Saturating Stable Matchings

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Abstract

I relate bipartite graph matchings to stable matchings. I prove a necessary and sufficient condition for the existence of a saturating stable matching, where every agent on one side is matched, for all possible preferences. I extend my analysis to perfect stable matchings, where every agent on both sides is matched.

1 Introduction

A bipartite graph is a graph G with two disjoint vertex sets, X and Y, and an edge set E, such that there is no edge connecting two vertices in the same set. A common goal in bipartite graphs is to connect these two sets in a *matching*, defined as a subset of E such that no two edges share a vertex. If a vertex is the endpoint of an edge in M, we say it is *matched*; otherwise it is unmatched. [West, 1996]

Hall [1935] gives a necessary and sufficient condition for a bipartite graph to have an X-saturating matching, where every vertex $x \in X$ is matched.

When we imagine vertices as agents and allow them to have preferences over the other side, we have Gale and Shapley [1962]'s classic stable marriage problem. A matching is stable if there does not exist a vertex pair $(x, y) \in$ $X \times Y$ which are not matched together but prefer each other to their partner (note that their partner may be no one - i.e. they are unmatched) under that matching; we call this a *blocking pair*.

Gale and Shapley [1962] prove there always exists a stable matching, giving a constructive proof by developing their famed deferred acceptance algorithm. Since their paper, the study of matchings has been taken on by

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economists in a rather different way than they are studied in graph theory, focusing on stability rather than combinatorics [Roth and Sotomayor, 1992].

So, we have two major classical theorems looking at matchings from two different perspectives. Hall [1935] gives a theorem for the existence of a saturating matching (not necessarily stable), while Gale and Shapley [1962] give a theorem for the existence of a stable matching (not necessarily saturating).

As matching applications proliferate, it may be desirable from a social perspective to have every agent matched (e.g., match everyone to a vaccine or match every medical student to a residency). Hence the question arises: can we find *saturating stable* matchings?

2 Main result

Consider a bipartite graph G = (X + Y, E). The neighborhood N(x) of a vertex x is the set of vertices adjacent to, or sharing an edge with, x, and the neighborhood of a set of vertices is the union of each vertex's neighborhood. The degree deg(x) is the number of adjacent vertices, or |N(x)|.

Define P to be a set of preference relations, the elements of which are each vertex's strict preference relation over the vertices in its neighborhood (it's *acceptable* partners). If x_1 prefers y_1 to y_2 , we write $x_1 : y_1 \succ y_2$. Every vertex prefers an acceptable vertex to being unmatched, and prefers being unmatched to an unacceptable vertex. The set of all possible preference instances is \mathbb{P} . SM means stable matching below.

Define the following two conditions for some vertex x:

$$|N(N(x))| \le |N(x)| \tag{1}$$

$$\exists y \in N(x)(deg(y) = 1) \tag{2}$$

Lemma 1. If some $x \in X$ satisfies condition (1) or (2) (or both), then it is matched in all SMs in all preference instances.

Proof. For contradiction's sake, assume that this x is unmatched in some SM. By assumption, this x satisfies $|N(N(x))| \le |N(x)|$ or $\exists y \in N(x)(deg(y) = 1)$, or both.

Case 1: $\exists y \in N(x)(\deg(y) = 1)$ Observe that this y is unmatched, as it only has one acceptable vertex, and that is x, which is unmatched. This x and y form a blocking pair so this matching is unstable, hence contradiction.

Case 2: $\nexists y \in N(x)(deg(y) = 1) x$ must satisfy $|N(N(x))| \leq |N(x)|$. Observe that x is only unmatched if all of the vertices in N(x) are already matched. Since vertices N(x) can only be matched to vertices in N(N(x)), this means that |N(N(x))| - 1 (the number of vertices in N(N(x)) excluding x itself) vertices are matched to |N(x)| vertices. By the pigeonhole principle, this is a contradiction.

This exhausts all the cases.

Lemma 2. If some $x \in X$ does not satisfy condition (1) nor (2), then there exists some preference instance $P \in \mathbb{P}$ under which it is unmatched in every SM.

Proof. Observe that this x is such that |N(N(x))| > |N(x)| and $(\forall y \in N(x))(deg(y) > 1)$.¹

By the assumption, there exists some $x \in X$ that satisfies |N(N(x))| > |N(x)| and $\forall y \in N(x)(deg(y) > 1)$. It suffices to show there exists a P which leaves x unmatched.

Consider the following preference instance P:

- $\forall a \in N(x) : i \succ x$, where $i \in N(N(x)) x$
- $\forall b \in N(N(x)) x : j \succ k$, where $j \in N(x)$ and $k \in N(b) N(x)$
- all other preference relations are allowed to vary

Roughly speaking, P states that all of x's options prefer all their other acceptable vertices to x itself, and that all of x's competitors prefer to be matched to a vertex in N(x) over any other vertex that they find acceptable.

Now we can show that x is unmatched in all SMs. Assume for contradiction's sake that there is some SM M in which x is matched to some vertex in N(x). Observe that $\forall v \in N(N(x)) - x$ are matched to a vertex in N(x). If some v wasn't, then, due to the stated preferences, and the fact that all vertices in N(x) have deg > 1 by the second part of the assumption, said v would form a blocking pair with some vertex in N(x).

However, this means that |N(N(x))| vertices (all of x's competitors plus x itself) are matched to |N(x)| vertices. But, since |N(N(x))| > |N(x)|, by the pigeonhole principle, this is a contradiction.

Therefore, x is unmatched in all SMs under P.

¹Note that the negation of deg(y) = 1 is deg(y) > 1, because it is in some vertex's neighborhood, and so deg(y) > 0.

Now the main result.

Theorem 1. Every SM is X-saturating for all preference instances if and only if for all $x \in X$, conditions (1) or (2) (or both) hold.

Proof. First, the if direction. For all $x \in X$, conditions (1) or (2) or both hold. By Lemma 1, the desired statement holds.

Next, the only if direction. The contrapositive, where some x satisfies neither condition, holds by Lemma 2.

2.1 Equivalent statements

For the market designer, it is not necessarily important if *all* SMs are saturating, but if at least one *exists*. Further, in a real-world matching market that uses the Gale-Shapley algorithm, a very particular SM is yielded, which is either the X-optimal or Y-optimal SM (depending on the algorithm configuration)², so the designer may be particularly interested in whether the outputted SM is saturating.

Interestingly, these are really all the same question, thanks to the following.

Theorem (McVitie and Wilson [1970]). "In a marriage problem of n men and k women if any person is unmarried in one stable marriage solution he or she will be unmarried in all the stable solutions."

So, if a single SM is X-saturating (no one is "unmarried"), then any other SM is also X-saturating (including the X-optimal one, and the X-pessimal one), and indeed all of them.

Lemma 3. For a given preference instance P, an arbitrary SM is X-saturating if and only if all SMs are X-saturating.

Proof. Follows from McVitie and Wilson [1970].

Corollary 1. For a given preference instance P, let the set of all SMs be \mathbb{M} . Then, for all preference instances, an arbitrary $M \in \mathbb{M}$ is X-saturating if and only if for all $x \in X$, at least one of conditions (1) and (2) hold.

Proof. Follows from Theorem 1 and Lemma 3.

 \square

²The X-optimal SM is such that $\forall x \in X$ prefers it to every other SM. The X-pessimal SM is such that $\forall x \in X$ prefers every other SM to it. The X-pessimal SM is also the Y-optimal SM [Roth and Sotomayor, 1992].

Figure 1: Examples of bipartite graphs



Thus, the biconditional in Theorem 1 is the same for the existence of an X-saturating matching, the X-optimal SM being X-saturating, or for any arbitrary SM the market designer is interested in. This equivalence holds for subsequent results in this paper, per Lemma 3.

3 Applications

3.1 Demonstrative examples

Looking at Figure 1a, conditions (1) and (2) are violated for x_2 , since $|N(N(x_2))| = 2 > |N(x_2)| = 1$, so X-saturating SMs do not exist for all preference instances.

In Figure 1b, which simply added one vertex to 1a, both conditions now hold for x_2 .

Lastly, in Figure 1c, condition (1) holds for x_1 and x_2 . While x_3 violates condition (1), it does satisfy (2), as $deg(y_3) = 1$ and $y_3 \in N(x_3)$.

3.2 Perfect matchings

Gale and Shapley [1962] considered |X| = |Y| = n and every $x \in X$ is acceptable to $y \in Y$ (and vice versa) (i.e. preferences are complete). In graph theory, this is a *complete bipartite graph*. Gale and Shapley [1962] say that no vertex is unmatched after the execution of their algorithm. In graph-theoretic terms, for all preference instances, the SM given by their algorithm is *perfect*, meaning that every vertex is matched.

This is actually implied by Theorem 1, as $\forall x \in X$ and $\forall y \in Y$ satisfy condition (1): $\forall x \in X$, |N(x)| = n and |N(N(x))| = n (due to being a complete bipartite graph) so condition (1) is fulfilled, and similarly $\forall y \in$





Y. Therefore, by Theorem 1, for all preference instances, every SM is X-saturating and Y-saturating, and hence perfect.

In fact, if our matching market is in one "piece", then the only way to obtain a perfect matching if |X| = |Y| = n is if preferences are complete. In Figure 2b, we can visually see that there are two different pieces - called *components* in graph theory. Even though preferences are incomplete (e.g., x_3 does not find y_1 acceptable), clearly a perfect SM will always exist.

I first consider graphs with only one component, like Figure 2a. A graph with only one component is called a *connected graph*, defined by a path existing between any two vertices [West, 1996].

Theorem 2. Given a connected bipartite graph G = (X + Y, E) with |X| = |Y| = n, all SMs are perfect for all preference instances if and only if G is a complete bipartite graph.

Proof. First, the if direction. If G is a complete bipartite graph, then $\forall x \in X$ and $\forall y \in Y$ satisfy condition (1), as discussed above, so by Theorem 1, all SMs in all preference instances are X-saturating and Y-saturating, meaning perfect.

In the other direction, proceed by induction on n. For the base case n = 1, there is one vertex each in X and Y, and they have an edge as G is connected. Clearly, this is a complete bipartite graph.

Next, assume that for some n = k, if all SMs are perfect for all preference instances, then G_k is a complete bipartite graph.

We wish to show that for n = k + 1, G_{k+1} is also a complete bipartite graph. G_{k+1} is formed by adding a vertex to each X and Y, called x and

y respectively. Because G_{k+1} is connected, at least one of x or y must be connected to a vertex other than y or x respectively. Without loss of generality, say it is x, which is connected to some not-y vertex $v \in Y$.

Observe that |N(v)| = k + 1, and so |N(N(x))| = k + 1. x must be matched in all SMs in all preference instances. By the contrapositive of Lemma 2, x must satisfy $k + 1 \ge |N(x)| \ge |N(N(x))| = k + 1$, and so |N(x)| = k + 1, which means x is connected to every vertex in Y.

By similar reasoning, y is also connected to every vertex in X, and hence we have a complete bipartite graph.

Thus, by induction, the "only if" statement holds.

I now extend Theorem 2 for the case of all graphs, not just connected ones. Looking at Figure 2b, the two components individually exhibit complete preferences over vertices in the same component. There is a special name for such components: these are called *bicliques* [West, 1996]. In a biclique, every vertex is connected to every vertex in the other set (a generalization of cliques to bipartite graphs). A complete bipartite graph is itself a biclique.

Corollary 2. Given a bipartite graph G = (X + Y, E) with |X| = |Y|, all SMs are perfect for all preference instances if and only if every component of G is a biclique.

Proof. Follows by applying Theorem 2 to each component of the graph. \Box

3.3 Matching with compatibility constraints

Maaz and Papanastasiou [2020] developed the matching with compatibility constraints problem by studying the Canadian medical residency match and point out that positions are designated as either for English speakers or French speakers; some students, being bilingual, can apply to either. Theorem 1 allows us to easily generalize their model to n compatibility classes.

The vertex set X is divided into n possibly overlapping sets $X = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$. The set $A_i - \bigcup_{j \neq i} A_j$ must be nonempty for all $i, j \in [1, n]$, meaning that there must be at least one vertex in each class that is not in any other class³. The set Y is partitioned into n disjoint subsets $Y = B_1 \cup B_2 \ldots \cup B_n$. See Figure 3 for a schematic. A vertex can not find another vertex acceptable if they do not belong to the same class; otherwise they may, but not necessarily. In the special case of compatibility-wise complete

(or CW-complete) preferences that Maaz and Papanastasiou [2020] study, every vertex finds every vertex in the same class acceptable.

Theorem 3. With n compatibility classes, every SM is X-saturating in all instances of CW-complete preferences if and only if $|B_i| \ge |A_i|$ for all $1 \le i \le n$.

Proof. First, the if direction. Take an arbitrary $x \in X$. It belongs to one or more compatibility classes; let this list of classes be stored in the vector q. Then $|N(x)| = \sum_{i \in q} |B_i|$. Further, observe that N(N(x)) is the set of all vertices in X that also belong to the same compatibility classes, including xitself. Thus, $|N(N(x))| = \sum_{i \in q} |A_i|$. Because $|B_i| \ge |A_i|$ for all $1 \le i \le n$, condition (1) holds for this x, and indeed for all x. By Lemma 1, the result follows.

Next, the "only if" direction. Assume for contradiction's sake that there exists a compatibility class i such that $|B_i| < |A_i|$. There exists at least one vertex $x \in X$ that is in the *i*th compatibility class and not in any other class. Then, $|N(x)| = |B_i| < |A_i| = |N(N(x))|$, so it does not fulfill condition (1). And, it does not fulfill condition (2) either because it cannot be connected to a vertex y with degree of 1, as y would violate CW-complete preferences, unless x is the only vertex, but that violates $|B_i| < |A_i|$. By Lemma 2, there exists a preference instance with a SM that is not X-saturating, which is a contradiction.

³This is a generalization of Maaz and Papanastasiou [2020]'s restriction that there must be a non-zero amount of students that speak only English and a non-zero amount that speak only French.

Figure 3: Matching with compatibility constraints with 2 and 3 classes. Lines indicate compatibility between every vertex in the two subsets touched by the line's endpoints; under CW-complete preferences, lines also indicate acceptability.



(b) 3 compatibility classes

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