

On the Complexity of Recognizing Integrality and Total Dual Integrality of the $\{0, 1/2\}$ -Closure

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Abstract

The $\{0, \frac{1}{2}\}$ -closure of a rational polyhedron $\{x: Ax \leq b\}$ is obtained by adding all Gomory-Chvátal cuts that can be derived from the linear system $Ax \leq b$ using multipliers in $\{0, \frac{1}{2}\}$. We show that deciding whether the $\{0, \frac{1}{2}\}$ -closure coincides with the integer hull is strongly NP-hard. A direct consequence of our proof is that, testing whether the linear description of the $\{0, \frac{1}{2}\}$ -closure derived from $Ax \leq b$ is totally dual integral, is strongly NP-hard.

1 Introduction

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ be a rational polyhedron. The integer hull of P is denoted by $P_I = \text{conv}(P \cap \mathbb{Z}^n)$. Any inequality of the form $u^T Ax \leq \lfloor u^T b \rfloor$ where $u \in \mathbb{R}_{\geq 0}^m$ and $u^T A \in \mathbb{Z}^n$ is valid for P_I . Inequalities of this kind are called *Gomory-Chvátal cuts* for P [5, 16]. The intersection of all halfspaces corresponding to Gomory-Chvátal cuts yields the *Gomory-Chvátal closure* P' of P . In fact, $[0, 1]$ -valued multipliers u suffice (see, e.g., [7]), i.e.,

$$P' = \{x \in P : u^T Ax \leq \lfloor u^T b \rfloor, u \in [0, 1]^m, u^T A \in \mathbb{Z}^n\}.$$

Caprara and Fischetti [4] introduced the family of Gomory-Chvátal cuts with multipliers $u \in \{0, \frac{1}{2}\}^m$. We refer to them as $\{0, \frac{1}{2}\}$ -cuts. The $\{0, \frac{1}{2}\}$ -closure of P is defined as

$$P_{\frac{1}{2}}(A, b) := \{x \in P : u^T Ax \leq \lfloor u^T b \rfloor, u \in \{0, \frac{1}{2}\}^m, u^T A \in \mathbb{Z}^n\}.$$

Note that $P_{\frac{1}{2}}(A, b)$ depends on the system $Ax \leq b$ defining the polyhedron P . From the definition, it follows that $P_I \subseteq P' \subseteq P_{\frac{1}{2}}(A, b) \subseteq P$.

$\{0, \frac{1}{2}\}$ -cuts are prominent in polyhedral combinatorics; examples of classes of inequalities that can be derived as $\{0, \frac{1}{2}\}$ -cuts include the blossom inequalities for the matching polytope [5, 11] and the odd-cycle inequalities for the stable set polytope [14]. Both classes of inequalities can be separated in polynomial time [14, 20]. In general, though, separation (and, thus, optimization) over the $\{0, \frac{1}{2}\}$ -closure of polyhedra is NP-hard: Caprara and Fischetti [4] show that the following *membership problem* for the $\{0, \frac{1}{2}\}$ -closure is strongly coNP-complete (see also [13, Theorem 2]).

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $\hat{x} \in \mathbb{Q}^n$ such that $\hat{x} \in P := \{x \in \mathbb{R}^n : Ax \leq b\}$, decide whether $\hat{x} \in P_{\frac{1}{2}}(A, b)$.

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The membership problem remains strongly coNP-complete even when $Ax \leq b$ defines a polytope in the 0/1 cube, as shown by Letchford, Pokutta and Schulz [19]. It is, however, well known that testing membership in the Gomory-Chvátal closure belongs to $\text{NP} \cap \text{coNP}$ if restricted to polyhedra P with $P' = P_I$ (see, e.g., [1]), which naturally includes all polyhedra P whose $\{0, \frac{1}{2}\}$ -closure coincides with P_I . For instance, the relaxation of the matching polytope given by nonnegativity and degree constraints has this property: If we add the blossom inequalities, the resulting linear system is sufficient to describe the integer hull [11], and it is even totally dual integral (TDI) [10]. This motivates the following research questions that are the subject of this paper: What is the computational complexity of recognizing rational polyhedra whose $\{0, \frac{1}{2}\}$ -closure coincides with the integer hull, and of deciding whether adding all $\{0, \frac{1}{2}\}$ -cuts produces a TDI system?

Related questions for the Gomory-Chvátal closure have been studied by Cornuéjols and Li [9]. They prove that, given a rational polyhedron P with $P_I = \emptyset$, deciding whether $P' = \emptyset$ is weakly NP-complete. This immediately implies weak NP-hardness of verifying $P' = P_I$. Cornuéjols, Lee and Li [8] extend these hardness results to the case when P is contained in the 0/1 cube. Moreover, they show that deciding whether a constant number of Gomory-Chvátal inequalities is sufficient to obtain the integer hull is weakly NP-hard, even for polytopes in the 0/1 cube. In this paper, we establish analogous hardness results for the $\{0, \frac{1}{2}\}$ -closure. Our main result is the following theorem, where $\mathbb{1}$ denotes the all-one vector.

Theorem 1. *Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ with $P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n$, deciding whether $P_{\frac{1}{2}}(A, b) = P_I$ is strongly NP-hard, even when the inequalities $-x \leq 0$ and $x \leq \mathbb{1}$ are part of the system $Ax \leq b$.*

We give a proof of this theorem in the next section. Our proof implies several further hardness results, which we explain in Section 3. In particular, deciding whether adding all $\{0, \frac{1}{2}\}$ -cuts to a given linear system $Ax \leq b$ produces a TDI system, is strongly NP-hard. We also establish strong NP-hardness of the following problems: deciding whether the $\{0, \frac{1}{2}\}$ -closure coincides with the Gomory-Chvátal closure; deciding whether a constant number of $\{0, \frac{1}{2}\}$ -cuts suffices to obtain the integer hull. Finally, we give a hardness result for the membership problem for the $\{0, \frac{1}{2}\}$ -closure, which is slightly stronger than the one of Letchford, Pokutta and Schulz [19].

2 Proof of Theorem 1

Proof of Theorem 1. We reduce from STABLE SET:

Let $G = (V, E)$ be a graph and $k \in \mathbb{N}, k \geq 2$. Does G have a stable set of size at least k ?

It is well known that STABLE SET is strongly NP-hard [18]. Note that the problem remains strongly NP-hard if restricted to graphs with minimum degree at least 2: Given an instance of STABLE SET specified by G and k , we construct a new graph G' by adding two dummy nodes to G as well as all edges with at least one endpoint being a dummy node. Every node in G' has degree at least 2, and every stable set in G' of size $k \geq 2$ is a stable set in G of the same size.

Consider an instance of STABLE SET given by $G = (V, E)$ and $k \geq 2$. By the above observation, we may assume that every node in V has degree at least 2. Note that $|V| =: n \geq 3$ and $|E| =: m \geq 3$ in this case. Let $A := 2 \cdot \mathbb{1}\mathbb{1}^T - M^T$ where $M \in \{0, 1\}^{m \times n}$ denotes the edge-node incidence matrix of G and $\mathbb{1}$ is the all-one vector of appropriate dimension. We

define a polytope $P \subseteq \mathbb{R}^m$ by the following system of inequalities:

$$0 \leq x \leq \mathbb{1} \tag{1}$$

$$Ax \leq 2 \cdot \mathbb{1} \tag{2}$$

$$(2k-3)\mathbb{1}^T x \geq 2k-3 \tag{3}$$

Claim 1. $P_I = \{x \in P: \mathbb{1}^T x = 1\}$.

Proof of Claim 1. If we add all inequalities in (2), we obtain the valid inequality $2(n-1)\mathbb{1}^T x \leq 2n$. Every integral point x in P therefore satisfies $\mathbb{1}^T x = 1$. Since $A \in \{1, 2\}^{n \times m}$, it is easy to check that every unit vector is indeed contained in P . We conclude that

$$P_I = \{x \in [0, 1]^m: \mathbb{1}^T x = 1\} \supseteq \{x \in P: \mathbb{1}^T x = 1\} \supseteq P_I. \quad \diamond$$

The $\{0, \frac{1}{2}\}$ -cuts that can be derived from (1)–(3) are all the inequalities of the following two types with $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$:

$$\sum_{i=1}^m (2u^T \mathbb{1} + \lfloor v_i - (Mu)_i \rfloor) x_i \leq 2u^T \mathbb{1} + \lfloor v^T \mathbb{1} \rfloor \tag{4}$$

$$\sum_{i=1}^m (2u^T \mathbb{1} - (k-1) + \lfloor \frac{1}{2} + v_i - (Mu)_i \rfloor) x_i \leq 2u^T \mathbb{1} - (k-1) + \lfloor \frac{1}{2} + v^T \mathbb{1} \rfloor \tag{5}$$

The first type (4) defines all cuts that are derived only from (1) and (2), whereas the second type (5) also uses inequality (3). The vector u is the vector of multipliers for inequalities (2) while v collects the multipliers for the upper bounds in (1).

In what follows, $P_{\frac{1}{2}}$ denotes the $\{0, \frac{1}{2}\}$ -closure of P defined by (1)–(3) together with (4) and (5) for all $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$.

Claim 2. $P_{\frac{1}{2}} = P_I$ if and only if there is a $\{0, \frac{1}{2}\}$ -cut equivalent to $\mathbb{1}^T x \leq 1$.

Proof of Claim 2. If there is such a cut, then $P_{\frac{1}{2}} \subseteq \{x \in P: \mathbb{1}^T x \leq 1\} = P_I$ by Claim 1. To see the “only if” part, consider the vector $y = (\frac{1}{n} + \varepsilon)\mathbb{1}$ for some small $\varepsilon > 0$. Clearly, $y \notin P_I$ since $\mathbb{1}^T y > 1$. We claim that there is a choice for ε such that $y \in P$ and y satisfies all $\{0, \frac{1}{2}\}$ -cuts except those that are equivalent to $\mathbb{1}^T x \leq 1$. First observe that every cut (of either type (4) or (5)) as well as every inequality in (2) and (3) may be written as $a^T x \leq \alpha$ for some $a \in \mathbb{Z}^m$, $\alpha \in \mathbb{Z}$ where $a_i \leq \alpha$ for all $i \in [m]$ and $\alpha \leq m+n$. If $\alpha \leq 0$, we clearly have $a^T y \leq \alpha$ since $y \geq \frac{1}{m}\mathbb{1}$. If $\alpha > 0$ and $a^T x \leq \alpha$ is not equivalent to $\mathbb{1}^T x \leq 1$, then $a_i < \alpha$ for at least one $i \in [m]$. It follows that $a^T y \leq \alpha - \frac{1}{m} + \varepsilon(m\alpha - 1)$. For instance, taking $\varepsilon := \frac{1}{m^2(m+n)}$ yields $a^T y \leq \alpha$ as desired. \diamond

In particular, the proof of Claim 2 shows that the inequality $\mathbb{1}^T x \leq 1$ is not valid for P .

Claim 3. No cut of type (4) is equivalent to $\mathbb{1}^T x \leq 1$.

Proof of Claim 3. Let $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$. If $u = 0$, (4) is dominated by the sum of the inequalities $\lfloor v_i - (Mu)_i \rfloor x_i \leq 0$ for all $i \in [m]$. Note that these are valid for P since $\lfloor v_i - (Mu)_i \rfloor \leq 0$ for all $i \in [m]$. If $v = 0$, the cut (4) is a trivial cut which is only derived from inequalities in the description of P with even right-hand sides. Hence, we may assume that both $u \neq 0$ and $v \neq 0$. It suffices to show that $\lfloor v_i - (Mu)_i \rfloor < \lfloor v^T \mathbb{1} \rfloor$ for at least one $i \in [m]$. If $v^T \mathbb{1} \geq 1$, there is nothing to show. Now let $v^T \mathbb{1} = \frac{1}{2}$ and suppose for the sake of contradiction that $\lfloor v_i - (Mu)_i \rfloor \geq 0$ for all $i \in [m]$. It follows that $Mu \leq v$. Since every column of M has at least two nonzero entries by assumption, we obtain $u = 0$, a contradiction. \diamond

Claim 4. A cut of type (5) induced by $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$ is equivalent to $\mathbb{1}^T x \leq 1$ if and only if $v = 0$, $2Mu \leq \mathbb{1}$, and $2u^T \mathbb{1} \geq k$.

Proof of Claim 4. Suppose first that $v \neq 0$. Then, for every $i \in [m]$, we have $\lfloor \frac{1}{2} + v_i - (Mu)_i \rfloor \leq 1 \leq \lfloor \frac{1}{2} + v^T \mathbb{1} \rfloor$. This holds with equality for all $i \in [m]$ simultaneously only if $v_i = \frac{1}{2}$ and $v^T \mathbb{1} \leq 1$, contradicting $m \geq 3$. Thus, no inequality of the form (5) with $v \neq 0$ has identical coefficients that coincide with the right-hand side. We may therefore assume that $v = 0$.

If $2u^T \mathbb{1} \leq k - 1$, inequality (5) is redundant: It is the sum of the inequalities $(2u^T \mathbb{1} - (k - 1))\mathbb{1}^T x \leq 2u^T \mathbb{1} - (k - 1)$ and $\lfloor \frac{1}{2} - (Mu)_i \rfloor x_i \leq 0$ for all $i \in [m]$, all of which are valid for P . Assuming that $2u^T \mathbb{1} \geq k$, inequality (5) is equivalent to $\mathbb{1}^T x \leq 1$ if and only if $(Mu)_i \leq \frac{1}{2}$ for all $i \in [m]$. \diamond

Putting together Claims 2 to 4, we conclude that $P_{\frac{1}{2}} = P_I$ if and only if there exists some $u \in \{0, \frac{1}{2}\}^n$ such that $2u$ is the incidence vector of a stable set in G of size at least k . \square

3 Further hardness results

A careful analysis of the proof of Theorem 1 shows that, if the polytopes P constructed in the reduction satisfy $P_{\frac{1}{2}} = P_I$, there is a single $\{0, \frac{1}{2}\}$ -cut that certifies this (see Claim 2). This observation immediately implies the following corollary.

Corollary 1. Let $k \in \mathbb{N}$ be a fixed constant. Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ with $P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n$, deciding whether one can obtain P_I by adding at most k $\{0, \frac{1}{2}\}$ -cuts is strongly NP-hard, even when $k = 1$, and $-x \leq 0$ and $x \leq \mathbb{1}$ are part of the system $Ax \leq b$.

Moreover, let us remark that $P' = P_I$ for the polytopes P arising from the reduction. This follows from the fact that for $n \geq 3$, the inequality $\mathbb{1}^T x \leq \lfloor 2n/2(n-1) \rfloor = 1$ is a Gomory-Chvátal cut for P , see the proof of Claim 1.

Corollary 2. Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ with $P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n$, deciding whether $P_{\frac{1}{2}}(A, b) = P'$ is strongly NP-hard, even when $-x \leq 0$ and $x \leq \mathbb{1}$ are part of the system $Ax \leq b$.

The linear systems arising from our reduction have another interesting property. The inequality description (1)–(5) of $P_{\frac{1}{2}}$ in the proof of Theorem 1 is a TDI system if and only if $P_{\frac{1}{2}} = P_I$. This can be seen as follows. Since any polyhedron defined by a TDI system with integer right-hand sides is integral [12], it suffices to show the “if” part. Suppose that $P_{\frac{1}{2}} = P_I$. By the proof of Theorem 1, there exist vectors $u', u'' \in \{0, \frac{1}{2}\}^n$ such that $2Mu' \leq \mathbb{1}$, $2Mu'' \leq \mathbb{1}$, $2(u')^T \mathbb{1} = k$, and $2(u'')^T \mathbb{1} = k - 2 \geq 0$ (see Claim 4). The cuts of type (5) derived with u' and u'' (where we take $v = 0$) are the inequalities $\mathbb{1}^T x \leq 1$ and $-\mathbb{1}^T x \leq -1$, respectively. The system defined by these two inequalities and $x \geq 0$ is a subsystem of (1)–(5) that is sufficient to describe $P_{\frac{1}{2}}$ (see Claim 1) and that is readily seen to be TDI: Let $c \in \mathbb{Z}^m$. We can assume w.l.o.g. that c_1 is the largest coefficient of c . It follows that $\max\{c^T x : x \in P_{\frac{1}{2}}\} = c_1$. It suffices to show that the inequality $c^T x \leq c_1$ is a nonnegative integer linear combination of the selected subsystem. Indeed, it is the sum of $c_1 \mathbb{1}^T x \leq c_1$ (which is a nonnegative integer multiple of $\mathbb{1}^T x \leq 1$ or $-\mathbb{1}^T x \leq -1$) and $-(c_1 - c_i)x_i \leq 0$ for all $i \in [m]$. The above argument shows the following result.

Corollary 3. Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Deciding whether the system given by $Ax \leq b$ and all $\{0, \frac{1}{2}\}$ -cuts derived from it is TDI, is strongly NP-hard, even when $-x \leq 0$ and $x \leq \mathbb{1}$ are part of the system $Ax \leq b$.

Further note that the presence of the constraints $x \leq \mathbb{1}$ in (1) is not essential for our reduction in the proof of Theorem 1. In fact, the upper bounds are redundant: For every $i \in [m]$, consider a row of A such that the entry in column i is equal to 2. Such a row exists because $n \geq 3$. The corresponding inequality in (2) together with the nonnegativity constraints $-x_j \leq 0$ (possibly twice) for all $j \neq i$ yields $2x_i \leq 2$ for all $x \in P$. As the only relevant cuts among (4) and (5) are those with $v = 0$, we conclude that all of the above results still hold true when the upper bounds $x \leq \mathbb{1}$ are not part of the input.

Another byproduct of our proof of Theorem 1 is that the membership problem for the $\{0, \frac{1}{2}\}$ -closure of polytopes in the 0/1 cube is strongly coNP-complete. This has already been shown by Letchford, Pokutta and Schulz [19]. However, neither of the two different reductions given in [19] constructs linear systems that include both nonnegativity constraints and upper bounds on every variable. When these constraints are required to be part of the input, membership testing remains strongly coNP-complete, as the following result shows.

Corollary 4. *The membership problem for the $\{0, \frac{1}{2}\}$ -closure of polytopes contained in the 0/1 cube is strongly coNP-complete, even when the inequalities $-x \leq 0$ and $x \leq \mathbb{1}$ are part of the input.*

Proof. The problem clearly belongs to coNP. To show hardness, we use the same reduction from STABLE SET as in the proof of Theorem 1. The vector y defined in the proof of Claim 2 satisfies $y \notin P_{\frac{1}{2}}$ if and only if the instance of STABLE SET is a “yes” instance. The encoding length of y is polynomial in m and n if we choose ε as in Claim 2. \square

4 Concluding remarks

It is worth pointing out that the problem of recognizing integrality of the $\{0, \frac{1}{2}\}$ -closure is in coNP when the membership problem for the $\{0, \frac{1}{2}\}$ -closure can be solved in polynomial time: If $P = \{x : Ax \leq b\}$ is a rational polyhedron with $P_{\frac{1}{2}}(A, b) \neq P_I$, it suffices to exhibit a fractional vertex \hat{x} of $P_{\frac{1}{2}}(A, b)$ along with a corresponding basis. Then one can verify in polynomial time that $\hat{x} \in P_{\frac{1}{2}}(A, b)$ and that \hat{x} is indeed a vertex. This observation can be found in [17, Chapter 9] where it is stated in the context of recognizing t -perfect graphs. These are the graphs whose stable set polytope is determined by nonnegativity and edge constraints together with the odd-cycle inequalities [6]. In fact, the odd-cycle inequalities can be derived as $\{0, \frac{1}{2}\}$ -cuts from the other two classes of inequalities [14]. This means that a graph is t -perfect if and only if the $\{0, \frac{1}{2}\}$ -closure of the relaxation of its stable set polytope given by nonnegativity and edge constraints is integral. Since a separating odd-cycle inequality can be found in polynomial time [14], recognizing t -perfection is in coNP. Whether this problem is in NP or in P is not known (see [17, Chapter 9]). However, some classes of t -perfect graphs are known to be polynomial-time recognizable, including claw-free t -perfect graphs [2] and bad- K_4 -free graphs [15]. Interestingly, for these two classes of graphs, the linear system in [6] that determines the stable set polytope is TDI [3, 21]. It is not known whether this holds true for t -perfect graphs in general (see [22]).

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