# Substitute Valuations: Generation and Structure 

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#### Abstract

Substitute valuations (in some contexts called gross substitute valuations) are prominent in combinatorial auction theory. An algorithm is given in this paper for generating a substitute valuation through Monte Carlo simulation. In addition, the geometry of the set of all substitute valuations for a fixed number of goods $K$ is investigated. The set consists of a union of polyhedrons, and the maximal polyhedrons are identified for $K=4$. It is shown that the maximum dimension of the polyhedrons increases with $K$ nearly as fast as two to the power $K$. Consequently, under broad conditions, if a combinatorial algorithm can present an arbitrary substitute valuation given a list of input numbers, the list must grow nearly as fast as two to the power $K$.


Key words: substitute valuation, gross substitute, M concavity, auction theory

## 1 Introduction

Roughly speaking, one commodity is a substitute for another if the commodities are approximately interchangeable. In the economics literature, the notion of substitute valuations dates back to the work of Walras on equilibrium theory and is prominent in the development of general equilibrium theory (see, for example, $[1,2,32]$ ). Kelso and Crawford [20] formulated a version of substitute property for discrete goods, opening the doors to a generalization of the theory of pricing and ascending auctions that had been developed earlier for matching markets by Damange, Gale, Leanord, Shapley, Shubik, Sotomayor, and others (see [11]). More recently, new characterizations of substitute valuations have been found, and algorithms and auctions for finding efficient allocations and market clearing prices for economies with buyers having substitute valuations have been found $[3,4,18,19,21,23,26,29]$.

For readers unfamiliar with the concept of substitute valuations in economics, we introduce it by describing a special case of an auction algorithm given
in [20]. Suppose there are $K$ goods to be auctioned to $n$ buyers, using an ascending price auction with nonnegative integer prices, as follows. Initially the price for each good is zero and each good is provisionally assigned to some buyer. During each round of the auction, suppose that if a good $k$ is not provisionally assigned to a particular buyer, then the buyer can place a bid for the good at price $p_{k}+1$, where $p_{k}$ is the current price. If there are any bids made for a good in the round, then the good is provisionally assigned to one of the buyers placing a bid for the good, and the price of the good is increased to the price bid. If no new bids are made in a particular round, then the auction ends. Once a good is provisionally assigned to a buyer, if no higher bids are ever placed on that good, it is sold to the highest buyer at the price bid. Although all the goods are sold in such an auction, the outcome can be very inefficient. For example, suppose one of the buyers would greatly value receiving two particular goods, but that neither good by itself would be of any value to the buyer. For example, the goods could be two communication links in series. The buyer could place bids for both goods. But suppose a second buyer aggressively competes for one of the two goods, eventually outbidding the first buyer for that good. Then the first buyer could be stuck buying the other good, even though that good alone has no value to the buyer. If all buyers have substitute valuations, however, and if all bid in a straight-forward manner based on those valuations, then the auction is indeed efficient. The substitutes property is that, if a buyer prefers a particular bundle of goods for one set of prices, and then if the prices of some goods are increased, then there is a new bundle preferred by the buyer which includes all items in the original bundle that did not have a price increase. Hence, if the valuations of all buyers satisfy the substitute condition, the vector of final prices for the goods is such that the supply of goods (i.e. one of each type) is matched by the demand. In summary, the substitute valuation property is precisely what is needed to make ascending price auctions efficient.

Auctions have been implemented for sale of such diverse resources as: wireless spectrum licenses, gate access at airport terminals, truckload transportation, bus routes, and polution permits $[9,10]$. Large sums of money can be involved and it may be very expensive to rely on learning from experience. There is thus a need to be able to produce simulations with buyers having realistic valuations. Design of realistic models for valuations is an art that involves spatial and/or temporal dependencies among the goods and buyers, which depend heavily on the particular market addressed. This topic is beyond the scope of this paper, but we refer to [22] for background, and presentation of a graph based approach to modeling dependencies.

Substitute valuations (a.k.a. gross substitute valuations) play a central role in the theory of auctions because they (1) are in a sense necessary for existence of market clearing prices $[18,24]$, (2) are related to existence of monotone price auctions, (3) are related to desirable monotonicity of prices and immunity from
strategic behavior by subgroups of buyers for the important family of Vickrey auctions [6]. In practice, the valuations of buyers in a given auction might not have the substitutes property. However, designing algorithms that work well in particular for substitute valuations, but also have reasonable behavior for the other valuations one is likely to encounter in a given market, is a reasonable approach to practical auction design [25].

Given the importance of the class of substitute valuations, it is useful to be able to:

- Check whether a given valuation $v$ has the substitute property.
- Generate random substitute valuations by Monte Carlo simulation, for the purposes of testing auction algorithms and exploring the structure of substitute valuations.
- Determine the size of the class of substitute valuations, in various senses, for example to determine how much information must be generated, transmitted, or stored in connection with the use of the valuations.

There exist nice results addressing the first or these items, including the result of $[23,29]$ given as Proposition 4 below, and the connection to matroid theory mentioned at the end of the next section. Also, a valuation has the substitute property if and only if its dual is submodular [5, Theorem 10].

The contributions of this paper are focused on the second and third items. First, a complete parameterization of all (nondecreasing) substitute valuations on four items is given in Section 3. Our motivation for this is threefold: (1) Gain intuition on the family of all substitute valuations, (2) Illustrate the geometry of the space of substitute valuations as a union of maximal polyhedrons, and identify the dimension of the polyhedrons. These concepts are used in Section 6. (3) Point out a lemma that will be used in the proof of convergence of the Monte Carlo algorithm.

Second, a Monte Carlo simulation algorithm for generating substitute valuations is given in Section 4. The first step of the algorithm is to generate an arbitrary valuation, and then the algorithm performs a finite number of modifications leading to a substitute valuation. A proof of convergence is given which involves the characterization of a substitute valuation on the class of sets of a given size by a local exchange property.

The third contribution of the paper is to give an indication of how much richer the class of substitute valuations is than the subclass arising from the assignment problem. Section 5 shows that the assignment valuations comprise a useful and interesting set of substitute valuations, which covers much of the set of all substitute valuations in the case of four goods. In particular, a subclass of assignment allocations is reviewed in Section 5.1 for which essentially no computation is needed to compute the value of a given bundle. This section
gives further intuition about the use of the dimension of the space of allocation valuations leading to a better understanding of the result of Section 6. Section 5.2 examines how much of the space of substitute valuations is covered by the assignment valuations for the case of four goods. This naturally ties together Sections 3 and 5.1.

Fourth, Section 6 displays a set of substitute valuations that, for a large number of goods, is markedly different from the set of assignment valuations. The existence of those valuations implies that for a large class of algorithms for presenting substitute valuations, the number of real-valued inputs must grow exponentially in the number of goods. Section 6 shows that for any $\epsilon>0$, if $K$ is sufficiently large, then the space of substitute valuations contains polyhedrons with dimension at least $2^{(1-\epsilon) K}$. As explained, this has a negative implication regarding the existence of presentation algorithms capable of producing arbitrary substitute valuations.

The results of this paper are interrelated as follows. As mentioned above, we present an algorithm for generating a substitute valuation in Section 4. A related problem is to find algorithms for presenting substitute valuations. The number of values of a valuation, $2^{K}$, grows quickly with $K$. The idea of a presentation algorithm is to ask a buyer to specify a smaller list of numbers, and to use a polynomial complexity algorithm which, given the input list from the buyer and a bundle of goods, can determine the value of the bundle. The use of allocation valuations, discussed in Section 5, is an excellent prototype of a presentation algorithm. For allocation valuations, the list of numbers specified by a buyer are the weights on a bipartite graph, and the value of a bundle of goods is given by the maximum weight of the matchings in a subgraph determined by the bundle. More complex but related presentation algorithms are given in [7]. However, there are no known presentation algorithms which cover the entire set of substitute valuations. If such an algorithm did exist, one could use it to generate random valuations by inputting to the algorithm a list of random numbers, offering a perhaps more attractive alternative to our generation algorithm in Section 4. Thus, the question of whether there exists a presentation algorithm that can cover the set of all substitute valuations is naturally related to the problem of simulating substitute valuations. This question is addressed in this paper in two ways: First, by giving the structure of the space of all substitute valuations for four goods. Secondly, by showing (Section 6) that, for large numbers of goods, under fairly general conditions, for any $\epsilon>0$, the list of numbers a buyer would need to input to specify arbitrary substitute valuations must have length at least $2^{(1-\epsilon) K}$ for large $K$. This negative result about possibilities for presentation algorithms strengthens the case to search for good generation algorithms. A final connection between sections is that we used the algorithm of Section 4 to help discover the class of valuations presented in Section 6.

Section 2 gives notation and background, states some useful known properties of substitute valuations which can be used to build up families of such valuations, and briefly points to where substitute valuations can be found under a different name in some recent literature on matroid theory.

## 2 Notation and background

Suppose $K$ goods, represented by $\mathcal{K}=\{1, \ldots, K\}$, are to be allocated. The set of possible bundles of goods (i.e. subsets of $\mathcal{K}$ ) is denoted by $2^{\mathcal{K}}$. If $A$ is a bundle and $i$ is a good not in $A$, we write $A i$ for the set $A \cup i$. If $A$ is a bundle and $i$ and $j$ are distinct goods not in $A$, we write $A i j$ for $A \cup\{i, j\}$. Similarly, if $i$ and $j$ are distinct goods, we sometimes write $i j$ to denote the set $\{i, j\}$. The notation $|A|$ denotes the cardinality of $A$.

A function $f: 2^{\mathcal{K}} \rightarrow \mathbb{R}$ is submodular if for any bundles $A$ and $B, f(A \cup B)-$ $f(A)-f(B)+f(A \cap B) \leq 0$. An equivalent condition is that $f(A i j)-f(A i)-$ $f(A j)+f(A) \leq 0$, whenever $A$ is a bundle and $i$ and $j$ are goods not in $A$ (see [31]). A function $f$ is supermodular if $-f$ is submodular.

Definition 1 A triplet of numbers $(x, y, z)$ has the double maximum property if at least two of the numbers are equal to the maximum of the three numbers, or equivalently, $x \leq \max \{y, z\}$ and $y \leq \max \{x, z\}$ and $z \leq \max \{x, y\}$. A triplet of numbers $(x, y, z)$ has the double minimum property if at least two of the numbers are equal to the minimum of the three numbers, or equivalently, if $x \geq \min \{y, z\}$ and $y \geq \min \{x, z\}$ and $z \geq \min \{x, y\}$.

Following terminology common in the economics literature, a valuation $v$ is a mapping from $2^{\mathcal{K}}$ to $\mathbb{R}$. Throughout this paper, we require that valuations be normalized to assign value zero to the empty set. We consider quasi-linear payoffs. Thus, given a price vector $p$, by which we mean an element of $\mathbb{R}_{+}^{K}$, the payoff function of a buyer with valuation $v$ is $v(A)-p \cdot A$ for $A \in 2^{\mathcal{K}}$, where the notation $p \cdot A=\sum_{k \in A} p_{k}$ is used. The demand correspondence $D$ for a buyer with valuation $v$ is defined by $D(p)=\arg \max _{A} v(A)-p \cdot A$.

Definition 2 (Substitute valuation [20]) A valuation v for $K$ distinct goods is $a$ substitute valuation if, for any price vectors $p$ and $q$ such that $p \leq q$ and any $A \in D(p)$, there exists a bundle $A^{\prime} \in D(q)$ such that $\left\{k \in A: p_{k}=q_{k}\right\} \subset A^{\prime}$.

The following condition is a key to checking whether a given valuation has the substitute property. (A brief intuitive explanation is given after Fact 8 below.)

Definition 3 (Properties $S 3(L)$ and $S 3$ ) Let $2 \leq L \leq K-1$. A valuation $v$ has property $S 3(L)$ if $(v(A i j)+v(A k), v(A i k)+v(A j), v(A j k)+v(A i))$ has
the double maximum property whenever $A$ is a bundle with $|A|=L-2$, and $i, j, k$ are distinct goods not in $A$. The valuation $v$ is said to satisfy $S 3$ if it satisfies $S 3(L)$ for $2 \leq L \leq K-1$.

Proposition 4 [23, 29] A nondecreasing valuation $v$ is a substitute valuation if and only if $v$ is submodular and $v$ has property $S 3$.

It is useful to represent a valuation as a linear function minus an interaction function, defined as follows.

Definition 5 An interaction function is a function $\theta: 2^{\mathcal{K}} \rightarrow \mathbb{R}$ such that $\theta(A)=0$ for bundles $A$ with $|A| \leq 1$.

Given a valuation $v$, let $\mu$ be the vector of valuations of singleton sets, so $\mu(k)=v(\{k\})$ for $1 \leq k \leq K$. Then the interaction function of $v$ is defined by $\theta(A)=\mu \cdot A-v(A)$ for all bundles $A$. Obviously, $v(A)=\mu \cdot A-\theta(A)$. In order to specify a substitute valuation $v$, it suffices to specify $\mu$ and $\theta$. We work with the interaction function rather than always working with the original valuation in this paper mainly for two reasons. First, in the case of four goods ( $\mathrm{K}=4$, Sections 3 and 5.2), it is easier to deal with the 11 nonzero values of the interaction function than with the 15 nonzero values of the original valuation. Second, in the generation algorithm we present in Section 4, the sequence of interaction functions converging to the final output is monotone nondecreasing. This could be written in terms of the original valuations, but then there would be difficulty with the valuations going negative or not being monotone nondecreasing. The properties $S 3$ and $S 3(L)$ can be expressed in terms of $\theta$ or in terms of another related function, as described next.

Definition 6 (Properties $S 3_{\theta}(L)$ and $S 3_{\theta}$ ) Let $2 \leq L \leq K-1$. An interaction function $\theta$ has property $S 3_{\theta}(L)$ if $(\theta(A i j)+\theta(A k), \theta(A i k)+\theta(A j), \theta(A j k)+$ $\theta(A i))$ has the double minimum property whenever $A$ is a bundle with $|A|=$ $L-2$, and $i, j, k$ are distinct goods not in $A$. An interaction function $\theta$ is said to satisfy $S 3_{\theta}$ if it satisfies $S 3_{\theta}(L)$ for $2 \leq L \leq K-1$.

The two-point conditional interaction function $\delta$ for a valuation $v$ is defined as follows. For any bundle $A$ and goods $i$ and $j$ not contained in $A, \delta_{i j \mid A}=$ $(v(A i)-v(A))+(v(A j)-v(A))-(v(A i j)-v(A))$, so that, intuitively, $\delta_{i j \mid A}$ is the penalty in value for the buyer acquiring both $i$ and $j$, given the buyer has already acquired the bundle $A$. The definition simplifies to $\delta_{i j \mid A}=v(A i)+$ $v(A j)-v(A i j)-v(A)$, or it can be written in terms of $\theta$ as $\delta_{i j \mid A}=\theta(A i j)-$ $\theta(A i)-\theta(A j)+\theta(A)$. For brevity we write $\delta_{i j}$ instead of $\delta_{i j \mid \emptyset}$. Note that $\delta_{i j}=\theta(i j)$ for distinct goods $i$ and $j$.

Definition 7 (Properties $S 3_{\delta}(L)$ and $S 3_{\delta}$ ) Let $2 \leq L \leq K-1$. A two-point conditional interaction function $\delta$ has property $S 3_{\delta}(L)$ if $\left(\delta_{i j \mid A}, \delta_{i k \mid A}, \delta_{j k \mid A}\right)$ has the double minimum property whenever $A$ is a bundle with $|A|=L-2$, and
$i, j, k$ are distinct goods not in $A$. The function $\delta$ is said to satisfy $S 3_{\delta}$ if it satisfies $S 3_{\delta}(L)$ for $2 \leq L \leq K-1$.

The following facts are obvious.
Fact 8 If $v$ is a valuation with interaction function $\theta$ and two-point conditional interaction function $\delta$, then:
(a) The following are equivalent: $v$ is submodular, $\theta$ is supermodular, $\delta$ is nonnegative.
(b) $v$ is nondecreasing if and only if $\mu(k) \geq \max _{A: k \notin A} \theta(A k)-\theta(A)$ for all $k$.
(c) For $2 \leq L \leq K-1$, the following are equivalent: $v$ satisfies $S 3(L)$, $\theta$ satisfies $S 3_{\theta}(L), \delta$ satisfies $S 3_{\delta}(L)$.
(d) The following are equivalent: v satisfies $S 3, \theta$ satisfies $S 3_{\theta}, \delta$ satisfies $S 3_{\delta}$.

For the reader unfamiliar with Proposition 4, we give a brief explanation for why substitute valuations must be submodular (we'll show for $K=2$ ) and why they must satisfy condition $S 3$ (we'll show $S 3_{\delta}(2)$ holds for $K=3$ ). See [23] for a complete, direct proof of the general result of Proposition 4. If $K=2$ and $v(i j)>v(i)+v(j)$ (violating submodularity) then prices $p_{i}$ and $p_{j}$ could be selected so that $v(i j)>p_{i}+p_{j}, v(i)<p_{i}$, and $v(j)<p_{j}$. Then $\{i, j\} \in D(p)$. But if $p_{i}$ is increased enough, the demand set shrinks to $\emptyset$, instead of including $j$, so $v$ is not a substitute valuation. Moving to $K=3$, suppose $v$ is submodular, but that the condition $\delta_{i j} \geq \min \left\{\delta_{i k}, \delta_{j k}\right\}$ fails to hold. Then the price vector $p$ can be selected so that

$$
0 \leq \delta_{i j}<\left(v(j)-p_{j}\right)<\left(v(i)-p_{i}\right)=\left(v(k)-p_{k}\right)<\min \left\{\delta_{i k}, \delta_{j k}\right\}
$$

Then $\{i, j\} \in D(p)$. Indeed, $\{i\}$ yields the same payoff as $\{k\}$ and a greater payoff than either $\{j\}$ or $\emptyset$. Since $\delta_{i j}<v(j)-p_{j},\{i, j\}$ has a larger payoff than $\{i\}$. By the same reasoning, $\{i, k\}$ and $\{j, k\}$ have smaller payoffs than $\{k\}$, or equivalently, $\{i\}$. So $\{i, j\}$ has a larger payoff than any other bundle of zero, one, or two goods. Finally, by submodularity, the change in payoff for adding $k$ to $\{i, j\}$ is less than or equal to the change in payoff for adding $k$ to $\{j\}$ alone, which is negative. Thus, $\{i, j\} \in D(p)$. But if $p_{i}$ is greatly increased, the unique new demand set is $\{k\}$, which does not include $j$ as required by the substitute condition.

Next, we collect together some facts about substitute valuations which are useful for building up interesting classes of valuations. For example, these properties are used in the construction of presentation algorithms in [7]. A valuation $v$ is linear if $v(A)=\mu \cdot A$ for a vector $\mu$ of nonnegative weights. A valuation $v$ is additively concave if there is a partition of $\mathcal{K}$ into (disjoint) subsets $S_{1}, \ldots, S_{J}$, and if there are nonnegative concave functions $\phi_{1}, \cdots, \phi_{J}$, such that $v(A)=\sum_{j} \phi_{j}\left(\left|A \cap S_{j}\right|\right)$. Both linear valuations and additively concave valuations are substitute valuations, as can be seen directly from the definition
of substitute valuations [18].
The aggregate, or max convolution, of two valuations, $v_{1}$ and $v_{2}$, is the valuation $v_{1} * v_{2}$, defined by:

$$
v_{1} * v_{2}(A)=\max _{B \subset A} v_{1}(A-B)+v_{2}(B)
$$

The value $v_{1} * v_{2}(A)$ is the sum of values for two buyers if the bundle $A$ is optimally split between them. The family of substitute valuations is closed under aggregation [21,28]. A valuation $v$ is called a single unit valuation if there exist nonnegative weights $\left(w_{k}: k \in \mathcal{K}\right)$ such that $v(A)=\max \left\{w_{k}\right.$ : $k \in A\}$. Single unit valuations are substitute valuations. Section 5 discusses assignment valuations, which arise as the aggregation of multiple single unit valuations.

Following Gul and Stacchetti [18], given $L$ with $0 \leq L \leq K$, the $L$-satiation of a valuation $v$ is the valuation $\widehat{v}$ defined by

$$
\widehat{v}(A)=\max _{B \subset A,|B| \leq L} v(B)
$$

The $L$-satiation of a substitute valuation is also a substitute valuation. This result was obtained by Gul and Stacchetti [18] for linear or additively concave valuations, and in general by Bing et al. [7]. Another proof is given below (see Corollary 19).

For completeness, we point out that the class of substitute valuations forms a natural link between the theory of matroids and auction theory. With the exception of Remark 18, the terminology in this paragraph is not used elsewhere in this paper. Fujishige and Yang [15] showed that, within the class of monotone valuations, the class of substitute valuations is equivalent to the class of $M^{\natural}$-concave (read " $M$ natural concave") functions introduced by Murota and Shioura [28]. The notion of $M^{\natural}$-concavity is an extension of the notion of $M$ concavity, introduced by Murota [27]. In turn, $M$-concavity is a generalization of the notion of valuated matroids introduced by Dress and Wenzel [12, 13], and valuated matroids are generalizations of matroid rank functions. As pointed out by Gale [16], matroids are intertwined with the theory of problems for which the greedy algorithm is optimal. Substitute valuations are associated with markets such that a simple ascending price auction is optimal. Furthermore, the assignment problem (a.k.a. weighted bipartite matching problem) is a prototype of both the theory of algorithms in matroid theory, and equilibrium theory in economics. So the connection between matroid theory and auction theory is a strong one.

## 3 Parameterization of substitute valuations on four goods

Let $\mathcal{S}_{K}$ denote the set of all nondecreasing substitute valuations on $\mathcal{K}$ (with value zero at $\emptyset$ ), viewed as a subset of $\mathbb{R}^{2^{\mathcal{K}}}$. Recall that a polyhedron (or polyhedral set) is a nonempty set that can be represented as the intersection of finitely many half-spaces. The following is a corollary of Proposition 4:

Corollary 9 For $K \geq 1$, the set $\mathcal{S}_{K}$ can be represented as the union of finitely many polyhedrons.

Proof. By Proposition 4, the set $\mathcal{S}_{K}$ is the subset of $\mathbb{R}^{2^{\mathcal{K}}}$, satisfying the normalization at $\emptyset$, monotonicity, submodularity, and $S 3$ conditions. The normalization constraint, namely $v(\emptyset)=0$, requires that $v$ be in the intersection of the two half-spaces, $\{v(\emptyset) \leq 0\}$ and $\{v(\emptyset) \geq 0\}$. Also, each constraint in the definition of monotonicity or submodularity is equivalent to constraining $v$ to be in a half-space. Condition $S 3$ is equivalent to the requirement that for any bundle $A$ and ordered set of goods $i, j, k$ not in $A$, at least one of the following two constraints holds:

$$
v(A i j)+v(A k) \geq v(A i k)+v(A j) \text { or } v(A i j)+v(A k) \geq v(A j k)+v(A i)
$$

That is, $v$ must be in one of two half-spaces. Making a particular choice of halfspace for each such $A, i, j, k$, thus specifies a subset of substitute valuations forming a polyhedron. The union of such polyhedrons, over all choices for the half-space for each $A, i, j, k$, is $\mathcal{S}_{K}$.

The maximal polyhedral subsets of $\mathcal{S}_{K}$ are the polyhedral subsets of $\mathcal{S}_{K}$ which are not proper subsets of any other polyhedral subsets of $\mathcal{S}_{K}$. Corollary 9 implies that $\mathcal{S}_{K}$ is equal to the union of its maximal polyhedral subsets. The dimension of a polyhedron is the dimension of the smallest affine subspace containing the polyhedron. In this section we identify the maximal polyhedral subsets of $\mathcal{S}_{K}$ for $1 \leq K \leq 4$, and note their dimensions. The emphasis is on the case $K=4$, but the other cases help build intuition.

## $3.1 K=1$

Clearly, $\mathcal{S}_{1}=\{(v(\emptyset), v(1)): v(\emptyset)=0, v(1) \geq 0\}$, so that $\mathcal{S}_{1}$ itself is a polyhedron, which is one dimensional.

## $3.2 K=2$

The set $\mathcal{S}_{2}$ consists of all four-vectors $(v(\emptyset), v(1), v(2), v(12))$ satisfying the normalization constraint: $v(\emptyset)=0$, the monotonicity constraints: $v(1) \geq$ $0, v(2) \geq 0, v(12) \geq v(1), v(12) \geq v(2)$, and the submodularity constraint $v(12)-v(1)-v(2)+v(\emptyset) \leq 0$. Note that condition S 3 is vacuous for $K=2$. Therefore, $\mathcal{S}_{2}$ itself is a polyhedron, which is three dimensional.

## $3.3 K=3$

The set $\mathcal{S}_{3}$ consists of vectors of length eight, so it is tedious to write out the constraints directly in terms of the values of $v$. We instead use the representation $v(A)=\mu \cdot A-\theta(A)$. Let $v$ be a substitute valuation on $\mathcal{K}=\{1,2,3\}$. Let $a=\min \{\theta(12), \theta(13), \theta(23)\}$. By the double minimum property of $\theta, S 3_{\theta}$, there is a permutation $(i, j, k)$ of $(1,2,3)$ so that $\theta(i j)=\theta(i k)=a$. Let $b=\theta(j k)$ and $c=\theta(i j k)$. We claim that the following conditions are satisfied by the six parameters $a, b, c, \mu_{i}, \mu_{j}, \mu_{k}$ :

$$
\begin{equation*}
0 \leq a \leq b, \quad a+b \leq c, \quad \mu_{i} \geq c-b, \quad \mu_{j} \geq c-a, \quad \mu_{k} \geq c-a \tag{1}
\end{equation*}
$$

We have $0 \leq a$ by the supermodularity of $\theta$ (or equivalently the submodularity of $v$ ), and $a \leq b$ by the choice of $a$ and $b$. The next inequality, $a+b \leq c$, also follows from the supermodularity of $\theta$. The last three inequalities in (1) result from the monotonicity of $v$ : they insure that adding a third good to the other two does not decrease $v$. So any substitute valuation for $K=3$ can be represented as above as claimed. Conversely, if $(i, j, k)$ is a permutation of $(1,2,3)$ and the six parameters $a, b, c, \mu_{i}, \mu_{j}, \mu_{k}$ satisfy (1), then the interaction function $\theta$ with $\theta(i j)=\theta(i k)=a, \theta(j k)=b$, and $\theta(i j k)=c$, together with $\mu$, determines a substitute valuation. (Since submodularity is insured by the first two sets of inequalties, monotonicity of $v$ when going from two goods to three implies monotonicity in general.) The set of all substitute valuations obtained this way, for the permutation $(i, j, k)$ fixed, specifies a six dimensional polyhedral subset of $\mathcal{S}_{3}$. Goods $j$ and $k$ play a symmetric role in the above, so that the same polyhedron results if $j$ and $k$ are swapped. Thus, $\mathcal{S}$ is the union of three six-dimensional polyhedrons, determined as above for $i=1$, $i=2$, or $i=3$. The three polyhedrons can also be compactly expressed as $\mathcal{S}_{3} \cap\left\{\delta_{12}=\delta_{13}\right\}, \mathcal{S}_{3} \cap\left\{\delta_{12}=\delta_{23}\right\}$, and $\mathcal{S}_{3} \cap\left\{\delta_{13}=\delta_{23}\right\}$. The intersection of any two of these three maximal polyhedrons is the five-dimensional polyhedron $\mathcal{S}_{3} \cap\left\{\delta_{12}=\delta_{13}=\delta_{23}\right\}$.

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3.4 K=4
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Let $v$ be a valuation on $\mathcal{K}=\{1,2,3,4\}$. We can focus on identifying the possible values of the interaction function $\theta(A)$ on bundles with $|A|=2$ and $|A|=3$. Indeed, suppose $\theta$ is specified on such sets consistently with supermodularity (i.e. so that $\delta_{i j} \geq 0$ and $\delta_{i j \mid k} \geq 0$ for distinct goods $i, j, k$ ). Then, $\theta$ will be supermodular if $\theta(1234)$ is large enough, and $v$ will be nondecreasing if the components of $\mu$ are large enough. The resulting $v$ will be a nondecreasing substitute valuation if and only if $\theta$ satisfies conditions $S 3_{\theta}(2)$ and $S 3_{\theta}(3)$, or equivalently, $\delta$ satisfies conditions $S 3_{\delta}(2)$ and $S 3_{\delta}(3)$.

Suppose that $v$ is a substitute valuation. Consider the complete undirected graph with vertex set $\mathcal{K}$ and associate with an edge $i j$ the value $\delta_{i j}$. By condition $S 3_{\delta}(2)$, the $\delta$ 's around any triangle in the graph have the double minimum property. Let $a$ be the minimum value of the $\delta$ 's for the six edges. At least one vertex has two incident edges with $\delta$ 's equal to $a$. Denote by Case 1 the case that some vertex $i$ has three edges incident with $\delta$ 's equal to $a$. In Case 1 , let $b$ denote the smallest $\delta$ value on the triangle $j k l$. By permuting the vertices if necessary, we assume that $\delta_{j k}=\delta_{j l}=b$. Let $c=\delta_{k l}$. Then $a \leq b \leq c$, and it is possible that $a<b<c$. Denote by Case 2 the case that there is a cycle of length four such that all four $\delta$ 's around the cycle are equal to $a$. By renumbering the vertices, if necessary, we can assume in Case 2 that $\delta_{i j}=\delta_{j k}=\delta_{k l}=\delta_{i l}=a$. Let $\delta_{i k}=b$ and $\delta_{j l}=c$. Then $a \leq \min \{b, c\}$, and it is possible that $a<\min \{b, c\}$.

The two cases are indicated in Figure 1. On one hand, at least one of the cases must hold (for a suitable labeling of the vertices). On the other hand, both cases hold if and only if at least five of the $\delta$ 's are equal to $a$. By checking


Case 1


Case 2

Fig. 1. Possibilities for the $\delta$ values for a substitute valuation with $K=4$.
each of the two cases, the following lemma, used in the next section, is easily verified.

Lemma 10 Ifv satisfies $S 3_{\delta}(2)$ then $\left(\delta_{i j}+\delta_{k l}, \delta_{i k}+\delta_{j l}, \delta_{i l}+\delta_{j k}\right)$ has the double minimum property for any distinct goods $i, j, k$, and $l$.

Condition $S 3_{\delta}(3)$ is summarized in Table 1. Each row of the table lists a
single good, followed by the three $\delta$ 's given that good. For example, the first row begins with good $i$, and the three values following it are equal to $\delta_{k l i}, \delta_{j l i}$, and $\delta_{j k \mid i}$. Here $\theta_{-i}=\theta_{j k l}=v(j)+v(k)+v(l)-v(j k l)$, and $\theta_{-j}, \theta_{-k}$, and $\theta_{-l}$ are defined similarly, and we use the fact $\delta_{i j \mid k}=\theta_{-l}-\delta_{i k}-\delta_{j k}$. Condition $S 3_{\delta}(3)$ means that the three quantities in each row of the table have the double minimum property.
Table 1
Summary of Condition $S 3_{\delta}(3)$ for $K=4$.

$$
\begin{array}{r|cccc}
i & & \theta_{-j}-\delta_{i k}-\delta_{i l} & \theta_{-k}-\delta_{i j}-\delta_{i l} & \theta_{-l}-\delta_{i j}-\delta_{i k} \\
j & \theta_{-i}-\delta_{j k}-\delta_{j l} & & \theta_{-k}-\delta_{i j}-\delta_{j l} & \theta_{-l}-\delta_{i j}-\delta_{j k} \\
k & \theta_{-i}-\delta_{j k}-\delta_{k l} & \theta_{-j}-\delta_{i k}-\delta_{k l} & & \theta_{-l}-\delta_{i k}-\delta_{j k} \\
l & \theta_{-i}-\delta_{j l}-\delta_{k l} & \theta_{-j}-\delta_{i l}-\delta_{k l} & \theta_{-k}-\delta_{i l}-\delta_{j l} &
\end{array}
$$

Table 2
Specialization of Table 1 to Case 1, with the main subcase indicated by boxes.


Let us now examine Case 1 further. If Case 1 holds, Table 1 becomes Table 2. (Ignore the boxes in Table 2 for a moment.) It turns out that if $a=b$ or $b=c$, or if the three entries in a row of the table are all equal, then the substitute valuation $v$ is in the intersection of multiple maximal polyhedrons of $\mathcal{S}_{4}$. So, for the purposes of identifying individual maximal polyhedrons, assume that $a<b<c$, and seek values of $\theta_{-i}, \theta_{-j}, \theta_{-k}$, and $\theta_{-l}$ so that precisely two entries in each row achieve the row minimum. A priori, there are 81 possible choices of two entries per row of Table 2 to be row minimums, but as we will see, the condition $a<b<c$ greatly reduces the number of possibilities. One possibility is indicated by the boxes in Table 2, showing which entries in each row are equal to the minimum value in the row. We call this the main subcase of Case 1. The two boxed terms in each row are equal, and the third term in each row is strictly larger than the two equal terms, if and only if the following two conditions are satisfied: $\theta_{-k}=\theta_{-l}, \theta_{-i}=\theta_{-j}+b-a$, and $a-b>\theta_{-k}-\theta_{-i}>a-c$. Looking for more possibilities, notice that if the last two entries in the first row are boxed as in the main subcase of Case 1, then the same must be true in the second row. Similarly, the choice in the third row forces the choice in the fourth row. Suppose we change the choice in the third row to boxing the second and third terms. This forces $\theta_{-i}+a-b>\theta_{j}=\theta_{-l}+c-b$, and therefore forces the second and third terms in the fourth row to be boxed. This gives
rise to the case shown in Table 3, in which no term involving $\theta_{-i}$ is boxed. We call this Case 1i. There are three more subcases of Case 1, which we denote by Case $1 j$, Case $1 k$, and Case $1 l$. Case $1 i$, ( $1 j, 1 k$, or $1 l$, respectively), corresponds to $\theta_{-i}\left(\theta_{-j}, \theta_{-k}\right.$, or $\theta_{-l}$, respectively) being so large that none of the terms in Table 2 involving it (i.e. all terms in one column of the table) are a minimum in their row. Due to the assumption $a<b<c$, the choice of boxed terms in Case $1 i$ is unique even for the first row of the table. The main subcase and cases $1 i$ through $1 l$ comprise all possibilities. Each of the
Table 3
Case $1 i$ is indicated by boxes.

five subcases of Case 1 corresponds to a maximal polyhedron contained in the set of all substitute valuations. All five subcases of Case 1 can be combined into the following set of conditions:

Case 1:
$a-b \geq \min \left\{\theta_{-k}, \theta_{-l}\right\}-\min \left\{\theta_{-i}, \theta_{-j}+b-a\right\} \geq a-c$, where either the first inequality holds with equality, or $\theta_{-k}=\theta_{-l}$, and either the second inequality holds with equality, or $\theta_{-i}=\theta_{-j}+b-a$. $\min \left\{\theta_{-k}, \theta_{-l}\right\} \geq a+b$ (needed for submodularity) $\min \left\{\theta_{-i}, \theta_{-j}+b-a\right\} \geq b+c$ (needed for submodularity)

For Case 2 we assume that $a<\min \{b, c\}$, for, as can be checked at the end, if this assumption does not hold, then $v$ is in multiple maximal polyhedrons. With this asumption, Case 2 can similarly be divided into five subcases, with each subcase corresponding to a maximal polyhedron in $\mathcal{S}_{4}$. In the main subcase of Case 2, the boxed items in Table 4 are the minimums in their rows. All five subcases of Case 2 can be combined into the following:
Case 2:
$b-a \geq \min \left\{\theta_{-j}, \theta_{-l}\right\}-\min \left\{\theta_{-i}, \theta_{-k}\right\} \geq a-c$, where either the first inequality holds with equality, or $\theta_{-j}=\theta_{-l}$, and either the second inequality holds with equality, or $\theta_{-i}=\theta_{-k}$.
$\min \left\{\theta_{-j}, \theta_{-l}\right\} \geq a+b$ (needed for submodularity)
$\min \left\{\theta_{-i}, \theta_{-k}\right\} \geq a+c$ (needed for submodularity)
Proposition 11 There are 75 maximal polyhedrons comprising $\mathcal{S}_{4}$, and each is ten-dimensional. Of the 75 maximal polyhedrons, there are 12 corresponding to each of the five subcases of Case 1, and three corresponding to each of the

Table 4
Specialization of Table 1 to Case 2, with the main subcase indicated by boxes.

five subcases of Case 2.
Proof. If Case 1 holds and $a<b<c$, then vertices $i$ and $j$ are uniquely identified, but if vertices $k$ and $l$ are swapped, Case 1 still holds. Consequently, there are 12 ways that Case 1 can hold, due to four possibilities for which vertex to label $i$, and then three possibilities of which vertex to label $j$, and, as described above, five subcases for each labeling of the vertices. If $a<\min \{b, c\}$, then up to exchanging values of $b$ and $c$, there are three ways that Case 2 can hold, corresponding to the three ways to select the pair of non-overlapping edges having values other than $a$. And there are five subcases of Case 2 for each possibility. The maximal polyhedrons comprising $\mathcal{S}_{4}$ are constrained by three equalities among the $\delta_{i j}$ 's, two equalities among the $\theta_{-i}$ 's, and the equality $v(\emptyset)=0$, leaving ten out of 16 dimensions remaining.

Remark 12 The above parameterization suggests a simple way to generate substitute valuations on four goods. First choose $i, j, k, l$ to be a random permutation of $1,2,3,4$. Then select which of the ten subcases should hold, and then generate an element of the corresponding polyhedron.

## 4 Generation of substitute valuations by Monte Carlo simulation

An algorithm for generating a nondecreasing substitute valuation on a computer using a random number generator is presented in this section. The algorithm is formulated using the representation $v(A)=\mu \cdot A-\theta(A)$, in terms of $v$ 's interaction function $\theta$ and vector of valuations of singleton sets, $\mu$. That is, the algorithm produces $\left(\theta(A): A \in 2^{\mathcal{K}}\right)$ with $\theta(A)=0$ for $|A| \leq 1$ so that for all bundles $A$ and distinct goods $i, j$ not in $A$ :

$$
\begin{equation*}
\theta(A i j) \geq \theta(A i)+\theta(A j)-\theta(A) \tag{2}
\end{equation*}
$$

and for all bundles $A$ and distinct goods $i, j, k$ not in $A$ :

$$
\begin{equation*}
\theta(A i j)+\theta(A k) \geq \min \{\theta(A i k)+\theta(A j), \theta(A j k)+\theta(A i)\} \tag{3}
\end{equation*}
$$

Algorithm 1. Substitute Valuation Generation Algorithm
Step 1 Generate $\theta_{o}$ and $\mu_{o}$. Initialize $\theta$ to $\theta_{o}$.
Step 2 for $2 \leq L \leq K$
for bundles $A$ with $|A|=L-2$ and distinct $i, j \notin A$
increase $\theta$ (Aij) by the minimum amount so (2) holds.
while changes, for bundles $A$ with $|A|=L-2$ and distinct $i, j, k \notin A$ increase $\theta($ Aij $)$ by the minimum amount so (3) holds.
Step 3 for $1 \leq k \leq K$
$\mu(k):=\max \left\{\mu_{o}(k), \max _{A: k \notin A} \theta(A)-\theta(A k)\right\}$.
and it produces $\mu$ so that for all goods $k$ :

$$
\begin{equation*}
\mu(k) \geq \max _{A: k \notin A} \theta(A k)-\theta(A) \tag{4}
\end{equation*}
$$

The algorithm is shown in brief in the box, and is now explained in more detail. In Step 1, nominal values $\theta_{o}$ and $\mu_{o}$ are generated. To avoid problems with roundoff error and to insure convergence, we require $\theta_{o}$ and $\mu_{o}$ to be integer valued. The $\theta, \mu$, and $v$ produced by the algorithm will also be integer valued. The nominal valuation $v_{o}=\mu_{o} \cdot A-\theta_{o}(A)$ need not have the substitute property. For example, we could take the value $\theta_{o}(A)$ for each bundle $A$ to be a random variable which is uniformly distributed over the interval of integers $[0, m|A|]$ for some constant $m$, or the sum of $|A|$ independent random variables, each uniformly distributed over the interval $[0, m]$ for some $m$. These suggestions are rather arbitrary, but they give larger means and variances for larger bundles. In a particular application, there may be a priori knowledge about the typical distribution of the valuations which could be incorporated into this step. The nominal values $\theta_{o}(A)$ need not be independent.

Step 2 of the algorithm produces $\theta$ in $K-1$ phases, indexed by $L$ running from 2 to $K$, and each phase has two parts. In the first part of phase $L$, for each bundle $A$ with $|A|=L-2$, and each choice of distinct goods $i$ and $j$ not in $A$, the following statement is executed:

$$
\begin{equation*}
\theta(A i j):=\max \{\theta(A i j), \theta(A i)+\theta(A j)-\theta(A)\} \tag{5}
\end{equation*}
$$

This insures supermodularity of $\theta$ up to level $L$. There is no need to visit a particular $A, i, j$ more than once in this part of the phase. The second part of phase $L$ consists of one or more iterations. In each iteration, for each bundle $A$ with $|A|=L-2$, and each choice of distinct goods $i, j, k$ not in $A$, the following statement is executed:

$$
\begin{equation*}
\theta(A i j):=\max \{\theta(A i j), \min \{\theta(A i k)+\theta(A j), \theta(A j k)+\theta(A i)\}-\theta(A k)\} \tag{6}
\end{equation*}
$$

Multiple iterations may be needed in the second part of phase $L$ because of the terms $\theta(A i k)$ and $\theta(A j k)$, which also involve $\theta$ evaluated on sets of cardinality $L$, in the righthand side of (6). These terms may be increased after the statement for $\theta(A i j)$ is executed, during the same iteration, which can require that $\theta(A i j)$ be increased again later. The inclusion of the term $\theta(A i j)$ on the righthand sides of (5) and (6) insures that the $\theta$ values are nondecreasing during execution of the algorithm. If there are no changes during an iteration, then the second part of phase $L$ of the algorithm is complete. This happens if and only if $\theta$ satisfies condition $S 3_{\theta}(L)$ before the iteration. By the nature of the algorithm, the proof of correctness comes down to a proof that the number of iterations needed in each phase of Step 2 is finite.

Step 3 of the algorithm sets $\mu$ to the smallest vector greater than or equal to $\mu_{o}$ such that (4) is satisfied. The description of the algorithm is complete.

Note that randomization is used only in Step 1 of the algorithm. The deterministic portion of the algorithm, Steps 2 and 3, insure that the $\theta$ and $\mu$ produced satisfy the substitutes and monotonicity conditions. There may be other applications for the deterministic portion of the algorithm. For example, the true valuation of a buyer may not quite be a substitute valuation, and participation in a particular auction may require an input that is a substitute valuation. Then the deterministic portion of the algorithm could be run to find a substitute valuation close to the original valuation.

We define a partial ordering " $\prec$ " on the set of interaction functions, which is pointwise within levels and lexicographic among levels, as follows. Write $\theta^{\prime} \prec \theta$ if $\theta^{\prime}=\theta$ or if there exists $L$ with $2 \leq L \leq K$ such that $\theta(A)=\theta^{\prime}(A)$ if $|A|<L, \theta(A) \leq \theta^{\prime}(A)$ if $|A|=L$, and $\theta\left(A_{o}\right)<\theta^{\prime}\left(A_{o}\right)$ for some $A_{o}$ with $\left|A_{o}\right|=L$.

Proposition 13 The algorithm terminates in finite time, and the corresponding valuation $v(A)=\mu \cdot A-\theta(A)$ is a nondecreasing substitute valuation. The interaction function $\theta$ produced is minimal in the " $\prec$ " order, among all supermodular interaction functions satisfying $S 3_{\theta}$ which pointwise dominate $\theta_{0}$.

A proof of Proposition 13 is given in this section, but first some properties connected with substitute valuations are given.

Definition 14 (Property $\left.F 4_{\theta}(L)\right)$ Let $2 \leq L \leq K-2$. An interaction function $\theta$ is said to have property $F 4_{\theta}(L)$ if $(\theta(A i j)+\theta(A k l), \theta(A i k)+\theta(A j l), \theta(A i l)+$ $\theta(A j k))$ has the double minimum property whenever $A$ is a bundle with $|A|=$ $L-2$, and $i, j, k, l$ are distinct goods not in $A$.

Lemma 15 Let $2 \leq L \leq K-2$. If $\theta$ satisfies $S 3_{\theta}(L)$ then it satisfies $F 4_{\theta}(L)$.
Proof. This result for $K=4$ was already stated as Lemma 10. That is, if
$\mathcal{K}=\{i, j, k, l\}$, then the fact that the double minimum property is satisfied by any three of the six numbers $\delta_{i j}, \delta_{i k}, \delta_{i l}, \delta_{j k}, \delta_{j l}, \delta_{k l}$ corresponding to a triangle, implies the double minimum property for the triple $\left(\delta_{i j}+\delta_{k l}, \delta_{i k}+\delta_{j l}, \delta_{i l}+\delta_{j k}\right)$. In general, for $2 \leq L \leq K-2$, for a bundle $A$ fixed with $|A|=L-2$, and for distinct goods $i, j, k, l$ not in $A$, condition $S 3_{\theta}(L)$, which is equivalent to condition $S 3_{\delta}(L)$, implies that the double minimum property is satisfied by any three of the six numbers $\delta_{i j \mid A}, \delta_{i k \mid A}, \delta_{i l \mid A}, \delta_{j k \mid A}, \delta_{j l \mid A}, \delta_{k l \mid A}$ corresponding to a triangle. So by the same reasoning used for Lemma $10, S 3_{\theta}(L)$ implies that $\left(\delta_{i j \mid A}+\delta_{k l \mid A}, \delta_{i k \mid A}+\delta_{j l \mid A}, \delta_{i l \mid A}+\delta_{j k \mid A}\right)$ satisfies the double minimum property, and thus that $F 4_{\theta}(L)$ holds.

Lemma 16 Suppose either $L=1$, or $2 \leq L \leq K-1$ and $(\theta(A):|A|=L)$ satisfies $F 4_{\theta}(L)$. Also, suppose $\mu \in \mathbb{R}^{\mathcal{K}}$, and suppose $\theta$ is determined on sets $A$ with $|A|=L+1$ as follows:

$$
\begin{equation*}
\theta(A)=\min _{i \in A}(\theta(A-i)+\mu(i)) \tag{7}
\end{equation*}
$$

Then $\theta$ satisfies $S 3_{\theta}(L+1)$.
Proof. Let $|A|=L-1$ and suppose $i, j, k$ are distinct goods not in $A$. Let $l^{*}$ be a good in $A i j$ such that $\theta(A i j)=\theta\left(A i j-l^{*}\right)+\mu\left(l^{*}\right)$. It must be shown that

$$
\begin{align*}
& \theta\left(A i j-l^{*}\right)+\mu\left(l^{*}\right)+\theta(A k) \geq  \tag{8}\\
& \min \left\{\min _{l^{\prime} \in A i k} \theta\left(A i k-l^{\prime}\right)+\mu\left(l^{\prime}\right)+\theta(A j), \min _{l^{\prime \prime} \in A i k} \theta\left(A j k-l^{\prime \prime}\right)+\mu\left(l^{\prime \prime}\right)+\theta(A i)\right\} .
\end{align*}
$$

If $l^{*}=i$, then trivially,

$$
\theta\left(A i j-l^{*}\right)+\mu\left(l^{*}\right)+\theta(A k)=\theta\left(A i k-l^{*}\right)+\mu\left(l^{*}\right)+\theta(A j) .
$$

Since $l^{*}$ is a possible value of $l^{\prime}$ in (8), (8) follows. Similarly, (8) is true if $l^{*}=j$. So for the remainder of this proof we suppose that $l^{*} \in A$, which can happen only if $L \geq 2$. Let $B=A-l^{*}$. By $F 4_{\theta}(L)$,

$$
\theta(B i j)+\theta\left(B l^{*} k\right) \geq \min \left\{\theta(B i k)+\theta\left(B j l^{*}\right), \theta(B j k)+\theta\left(B i l^{*}\right)\right\}
$$

which is equivalent to

$$
\left.\begin{array}{rl}
\theta\left(A i j-l^{*}\right) & +\mu\left(l^{*}\right)+\theta(A k)
\end{array} \quad \text { } \quad \begin{array}{rl} 
& \min \left\{\theta\left(A i k-l^{*}\right)\right.
\end{array}+\mu\left(l^{*}\right)+\theta(A j), \theta\left(A j k-l^{*}\right)+\mu\left(l^{*}\right)+\theta(A i)\right\} .
$$

Since $l^{*}$ is a possible value for either $l^{\prime}$ or $l^{\prime \prime}$, this implies (8).

Proof of Proposition 13. Step 3 insures that the output valuation is nondecreasing, so it is sufficient to show that Step 2 of the algorithm eventually terminates, and the interaction function $\theta$ produced satisfies the properties advertised in the proposition. Suppose $\theta^{\prime}$ is any supermodular interaction function satisfying $S 3_{\theta}$ which pointwise dominates $\theta_{o}$. During phase $L$, the algorithm only modifies values of $\theta(B)$ with $|B|=L$, and it does so by increasing the values the smallest amount possible, so as to satisfy the supermodularity and $S 3_{\theta}(L)$ conditions. If phase $L$ is completed, $\theta$ must be supermodular up to level $L$ and satisfy $S 3_{\theta}(L)$. Moreover, if $\theta(B)=\theta^{\prime}(B)$ for all sets $B$ with $|B|<L$, then $\theta(B) \leq \theta^{\prime}(B)$ for all sets $B$ with $|B|=L$. Therefore, if the algorithm terminates, the interaction function $\theta$ produced must have the desired properties.

It remains to show that the algorithm terminates. To that end, it will be shown by induction on $L$ that, for $1 \leq L \leq K$, the following statement is true: Either $L=1$ or the algorithm completes phase $L$. The statement is trivially true for the base case $L=1$. For the sake of argument by induction, suppose the statement is true for some $L$ with $1 \leq L \leq K-1$. If $L \geq 2$, then at the end of execution of phase $L, \theta$ satisfies $S 3_{\theta}(L)$, and hence also $F 4_{\theta}(L)$, by Lemma 15. Thus, either $L=1$, or $2 \leq L \leq K-1$ and $\theta$ satisfies $F 4_{\theta}(L)$. Let

$$
\bar{\theta}(A)=\left\{\begin{array}{cl}
\theta(A) & \text { if }|A| \leq L \\
c+\min _{i \in A} \theta(A-i) & \text { if }|A|=L+1
\end{array}\right.
$$

where $c$ is a constant chosen large enough that $\bar{\theta}$ is supermodular up to level $L+1$, and $\bar{\theta}(A) \geq \theta_{o}(A)$ for all $A$ with $|A|=L+1$. By Lemma 16 with $\mu(i)=c$ for all $i$, it follows that $\bar{\theta}$ satisfies $S 3_{\theta}(L+1)$. It follows that $(\theta(A):|A|=L+1)$ is bounded above by $(\bar{\theta}(A):|A|=L+1)$ throughout execution of phase $L+1$ of the algorithm. Since some entry in $(\theta(A):|A|=L+1)$ strictly increases each time the algorithm finds a violation of $S 3_{\theta}$, since $(\theta(A):|A|=L+1)$ is bounded above, and since all the values are integers, execution of phase $L+1$ must terminate in a finite number of steps. Therefore, the induction statement is true for $L+1$, and hence for all $L$ in the range $1 \leq L \leq K$, as required.

Remark 17 The number of iterations required by the algorithm was proved to be finite by appealing to the assumption that integer values are used. Unfortunately, the number of iterations of the algorithm is not bounded by a function of $K$ alone. Indeed, if $K=6$, and the nonzero values of $\theta_{o}$ are $\theta_{o}(111100)=$ $\theta_{o}(110011)=\theta_{o}(001111)=m \geq 1$ and $\theta_{o}(000111)=\theta_{o}(010111)=\theta_{o}(100111)=$ 1 , then $\theta_{o}$ satisfies $S 3_{\theta}(2)$ and $S 3_{\theta}(3)$ and $\theta_{o}$ is submodular up to level $L=4$. Thus, at the beginning of the second half of phase $L=4$ of Step $2, \theta=\theta_{o}$. Starting from that point, for $1 \leq j \leq m$, after $j$ iterations, $\theta(A)=j$ for all $A$ with $|A|=4$ and $A \notin\{111100,110011,001111\}$, and $\theta$ is still equal to $m$ on $\{111100,110011,001111\}$. After $m$ iterations, $\theta(A)=m$ for all $A$ with
$|A|=m$, and no changes are made during the $(m+1)^{\text {th }}$ iteration. Therefore, $m+1$ iterations are needed. Since $m$ is an arbitrary positive integer, the number of iterations is thus not bounded by a function of $K$ alone. However, for all distributions of $\theta_{o}$ that we tried along the lines of those we suggested for Step 1, the average number of iterations per level was less than 2.1 per level, for up to $K=20$. It was only through running millions of examples that we discovered the behavior exhibited in this example.

Remark 18 Condition $F 4_{\theta}(L)$ for $L$ fixed, when translated to a condition on $v$, is a special case of the local exchange property $E X C_{l o c}$ introduced by Murota [27, p. 282]. As noted at the end of Section 2, substitute valuations are $M^{\natural}$ concave functions. It follows that valuations restricted to a single level $\left\{A \in 2^{\mathcal{K}}:|A|=L\right\}$ are $M$-concave (in fact they are valuated matroids). Murota [27] showed that the local exchange property implies the seemingly more general exchange property in the definition of $M$-concavity.

As a by-product of the proof of convergence of Proposition 13, we recover the following result about $L$-satiation (defined in Section 2):

Corollary 19 [7] Let $0 \leq L \leq K$. Then the L-satiation of a substitute valuation is also a substitute valuation.

Proof. Let $\widehat{v}$ be a substitute valuation, and let $v$ denote its $L$-satiation. To avoid trivialities, assume $1 \leq L \leq K-1$. Since $v$ equals $\widehat{v}$ up to level $L$, $v$ is nondecreasing, is submodular, and satisfies $S 3$, all up to level $L$. That is,
(1) $v(A) \leq v(A i)$ whenever $A$ is a bundle with $|A| \leq L-1$ and $i$ is a good not in $A$
(2) $v(A i j)-v(A i)-v(A j)+v(A) \leq 0$ whenever $A$ is a bundle with $|A| \leq L-2$ and $i$ and $j$ are distinct goods not in $A$
(3) $S 3\left(L^{\prime}\right)$ holds for $0 \leq L^{\prime} \leq L$.

It suffices to show that $v$ is nondecreasing, submodular, and satisfies $S 3$, all up to level $L+1$, for then an obvious proof by induction can be used to show that these properties hold for all levels, implying that $v$ is a substitute valuation.

Clearly, $v$ is nondecreasing up to level $L+1$. To prove that $v$ is submodular up to level $L+1$, let $A$ be any bundle with $|A|=L-1$, and let $i$ and $j$ be goods not in $A$. It must be shown that for any item $k \in \operatorname{Aij}, v(A i j-k)-$ $v(A i)-v(A j)+v(A) \leq 0$. If $k \in\{i, j\}$ this inequality is true because $v$ is nondecreasing. If $k \in A$, then with $B=A-k$, the inequality reduces to

$$
\begin{equation*}
v(B i j)-v(B i k)-v(B j k)+v(B k) \leq 0 \tag{9}
\end{equation*}
$$

But, by property $S 3(L)$, either $v(B i j)+v(B k) \leq v(B i k)+v(B j)$ or $v(B i j)+$ $v(B k) \leq v(B j k)+v(B i)$. Either one of these inequalities and the fact $v$ is
nondecreasing implies (9). Therefore, $v$ is submodular up to level $L+1$.
By Lemma 15, the interaction function of $v$ satisfies $F 4_{\theta}(L)$. Let the vector $\mu$ in the construction (7) of Lemma 16 denote the single item price vector for $v$. Then, with $\theta$ denoting the interaction function of $v,(7)$ is equivalent to $v(A)=\max _{i \in A} v(A-i)$. That is, the extension formula (7) is the same as the definition of $v$ on level $L+1$ as the $L$-satiation of $\widehat{v}$. Therefore, Lemma 16 implies that $v$ has property $S 3(L+1)$.

Remark 20 Submodularity by itself is not necessarily preserved by L-satiation. For example, the following valuation is nondecreasing and submodular (but is not a substitute valuation):

Its 2-satiation satisfies $v(i j k l)-v(i j k)-v(i j l)+v(i j)>0$, and so it is not submodular.

## 5 Assignment valuations

Assignment valuations form an important subclass of substitute valuations, and they can be described as follows. An economy of $n$ buyers, each with a single unit valuation, can be represented by an $n \times K$ weight matrix $W$, with entry $w_{i, k}$ denoting the value of good $k$ to buyer $i$. The aggregate valuation $v=v_{1} * \cdots * v_{n}$ for such a set of buyers is determined by an assignment problem for each bundle $A$, described as follows. For notational convenience, let $\triangle$ be a null item, not in $\mathcal{K}$, and let $w_{i \Delta}=0$ for any buyer $i$. For any $A \subset \mathcal{K}$, an assignment of the goods in $A$ to the $n$ buyers is given by a mapping $\sigma:\{1, \cdots, n\} \rightarrow A \cup\{\triangle\}$ such that $\sigma_{i}=\sigma_{j}$ only if $\sigma_{i}=\sigma_{j}=\triangle$. The interpretation is that buyer $i$ is allocated good $\sigma_{i}$. All goods in $A$ need not be allocated. Let $T(A)$ denote the set of such assignments. Then $v(A)=$ $\max \left\{\sum_{i=1}^{n} w_{i, \sigma_{i}}: \sigma \in T(A)\right\}$. A valuation $v$ arising from a weight matrix $W$ in this way is called an assignment valuation. Assignment valuations are an important subclass of substitute valuations. Special cases of assignment valuations include the linear valuations and the separable concave valuations, mentioned in Section 1.

### 5.1 Assignment valuations with monotone assignments

If the weight matrix $W$ for an assignment valuation $v$ satisfies certain conditions, then there is a very simple formula for the valuation. Specifically, suppose $W$ has dimension $K \times K$ with the following properties:
(1) (Nonnegative) $w(i, k) \geq 0$ for $1 \leq i \leq K, 1 \leq k \leq K$,
(2) (Nonincreasing in $i) w(i+1, k) \leq w(i, k)$ for $1 \leq k \leq K, 1 \leq i \leq K-1$,
(3) (Supermodular) $w(i+1, k+1)-w(i, k+1)-w(i+1, k)+w(i, k) \geq 0$ for $1 \leq i \leq K-1,1 \leq k \leq K-1$.

An example for $K=5$ is as follows:

$$
W=\left(\begin{array}{ccccc}
32 & 35 & 25 & 26 & 22 \\
24 & 30 & 20 & 21 & 19 \\
16 & 22 & 14 & 16 & 14 \\
9 & 15 & 7 & 9 & 9 \\
2 & 8 & 1 & 4 & 5
\end{array}\right)
$$

Writing a bundle $A$ as $A=\left\{k_{1}, \ldots, k_{L}\right\}$ with $k_{1}<\cdots<k_{L}$, we claim that an optimal assignment for $A$ is given by $\sigma_{i}=k_{i}$ for $1 \leq i \leq L$. That is, for $1 \leq i \leq L$, the $i^{\text {th }}$ buyer is allocated the $i^{t h}$ good in $A$. To see this, note by the monotonicity and nonnegativity that the goods should be assigned to buyers 1 through $L$, and then the supermodularity implies that if $1 \leq i<i^{\prime} \leq L$ and if buyer $i$ is assigned a good with a higher index than the good assigned to buyer $i^{\prime}$, then the value of the assignment is not decreased if the goods are swapped.

Hence, the valuation $v$ can be expressed in terms of $W$ as follows:

$$
v\left(\left\{k_{1}, \cdots, k_{L}\right\}\right)=\sum_{i=1}^{L} w\left(i, k_{i}\right) .
$$

An interpretation is that there is a variable value for each item $k$ included in the assigned set. The variable value for an item $k$ is $w(i, k)$ if $k$ is the $i^{\text {th }}$ good in the bundle.

Since the maximum matchings do not involve $w(i, k)$ for $k<i$, the same matchings are still maximum weight matchings with the same values if $W$ is
changed by setting $w(i, k)=0$ for $k<i$, to obtain a matrix of the form:

$$
\widehat{W}=\left(\begin{array}{ccccc}
32 & 35 & 25 & 26 & 22 \\
0 & 30 & 20 & 21 & 19 \\
0 & 0 & 14 & 16 & 14 \\
0 & 0 & 0 & 9 & 9 \\
0 & 0 & 0 & 0 & 5
\end{array}\right)
$$

The matrix $\widehat{W}$ satisfies the following conditions:
( $\widehat{0}$ ) (Upper triangular) $\widehat{w}(i, k)=0$ for $1 \leq k<i \leq K$,
( $\widehat{1})$ (Nonnegative) $\widehat{w}(i, k) \geq 0$ for $1 \leq i \leq K, 1 \leq k \leq K$,
(2) (Supermodular on upper triangle) $\widehat{w}(i+1, k+1)-\widehat{w}(i, k+1)-\widehat{w}(i+$ $1, k)+\widehat{w}(i, k) \geq 0$ for $1 \leq i \leq K-1, i<k \leq K-1$,
$(\widehat{3})$ (Nonincreasing in $i$ for $k=K) \widehat{w}(i+1, K) \leq \widehat{w}(i, K)$ for $1 \leq i \leq K-1$,
( $\widehat{4})$ (Extension condition) $(\widehat{w}(k, k)+\widehat{w}(k+1, k+1) \cdots+\widehat{w}(K, K))-$

$$
(\widehat{w}(k, k+1)+\widehat{w}(k+1, k+2)+\cdots+\widehat{w}(K-1, K)) \geq 0 \text { for } 1 \leq k \leq K-1 .
$$

Moreover, in general, the reverse direction can be taken:
Proposition 21 Suppose a weight matrix $\widehat{W}$ satisfies the conditions $\widehat{0}-\widehat{4}$ above. Then an optimal assignment for any bundle $A=\left\{k_{1}, \ldots, k_{L}\right\}$ with $k_{1}<\cdots<k_{L}$, is given by $\sigma_{i}=k_{i}$ for $1 \leq i \leq L$.

Proof. It suffices to show that $\widehat{W}$ can be modified in positions $i>k$ so that the modification $W$ satisfies the original conditions 1-3. Working from right to left, it is clear that if $\widehat{W}$ is to be modified for indices $i>k$ so that the modification is supermodular everywhere, then the maximal such modification is such that $\widehat{w}(i+1, k+1)-\widehat{w}(i, k+1)-\widehat{w}(i+1, k)+\widehat{w}(i, k)=0$ for $1 \leq k<i \leq K$. Such a supermodular modification of $\widehat{W}$ will be nonincreasing in $i$ for all $k$ because of the supermodularity and condition $\widehat{3}$. It remains to check that the modification is nonnegative. However, the quantities in condition $\widehat{4}$ are the first $K-1$ entries of the last row of the modified matrix. Thus, the modification $W$ of $\widehat{W}$ satisfies conditions 1-3, as claimed.

See [31, Section 3.2] for much more general versions of this monotonicity result.
Just as the set of all substitute valuations $\mathcal{S}_{K}$ can be represented as a union of finitely many polyhedrons, the same is true for the set of assignment valuations. Indeed, there is a finite number of ways to select an assignment $\sigma$ for each set of goods $A$. Some selections are the optimal ones for a nonempty set of weight matrices $W$, which forms a polyhedron within the set of weight matrices. The corresponding valuations for the fixed selections thus form a
polyhedral subset of $\mathcal{S}_{K}$. The union of such sets is precisely the set of assignment valuations.

There are $K(K+1) / 2$ degrees of freedom in the choice of $\widehat{W}$ for the weight matrices described in Proposition 21. Thus, $K(K+1) / 2$ is a lower bound on the largest dimension of a polyhedral subset of the set of assignment valuations. An upper bound is $K^{2}-(K-1)$, which can be seen as follows. For each good, we can form a list of the buyers according to decreasing weight, with ties broken arbitrarily. If a given good is assigned to some buyer, the other buyers higher on the list for that good should also be assigned to some good. It can be seen that at most one good would ever be forced to be assigned to its $K^{t h}$ choice of buyer.

### 5.2 Assignment valuations for four goods

The following proposition shows the relationship between the maximal polyhedrons of the set of substitute valuations and the maximal polyhedrons of the set of assignment valuations, for four goods.

Proposition 22 All five subcases of Case 1 for $K=4$ correspond to assignment valuations, and in particular the valuations of Case $1 i$ are those that arise from Proposition 21 for $K=4$, but none of the valuations of Case 2 with $0<a<\min \{b, c\}$ are assignment valuations. That is, the maximal polyhedrons of assignment valuations in $\mathcal{S}_{4}$ are precisely the 60 maximal polyhedrons of $\mathcal{S}_{4}$ corresponding to Case 1.

Proof. The valuations in the main subcase of Case 1 correspond to assignment valuations for the ordering of states $(i, j, k, l)=(1,2,3,4)$ and weight matrices of the form

$$
W=\left(\begin{array}{cccc}
\mu_{1} & \mu_{2} & \mu_{3} & \mu_{4} \\
\mu_{1}-a & \mu_{2}-b & \mu_{3}-c & 0 \\
0 & \mu_{2}-d & \mu_{3}-e & 0 \\
0 & 0 & \mu_{3}-f & 0
\end{array}\right)
$$

under the following constraints on the constants involved. The constants $a, b, c$ and the vector $\mu$ have the same significance as in Case 1. The constraints on $a, b, c, d, e, f$ are $0 \leq a \leq b \leq c \leq e \leq f$ and $0 \leq e-d \leq c-b$, while the constants $d, e, f$ parameterize the remaining three degrees of freedom in the choice of $\theta$. The vector $\mu$ must be large enough so that for any bundle $A$, it is optimal to assign all goods in $A$ for the purposes of computing $v(A)$. Equivalently, if only such full matchings are considered, the resulting valuation should be nondecreasing in $A$. In this case, it means that $\mu$ should satisfy the
constraints:

$$
\mu_{1} \geq a+d-b+f-e, \mu_{2} \geq d+f-e, \mu_{3} \geq f, \mu_{4} \geq f
$$

In the description of other cases below, the constraints on $\mu$ are determined similarly, without comment. Under these constraints, the assignment valuation falls into the main subcase of Case 1 with $\theta_{-l}=\theta_{-k}=a+d, \theta_{-j}=a+e$, $\theta_{-i}=b+e$, and $\theta(1234)=a+d+f$. It is not difficult to show that all valuations in the main subcase of Case 1 can be so obtained.

The same weight matrix $W$ covers Case $1 j$ under the conditions $0 \leq a \leq b \leq c$, $e \leq f$, and $0 \leq d-b \leq e-c$.

The same weight matrix $W$ covers Case $1 k$ under the conditions $0 \leq a \leq b \leq$ $c \leq e \leq \min \{d, f\}$. Since $k$ and $l$ play a symmetric role in Case 1 , we can cover Case $1 l$ by using $W$ with the third and fourth columns interchanged.

Finally, to obtain Case $1 i$ we can use the weight matrix

$$
W^{\prime}=\left(\begin{array}{lrr}
\mu_{1} & \mu_{2} & \mu_{3}
\end{array} \mu_{4}+\begin{array}{llr}
\mu_{1}-a \mu_{2}-b \mu_{3}-c & 0 \\
\mu_{1}-d \mu_{2}-e & 0 & 0 \\
\mu_{1}-f & 0 & 0
\end{array} 00 .\right.
$$

with the conditions $a \leq b \leq c, \quad e-b \geq d-a \geq 0$, and $f \geq d$. The set of possible values of the matrix $W^{\prime}$ as the ten constants vary (including the constraints on the $\mu_{k}$ 's) corresponds to the set of matrices $\widehat{W}$ given by Proposition 21, except with the columns listed in reverse order. That is, for $K=4$, Proposition 21 refers to the valuations of Case 1i.

Lehmann et al. [21, Example 1] gives an example of a substitute valuation for $K=4$ which is not an assignment valuation. The example falls into Case 2 with $a=1$ and $b=c=5$. The following argument, also used in [21], shows that none of the valuations of Case 2 with $0<a<\min \{b, c\}$ are assignment valuations. For the sake of argument by contradiction, suppose $v$ is a valuation in Case 2 with $0<a<\min \{b, c\}$, and that $v$ is an assignment valuation for a weight matrix $W$. Since all the pairwise $\delta$ 's are strictly positive, some row of $W$ must equal $(v(i), v(j), v(k), v(l))$, and each of these entries is the maximum entry in its respective column. Since $\delta_{i k}>a$, any other entry of column $k$ must be strictly less than $v(k)-a$. Similarly, since $\delta_{j l}>a$, any other entry of column $j$ must be strictly less than $v(j)-a$. But then it is impossible that $\delta_{j k}=a$. Therefore, as claimed, none of the valuations of Case 2 with $0<a<\min \{b, c\}$ are assignment valuations.

## 6 Speckled valuations

It turns out that some substitute valuations are significantly different from assignment valuations, and they cover polyhedrons with substantially larger dimension.

Proposition 23 There is a polyhedron contained in $\mathcal{S}_{16}$ with dimension 2, 727, and a polyhedron contained in $\mathcal{S}_{24}$ with dimension 424,607. For any $K \geq 2$, there is a polyhedron contained in $\mathcal{S}_{K}$ with dimension at least $2 K-1+\frac{2^{K-1}-2}{K}$.

Before getting to the proof of the proposition, we shall introduce some terminology from the theory of binary codes. A codeword of length $K$ is a binary sequence of length $K$, and the weight of a codeword is the number of one's in the codeword. A bundle $A$ naturally corresponds to the codeword with a one in the $k^{\text {th }}$ position if and only if good $k$ is in $A$, for $1 \leq k \leq K$. Let $d_{H}\left(A, A^{\prime}\right)$ denote the Hamming distance between two sets: i.e. $d_{H}\left(A, A^{\prime}\right)=\left|A \backslash A^{\prime}\right|+\left|A^{\prime} \backslash A\right|$.

The proof of the proposition is based on the following construction, valid for $K \geq 2$. Let $\alpha_{1}, \cdots, \alpha_{K}$ and $\beta_{2}, \beta_{3}, \ldots \beta_{K}$ be constants in the interval $[0,1]$. Let $\beta_{0}=\beta_{1}=0$. Let $\mathcal{C}$ denote a collection of bundles such that
(1) For all $A \in \mathcal{C}, 2 \leq|A| \leq K-1$ and $|A|$ is even.
(2) If $A, A^{\prime} \in \mathcal{C}$ and $|A|=\left|A^{\prime}\right|$ then $d_{H}\left(A, A^{\prime}\right) \geq 4$.

Let $\gamma_{A}$ be an element of $[0,1]$ for any $A \in \mathcal{C}$. Define the interaction function $\theta$ by

$$
\theta(A)=\beta_{|A|}+I_{\{A \in \mathcal{C}\}} \gamma_{A}+\phi(|A|)
$$

where $\phi(L)=(1.5) L(L-1)$, and define the vector $\mu$ by $\mu_{k}=(3 K-1)+\alpha_{k}$. Let $v$ be the corresponding valuation: $v(A)=\mu \cdot A-\theta(A)$. We call the valuations of this form speckled valuations, thinking of the many values of $\gamma_{A}$ for $A \in \mathcal{C}$ as specks, or small spots, on the valuation.

Lemma 24 The valuation $v$ is a nondecreasing substitute valuation. For $\mathcal{C}$ fixed, there are $2 K-1+|\mathcal{C}|$ degrees of freedom in the choice of $v$.

Proof. It suffices to show that $\theta$ has property $S 3_{\theta}$, that $\theta$ is submodular, and that $v$ is nondecreasing. Property $S 3_{\theta}$ amounts to showing that $(\theta(A i j)+$ $\theta(A k), \theta(A i k)+\theta(A j), \theta(A j k)+\theta(A i))$ has the double minimum property whenever $A$ is a bundle and $i, j, k$ are distinct goods not in $A$. For fixed $A, i, j, k$ with $|A|=L-2$, this condition involves $\theta$ evaluated on three sets of cardinality $L-1$ and three sets of cardinality $L$. Moreover, the three sets of cardinality $L-1$ each have Hamming distance two to the other two sets of cardinality $L-1$. Likewise, the three sets of cardinality $L$ each have Hamming distance two to the other two sets of cardinality $L$. Therefore, at most one of the six sets involved is in $\mathcal{C}$. If none of the six sets is in $\mathcal{C}$, then the three
values $\theta(A i j)+\theta(k), \theta(A i k)+\theta(j), \theta(A j k)+\theta(A i)$ are equal. If one of the six sets is in $\mathcal{C}$, then $(\theta(A i j)+\theta(k), \theta(A i k)+\theta(j), \theta(A j k)+\theta(A i))$ still has the double minimum property. So $\theta$ has property $S 3_{\theta}$.

To see that $\theta$ is supermodular, let $2 \leq L \leq K$, let $A$ be a bundle with cardinality $L-2$, and let $i, j$ be goods not in $A$. Note that $\phi(L)-2 \phi(L-1)+$ $\phi(L-2)=3$, and also that at most one of $A i$ and $A j$ are in $\mathcal{C}$. These observations and the fact that the $\beta$ 's and $\gamma$ 's are in the interval $[0,1]$ imply that $\theta(A i j)-\theta(A i)-\theta(A j)+\theta(A) \geq 3-2 \beta_{L-1}-1 \geq 0$, so that $\theta$ is supermodular.

As for the monotonicity of $v$, note that for $1 \leq L \leq K, \phi(L)-\phi(L-1)=$ $3(L-1) \leq 3(K-1)$. So if $A$ is a bundle with some cardinality $L-1$, and $i$ is a good not in $A$, then

$$
v(A i)-v(A) \geq 3 K-1-\beta_{L-1}-I_{\{A \in \mathcal{C}\}} \gamma_{A}-3(K-1) \geq 0
$$

Thus $v$ is nondecreasing.
For fixed $\mathcal{C}$, there are $2 K-1+|\mathcal{C}|$ degrees of freedom in the choice of the $\alpha$ 's, $\beta$ 's and $\gamma$ 's. It is easy to check that the mapping from these variables to $v$ is linear and invertible.

Proof of Proposition 23. For a fixed $\mathcal{C}$, the dimension of the set of valuations constructed above is $2 K-1+|\mathcal{C}|$, so it remains to show that $|\mathcal{C}|$ can be taken large enough. The maximum possible cardinality of $\mathcal{C}$ subject to the above conditions can be expressed as follows:

$$
\begin{equation*}
|\mathcal{C}|=\sum_{L: L \text { even, } 2 \leq L \leq K-1} A(K, 4, L), \tag{10}
\end{equation*}
$$

where $A(K, 4, L)$ denotes the maximum possible cardinality of a set of weight $L$ binary codewords of length $K$ with Hamming distance at least 4 between any two codewords. By symmetry, $A(K, 4, L)=A(K, 4, K-L)$. Tables in [8] show that $A(16,4,2)=8, A(16,4,4) \geq 140, A(16,4,6) \geq 615$, and $A(16,4,8) \geq$ 1170 , so for $K=16$ it is possible that $|\mathcal{C}|=2(8+140+615)+1170=2696$, giving the bound for $K=16$ in Proposition 23. Similarly, for $K=24$, it is possible that $|\mathcal{C}|=2 *(12+498+7084+34914+96496)+146552=424560$.

It is shown in [17] that $A(K, 4, L) \geq \frac{1}{K}\binom{K}{L}$. This, combined with (10) and the fact $\sum_{L: L}$ even $\binom{K}{L}=2^{K-1}$, implies that $\mathcal{C}$ can be selected with cardinality at least $\frac{2^{K-1}-2}{K}$, which implies the last statement of the proposition.

Remark 25 The existence of speckled valuations has negative implications for the problem of finding a computationally efficient way to present arbitrary
substitute valuations. Suppose, for example, that an algorithm for presenting substitute valuations takes as input a string $x$ of real numbers, and then the algorithm determines $v(A)$ for any bundle $A$ as a linear transformation of $x$, with coefficients depending on $A$ and $x$. Symbolically, we can write this as $v(A)=\sum_{a} h(A, x, a) x_{a}$. Suppose further that for each bundle $A$ and index $a$, there are only finitely or countably infinitely many possible values of the coefficient $h(A, x, a)$ as $x$ varies. For example, the assignment valuations described in Section 5 can be put into this form, with the coefficients $h(A, x, a)$ taking values $\{0,1\}$. The same is true of the $S$-presentations and $H$-presentations given in [7]. If for every substitute valuation on $K$ items, there is a choice of input $x$ so that the algorithm outputs the substitute valuation, then the possible outputs of the algorithm must cover the polyhedrons in $\mathcal{S}_{K}$ consisting of speckled valuations. But the set of possible output valuations is a finite or countably infinite union of sets of dimension less than or equal to the length of the input vector $x$. Therefore, in view of Proposition 23, the length of $x$ must be greater than or equal to $2 K-1+\frac{2^{K-1}-2}{K}$. For any $\epsilon>0$, this lower bound exceeds $2^{(1-\epsilon) K}$ for sufficiently large $K$.

## 7 Discussion

This paper addresses valuations for single-unit markets, for which each of the $K$ goods is distinct. In a multi-unit market, there may be multiple goods of the same type, and the valuation should be invariant with respect to substituting one good with another of the same type. This poses additional constraints on $v$. The Monte Carlo algorithm we presented extends immediately to this case, because if the nominal function $\theta_{o}$ satisfies invariance under swapping goods of the same type, then the resulting $\theta$ constructed by the algorithm will be similarly invariant. That is, to use the terminology of [26], the algorithm can be used to generate strong substitute valuations for multi-unit auctions. Note that by mapping from single- to mulit-unit auctions in this way, different goods of each type can have different prices. Another class of valuations, called weak substitute valuations in [26], are defined as in Definition 2, with the prices of all goods of the same type being the same. It would be interesting to find a method to generate weak substitute valuations.

It would be interesting to find an algorithm for generating substitute valuations such that the running time is bounded by a function of $K$ alone. As mentioned in Remark 17, there is no such bound for our algorithm. Another topic for additional work is to see how well a generation algorithm, either the one we suggested or a new one, can produce valuations with given distributions. Roughly speaking, if the valuations generated in Step 1 of our algorithm have a specified distribution, and if the valuations aren't changed too much by steps 2 and 3 , then the output valuation should approximately have the given
distribution. As noted in the introduction, there is also much interest in generating realistic valuations for various practical settings, and such valuations typically do not have the substitute property.

The examination of the richness of the class of assignment valuations and the class of all substitute valuations in this paper is purely mathematical, rather than based on the valuations that arise in practice. While the existence of the speckled valuations shows that the set of substitute valuations is much richer than the set of assignment valuations, it is not clear whether the extra richness has practical value. Further, there are important simple examples of valuations which are not substitute valuations. The most prominent of them is the case of two complementary goods: $(v(\emptyset), v(1), v(2), v(12))=(0,0,0,1)$.

Echenique [14] pursued a different approach to determining the richness of the set of substitute valuations. The framework for his results is the notion of substitute introduced by Roth [30]. Roth's framework is more general than the original one of [20] (see [19]). Echenique [14] counts the number of substitute choice functions. The results of [14] are not directly comparable to those here, but the conclusions are somewhat similar.

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