On the Impossibility of Decomposing Binary Matroids

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Abstract

We show that there exist k-colorable matroids that are not (b, c)-decomposable when b and c are constants. A matroid is (b, c)-decomposable, if its ground set of elements can be partitioned into sets $X_1, X_2, \dots, X_l$ with the following two properties. Each set X_i has size at most ck. Moreover, for all sets Y such that $|Y \cap X_i| \leq 1$ it is the case that Y is b-colorable. A (b, c)-decomposition is a strict generalization of a partition decomposition and, thus, our result refutes a conjecture from [BSY19].

Keywords— Matroid, Matroid Coloring, Matroid Decomposition, Matroid Intersection

1 Introduction

Consider a matroid $M = (S, \mathcal{I})$ where S is the ground set of elements and $\mathcal{I}$ is the collection of independent sets. M is said to be k-colorable if S can be partitioned in k sets $C_1, C_2, \dots, C_k$ such that $C_i \in \mathcal{I}$ for all $i \in [k]$. The smallest k for which M is k-colorable is known as the coloring number of the matroid M. An optimal coloring of a matroid can be computed in polynomial time [Edm65]. This is not necessarily the case anymore if we consider, instead of a single matroid, the intersection of h matroids. Consider a collection of h matroids on the same ground set $M_i = (S, \mathcal{I}_i)$ for $i \in [h]$. The intersection of $M_1, M_2, \dots, M_h$ is said to be k-colorable if S can be partitioned in k sets $X_1, X_2, \dots X_k$ such that $X_j \in \bigcap_{i=1}^{h} \mathcal{I}_i$ for all j. That is, each X_j is independent in all of the h matroids. The coloring number of the intersection of $M_1, M_2, \dots, M_h$ is the smallest k for which the given intersection is k-colorable. Matroid intersection coloring is known to be NP-hard for $h \geq 3$ [OBS17].

[IMP20] showed that if each of the k-colorable matroids $M_1, \dots, M_h$ is (b, c)-decomposable, the intersection of these matroids can be colored with $k \cdot h \cdot c \cdot b^h$ colors.

Definition 1 ((b, c)-decomposable)**.** *A k-colorable matroid $M = (S, \mathcal{I})$ is (b, c)-decomposable if there is a partition $X = \{X_1, X_2, \dots, X_\ell\}$ of S such that:*

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- *For all $i \in [\ell]$, it is the case that $|X_i| \leq c \cdot k$, and*
- *every set $Y = \{v_1, \ldots, v_\ell\}$, such that $v_i \in X_i$, is b-colorable.*

We refer to X as a (b, c)-decomposition.

If $b = 1$ then $X = \{X_1, X_2, \ldots, X_\ell\}$ represents a partition matroid, and thus [BSY19] called the $(1, c)$-decomposition a partition reduction. Furthermore, [IMP20] showed that if the (b, c)-partitions are given for a collection of matroids on the same ground set, or can be efficiently computed, then the coloring of their intersection can be efficiently computed. Note that if h, b and c are all $O(1)$ then the resulting coloring is an $O(1)$-approximation to an optimal coloring as the coloring number for each individual matroid lower bounds the coloring number for the intersection.

Furthermore, [BSY19, IMP20, LMP21] showed that many common types of matroids, including transversal matroids, laminar matroids, graphic matroids and gammoids, have $(1, 2)$-decompositions. Moreover, they showed that these decompositions can be computed efficiently from the standard representations of these matroids. Thus [BSY19] reasonably conjectured that every matroid is $(1, 2)$-decomposable. If this conjecture held, and such decompositions could be found efficiently, then the result from [IMP20] would yield an efficient $O(1)$-approximation algorithm for coloring the intersection of $O(1)$ arbitrary matroids.

This paper's main result is that there are matroids that are not $(O(1), O(1))$-decomposable. This refutes the conjecture from [BSY19]. In particular, we show that the binary matroid, consisting of the $2^n - 1$ nonzero vectors of dimension n, is not $(O(1), O(1))$-decomposable.

Before proving our main result in Section 2, we review related work and basic definitions.

1.1 Other Related Work

[AB06] showed that for two matroids M_1 and M_2, with coloring numbers k_1 and k_2, the coloring number k of $M_1 \cap M_2$ is at most $2\max(k_1, k_2)$. The proof in [AB06] uses topological arguments that do not directly give an efficient algorithm for finding the coloring. [BSY19] also showed how to use the existence of $(1, c)$-decompositions to prove the existence of certain list colorings.

Motivated by applications to the matroid secretary problem, [AKKG21] independently showed that the same binary matroid that we consider is not $(1, O(1))$-decomposable.

1.2 Definitions

A *hereditary set system* is a pair $M = (S, \mathcal{I})$ where S is a universe of n elements and $\mathcal{I} \subseteq 2^S$ is a collection of subsets of S with the property that if $A \subseteq B \subseteq S$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$. The sets in $\mathcal{I}$ are called *independent*. A subset R of S is *k-colorable* if R can be partitioned into k independent sets. The *coloring number* of M is the smallest k such that S is k-colorable. The *rank* $r(X)$ of a subset X of S is the maximum cardinality of an independent subset of X. A *matroid* is an hereditary set system with the additional properties that $\emptyset \in \mathcal{I}$ and if $A \in \mathcal{I}$, $B \in \mathcal{I}$, and $|A| < |B|$ then there exists an $s \in B \setminus A$ such that $A \cup \{s\} \in \mathcal{I}$. The intersection of matroids $(S, \mathcal{I}_1), \ldots, (S, \mathcal{I}_h)$ is a hereditary set system with universe S where a set $I \subseteq S$ is independent if and only if for all $i \in [1, h]$ it is the case that $I \in \mathcal{I}_i$. A *flat* F of M is subset of S such that for all elements $y \in S \setminus F$ it is the case that adding y to F strictly increases the rank.

2 Main Result: Binary Matroids are Not Decomposable

This section focuses on showing that binary matroids are not (b, c)-decomposable for constants b and c.

Definition 2. *Let $M = (S, \mathcal{I})$ be the binary matroid where S consists of all n dimensional vectors with entries that are either 0 or 1, with the exception of the all zero vector. A subset R of S is independent if and only if the elements of R are linearly independent over the field with the elements 0 and 1 with addition and multiplication modulo 2.*

Note that S contains $2^n - 1$ elements and has rank n.

Lemma 3. *The coloring number of any rank d flat of M is $\lceil (2^d - 1)/d \rceil$. Thus, by taking $d = n$, the coloring number k of M is precisely $\lceil 2^n/n \rceil$.*

Proof. It is well known that a matroid can be colored with k colors if and only if for every subset R of elements, $k \cdot r(R) \geq |R|$, that is, k times the rank of R is at least the cardinality of R [Edm65]. The maximum value of $|R|/r(R)$ over subsets R of a rank d flat F occurs when $R = F$. Thus this maximum is $(2^d - 1)/d$. □

Lemma 4. *If $d \leq n/2$ then the number of distinct rank d flats of M is at least $\frac{2^{dn}}{2^{d^2+d}}$.*

Proof. Consider the process of picking one by one a collection of d vectors to form a basis of a rank d flat F. When considering the ith choice, there are $(2^n - 1) - (2^{i-1} - 1)$ choices of elements of S that are linearly independent from the previous choices. As the order of the d vectors chosen does not matter, the number possible collections of elements that form a basis of rank d flat is the following.

$$\frac{\prod_{i=1}^{d} \left((2^n - 1) - (2^{i-1} - 1)\right)}{d!}$$

Similarly for a particular rank d flat F there are

$$\frac{\prod_{i=1}^{d} \left((2^d - 1) - (2^{i-1} - 1)\right)}{d!}$$

collections of elements from F that form a basis for F. Thus there are

$$\frac{\prod_{i=1}^{d} \left((2^n - 1) - (2^{i-1} - 1)\right)}{\prod_{i=1}^{d} \left((2^d - 1) - (2^{i-1} - 1)\right)} = \prod_{i=1}^{d} \left(\frac{2^n - 2^{i-1}}{2^d - 2^{i-1}}\right)$$

flats of rank d. Lower bounding each term in the product in the numerator by $2^n - 2^d$, and upper bounding each term in the product in the denominator by 2^d, we can conclude that there are at least

$$\left(\frac{2^n - 2^d}{2^d}\right)^d$$

flats of rank d. Then if $d \leq n/2$, this is at least $\frac{2^{dn}}{2^{d^2+d}}$. □

Theorem 5. *If M admits a (b, c)-decomposition then it must be the case that $4c^2 2^{d^2+d} \geq n$, where d is the minimum integer such that $(2^d - 1)/d > b$. In particular, for sufficiently large n, M admits no $(O(1), O(1))$-decomposition.*

Proof. Consider an arbitrary (b, c)-decomposition $X = \{X_1, X_2, \ldots, X_\ell\}$ of M. As $(2^d - 1)/d > b$, a flat of rank d is not b-colorable by Lemma 3 . Thus for each rank d flat F, at least two elements of F must be in the same part in X. Otherwise, we get a contradiction to the definition of (b, c)-decomposability. To see this, consider setting Y to F in the definition of the (b, c)-decomposition. That is, each element of F is selected to be in Y as this includes at most one element in any part X_i. The resulting representatives would not be b-colorable by the above characterization of F. If two elements of a rank d flat F are in the same part $X_i \in X$ then we say that F is covered by part X_i.

Since X is a (b, c)-decomposition, the cardinality of each part of X is at most ck. Each pair of elements x, y in a part $X_i \in X$ can be contained in at most $\binom{2^n}{d-2}$ rank d flats. To see this note that each rank d flat F can be represented by d independent basis vectors in F, and since x and y are already specified, there are at most $d-2$ more choices for these basis vectors. There are at most $\binom{ck}{2}$ possible pairs of elements from a part $X_i \in X$, and X_i can cover at most $\binom{ck}{2}\binom{2^n}{d-2}$ different flats. Thus in aggregate, all the parts of X can cover at most $\ell\binom{ck}{2}\binom{2^n}{d-2}$ flats. Then using the fact that ℓ is at most n, k is at most $2 \cdot 2^n/n$, and upper bounding $\binom{x}{y}$ by x^y, we can conclude that in aggregate all the parts of X can cover at most $\ell\binom{ck}{2}\binom{2^n}{d-2} \leq n(ck)^2(2^n)^{d-2} \leq 4c^2 2^{nd}/n$ flats. Since each of the flats must be covered by some part of X, and since by Lemma 4 the number of rank d flats is at least $\frac{2^{nd}}{2^{d^2+d}}$, it must be the case that

$$4c^2 2^{nd}/n \geq \frac{2^{nd}}{2^{d^2+d}}$$

or equivalently $4c^2 2^{d^2+d} \geq n$. □

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