}$ has a tight tour $T$, the formula $\varphi$ is satisfiable, thus concluding the proof of Proposition 3.6. By Lemma 3.7(i) and (ii), for each clause $C_{j}$ of $\varphi$, the vertices $c_{j}^{1}$
and $c_{j}^{2}$ are matched to vertices in $V_{F T}$ by $M$. By Construction 3.3, $c_{j}^{1}$ can only be matched to $t_{i}^{6 \ell-3}$ or $f_{i}^{6 \ell-3}$ for some $i \in\{1, \ldots, n\}$ and $\ell \in\left\{1, \ldots, \mu_{i}\right\}$. By Lemma 3.9, if $c_{j}^{1}$ is matched to $t_{i}^{6 \ell-3}$, then $t_{i}^{6 \ell-2}$ is matched to $c_{j}^{2}$ and the edges $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\}$ are covered for all $\ell \in\left\{1, \ldots, \mu_{i}\right\}$, whereas $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ is not covered for any $\ell \in\left\{1, \ldots, \mu_{i}\right\}$. Thus, clause $C_{j}$ (and all other clauses containing $x_{i}$ ) can be satisfied by setting variable $x_{i}$ to "true". If, on the other hand, $c_{j}^{1}$ is matched to $f_{i}^{6 \ell-3}$, then $f_{i}^{6 \ell-2}$ is matched to $c_{j}^{2}$ and the edges $\left\{f_{i}^{6 \ell-3}, f_{i}^{6 \ell-2}\right\}$ are covered for all $\ell \in\left\{1, \ldots, \mu_{i}\right\}$, whereas $\left\{t_{i}^{6 \ell-3}, t_{i}^{6 \ell-2}\right\}$ is not covered for any $\ell \in\left\{1, \ldots, \mu_{i}\right\}$. Thus, clause $C_{j}$ (and all other clauses containing $\bar{x}_{i}$ ) can be satisfied by setting $x_{i}$ to "false".

## 4. Relation between HCPP(l) and the Rural Postman

Dror et al. [8] showed how to reduce $\operatorname{HCPP}(\mathrm{c}, 1)$ to polynomialtime solvable special cases of the following problem.

Problem 4.1 ( $s-t$-Rural Postman Path Problem, $s-t$-RPP).
Input: An undirected graph $G=(V, E)$, edge weights $\omega: E \rightarrow$
$\mathbb{N}$, vertices $s, t \in V$, and a subset $R \subseteq E$ of required edges.
Find: A walk $W^{*}$ of minimum total weight $\omega\left(W^{*}\right)$, starting in $s$, ending in $t$, and traversing all edges of $R$.

In general, $s$ - $t$-RPP is strongly NP-hard, as well as the better known Rural Postman Problem (RPP), where the goal is to find a closed walk [25]. Dror et al. [8] reduce $\operatorname{HCPP}(\mathrm{c}, 1)$ to multiple $s$ -$t$-RPP instances in which the subgraph $G\langle R\rangle$ is connected. Since this case of $s-t$-RPP is polynomially-time solvable, this yields a polynomial-time algorithm for $\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ [8].

We now show that, while applying the same construction to $\operatorname{HCPP}(1)$ does not yield polynomial-time solvable instances of $s$ - $t$-RPP, it allows to transfer running times, approximation factors, and error probabilities of $s-t$-RPP algorithms to $\mathrm{HCPP}(1)$. This is in contrast to a construction due to Cabral et al. [5], who showed a polynomial-time reduction of $\mathrm{HCPP}(1)$ to RPP that does not allow to transfer approximation factors: it introduces very heavy required edges, which always contribute to the goal function and thus make bad approximate solutions "look" good. We now describe the construction of Dror et al. [8].

Definition 4.2. In this section, we denote the edge classes of $\operatorname{HCPP}(\mathrm{l})$ instances $(G, \omega, \mathcal{P}, \prec)$ by $E_{1}, \ldots, E_{k}$, where $E_{i}<E_{j}$ if and only if $1 \leq i<j \leq k$.


Figure 4: Illustration for Construction 4.3: from a graph $G$ with $k=3$ edge classes (on the left), Construction 4.3 constructs a graph $\Gamma$ with $k+1$ layers (on the right). Note that, for example, vertex $b \in V(G)$ is the only vertex in $V\left(E_{2}\right)$ incident to edges of previous classes, thus, its copy $b_{2} \in V(\Gamma)$ is the only vertex in layer $V_{2}$.

By $R[u, v, i]$, we denote the $s-t$-RPP instance of finding a minimum-weight walk between the vertices $u$ and $v$ in $G\left\langle E_{1} \cup\right.$ $\left.\cdots \cup E_{i}\right\rangle$ traversing all edges in $E_{i}$. By $P[u, v, i]$, we denote an arbitrary optimal solution to $R[u, v, i]$.

Construction 4.3. From a $\operatorname{HCPP}(1)$ instance ( $G, \omega, \mathcal{P}, \prec$ ), construct a directed arc-weighted graph $\Gamma=\left(V_{\Gamma}, A_{\Gamma}\right)$ as illustrated in Figure 4: The vertex set $V_{\Gamma}=\bigcup_{i=1}^{k+1} V_{i}$ is a union of layers $V_{i}$. For each $i \in\{2, \ldots, k\}$, layer $V_{i}$ contains a copy of each vertex in $G$ that is incident to an edge of $E_{i}$ and of any predecessor class. Namely,

$$
\begin{array}{ll}
V_{1}=\left\{u_{1} \mid u \in V\left(E_{1}\right)\right\}, & V_{k+1}=\left\{u_{k+1} \mid u \in V\left(E_{1}\right)\right\} \\
V_{i}=\left\{u_{i} \mid u \in V\left(E_{i}\right) \cap \bigcup_{j=1}^{i-1} V\left(E_{j}\right)\right\} & \text { for } i \in\{2, \ldots, k\}
\end{array}
$$

For each pair of vertices $u_{i} \in V_{i}$ and $v_{i+1} \in V_{i+1}$, where $i \in$ $\{1, \ldots, k\}$, there is an $\operatorname{arc}\left(u_{i}, v_{i+1}\right) \in A_{\Gamma}$ of weight $\omega_{\Gamma}\left(u_{i}, v_{i+1}\right)=$ $\omega(P[u, v, i])$. If $P[u, v, i]$ does not exist, there is no $\operatorname{arc}\left(u_{i}, v_{i+1}\right)$.

Proposition 4.4 (Dror et al. [8]). Let $I:=(G, \omega, \mathcal{P}, \prec)$ be an $\operatorname{HCPP}(1)$ instance and $\Gamma$ be constructed from $I$ by Construction 4.3. Then, the weight of an optimal solution to $I$ coincides with the weight of a least-weight layer path in $\Gamma$, where a layer path in $\Gamma$ is a path from $v_{1} \in V_{1}$ to $v_{k+1} \in V_{k+1}$ such that $v_{1}$ and $v_{k+1}$ are copies of the same vertex $v \in V\left(E_{1}\right)$.

In particular, each layer path in $\Gamma$ has the form $J=\left(v_{1}, y_{2}^{2}, y_{3}^{3}\right.$, $\left.\ldots, y_{k}^{k}, v_{k+1}\right)$, where $y_{i}^{i} \in V_{i}$ for $i \in\{2, \ldots, k\}$ and concatenating the corresponding walks $P\left[v, y^{2}, 1\right], P\left[y^{2}, y^{3}, 2\right], \ldots, P\left[y^{k}, v, k\right]$ yields a feasible solution $W_{J}$ of weight $\omega\left(W_{J}\right)=\omega_{\Gamma}(J)$ for $I$.

Construction 4.3 can be used to solve $\operatorname{HCPP}(\mathrm{c}, \mathrm{l})$ in $O\left(k n^{5}\right)$ time: $\Gamma$ has at most $k n^{2}$ arcs, the weight of each is computed by solving an $s-t$-RPP instance $R[u, v, i]$, which works in $O\left(n^{3}\right)$ time since the set $E_{i}$ of required edges is connected [8]. It remains to find a layer path in $\Gamma$. This can be done in $O\left(k n^{3}\right)$ time by $n$ times calling a linear-time single-source shortest-path algorithm for directed acyclic graphs.

However, when applied to $\operatorname{HCPP}(1)$, Construction 4.3 gets to solve $s$ - $t$-RPP instances $R[u, v, i]$ where the set of required edges $E_{i}$ might be disconnected. Since we do not know how to solve them in polynomial time, in Sections 5 and 6, we will solve them using approximation algorithms and randomized fixed-parameter algorithms. Their performance guarantees carry over to $\operatorname{HCPP}(1)$ as follows.

Lemma 4.5. Let $I=(G, \omega, \mathcal{P},<)$ be an $\operatorname{HCPP}(1)$ instance. Assume that there is an algorithm running in $\tau$ time that, given any $s$ - $t$-RPP instance $R[u, v, i]$ (cf. Definition 4.2), outputs an $\alpha$-approximate solution for $R[u, v, i]$ with probability at least $1-p$.

Then, there is an algorithm running in $O\left(k n^{2} \tau+k n^{3}\right)$ time that returns an $\alpha$-approximate solution for $I$ with probability at least $1-p k$.
Proof. Let $\mathcal{A}$ denote the assumed randomized approximation algorithm for solving $s-t$-RPP instances $R[u, v, i]$. Since we can check the feasibility of any solution returned by $\mathcal{A}$ in linear time, we can assume that $\mathcal{A}$ makes only one-sided errors: For an infeasiable instance $R[u, v, i]$, it returns nothing. For a feasible instance $R[u, v, i]$, with probability at most $p$, it may return nothing or produce a solution that is more expensive than an $\alpha$-approximate solution. Moreover, since feasibility of $I$ is easy to check [8], we will assume that $I$ has a feasible solution. Then we compute a solution to $I$ as follows.

Construct an arc-weighted directed graph $\widetilde{\Gamma}=\left(\widetilde{V}_{\Gamma}, \widetilde{A}_{\Gamma}\right)$ from $G$ as described in Construction 4.3, yet for each $i \in\{1, \ldots, k\}$ and every $u_{i} \in V_{i}$ and $v_{i+1} \in V_{i+1}$, the weight $\widetilde{\omega}_{\Gamma}\left(u_{i}, v_{i+1}\right)=$ $\omega(\widetilde{P}[u, v, i])$, where $\widetilde{P}[u, v, i]$ is computed by applying $\mathcal{A}$ to the $s$ - $t$-RPP instance $R[u, v, i]$ (if $\mathcal{A}$ fails to produce a solution, then let there be no $\operatorname{arc}\left(u_{i}, v_{i+1}\right)$ in $\left.\widetilde{\Gamma}\right)$. Finally, try to compute a leastweight layer path $J$ in $\widetilde{\Gamma}$. If it exists, then the corresponding closed walk $W_{J}$ is a feasible solution of weight $\omega\left(W_{J}\right)=\widetilde{\omega}_{\Gamma}(J)$ for $I$. The running time of the whole procedure is $O\left(k n^{2} \tau+k n^{3}\right)$ since the graph $\widetilde{\Gamma}$ has $O\left(k n^{2}\right)$ arcs, the weight of each can be computed in $\tau$ time, and the least-weight layer path in $\Gamma$ can finally be found by $n$ times applying a single-source shortest-path algorithm for directed acyclic graphs. It remains to analyze the probability that the procedure returns an $\alpha$-approximate solution for $I$.

To this end, let $W^{*}$ be an optimal solution to $I, \Gamma=\left(V_{\Gamma}, A_{\Gamma}\right)$ be constructed by Construction 4.3 from $I$, and $J^{*}=\left(x_{1}, y_{2}^{2}, y_{3}^{3}, \ldots\right.$, $y_{k}^{k}, x_{k+1}$ ) be a least-weight layer path in $\Gamma$. First, assume that $\mathcal{A}$ indeed produced an $\alpha$-approximate solution for each instance $R[u, v, i]$ corresponding to any $\operatorname{arc}\left(u_{i}, v_{i+1}\right)$ on $J^{*}$. Then, for each $\operatorname{arc}\left(u_{i}, v_{i+1}\right)$ on $J^{*}$,

$$
\widetilde{\omega}_{\Gamma}\left(u_{i}, v_{i+1}\right)=\omega(\widetilde{P}[u, v, i]) \leq \alpha \omega(P[u, v, i])=\alpha \omega_{\Gamma}\left(u_{i}, v_{i+1}\right)
$$

and $J^{*}$ witnesses the existence of the computed least-weight layer path $J$ in $\widetilde{\Gamma}$. Thus, the weight $\omega\left(W_{J}\right)=\widetilde{\omega}_{\Gamma}(J)$ is at most

$$
\begin{aligned}
\widetilde{\omega}_{\Gamma}\left(J^{*}\right) & =\widetilde{\omega}_{\Gamma}\left(x_{1}, y_{2}^{2}\right)+\widetilde{\omega}_{\Gamma}\left(y_{2}^{2}, y_{3}^{3}\right)+\cdots+\widetilde{\omega}_{\Gamma}\left(y_{k}^{k}, x_{k+1}\right) \\
& \leq \alpha \omega_{\Gamma}\left(x_{1}, y_{2}^{2}\right)+\alpha \omega_{\Gamma}\left(y_{2}^{2}, y_{3}^{3}\right)+\cdots+\alpha \omega_{\Gamma}\left(y_{k}^{k}, x_{k+1}\right) \\
& =\alpha \omega_{\Gamma}\left(J^{*}\right)=\alpha \omega\left(W^{*}\right) .
\end{aligned}
$$

If the described procedure fails to produce an $\alpha$-approximate solution for $I$, then, by contraposition, $\mathcal{A}$ failed to produce an $\alpha$ approximate solution for at least one $s$ - $t$-RPP instance $R[u, v, i]$ corresponding to an arc $\left(u_{i}, v_{i+1}\right)$ on $J^{*}$. Since $J^{*}$ has $k$ arcs, this happens with probability at most $k p$ by the union bound.

## 5. A $5 / 3$-approximation algorithm for $\mathrm{HCPP}(\mathrm{l})$

We now show a polynomial-time 5/3-approximation algorithm for $s-t$-RPP, which, by Lemma 4.5, carries over to $\operatorname{HCPP}(1)$. The algorithm is an adaption of the Christofides-Serdyukov-like $3 / 2$-approximation algorithm from RPP [3,10] to $s$ - $t$-RPP. It closely follows Hoogeveen's [17] adaption of the ChristofidesSerdyukov 3/2-approximation algorithm from metric TSP [4, 7, 29] to metric $s-t$-TSP.

Theorem 5.1. The $s-t$-RPP is 5/3-approximable in $O\left(n^{3}\right)$ time.
Proof. We assume $s \neq t$ (otherwise, one can add a dummy vertex $s \neq t$ and an edge $\{s, t\}$ of zero weight to the initial graph). We only show the $5 / 3$-approximation algorithm for $s-t$-RPP instances $I:=(G, R, \omega, s, t)$ such that $G=(V, E)$ is a complete graph on the vertex set $V=V(R) \cup\{s, t\}$ and such that the weight function $\omega$ satisfies the triangle inequality. This is enough, since the general case reduces to this special case in $O\left(n^{3}\right)$ time and any $\alpha$-approximation for the special case yields an $\alpha$-approximation for the general case [3]. The 5/3-approximation algorithm works in four steps.

Step 1. Compute a set $T \subseteq E$ of edges of minimum total weight such that $R \cup T$ forms a spanning connected subgraph of $G$ (for example, using Kruskal's algorithm [22]).

Step 2. Let $S \subseteq V$ be the set of vertices in $V \backslash\{s, t\}$ that are imbalanced in $G\langle R \cup T\rangle$ and of those vertices in $\{s, t\}$ that are balanced in $G\langle R \cup T\rangle$. Note that $|S|$ is even: Indeed, consider the set $S^{\prime} \subseteq V$ of all vertices that are imbalanced in $G\langle R \cup T\rangle$. Clearly, $\left|S^{\prime}\right|$ is even. Now, if $s, t \in S^{\prime}$, then $S=S^{\prime} \backslash\{s, t\}$. If $s, t \notin S^{\prime}$, then $S=S^{\prime} \cup\{s, t\}$. If $s \in S^{\prime}$ and $t \notin S^{\prime}$ (or vice versa), then $S=S^{\prime} \cup\{t\} \backslash\{s\}$ (or $S=S^{\prime} \cup\{s\} \backslash\{t\}$ ). Thus, $|S|$ is even.

Step 3. Construct a minimum-weight perfect matching $M \subseteq$ $E$ on the vertices of $S$ in $G$ (for example, using Lawler's algorithm [24, Section 6.10]).

Step 4. Return an Euler walk $P$ in $G\langle R \uplus T \uplus M\rangle$. Note that $P$ exists (and can be computed using Hierholtzer's algorithm [11, 16]) since $G\langle R \uplus T \uplus M\rangle$ is connected and all its vertices except for $s$ and $t$ are balanced. Thus, the endpoints of $P$ are $s$ and $t$ and $P$ is a feasible solution to $I$.

All steps can be carried out in $O\left(n^{3}\right)$ time. It remains to prove that $P$ is a $5 / 3$-approximation. To this end, let $P^{*}$ be an optimal solution for $I$. Obviously, $\omega(R \cup T) \leq \omega\left(P^{*}\right)$. Thus, it remains to show $\omega(M) \leq 2 / 3 \cdot \omega\left(P^{*}\right)$. To this end, consider $Q=$ $E\left(P^{*}\right) \uplus R \uplus T$. We will construct three perfect matchings $M_{1}, M_{2}$, and $M_{3}$ on $S$ in $G$ such that $\omega\left(M_{1}\right)+\omega\left(M_{2}\right)+\omega\left(M_{3}\right) \leq \omega(Q)$, and thus $\omega(M) \leq 1 / 3 \cdot \omega(Q) \leq 2 / 3 \cdot \omega\left(P^{*}\right)$.

Since the imbalanced vertices of $G\left\langle P^{*}\right\rangle$ are exactly $s$ and $t$, the imbalanced vertices in $G\langle Q\rangle$ are exactly those in the set $S$. Let the vertices of $S=\left\{v_{1}, v_{2}, \ldots, v_{2 \ell}\right\}$ be numbered in the order
of their first occurrence on $P^{*}$ and let $P_{i}^{*}$ be the subwalk of $P^{*}$ between the vertices $v_{2 i-1} \in S$ and $v_{2 i} \in S$ for all $i \in\{1, \ldots, \ell\}$. Let

$$
E_{1}:=\biguplus_{i=1}^{\ell} E\left(P_{i}^{*}\right)
$$

By shortcutting each path $P_{i}^{*}$ to one edge, one gets a perfect matching $M_{1}$ on the vertices of $S$ such that $\omega\left(M_{1}\right) \leq \omega\left(E_{1}\right)$.

The subgraph $G\left\langle Q \backslash E_{1}\right\rangle$ is Eulerian: it is connected since $R \uplus T \subseteq Q \backslash E_{1}$ and it is balanced since the imbalanced vertices of $G\left\langle E_{1}\right\rangle$ are exactly those of $G\langle Q\rangle$, that is, $S$. Its Euler cycle can be shortcut to a simple cycle on $S$, which can be partitioned into two perfect matchings $M_{2}$ and $M_{3}$ on $S$. Thus,

$$
\begin{aligned}
\omega(P) & =\omega(R \cup T)+\omega(M) \\
& \leq \omega\left(P^{*}\right)+\left(\omega\left(M_{1}\right)+\omega\left(M_{2}\right)+\omega\left(M_{3}\right)\right) / 3 \\
& \leq \omega\left(P^{*}\right)+\left(\omega\left(E_{1}\right)+\omega\left(Q \backslash E_{1}\right)\right) / 3 \\
& \leq \omega\left(P^{*}\right)+\omega(Q) / 3 \leq 5 / 3 \cdot \omega\left(P^{*}\right),
\end{aligned}
$$

where the second inequality is due to the metric weights $\omega$.
Plugging Theorem 5.1 into Lemma 4.5, we immediately get:
Corollary 5.2. $\mathrm{HCPP}(\mathrm{l})$ is $5 / 3$-approximable in $O\left(k n^{5}\right)$ time.

## 6. Parameterized algorithms for $\mathbf{H C P P}(\mathbf{l})$

Lemma 4.5 allows us to easily transfer well-known parameterized algorithms from RPP to $\operatorname{HCPP}(1)$ to show:

Theorem 6.1. Let $\omega_{\max }$ be the maximum edge weight and $c$ be the maximum number of connected components in any edge class of an $\operatorname{HCPP}(1)$ instance. Then, $\operatorname{HCPP}(1)$ is
i) polynomial-time solvable for constant $c$ and
ii) solvable in $2^{c} \cdot \operatorname{poly}\left(\omega_{\max }, n\right)$ time with exponentially decreasing error probability.

Proof. To prove the theorem, it is enough to show that the known RPP algorithms can also be used for $s-t$-RPP. To this end, we reduce $s$ - $t$-RPP to RPP. We assume that $s \neq t$ and that $s$ and $t$ are non-adjacent in $s$ - $t$-RPP instances (otherwise, we can add a new source $s^{\prime}$ and a required weight-zero edge $\left\{s^{\prime}, s\right\}$ ).

Now, note that an $s$ - $t$-RPP instance $I:=(G, R, \omega, s, t)$ can be reduced to an RPP instance $I^{\prime}:=\left(G^{\prime}, R^{\prime}, \omega^{\prime}\right)$ where $I^{\prime}$ is obtained from $I$ by adding an edge $\{s, t\}$ of weight $2 \omega(E)$ to both $E$ and $R$. Then, an optimal solution $P$ for $I$ yields a solution of weight $\omega(P)+2 \omega(E)$ for $I^{\prime}$. Moreover, an optimal solution $P$ for $I^{\prime}$ uses the edge $\{s, t\}$ exactly once: if $P$ traversed it multiple times, then it would be cheaper to replace the second traversal of $\{s, t\}$ by any other $s$ - $t$-path in $G$. Thus, $P$ can be turned into a solution of weight $\omega(P)-2 \omega(E)$ for $I$. That is, optimal solutions translate between $I$ and $I^{\prime}$ (yet approximate solutions do not).

Moreover, if the number of connected components in $G\langle R\rangle$ is $c^{\prime}$, then the number of connected components in $G^{\prime}\left\langle R^{\prime}\right\rangle$ is at most $c^{\prime}+1$. Thus, since RPP is solvable in polynomial time for constant $c^{\prime}[3,12]$, so is $s-t$-RPP. And since RPP is solvable in $2^{c^{\prime}} \cdot \operatorname{poly}\left(\omega_{\max }, n\right)$ with exponentially decreasing error probability [15], so is $s-t$-RPP. To conclude the proof of the theorem, it is
enough to apply Lemma 4.5 and to observe that, for any $s-t$-RPP instance $P[u, v, i]$ solved, the subgraph $G\left\langle E_{i}\right\rangle$ induced by the required edges $E_{i}$ has at most $c$ connected components.

## 7. Conclusion

Our work leaves open several questions. First, what is the computational complexity of $\operatorname{HCPP}(\mathrm{c})$ with a constant number of edge classes? It has been conjectured to be polynomial-time solvable [3], yet no polynomial-time algorithm is known even for the case with three classes.

Second, can one close the gap between our 5/3-approximation for $s$ - $t$-RPP and the known 3/2-approximation for RPP [3, 10]? For example, recently, a 3/2-approximation for metric $s-t$ TSP has been shown [31], matching the approximation factor of the Christofides-Serdyukov algorithm for metric TSP [4, 7, 29]. It is not obvious whether the used approaches carry over to $s$ -$t$-RPP, yet closing the gap between $s-t$-RPP and RPP would immediately give a $3 / 2$-approximation for $\operatorname{HCPP}(1)$.

Our fixed-parameter algorithm for $\mathrm{HCPP}(1)$ parameterized by the maximum number $c$ of connected components in any edge class raises the question whether and how lossy kernelization results for RPP parameterized by $c$ [2] carry over to HCPP(1).

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