The Hierarchical Chinese Postman Problem: the slightest disorder makes it hard, yet disconnectedness is manageable

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Abstract

The Hierarchical Chinese Postman Problem is finding a shortest traversal of all edges of a graph respecting precedence constraints given by a partial order on classes of edges. We show that the special case with connected classes is NP-hard even on orders decomposable into a chain and an incomparable class. For the case with linearly ordered (possibly disconnected) classes, we get 5/3-approximations and fixed-parameter algorithms by transferring results from the Rural Postman Problem.

Keywords: approximation algorithm, fixed-parameter algorithm, NP-hardness, arc routing, rural postman problem, temporal graphs

The following NP-hard arc routing problem arises in snow plow-

given by a partial order on classes of edges. We show that the decomposable into a chain and an incomparable class. For the 5/3-approximations and fixed-parameter algorithms by transferr *Keywords:* approximation algorithm, fixed-parameter algorithms **1. Introduction 1. Introduction 1. Introduction Problem 1.1** (Hierarchical Chinese Postman Problem, HCPP). *Input:* An undirected graph *G* = (*V*, *E*), edge weights *ω*: *E* → N, a partition *P* of *E* into *k* classes, a partial order < on *P*. *Find:* A least-weight closed walk traversing each edge in *E* at least once such that each edge *e* in a class *E'* is traversed only after all edges in all classes *E''* < *E'* are traversed.
The case *k* = 1 is the *Chinese Postman Problem* (*CPP*), which reduces to a minimum-weight perfect matching problem [4, 6, 9, 28]. We study the following special cases of HCPP: HCPP(c): each edge class induces a connected subgraph, HCPP(c): both of the above restrictions.
HCPP(1) and HCPP(c,1) can also be understood as variants of the Travelling Salesman Problem (TSP) in temporal graphs [26], with the difference that it is required to explore all edges instead of all vertices and that edges never disappear from the graph. HCPP(c,1) is polynomial-time solvable [8, 14, 21]. This naturally raises two questions about HCPP(c) and HCPP(1): (a) Is HCPP(c) effectively solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like the term of the traveling solvable on other order types, like term of the term of the traveling solvable to the term of the traveling the term of the traveling solvable to the term of the traveling the t HCPP(1) and HCPP(c,1) can also be understood as variants of the Travelling Salesman Problem (TSP) in temporal graphs [26], with the difference that it is required to explore all edges instead of all vertices and that edges never disappear from the graph.

(a) Is HCPP(c) effectively solvable on other order types, like several scheduling problems on, for example, tree orders [18], interval orders [27], and bounded-width orders [1, 30]?

(b) Is HCPP(l) effectively solvable when the number of connected components in each class is sufficiently small? If the number of connected components is unbounded, HCPP(1) is NP-hard already for k = 2 [3].

Our contributions. In Section 3, we show that HCPP(c) is NPhard even on partial orders that are decomposable into a linear order and a class that is incomparable to all others classes, thus negatively answering (a) for all order types mentioned there. The remaining sections are dedicated to question (b).

In Section 4, we revisit a construction that reduces HCPP(c,l) to the *s*-*t*-Rural Postman Path Problem (*s*-*t*-RPP, Problem 4.1) [8]. We show that, when applied to HCPP(1), the construction transfers performance guarantees of approximation and randomized algorithms from s-t-RPP to HCPP(1).

In Section 5, we show a 5/3-approximation algorithm for HCPP(1). This contrasts TSP in temporal graphs, which is not better than 2-approximable unless P = NP [26]. To get 5/3-approximations for HCPP(1), we use the construction from Section 4 and show a 5/3-approximation algorithm for *s*-*t*-RPP analogously to that for s-t-TSP [17]. Any better approximation factor for *s*-*t*-RPP will directly carry over to HCPP(1).

In Section 6, we use the construction from Section 4 and known algorithms for the Rural Postman Problem [3, 10, 15] to show that HCPP(1) is solvable in polynomial-time if each class induces a constant number c of connected components. When the edge weights are polynomially bounded, one can even obtain randomized fixed-parameter algorithms with respect to c.

2. Preliminaries

By \mathbb{N} , we denote the set of natural numbers, including zero. For two multisets A and B, $A \uplus B$ is the multiset obtained by adding the multiplicities of elements in A and B. By $A \setminus B$ we denote the multiset obtained by subtracting the multiplicities of elements in B from the multiplicities of elements in A. Finally, given some weight function $\omega: A \to \mathbb{N}$, the *weight* of a multiset A is $\omega(A) := \sum_{e \in A} v(e)\omega(e)$, where v(e) is the multiplicity of *e* in *A*.

We mostly consider simple undirected graphs G = (V, E)with a set V(G) := V of vertices and a set $E(G) := E \subseteq \{\{u, v\} \mid$ $u, v \in V, u \neq v$ of *edges*. Unless stated otherwise, *n* denotes the

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number of vertices and *m* denotes the number of edges. Within proofs, there may occur *multigraphs*, where *E* is a *multiset*, and directed graphs G = (V, A) with a set of *arcs* $A \subseteq V^2$. The *degree* of a vertex in an undirected multigraph is its number of incident edges. We call a vertex *balanced* if it has even degree. We call a graph *balanced* if all its vertices are balanced. For a multiset *R* of edges, we denote by V(R) the set of their incident vertices. For a multiset *R* of edges of $G, G\langle R \rangle := (V(R), R)$ is the (multi)graph *induced by the edges in R*.

A walk from v_0 to v_ℓ in a graph G is a sequence $w = (v_0, e_1, e_1)$ $v_1, e_2, v_2, \dots, e_\ell, v_\ell$ such that $e_i = \{v_{i-1}, v_i\}$ (if G is undirected) or $e_i = (v_{i-1}, v_i)$ (if G is directed) for each $i \in \{1, \dots, \ell\}$. When there is no ambiguity (like in simple graphs), we will also specify walks simply as a list of vertices. If $v_0 = v_\ell$, then we call w a closed walk. If all vertices on w are pairwise distinct, then w is a path. If only its first and last vertex coincide, then w is a cycle. A subwalk w' of w is any subsequence w' of w that is itself a walk. By E(w), we denote the multiset of edges on w, that is, each edge appears on w and in E(w) equally often. The weight of walk w is $\omega(w) := \sum_{\ell=1}^{\ell} \omega(e_{\ell})$. For a walk w, we denote $G\langle w \rangle := G\langle E(w) \rangle$. Note that $G\langle R \rangle$ and $G\langle w \rangle$ do not contain isolated vertices yet might contain edges with a higher multiplicity than G and, therefore, are not necessarily sub(multi)graphs of G. An Euler walk for G is a walk that traverses each edge or arc of G exactly as often as it is present in G. An Euler tour is a closed Euler walk. A graph is Eulerian if it allows for an Euler tour. A connected undirected graph is Eulerian if and only if all its vertices are balanced.

For any $\alpha \ge 1$, an α -approximate solution for a minimization problem is a feasible solution whose weight does not exceed the weight of an optimal solution by more than a factor of α , called the *approximation factor* [13]. The Exponential Time Hypothesis (ETH) is that 3-SAT (Problem 3.2) with *n* variables is not solvable in $2^{o(n)}$ time [19].

3. NP-hardness for HCPP(c) with one incomparable class

HCPP(c,l) is polynomial-time solvable [8, 14, 21]. We prove that adding a single incomparable class makes the problem NP-hard.

Theorem 3.1. Even on edges with weight one and orders that are decomposable into a linear order and a class that is incomparable to all others classes,

- (i) HCPP(c) is NP-hard and
- (ii) not solvable in $2^{o(n+m+k)}$ time unless ETH fails.

To prove Theorem 3.1, we use a polynomial-time many-one reduction from 3-SAT, which is NP-hard [20] and, unless the ETH fails, is not solvable in $2^{o(n+m)}$ time [19].

Problem 3.2 (3-SAT).

Input: A formula φ in conjunctive normal form with *n* variables and *m* clauses, each containing at most three literals.

Question: Is there a assignment to the variables satisfying φ ?

The reduction is carried out by the following construction, which is illustrated in Figure 1.

Construction 3.3. Let φ be an instance of 3-SAT. First, delete each clause containing both *x* and \bar{x} : they are always satisfied. Consider now the variables x_1, \ldots, x_n and clauses C_1, \ldots, C_m . For each $i \in \{1, \ldots, n\}$, let μ_i denote the number of clauses containing either x_i or \bar{x}_i . We describe an instance $I_{\varphi} = (G, \omega, \mathcal{P}, \prec)$ of HCPP(c). Each edge will have weight one.

Graph *G* contains a path (c_j^1, c^*, c_j^2) for each $j \in \{1, ..., m\}$ and, for each $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., 6\mu_i\}$, a path $P_i^{\ell} := (t_i^{\ell}, z_i^{\ell}, f_i^{\ell})$. For each $i \in \{1, ..., n\}$, it contains a cycle

$$X_i := (t_i^1, t_i^2, \dots, t_i^{6\mu_i-1}, t_i^{6\mu_i}, f_i^{6\mu_i}, f_i^{6\mu_i-1}, \dots, f_i^2, f_i^1, t_i^1).$$

For each $i \in \{1, ..., n\}$ and $i' = i \mod n + 1$, cycles X_i and $X_{i'}$ are connected to each other via a cycle

$$Y_{ii'} = (t_i^{6\mu_i}, f_{i'}^1, f_i^{6\mu_i}, t_{i'}^1, t_i^{6\mu_i}).$$

For each literal x_i in a clause C_i , graph G contains a cycle

$$Z_{ij} := (t_i^{6\ell-3}, c_j^1, a_{ij}, c_j^2, t_i^{6\ell-2}, b_{ij}, t_i^{6\ell-3}),$$

where $\ell \le \mu_i$ is such that C_j is the ℓ -th clause containing x_i or \bar{x}_i . For each literal \bar{x}_i in a clause C_j , graph G contains a cycle

$$\bar{Z}_{ij} := (f_i^{6\ell-3}, c_j^1, a_{ij}, c_j^2, f_i^{6\ell-2}, b_{ij}, f_i^{6\ell-3}),$$

where $\ell \le \mu_i$ is such that C_j is the ℓ -th clause containing x_i or \bar{x}_i .

The edges are ordered as follows. For each $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., 6\mu_i\}$, $E_i^{\ell} = E(P_i^{\ell})$ is a connected class. They are lexicographically ordered, that is,

$$E_i^{\ell} \prec E_{i'}^{\ell'} \iff (i < i') \lor (i = i' \land \ell \le \ell').$$

They are preceded by the connected class

$$E_0 = \bigcup_{i=1}^n E(X_i) \cup \bigcup_{i=1}^n E(Y_{i,i \bmod n+1}) \cup \bigcup_{x_i \in C_j} E(Z_{ij}) \cup \bigcup_{\bar{x}_i \in C_j} E(\bar{Z}_{ij}),$$

Finally, the edge set E^* consisting of all edges incident to c^* forms a connected class incomparable to all other classes.

For convenience, we collect the vertices of I_{φ} of the form t_i^{ℓ} and f_i^{ℓ} and of the form c_i^1 and c_i^2 into sets

$$V_{FT} := \bigcup_{i=1}^{n} \bigcup_{\ell=1}^{6\mu_i} \{t_i^{\ell}, f_i^{\ell}\}$$
 and $V_C := \bigcup_{j=1}^{m} \{c_j^1, c_j^2\}.$

Observation 3.4. Let $I_{\varphi} = (G, \omega, \mathcal{P}, \prec)$ be the HCPP(c) instance constructed by Construction 3.3 from a 3-SAT instance φ with *n* variables and *m* clauses. Then,

- (i) the subgraph $G\langle E_0 \rangle$ is Eulerian: it is connected and the union of pairwise edge-disjoint cycles, that is, balanced,
- (ii) the imbalanced vertices of G are therefore $V_{FT} \cup V_C$, and
- (iii) the number of vertices, edges, and classes is O(n + m), since $\sum_{i=1}^{n} \mu_i \leq 3m$ in any 3-SAT formula.

In the following, we will show that the HCPP(c) instance I_{φ} allows for a *tight* tour if and only if φ is satisfiable:



Figure 1: Illustration of Construction 3.3. The graph is generated from the formula $\varphi = (\bar{x}_1 \lor x_4) \land (\bar{x}_2 \lor x_3)$, that is, $C_1 = (\bar{x}_1 \lor x_4)$ and $C_2 = (\bar{x}_2 \lor x_3)$. The dotted edges form the edge set E_0 . The solid edges form the paths P_i^ℓ and (c_j^1, c^*, c_j^2) . The gray areas illustrate the types of cycles introduced in the construction: they consist of the dashed edges enclosed in the gray areas. Note that E_0 (the dotted edges) forms an Eulerian subgraph: it is the union of cycles and thus each vertex has an even number of incident edges in E_0 . Thus, each vertex of the form t_i^ℓ , f_i^ℓ , c_j^1 and c_j^2 is imbalanced and they are the only imbalanced vertices.



Figure 2: A minimum-weight closed walk for the graph in Figure 1 first visits each edge in E_0 exactly once (the dotted edges in Figure 1), and then follows the arrows as shown in this figure. The walk corresponds to $x_1 = 0$ and $x_2 = x_3 = x_4 = 1$.

Definition 3.5. A feasible solution for an HCPP(c) instance $(G, \omega, \mathcal{P}, \prec)$ with G = (V, E) is a *tight tour* if it has weight at most |E| + b/2, where b is the number of imbalanced vertices in G.

Proposition 3.6. The HCPP(c) instance $I_{\varphi} = (G, \omega, \mathcal{P}, \prec)$ created from a 3-SAT instance φ by Construction 3.3 allows for a tight tour if and only if φ is satisfiable.

In the rest of this section, it remains to prove Proposition 3.6, which, together with Observation 3.4(iii), yields Theorem 3.1.

Satisfiability of φ implies a tight tour in I_{φ} . Assume that φ is satisfiable. We show a tight tour T for I_{φ} . Without loss of generality, assume that x_1 is "true": otherwise, we can replace x_1 by \bar{x}_1 throughout the formula φ .

The tight tour *T* for I_{φ} then looks as follows (an example is shown in Figure 2). It starts in f_1^1 , first visits each edge of E_0 exactly once and returns to f_1^1 . This is possible by Observation 3.4(i). Then, it remains to traverse the paths $(t_i^{\ell}, z_i^{\ell}, f_i^{\ell})$ for each $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., 6\mu_i\}$ and the paths (c_j^1, c^*, c_j^2) for each $j \in \{1, ..., m\}$. This is done as follows. For *i* from 1 to *n*, if x_i is "true", *T* visits the vertices

$$f_i^1, z_i^1, t_i^1, t_i^2, z_i^2, f_i^2, f_i^3, z_i^3, t_i^3, t_i^4, z_i^4, f_i^4 \dots, t_i^{6\mu_i}, z_i^{6\mu_i}, f_i^{6\mu_i},$$

for some $\ell \in \{1, ..., \mu_i\}$ taking a detour through the vertices $t_i^{6\ell-3}, c_j^1, c^*, c_j^2, t_i^{\ell-2}$ if clause C_j contains x_i and (c_j^1, c^*, c_j^2) has not been traversed before. If x_i is "false", then T visits

$$t_i^1, z_i^1, f_i^1, f_i^2, z_i^2, t_i^2, t_i^3, z_i^3, f_i^3, f_i^4, z_i^4, t_i^4, \dots, f_i^{6\mu_i}, z_i^{6\mu_i}, t_i^{6\mu_i},$$

for some $\ell \in \{1, ..., \mu_i\}$ taking a detour through the vertices $f_i^{6\ell-3}, c_j^1, c^*, c_j^2, f_i^{6\ell-2}$ if clause C_j contains \bar{x}_i and (c_j^1, c^*, c_j^2) has not been traversed before. Finally, after $f_n^{6\mu_n}$ or $t_n^{6\mu_n}$, the walk *T* returns to f_1^1 . Note that this traversal is possible due to the cycle $Y_{i,i \mod n+1}$ for each $i \in \{1, ..., n\}$.

Observe that the closed walk *T* contains all edges and respects precedence constraints: For the edges in E_0 and all paths $(t_i^{\ell}, z_i^{\ell}, f_i^{\ell})$ for $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., 6\mu_i\}$, this is obvious. To see that the path (c_j^1, c^*, c_j^2) has been traversed for each $j \in \{1, ..., m\}$, observe that each clause C_j contains a true literal, so that a detour via c_j^1, c^*, c_j^2 is taken.

To see that *T* is tight, we check which edges are traversed a second time. When x_i is "true", the edges $\{f_i^{2\ell}, f_i^{2\ell+1}\} \in E_0$ for $\ell \in \{1, ..., \mu_i - 1\}$ are visited a second time, whereas each edge $\{t_i^{2\ell-1}, t_i^{2\ell}\} \in E_0$ for $\ell \in \{1, ..., \mu_i\}$ is traversed a second time or skipped by a detour that traverses the edges $\{t_i^{2\ell-1}, c_j^1\}$ and $\{t_i^{2\ell}, c_j^2\}$ a second time. Analogously when x_i is "false". Moreover, for each $i \in \{1, ..., n\}$, one edge of the cycle $Y_{i,i \mod n+1}$ is visited a second time: it joins the last vertex visited by T in X_i to the first vertex visited by T in $X_{i \mod n+1}$. We thus see that the edges visited a second time form a matching. Their endpoints are the b imbalanced vertices $V_{FT} \cup V_C$. Thus, T traverses not more than |E| + b/2 edges.

Tight tour for I_{φ} *implies satisfiability of* φ . Assume that I_{φ} allows for a tight tour *T*. We show that φ is satisfiable.

Lemma 3.7. Let $M = E(T) \setminus E$, that is, M is the multiset of the edges that the tight tour T traverses additionally to E (taking into account the multiplicity of additional visits). Then,

- (i) $M \subseteq E_0$ is a perfect matching on the vertices $V_{FT} \cup V_C$, in particular, *M* contains each edge at most once,
- (ii) each edge in M has an endpoint in V_{FT} ,
- (iii) $G\langle (E \setminus E_0) \cup M \rangle$ is connected.

Proof. (i) Since *T* is a closed walk, all vertices in $G\langle T \rangle$ are balanced, whereas its subgraph $G\langle E \rangle = G$ has *b* imbalanced vertices. Since *T* contains at most |E|+b/2 edges, the graph $G\langle T \rangle$ contains at most b/2 edges additionally to those in $G\langle E \rangle$. Thus, $G\langle T \rangle$ contains a set *M* of at most b/2 edges whose endpoints are the *b* imbalanced vertices of $G\langle E \rangle$. By Observation 3.4(ii), these are exactly the vertices $V_{FT} \cup V_C$. This is only possible if *M* is a perfect matching on $V_{FT} \cup V_C$. Since each edge of *G* that is not in E_0 has at least one balanced endpoint (namely, c^* or one of the z_i^{ℓ}), we easily get $M \subseteq E_0$.

(ii) The only edges in E_0 that have no endpoints in V_{FT} have one of the vertices of the form a_{ij} or b_{ij} as endpoints. Since these are only on the cycle Z_{ij} or \overline{Z}_{ij} , they are balanced. Thus, M cannot contain such edges.

(iii) Let T^* be a tight tour for $(G, \omega, \mathcal{P}, \prec)$ such that T^* visits exactly the edges in M twice and the minimal prefix T_1 of T^* traversing all edges in E_0 has minimum length (the tight tour Tand (i) witness the existence of T^*). Let T_2 be the rest of T^* . We will prove that T_1 is an Euler tour for $G\langle E_0 \rangle$. Then (iii) follows since $G\langle (E \setminus E_0) \cup M \rangle$ is even Eulerian: T_2 visits all edges in $(E \setminus E_0) \cup M$, since they are not visited by T_1 ; T_2 is closed since T^* and its prefix T_1 are; and, by choice of T^* , T_2 does not visit any edge in $(E \setminus E_0) \cup M$ more than once. It remains to prove that T_1 is indeed an Euler tour for $G\langle E_0 \rangle$.

We first prove that T_1 is a closed walk. By the minimality of T_1 and choice of \prec , T_1 does not end in c^* and does not contain edges from any class E_i^{ℓ} . It might contain edges from E^* . Assume, for the sake of a contradiction, that T_1 starts at some vertex *s* and ends at some vertex $t \neq s$. Then, *t* is not balanced in $G\langle T_1 \rangle$ but balanced in $G\langle E_0 \rangle$ by Observation 3.4(i). Thus, there is an edge $e = \{t', t\} \in E^* \cup M$ on T_1 . The graph $G\langle E(T_1) \setminus \{e\} \rangle$ contains E_0 , is connected, all its vertices except for *s* and *t'* are balanced, and it therefore has an Euler walk T'_1 . It follows that (T'_1, e, T_2) is another tight tour for *G* visiting all edges with the same multiplicity as T^* , yet its prefix T'_1 containing E_0 satisfies $|T'_1| < |T_1|$. This contradicts the choice of T^* .

We now show that T_1 traverses each edge $e \in E_0 \cup E^*$ at most once. Towards a contradiction, assume that it traverses $e = \{u, v\}$ twice. Then, $v \in V_{FT}$ by (ii). Thus, v is not incident to any edges in E^* and, because v is balanced in $G\langle E_0 \rangle$ by Observation 3.4(i), it is also balanced in $G\langle E_0 \cup E^* \rangle$. Since v is balanced in $G\langle E_0 \cup E^* \rangle$, balanced in $G\langle T_1 \rangle$, and T_1 traverses e twice, T_1 also traverses another edge incident to v twice, contradicting (i).

Finally, we prove that T_1 contains *only* edges of E_0 . Towards a contradiction, assume that T_1 contains an edge $\{c^*, c\} \in E^*$. Vertex $c \in V_C$ is balanced in $G\langle T_1 \rangle$, yet not balanced in its subgraph $G\langle E_0 \cup \{c^*, c\}\rangle$ by Observation 3.4(i). Thus, $G\langle T_1 \rangle$ contains some edge $e \in E_0 \cup E^*$ twice, which is impossible. \Box



Figure 3: The two cases in the proof of Lemma 3.9. Dotted edges are all the possibly present edges in E_0 available for inclusion in M by Lemma 3.7(i) (the edge $\{t_i^{6\ell-5}, f_i^{6\ell-5}\}$ is present if $\ell = 1$, the edge $\{t_i^{6\ell}, f_i^{6\ell}\}$ is present if $\ell = \mu_i$). Including $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ in M or excluding it from M force all the bold edges into M due to the fact that all vertices must be contained in some edge of M and that edges drawn above each other cannot both be part of M.

We now show that the matching M from Lemma 3.7 takes one of two possible forms in each variable cycle X_i . This will correspond to setting a variable to "true" or "false".

Definition 3.8. Let $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., \mu_i\}$. We call an edge $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ *covered* if $\{t_i^{6\ell-3}, t_i^{6\ell-2}\} \in M$ or if there is a $j \in \{1, ..., m\}$ such that both $\{t_1^{6\ell-3}, c_j^1\}$ and $\{t_1^{6\ell-2}, c_j^2\}$ are in *M*.

We call an edge $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ covered if $\{f_i^{6\ell-3}, f_i^{6\ell-2}\} \in M$ or if there is a $j \in \{1, ..., m\}$ such that both $\{f_1^{6\ell-3}, c_j^1\}$ and $\{f_1^{6\ell-2}, c_j^2\}$ are in M.

Lemma 3.9. For each $i \in \{1, ..., n\}$, either all $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ are covered or all $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ are covered for $\ell \in \{1, ..., \mu_i\}$.

Proof. For any $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., \mu_i\}$, we first show that exactly one of $\{t_i^{6\ell-2}, t_i^{6\ell-3}\}$ and $\{f_i^{6\ell-2}, f_i^{6\ell-3}\}$ is covered. Note that, by Construction 3.3, at most one of these pairs of vertices is attached to $\{c_j^1, c_j^2\}$ for any $j \in \{1, ..., m\}$. Without loss of generality, let this be $\{t_i^{6\ell-2}, t_i^{6\ell-3}\}$. The other case is symmetric.

Denote $R := E \setminus E_0$. By Lemma 3.7(iii), $G\langle R \cup M \rangle$ is connected. Thus, there is at least one edge of M leaving any subset of connected components of $G\langle R \rangle$ and, for each $h \in$ $\{6\ell - 5, \dots, 6\ell - 1\}$, only one of $\{t_i^h, t_i^{h+1}\}$ and $\{f_i^h, f_i^{h+1}\}$ is in M: otherwise, the matching M could not contain any edge leaving the set of connected components $\{\{t_i^h, z_i^h, f_i^h\}, \{t_i^{h+1}, z_i^{h+1}, f_i^{h+1}\}\}$. We also exploit that, by Lemma 3.7(i), all vertices in V_{FT} must be incident to an edge of M.

We now distinguish two cases, illustrated in Figure 3. First, assume that $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ is covered, that is, in M. Then all bold edges shown in Figure 3a are in M. Thus, the edge $\{t_i^{6\ell-2}, t_i^{6\ell-3}\}$ is not covered. If the edge $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ is *not* covered, that is, not in M, then all bold edges shown in Figure 3b are in M. To match the vertices $t_i^{6\ell-3}$ and $t_i^{6\ell-2}$, one either has $\{t_i^{6\ell-3}, t_i^{6\ell-2}\} \in M$ or $\{\{t_i^{6\ell-3}, c_j^1\}, \{t_i^{6\ell-2}, c_j^2\}\} \subseteq M$. That is, $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ is covered.

Finally, towards a contradiction, assume that there are ℓ, ℓ' such that $\{f_i^{6\ell-2}, f_i^{6\ell-3}\}$ and $\{t_i^{6\ell'-2}, t_i^{6\ell'-3}\}$ are covered. Then we can choose ℓ, ℓ' so that $|\ell - \ell'| = 1$. Assume $\ell' = \ell + 1$, the other case is symmetric. Then, as illustrated in Figure 3a, vertex $t_i^{6\ell}$ has to be matched to $t_i^{6\ell'-5}$ (there is no edge $\{t_i^{6\ell}, f_i^{6\ell}\}$ in this case by Construction 3.3, since $\ell < \mu_i$). However, vertex $t_i^{6\ell'-5}$ is already matched to $t_i^{6\ell'-4}$, so that this is impossible.

We can now easily prove that, since I_{φ} has a tight tour *T*, the formula φ is satisfiable, thus concluding the proof of Proposition 3.6. By Lemma 3.7(i) and (ii), for each clause C_i of φ , the vertices c_i^1 and c_j^2 are matched to vertices in V_{FT} by M. By Construction 3.3, c_j^1 can only be matched to $t_i^{6\ell-3}$ or $f_i^{6\ell-3}$ for some $i \in \{1, ..., n\}$ and $\ell \in \{1, ..., \mu_i\}$. By Lemma 3.9, if c_j^1 is matched to $t_i^{6\ell-3}$, then $t_i^{6\ell-2}$ is matched to c_j^2 and the edges $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ are covered for all $\ell \in \{1, ..., \mu_i\}$, whereas $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ is not covered for any $\ell \in \{1, ..., \mu_i\}$. Thus, clause C_j (and all other clauses containing x_i) can be satisfied by setting variable x_i to "true". If, on the other hand, c_j^1 is matched to $f_i^{6\ell-2}$, then $f_i^{6\ell-2}$ is matched to c_j^2 and the edges $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ are covered for all $\ell \in \{1, ..., \mu_i\}$, whereas $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ is not covered for any $\ell \in \{1, ..., \mu_i\}$, such that $t_i^{6\ell-2}$ is matched to c_j^2 and the edges $\{f_i^{6\ell-3}, f_i^{6\ell-2}\}$ are covered for all $\ell \in \{1, ..., \mu_i\}$, whereas $\{t_i^{6\ell-3}, t_i^{6\ell-2}\}$ is not covered for any $\ell \in \{1, ..., \mu_i\}$. Thus, clause C_j (and all other clauses containing \bar{x}_i) can be satisfied by setting \bar{x}_i to "false".

4. Relation between HCPP(l) and the Rural Postman

Dror et al. [8] showed how to reduce HCPP(c,l) to polynomialtime solvable special cases of the following problem.

Problem 4.1 (s-t-Rural Postman Path Problem, s-t-RPP).

Input: An undirected graph G = (V, E), edge weights $\omega \colon E \to \mathbb{N}$, vertices $s, t \in V$, and a subset $R \subseteq E$ of *required edges*.

Find: A walk W^* of minimum total weight $\omega(W^*)$, starting in *s*, ending in *t*, and traversing all edges of *R*.

In general, *s*-*t*-RPP is strongly NP-hard, as well as the better known Rural Postman Problem (RPP), where the goal is to find a *closed* walk [25]. Dror et al. [8] reduce HCPP(c,l) to multiple *s*-*t*-RPP instances in which the subgraph $G\langle R \rangle$ is connected. Since this case of *s*-*t*-RPP is polynomially-time solvable, this yields a polynomial-time algorithm for HCPP(c,l) [8].

We now show that, while applying the same construction to HCPP(l) does not yield polynomial-time solvable instances of *s*-*t*-RPP, it allows to transfer running times, approximation factors, and error probabilities of *s*-*t*-RPP algorithms to HCPP(l). This is in contrast to a construction due to Cabral et al. [5], who showed a polynomial-time reduction of HCPP(l) to RPP that does *not* allow to transfer approximation factors: it introduces very heavy required edges, which always contribute to the goal function and thus make bad approximate solutions "look" good. We now describe the construction of Dror et al. [8].

Definition 4.2. In this section, we denote the edge classes of HCPP(1) instances $(G, \omega, \mathcal{P}, \prec)$ by E_1, \ldots, E_k , where $E_i \prec E_j$ if and only if $1 \le i < j \le k$.



Figure 4: Illustration for Construction 4.3: from a graph G with k = 3 edge classes (on the left), Construction 4.3 constructs a graph Γ with k + 1 layers (on the right). Note that, for example, vertex $b \in V(G)$ is the only vertex in $V(E_2)$ incident to edges of previous classes, thus, its copy $b_2 \in V(\Gamma)$ is the only vertex in layer V_2 .

By R[u, v, i], we denote the *s*-*t*-RPP instance of finding a minimum-weight walk between the vertices *u* and *v* in $G\langle E_1 \cup \cdots \cup E_i \rangle$ traversing all edges in E_i . By P[u, v, i], we denote an arbitrary optimal solution to R[u, v, i].

Construction 4.3. From a HCPP(I) instance $(G, \omega, \mathcal{P}, \prec)$, construct a directed arc-weighted graph $\Gamma = (V_{\Gamma}, A_{\Gamma})$ as illustrated in Figure 4: The vertex set $V_{\Gamma} = \bigcup_{i=1}^{k+1} V_i$ is a union of *layers* V_i . For each $i \in \{2, \ldots, k\}$, layer V_i contains a copy of each vertex in *G* that is incident to an edge of E_i and of any predecessor class. Namely,

$$V_{1} = \{u_{1} \mid u \in V(E_{1})\}, \qquad V_{k+1} = \{u_{k+1} \mid u \in V(E_{1})\},$$
$$V_{i} = \left\{u_{i} \mid u \in V(E_{i}) \cap \bigcup_{j=1}^{i-1} V(E_{j})\right\} \quad \text{for } i \in \{2, \dots, k\}.$$

For each pair of vertices $u_i \in V_i$ and $v_{i+1} \in V_{i+1}$, where $i \in \{1, ..., k\}$, there is an arc $(u_i, v_{i+1}) \in A_{\Gamma}$ of weight $\omega_{\Gamma}(u_i, v_{i+1}) = \omega(P[u, v, i])$. If P[u, v, i] does not exist, there is no arc (u_i, v_{i+1}) .

Proposition 4.4 (Dror et al. [8]). Let $I := (G, \omega, \mathcal{P}, \prec)$ be an HCPP(1) instance and Γ be constructed from *I* by Construction 4.3. Then, the weight of an optimal solution to *I* coincides with the weight of a least-weight *layer path* in Γ , where a *layer path* in Γ is a path from $v_1 \in V_1$ to $v_{k+1} \in V_{k+1}$ such that v_1 and v_{k+1} are copies of the same vertex $v \in V(E_1)$.

In particular, each layer path in Γ has the form $J = (v_1, y_2^2, y_3^3, \dots, y_k^k, v_{k+1})$, where $y_i^i \in V_i$ for $i \in \{2, \dots, k\}$ and concatenating the corresponding walks $P[v, y^2, 1], P[y^2, y^3, 2], \dots, P[y^k, v, k]$ yields a feasible solution W_J of weight $\omega(W_J) = \omega_{\Gamma}(J)$ for I.

Construction 4.3 can be used to solve HCPP(c,l) in $O(kn^5)$ time: Γ has at most kn^2 arcs, the weight of each is computed by solving an *s*-*t*-RPP instance R[u, v, i], which works in $O(n^3)$ time since the set E_i of required edges is connected [8]. It remains to find a layer path in Γ . This can be done in $O(kn^3)$ time by *n* times calling a linear-time single-source shortest-path algorithm for directed acyclic graphs.

However, when applied to HCPP(l), Construction 4.3 gets to solve *s*-*t*-RPP instances R[u, v, i] where the set of required edges E_i might be disconnected. Since we do not know how to solve them in polynomial time, in Sections 5 and 6, we will solve them using approximation algorithms and randomized fixed-parameter algorithms. Their performance guarantees carry over to HCPP(l) as follows. **Lemma 4.5.** Let $I = (G, \omega, \mathcal{P}, \prec)$ be an HCPP(l) instance. Assume that there is an algorithm running in τ time that, given any *s*-*t*-RPP instance R[u, v, i] (cf. Definition 4.2), outputs an α -approximate solution for R[u, v, i] with probability at least 1 - p.

Then, there is an algorithm running in $O(kn^2\tau + kn^3)$ time that returns an α -approximate solution for I with probability at least 1 - pk.

Proof. Let \mathcal{A} denote the assumed randomized approximation algorithm for solving *s*-*t*-RPP instances R[u, v, i]. Since we can check the feasibility of any solution returned by \mathcal{A} in linear time, we can assume that \mathcal{A} makes only one-sided errors: For an infeasiable instance R[u, v, i], it returns nothing. For a feasible instance R[u, v, i], with probability at most p, it may return nothing or produce a solution that is more expensive than an α -approximate solution. Moreover, since feasibility of I is easy to check [8], we will assume that I has a feasible solution. Then we compute a solution to I as follows.

Construct an arc-weighted directed graph $\widetilde{\Gamma} = (\widetilde{V}_{\Gamma}, \widetilde{A}_{\Gamma})$ from *G* as described in Construction 4.3, yet for each $i \in \{1, ..., k\}$ and every $u_i \in V_i$ and $v_{i+1} \in V_{i+1}$, the weight $\widetilde{\omega}_{\Gamma}(u_i, v_{i+1}) = \omega(\widetilde{P}[u, v, i])$, where $\widetilde{P}[u, v, i]$ is computed by applying \mathcal{A} to the *s*-*t*-RPP instance R[u, v, i] (if \mathcal{A} fails to produce a solution, then let there be no arc (u_i, v_{i+1}) in $\widetilde{\Gamma}$). Finally, try to compute a least-weight layer path J in $\widetilde{\Gamma}$. If it exists, then the corresponding closed walk W_J is a feasible solution of weight $\omega(W_J) = \widetilde{\omega}_{\Gamma}(J)$ for I. The running time of the whole procedure is $O(kn^2\tau + kn^3)$ since the graph $\widetilde{\Gamma}$ has $O(kn^2)$ arcs, the weight of each can be computed in τ time, and the least-weight layer path in Γ can finally be found by n times applying a single-source shortest-path algorithm for directed acyclic graphs. It remains to analyze the probability that the procedure returns an α -approximate solution for I.

To this end, let W^* be an optimal solution to I, $\Gamma = (V_{\Gamma}, A_{\Gamma})$ be constructed by Construction 4.3 from I, and $J^* = (x_1, y_2^2, y_3^3, ..., y_k^k, x_{k+1})$ be a least-weight layer path in Γ . First, assume that \mathcal{A} indeed produced an α -approximate solution for each instance R[u, v, i] corresponding to any arc (u_i, v_{i+1}) on J^* . Then, for each arc (u_i, v_{i+1}) on J^* ,

$$\widetilde{\omega}_{\Gamma}(u_i, v_{i+1}) = \omega(\widetilde{P}[u, v, i]) \leq \alpha \omega(P[u, v, i]) = \alpha \omega_{\Gamma}(u_i, v_{i+1})$$

and J^* witnesses the existence of the computed least-weight layer path J in $\widetilde{\Gamma}$. Thus, the weight $\omega(W_J) = \widetilde{\omega}_{\Gamma}(J)$ is at most

$$\widetilde{\omega}_{\Gamma}(J^*) = \widetilde{\omega}_{\Gamma}(x_1, y_2^2) + \widetilde{\omega}_{\Gamma}(y_2^2, y_3^3) + \dots + \widetilde{\omega}_{\Gamma}(y_k^k, x_{k+1})$$

$$\leq \alpha \omega_{\Gamma}(x_1, y_2^2) + \alpha \omega_{\Gamma}(y_2^2, y_3^3) + \dots + \alpha \omega_{\Gamma}(y_k^k, x_{k+1})$$

$$= \alpha \omega_{\Gamma}(J^*) = \alpha \omega(W^*).$$

If the described procedure fails to produce an α -approximate solution for *I*, then, by contraposition, \mathcal{A} failed to produce an α -approximate solution for at least one *s*-*t*-RPP instance R[u, v, i] corresponding to an arc (u_i, v_{i+1}) on J^* . Since J^* has *k* arcs, this happens with probability at most kp by the union bound.

5. A 5/3-approximation algorithm for HCPP(l)

We now show a polynomial-time 5/3-approximation algorithm for *s*-*t*-RPP, which, by Lemma 4.5, carries over to HCPP(l). The algorithm is an adaption of the Christofides-Serdyukov-like 3/2-approximation algorithm from RPP [3, 10] to *s*-*t*-RPP. It closely follows Hoogeveen's [17] adaption of the Christofides-Serdyukov 3/2-approximation algorithm from metric TSP [4, 7, 29] to metric *s*-*t*-TSP.

Theorem 5.1. The *s*-*t*-RPP is 5/3-approximable in $O(n^3)$ time.

Proof. We assume $s \neq t$ (otherwise, one can add a dummy vertex $s \neq t$ and an edge $\{s, t\}$ of zero weight to the initial graph). We only show the 5/3-approximation algorithm for *s*-*t*-RPP instances $I := (G, R, \omega, s, t)$ such that G = (V, E) is a complete graph on the vertex set $V = V(R) \cup \{s, t\}$ and such that the weight function ω satisfies the triangle inequality. This is enough, since the general case reduces to this special case in $O(n^3)$ time and any α -approximation for the special case yields an α -approximation for the general case [3]. The 5/3-approximation algorithm works in four steps.

Step 1. Compute a set $T \subseteq E$ of edges of minimum total weight such that $R \cup T$ forms a spanning connected subgraph of *G* (for example, using Kruskal's algorithm [22]).

Step 2. Let $S \subseteq V$ be the set of vertices in $V \setminus \{s, t\}$ that are imbalanced in $G\langle R \cup T \rangle$ and of those vertices in $\{s, t\}$ that are balanced in $G\langle R \cup T \rangle$. Note that |S| is even: Indeed, consider the set $S' \subseteq V$ of all vertices that are imbalanced in $G\langle R \cup T \rangle$. Clearly, |S'| is even. Now, if $s, t \in S'$, then $S = S' \setminus \{s, t\}$. If $s, t \notin S'$, then $S = S' \cup \{s, t\}$. If $s \in S'$ and $t \notin S'$ (or vice versa), then $S = S' \cup \{t\} \setminus \{s\}$ (or $S = S' \cup \{s\} \setminus \{t\}$). Thus, |S| is even.

Step 3. Construct a minimum-weight perfect matching $M \subseteq E$ on the vertices of S in G (for example, using Lawler's algorithm [24, Section 6.10]).

Step 4. Return an Euler walk *P* in $G\langle R \uplus T \uplus M \rangle$. Note that *P* exists (and can be computed using Hierholtzer's algorithm [11, 16]) since $G\langle R \uplus T \uplus M \rangle$ is connected and all its vertices except for *s* and *t* are balanced. Thus, the endpoints of *P* are *s* and *t* and *P* is a feasible solution to *I*.

All steps can be carried out in $O(n^3)$ time. It remains to prove that *P* is a 5/3-approximation. To this end, let *P*^{*} be an optimal solution for *I*. Obviously, $\omega(R \cup T) \leq \omega(P^*)$. Thus, it remains to show $\omega(M) \leq 2/3 \cdot \omega(P^*)$. To this end, consider $Q = E(P^*) \uplus R \uplus T$. We will construct three perfect matchings M_1, M_2 , and M_3 on *S* in *G* such that $\omega(M_1) + \omega(M_2) + \omega(M_3) \leq \omega(Q)$, and thus $\omega(M) \leq 1/3 \cdot \omega(Q) \leq 2/3 \cdot \omega(P^*)$.

Since the imbalanced vertices of $G\langle P^* \rangle$ are exactly *s* and *t*, the imbalanced vertices in $G\langle Q \rangle$ are exactly those in the set *S*. Let the vertices of $S = \{v_1, v_2, \dots, v_{2\ell}\}$ be numbered in the order

of their first occurrence on P^* and let P_i^* be the subwalk of P^* between the vertices $v_{2i-1} \in S$ and $v_{2i} \in S$ for all $i \in \{1, ..., \ell\}$. Let

$$E_1 := \bigcup_{i=1}^{\ell} E(P_i^*)$$

By shortcutting each path P_i^* to one edge, one gets a perfect matching M_1 on the vertices of *S* such that $\omega(M_1) \le \omega(E_1)$.

The subgraph $G\langle Q \setminus E_1 \rangle$ is Eulerian: it is connected since $R \uplus T \subseteq Q \setminus E_1$ and it is balanced since the imbalanced vertices of $G\langle E_1 \rangle$ are exactly those of $G\langle Q \rangle$, that is, *S*. Its Euler cycle can be shortcut to a simple cycle on *S*, which can be partitioned into two perfect matchings M_2 and M_3 on *S*. Thus,

$$\omega(P) = \omega(R \cup T) + \omega(M)$$

$$\leq \omega(P^*) + (\omega(M_1) + \omega(M_2) + \omega(M_3))/3$$

$$\leq \omega(P^*) + (\omega(E_1) + \omega(Q \setminus E_1))/3$$

$$\leq \omega(P^*) + \omega(Q)/3 \leq 5/3 \cdot \omega(P^*),$$

where the second inequality is due to the metric weights ω . \Box

Plugging Theorem 5.1 into Lemma 4.5, we immediately get:

Corollary 5.2. HCPP(1) is 5/3-approximable in $O(kn^5)$ time.

6. Parameterized algorithms for HCPP(l)

Lemma 4.5 allows us to easily transfer well-known parameterized algorithms from RPP to HCPP(l) to show:

Theorem 6.1. Let ω_{max} be the maximum edge weight and *c* be the maximum number of connected components in any edge class of an HCPP(1) instance. Then, HCPP(1) is

- i) polynomial-time solvable for constant *c* and
- ii) solvable in $2^c \cdot \text{poly}(\omega_{\text{max}}, n)$ time with exponentially decreasing error probability.

Proof. To prove the theorem, it is enough to show that the known RPP algorithms can also be used for *s*-*t*-RPP. To this end, we reduce *s*-*t*-RPP to RPP. We assume that $s \neq t$ and that *s* and *t* are non-adjacent in *s*-*t*-RPP instances (otherwise, we can add a new source *s'* and a required weight-zero edge $\{s', s\}$).

Now, note that an *s*-*t*-RPP instance $I := (G, R, \omega, s, t)$ can be reduced to an RPP instance $I' := (G', R', \omega')$ where I' is obtained from *I* by adding an edge $\{s, t\}$ of weight $2\omega(E)$ to both *E* and *R*. Then, an optimal solution *P* for *I* yields a solution of weight $\omega(P) + 2\omega(E)$ for *I'*. Moreover, an optimal solution *P* for *I'* uses the edge $\{s, t\}$ exactly once: if *P* traversed it multiple times, then it would be cheaper to replace the second traversal of $\{s, t\}$ by any other *s*-*t*-path in *G*. Thus, *P* can be turned into a solution of weight $\omega(P) - 2\omega(E)$ for *I*. That is, optimal solutions translate between *I* and *I'* (yet approximate solutions do not).

Moreover, if the number of connected components in $G\langle R \rangle$ is c', then the number of connected components in $G'\langle R' \rangle$ is at most c' + 1. Thus, since RPP is solvable in polynomial time for constant c' [3, 12], so is *s*-*t*-RPP. And since RPP is solvable in $2^{c'} \cdot \text{poly}(\omega_{\text{max}}, n)$ with exponentially decreasing error probability [15], so is *s*-*t*-RPP. To conclude the proof of the theorem, it is enough to apply Lemma 4.5 and to observe that, for any *s*-*t*-RPP instance P[u, v, i] solved, the subgraph $G\langle E_i \rangle$ induced by the required edges E_i has at most *c* connected components.

7. Conclusion

Our work leaves open several questions. First, what is the computational complexity of HCPP(c) with a constant number of edge classes? It has been conjectured to be polynomial-time solvable [3], yet no polynomial-time algorithm is known even for the case with three classes.

Second, can one close the gap between our 5/3-approximation for *s*-*t*-RPP and the known 3/2-approximation for RPP [3, 10]? For example, recently, a 3/2-approximation for metric *s*-*t*-TSP has been shown [31], matching the approximation factor of the Christofides-Serdyukov algorithm for metric TSP [4, 7, 29]. It is not obvious whether the used approaches carry over to *s*-*t*-RPP, yet closing the gap between *s*-*t*-RPP and RPP would immediately give a 3/2-approximation for HCPP(1).

Our fixed-parameter algorithm for HCPP(l) parameterized by the maximum number c of connected components in any edge class raises the question whether and how lossy kernelization results for RPP parameterized by c [2] carry over to HCPP(l).

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