

Solving Poisson's equation for birth–death chains: Structure, instability, and accurate approximation*

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Abstract

Poisson's equation plays a fundamental role as a tool for performance evaluation and optimization of Markov chains. For continuous-time birth–death chains with possibly unbounded transition and cost rates as addressed herein, when analytical solutions are unavailable its numerical solution can in theory be obtained by a simple forward recurrence. Yet, this may suffer from numerical instability, which can hide the structure of exact solutions. This paper presents three main contributions: (1) it establishes a structural result (convexity of the relative cost function) under mild conditions on transition and cost rates, which is relevant for proving structural properties of optimal policies in Markov decision models; (2) it elucidates the root cause, extent and prevalence of instability in numerical solutions by standard forward recurrence; and (3) it presents a novel forward–backward recurrence scheme to compute accurate numerical solutions. The results are applied to the accurate evaluation of the bias and the asymptotic variance, and are illustrated in an example.

Keywords: Poisson's equation; Birth–death process; Relative cost function; Numerical instability; Linear recurrence; Backward recurrence

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1 Introduction.

Consider a stochastic system whose state evolution is modeled by an ergodic continuous-time birth–death Markov chain $\{X(t)\}_{t \geq 0}$ (see, e.g., [1, Ch. 6]) on the state space $\mathbb{N}_0 \triangleq \{0, 1, 2, \dots\}$, with birth and death rates $\lambda_n > 0$ and $\mu_n > 0$ for $n \in \mathbb{N} \triangleq \{1, 2, \dots\}$, and $\lambda_0 > 0 = \mu_0$, having steady-state probabilities p_n . Costs accrue at the state-dependent rates c_n , satisfying

$$\sum_{n=0}^{\infty} |c_n| p_n < \infty, \quad (1)$$

so that the mean steady-state cost $\zeta \triangleq \sum_{n=0}^{\infty} c_n p_n$ is well defined and finite.

In applications to queueing systems, the state n represents the number of jobs in the system, λ_n is the state-dependent arrival rate, and μ_n can incorporate both service and abandonment rates. The c_n model a possibly nonlinear cost structure, which may account for different types of costs, such as holding or abandonment costs. See the examples in Remark 1.1. There is an extensive literature on queueing models with costs, concerning their optimal dynamic control. See, e.g., [2] and references therein.

Birth, death and cost rates may all be *unbounded*. Thus, the present setting is strictly more general than a discrete-time one, as analysis of a discrete-time Markov chain can be reduced to that of a continuous-time chain, whereas the reverse is only true for *uniformizable* chains, having bounded transitions rates.

This paper addresses the exact and approximate numerical solutions to the *Poisson equation* for such a system, given by the second-order recurrence

$$\lambda_0 b_1 - \lambda_0 b_0 = \zeta - c_0, \quad \lambda_n b_{n+1} - (\lambda_n + \mu_n) b_n + \mu_n b_{n-1} = \zeta - c_n, \quad n \in \mathbb{N}. \quad (2)$$

Note that (2) is a *Poisson equation* in the usage of this term in the Markov chain literature, where it plays a fundamental role as a tool for performance evaluation and optimization. See, e.g., [3–5]. Ref. [6, pp. 458–459] surveys wide-ranging applications in applied probability, statistics and engineering, including performance bounds, analysis and variance reduction in simulation, *Markov decision processes* (MDPs) (see [7]), and perturbation theory. Note that the iterative solution of the dynamic programming optimality (Bellman) equations of an MDP model via Howard’s [8] *policy improvement* method entails solving Poisson equations corresponding to stationary deterministic policies.

Concerning exact solutions, we aim to exploit explicit expressions to establish *convexity* of functions $b: \mathbb{N}_0 \rightarrow \mathbb{R}$ solving (2), under mild conditions on transition and cost rates. Note that a standard approach to prove structural properties of optimal policies in MDP models relies on such a convexity property. See [9].

As for approximate numerical solutions, these are required when analytical solutions to (2) are unavailable. When ζ is not representable by a *machine number*, due to finite-precision arithmetic, and is approximated by $\hat{\zeta} \neq \zeta$, one may consider, ignoring other error sources, that the numerical recurrence actually solved is the modified Poisson equation (with $z = \hat{\zeta}$)

$$\lambda_0 b_1 - \lambda_0 b_0 = z - c_0, \quad \lambda_n b_{n+1} - (\lambda_n + \mu_n) b_n + \mu_n b_{n-1} = z - c_n, \quad n \in \mathbb{N}. \quad (3)$$

We will thus investigate (3), viewing $z \in \mathbb{R}$ as an *input parameter*, with the aim of elucidating how the approximation error in $\hat{\zeta}$ is amplified into corresponding errors in the \hat{b}_n solving (3).

From the structure of (3), it is evident that there exists a solution b for any choice of $z \in \mathbb{R}$, which can be constructed by setting b_0 arbitrarily, and generating the remaining b_n by the second-order linear forward recurrence

$$b_1 = \frac{z - c_0}{\lambda_0} + b_0, \quad b_{n+1} = \frac{z - c_n}{\lambda_n} + \frac{\lambda_n + \mu_n}{\lambda_n} b_n - \frac{\mu_n}{\lambda_n} b_{n-1}, \quad n = 1, 2, \dots \quad (4)$$

Clearly, for given z , the solutions b to (3) are unique up to an additive constant.

The present viewpoint contrasts with most work on Poisson’s equation, which has focused on the choice $z = \zeta$ (see, e.g., [5]). A major reason for this is the following. Suppose that birth–death rates satisfy a *Foster–Lyapunov drift condition* with *weight function* $w: \mathbb{N}_0 \rightarrow [1, \infty)$, and let $\mathbb{B}_w \triangleq \{f: \sup_n |f_n|/w_n < \infty\}$ be the Banach space of *w-bounded* functions. Then (see [10, Theorem 7.1] and [7, Prop. 7.11]): (i) the chain satisfies a strong form of ergodicity (it is *w-exponentially ergodic*); (ii) $\sum_{n=0}^{\infty} w_n p_n < \infty$; and (iii) for any $c \in \mathbb{B}_w$, (3),

seen as an equation in (z, b) , has a *unique* solution in $\mathbb{R} \times \mathbb{B}_w$, up to an additive constant for b . Furthermore, this solution is of the form $(z, b) = (\zeta, \beta + a)$ with $a \in \mathbb{R}$, where, writing as $\mathbb{E}_n[\cdot]$ the expectation starting from $X(0) = n$,

$$\beta_n \triangleq \mathbb{E}_n \left[\int_0^\infty (c_{X(t)} - \zeta) dt \right] \quad (5)$$

is the *bias* or *relative cost* of starting from n rather than from steady state. Note that β is characterized as the only $b \in \mathbb{B}_w$ solving (2) for which

$$\sum_{n=0}^{\infty} b_n p_n = 0. \quad (6)$$

As for interpretation, if b with $b_m = 0$ solves (2), then b_n is the *relative cost* of starting from n rather than m , so $b_n = \mathbb{E}[\int_0^\infty (c_{X^n(t)} - c_{X^m(t)}) dt]$, for realizations $\{X^n(t)\}_{t \geq 0}$ and $\{X^m(t)\}_{t \geq 0}$ of the chain starting from n and m .

We further consider the function $\varphi: \mathbb{N}_0 \rightarrow \mathbb{R}$ defined by

$$\varphi_n \triangleq \beta_{n+1} - \beta_n = b_{n+1} - b_n, \quad n \in \mathbb{N}_0, \quad (7)$$

for any b solving (2). We will refer to φ as the *marginal relative cost* function, as φ_n is the relative cost of starting from $n+1$ rather than n . Note that φ is determined by the following first-order linear recurrence, which reformulates (2):

$$\lambda \varphi_0 = \zeta, \quad \lambda \varphi_n - \mu_n \varphi_{n-1} = \zeta - c_n, \quad n \in \mathbb{N}. \quad (8)$$

Similarly to (3), when (8) is solved numerically substituting $z = \hat{\zeta}$ for ζ , the recurrence that is actually solved (ignoring other sources of error) is

$$\lambda f_0 = z, \quad \lambda f_n - \mu_n f_{n-1} = z - c_n, \quad n \in \mathbb{N}. \quad (9)$$

To illustrate, consider (cf. [11, §4.2]), the M/M/1+M queue with deadlines to the end of service, with $\lambda_n \triangleq \lambda$, $\mu_n \triangleq \mu + n\theta$ and $c_n \triangleq n\theta$, where $\lambda, \mu, \theta > 0$ are the arrival, service and abandonment rates. From (9), we obtain the recursion

$$f_0 = \frac{z}{\lambda}, \quad f_n = \frac{z - n\theta}{\lambda} + \frac{\mu + n\theta}{\lambda} f_{n-1}, \quad n = 1, 2, \dots \quad (10)$$

Consider now the instance with $\lambda = 0.9$, $\mu = 1$ and $\theta = 0.5$, for which

$$\zeta = \frac{1}{10} \left(\frac{81}{5e^{9/5} - 14} - 1 \right). \quad (11)$$

Computing (11) with MATLAB gives the double-precision floating-point number (see [12, Ch. 2]) $\hat{\zeta} \approx 0.398515613690624$. Table 1 shows, in the $\hat{\varphi}_n$ column, the first thirty f_n computed through (10) using $z = \hat{\zeta}$. After growing to 1 at first, $\hat{\varphi}_n$ diverges to *minus* infinity. The same behavior results when z is set to the next larger machine number, $\hat{\zeta} + 2^{-54}$. Yet, when the following machine number, $z = \tilde{\zeta} = \hat{\zeta} + 2^{-53}$, is used, the resulting f_n , written as $\tilde{\varphi}_n$, eventually diverge to *plus* infinity. Thus, approximate computation of the φ_n through (10), and hence of the b_n , suffers from an *explosive numerical instability* with respect to unavoidable errors in the approximation to the input ζ .

Experimentation with other models reveals that such instability is not exceptional. Thus, e.g., [13] reports numerical instabilities preventing accurate numerical solution of Poisson's equation in the queueing models considered there.

Table 1 further shows the probabilities p_n , to gain insight on the relation between these and the magnitudes of errors in the computed approximations to φ_n . The loss of significant digits of accuracy in the latter is evident only for unlikely states, with such inaccuracies growing steeply as the p_n get smaller. Note that the accurate estimation of performance metrics associated to very low-probability states, corresponding to *rare events*, is of considerable interest in a variety of areas, including computer-communication systems, reliability, finance, etc., and is the subject of major research attention. See, e.g., the survey [14].

Table 1: Approximate numerical computation of $\varphi_n = b_{n+1} - b_n$: explosive instability.

n	p_n	$\hat{\varphi}_n$	$\tilde{\varphi}_n$	n	p_n	$\hat{\varphi}_n$	$\tilde{\varphi}_n$
0	5.0×10^{-1}	0.44279513	0.44279513	15	1.9×10^{-11}	0.9388453	0.9388507
1	3.0×10^{-1}	0.62523145	0.62523145	16	1.9×10^{-12}	0.9423595	0.9424134
2	1.4×10^{-1}	0.72108723	0.72108723	17	1.8×10^{-13}	0.9454791	0.9460477
3	4.9×10^{-2}	0.77914855	0.77914855	18	1.6×10^{-14}	0.9481179	0.9544357
4	1.5×10^{-2}	0.81773474	0.81773474	19	1.4×10^{-15}	0.9486149	1.0223229
5	3.7×10^{-3}	0.84509690	0.84509690	20	1.1×10^{-16}	0.9258667	1.8267420
6	8.4×10^{-4}	0.86544800	0.86544800	21	8.9×10^{-18}	0.6066468	1.2×10^1
7	1.7×10^{-4}	0.88114626	0.88114626	22	6.6×10^{-19}	-3.7×10^0	1.5×10^2
8	3.0×10^{-5}	0.89360768	0.89360768	23	4.8×10^{-20}	-6.4×10^1	2.1×10^3
9	5.0×10^{-6}	0.90373094	0.90373094	24	3.3×10^{-21}	-9.3×10^2	3.0×10^4
10	7.4×10^{-7}	0.91211248	0.91211248	25	2.2×10^{-22}	-1.4×10^4	4.5×10^5
11	1.0×10^{-7}	0.91916304	0.91916304	26	1.4×10^{-23}	-2.2×10^5	7.0×10^6
12	1.3×10^{-8}	0.92517434	0.92517435	27	8.8×10^{-25}	-3.5×10^6	1.1×10^8
13	1.6×10^{-9}	0.93035909	0.93035915	28	5.3×10^{-26}	-5.8×10^7	1.9×10^9
14	1.8×10^{-10}	0.93487590	0.93487647	29	3.1×10^{-27}	-1.0×10^9	3.2×10^{10}

Regarding marginal relative costs φ_n , they are of direct interest in a major application of Poisson's equation: the *one-step policy improvement* (OSPI) method to the design of scalable heuristic policies for certain multidimensional MDPs, e.g., those concerning the optimal routing of a job stream to parallel service stations, each with its own queue. One starts with a tractable *static policy* (state-independent) under which the queues evolve independently. Then, a single step of Howard's *policy improvement* algorithm (see [8]) for MDPs gives a *dynamic policy* (state-dependent) with better cost performance.

The latter is an *index policy*, where the *index* for each station is precisely the marginal relative cost φ_n , and each arrival is routed to a station of currently lowest index value. See, e.g., [11, 15–21].

Intuition would suggest that, if such a policy routes an arrival to a station in a given system state, it should prescribe the same action if that station was less loaded, other things being equal. This would be ensured if one could prove that φ_n is *nondecreasing* in n or, equivalently, that β_n is *convex*. Note that, in Table 1, the computed φ_n increase at first, but, when numerical instabilities set in, such a monotonicity is lost, raising doubts on the behavior of the exact φ_n .

A related metric of interest is the *asymptotic variance* as $t \rightarrow \infty$ of the average cost up to time t , $\bar{C}(t) \triangleq (1/t) \int_0^t c_X(s) ds$, which is defined by

$$\sigma^2 \triangleq \lim_{t \rightarrow \infty} t \operatorname{Var} [\bar{C}(t)].$$

Under the aforementioned drift condition, and provided that $c^2 \in \mathbb{B}_w$, σ^2 is well defined and finite, being given by (see [5, Theorem 4.4])

$$\sigma^2 = 2 \sum_{n=0}^{\infty} b_n(c_n - \zeta)p_n = 2 \sum_{n=0}^{\infty} \beta_n c_n p_n \quad (12)$$

for any $b \in \mathbb{B}_w$ solving (2), and $\bar{C}(t)$ satisfies the *functional central limit theorem*

$$\sqrt{t} \frac{\bar{C}(t) - \zeta}{\sigma} \Longrightarrow N(0, 1) \text{ as } t \rightarrow \infty,$$

where \Longrightarrow denotes weak convergence and $N(0, 1)$ is the standard normal distribution. This result can be used, e.g., to set simulation run lengths to obtain confidence intervals for ζ . See, e.g., [13, 22, 23].

1.1 Goals and contributions

We address the following research goals: 1) identify mild conditions on transition and cost rates ensuring that the φ solving (3) is nondecreasing (equivalently, the b 's solving (2) are convex); 2) elucidate the root cause, extent and prevalence of numerical instability in computed solutions to Poisson's equation due to substituting

approximations $\hat{\zeta}$ for ζ ; and 3) obtain means of computing accurate approximate solutions when analytical solutions are not available.

Regarding the first goal, consider the following assumption, where we write $d_n \triangleq \mu_n - \lambda_n$ and use the backward difference notation $\Delta x_n \triangleq x_n - x_{n-1}$.

Assumption 1.1. (i) d is

- (i.a) *nondecreasing* ($\Delta d_n \geq 0$), with $\Delta d_1 > 0$; and
- (i.b) *concave* ($\Delta d_{n+1} \leq \Delta d_n$).

(ii) c is

- (ii.a) *nonnegative* ($c_n \geq 0$) and *nondecreasing* ($\Delta c_n \geq 0$); and
- (ii.b) *convex* ($\Delta c_{n+1} \geq \Delta c_n$).

We have the following result.

Theorem 1.2. *Under Assumption 1.1, φ is nondecreasing (β is convex).*

Remark 1.1. (a) Assumption 1.1 is mild and widely satisfied in birth–death queueing models. Thus, consider the following broad m -server model with possible customer balking and abandonment, which encompasses a variety of standard models: $\lambda_n = \lambda(1 - \alpha_n)$, with $0 \leq \alpha_n < 1$ the state-dependent balking probability, $\mu_n = \min(m, n)\mu + g(m, n)\theta$, where $\mu > 0$ and $\theta \geq 0$ are the service and abandonment rates, and $g(m, n) \triangleq (n - m)^+$ if customers can only abandon prior to entering service, while $g(m, n) \triangleq n$ otherwise. Further, $c_n = c^{\text{ab}}g(m, n)\theta + c_n^{\text{h}}$, where $c^{\text{ab}} \geq 0$ and $c_n^{\text{h}} \geq 0$ are the cost per abandonment and the holding cost rate. It is readily verified that this model satisfies Assumption 1.1 if the following holds: (i) α_n is nondecreasing and concave; (ii) in the case that customers can only abandon before entering service, $\theta \leq \mu$, i.e., the mean time to abandon ($1/\theta$) is not shorter than the mean service time ($1/\mu$); and (iii) c_n^{h} is nondecreasing and convex. These are all intuitively sound conditions.

(b) Theorem 1.2 is also relevant for birth–death models arising in population biology. Thus, consider the classic linear birth–death model with immigration, with $\lambda_n = n\lambda + \alpha$ and $\mu_n = n\mu$. In the case of a population whose size is costly, e.g., a pest population as in [24], one may take c_n to be the cost of having n individuals. Then, Assumption 1.1 clearly holds provided that c is nonnegative, nondecreasing and convex.

(c) Note that the conditions in Assumption 1.1 were first formulated in [25, Ass. 7.1], though with a different purpose: they were shown there to be sufficient conditions for existence of the *Whittle index* characterizing optimal policies in a broad birth–death admission control model.

As for the second goal, we explicitly identify and analyze the *error amplification factors* (see, e.g., [26] for early use of such a concept) that characterize how the error in the approximation $\hat{\zeta}$ to the input ζ propagates, in the standard *forward recurrence* scheme, to produce errors in the computed approximations \hat{b}_n and $\hat{\varphi}_n$ to the b_n and φ_n solving (2) and (8). See Proposition 5.1. Such results are further used to analyze the accuracy of computed approximations to the bias β and the asymptotic variance σ^2 . See Proposition 5.5.

Concerning the third goal, we analyze the approximation errors resulting from a *backward recurrence* scheme, which has not been previously considered for this model and has substantially improved accuracy for large states. See Propositions 6.4 and 6.5. We further propose a novel mixed *forward–backward recurrence* scheme that outperforms both forward and backward recurrence in terms of accuracy. See Propositions 7.1 and 7.3. The effectiveness of the proposed approach is demonstrated for the motivating example above. See §8.

1.2 Organization of the paper

The remainder of the paper proceeds as follows. §2 reviews related work. §3 presents results on birth–death chains, some of them new, that are required for subsequent analyses. §4 gives expressions for the exact solution to Poisson’s equation, as well as new expressions for the bias and the asymptotic variance in terms of marginal

relative costs; it further gives the proof of Theorem 1.2. §5 develops an error analysis of computed solutions to Poisson’s equation through forward recurrence. §6 analyzes a backward recurrence scheme with improved accuracy for large states. §7 presents a mixed forward–backward recurrence scheme, with improved accuracy with respect to the pure schemes. §8 illustrates the results in an example. §9 concludes. Appendix A lays the groundwork for the proof of Theorem 1.2.

Note that an early version of this work appeared in the proceedings [27].

2 Further related work

In addition to the work referred to in §1, we briefly review next two further related strands of research. The first one focuses on the Poisson equation for discrete-time Markov chains. [28] reviews structural results from a probabilistic interpretation viewpoint, while [29] establishes uniqueness of solutions to Poisson’s equation in such a setting. Another line of work aims to elucidate the structure of solutions to Poisson’s equation for discrete-time birth–death chains and extensions. See, e.g. [30, 31].

Another closely related relevant stream of work is the wide literature on numerical stability of computed solutions to linear recurrences, both in general and applied to the evaluation of special functions. In a classic study, Gautschi addressed in [26] the recursive numerical solution of a general first-order linear recurrence relation, assuming that the only source of error is the approximation of the initial condition. He considered the propagation of such an error to the computation of successive terms, as measured by *relative error amplification factors*, under two recursive schemes, standard *forward recurrence* and *backward recurrence*. The paper, which refers to earlier work on the effective use of the latter type of recurrence, elucidates in which cases each scheme is more accurate. Note however that, although (8) is a first-order linear recurrence, the analysis in [26] does not directly apply to it, as that paper considers that the source of error is in the initial condition, whereas here we consider that it lies in the approximation to ζ in the right-hand side of the equation.

Early work on second-order linear recurrence relations, regarding “*the possibility and the prevention of numerical instability*”, is reviewed in [32], which focuses on the homogeneous case, while, e.g., [33, 34] consider the non-homogeneous case. In such papers, numerically stable schemes, including backward recurrence, for accurately computing so-called *minimal solutions* are developed. Yet, again, such work does not directly apply to the analysis of Poisson’s equation (3), since the source of error addressed here is in the right-hand sides, which is not considered in the aforementioned papers.

For more recent work see, e.g., [35], which develops rounding-error bounds for the numerical solution of higher-order linear recurrence relations.

Even though, for the reasons outlined above, such work does not appear to be directly applicable to the recurrence relations herein, still, key ideas developed in that field such as error amplification factor analysis, and the use of backward recurrence to overcome numerical instability, play a central role in this paper.

Yet, such general ideas need to be adapted to the present setting, by exploiting the rich special structure and the probabilistic interpretation of the recurrences herein. To the best of the author’s knowledge, such an analysis has not been undertaken before, and is hence a novel contribution of this work.

3 Required results on birth–death Markov chains

This section presents results that are required for the ensuing analyses, mostly known, but also some new extensions (which the author has not found in the literature), on mean first-passage times and costs for birth–death chains.

Consider a birth–death process $\{X(t)\}_{t \geq 0}$ with costs as outlined in §1. Its *potential coefficients* π_n are the unnormalized state probabilities,

$$\pi_0 \triangleq 1, \quad \pi_n \triangleq \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \quad n \in \mathbb{N},$$

which satisfy the recurrence

$$\pi_0 = 1, \quad \lambda_{n-1}\pi_{n-1} = \mu_n\pi_n, \quad n \in \mathbb{N}. \tag{13}$$

We assume that

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \pi_n < \infty, \quad (14)$$

which (see [36, Theorem 2(a)]) is a necessary and sufficient condition for the process to be uniquely determined by its birth–death rates and ergodic. The following is a standard sufficient condition for (14), where $\rho_n \triangleq \lambda_n / \mu_n$:

$$\limsup_{n \rightarrow \infty} \rho_n < 1. \quad (15)$$

The steady-state probabilities are given by

$$p_n = \frac{\pi_n}{\sum_{j=0}^{\infty} \pi_j} = \pi_n p_0, \quad n \in \mathbb{N}_0. \quad (16)$$

We will consider the functions

$$P_n \triangleq \sum_{j=0}^n p_j, \quad C_n \triangleq \sum_{j=0}^n c_j p_j \quad \text{and} \quad Z_n \triangleq \frac{C_n}{P_n}, \quad n \in \mathbb{N}_0, \quad (17)$$

which satisfy

$$P_n \rightarrow 1, \quad C_n \rightarrow \zeta \quad \text{and} \quad Z_n \rightarrow \zeta \quad \text{as} \quad n \rightarrow \infty. \quad (18)$$

Thus, P_n is the cumulative distribution function of the p_n . We also consider the corresponding *tail* functions defined by

$$\bar{P}_n \triangleq 1 - P_n, \quad \bar{C}_n \triangleq \zeta - C_n \quad \text{and} \quad \bar{Z}_n \triangleq \zeta - Z_n, \quad n \in \mathbb{N}_0. \quad (19)$$

Letting $\tau_n \triangleq \min\{t \geq 0: X(t) = n\}$ be the *first-passage time to state n* , let $T_n^+ \triangleq \mathbb{E}_n[\tau_{n+1}]$ and $T_n^- \triangleq \mathbb{E}_n[\tau_{n-1}]$ be the mean first-passage times from n to $n+1$, and from n to $n-1$, and write as $H_n^+ \triangleq \mathbb{E}_n[\int_0^{\tau_{n+1}} c_{X(s)} ds]$ and $H_n^- \triangleq \mathbb{E}_n[\int_0^{\tau_{n-1}} c_{X(s)} ds]$ the corresponding mean costs. We further define

$$Z_n^+ \triangleq \frac{H_n^+}{T_n^+} \quad \text{and} \quad Z_{n+1}^- \triangleq \frac{H_{n+1}^-}{T_{n+1}^-}, \quad n \in \mathbb{N}_0. \quad (20)$$

The following recurrences follow from elementary probabilistic arguments:

$$\lambda_0 T_0^+ = 1, \quad \lambda_n T_n^+ - \mu_n T_{n-1}^+ = 1, \quad n \in \mathbb{N}, \quad (21)$$

which is given in [37, Eq. (1.5a)] (see its derivation in [1, Ch. 6.3]) and

$$\lambda_0 H_0^+ = c_0, \quad \lambda_n H_n^+ - \mu_n H_{n-1}^+ = c_n, \quad n \in \mathbb{N}. \quad (22)$$

In [37, Eq. (1.8)], it is shown that T_n^+ and P_n are linked by

$$T_n^+ = \frac{P_n}{\lambda_n p_n}, \quad (23)$$

and the same argument given there yields that

$$H_n^+ = \frac{C_n}{\lambda_n p_n}. \quad (24)$$

Note that it follows immediately from (23) that, as $n \rightarrow \infty$,

$$T_n^+ \rightarrow \infty \quad \text{if and only if} \quad \lambda_n p_n \rightarrow 0, \quad (25)$$

and

$$\sup_n T_n^+ = \infty \quad \text{if and only if} \quad \inf_n \lambda_n p_n = 0, \quad (26)$$

so the T_n^+ are bounded if and only if the $\lambda_n p_n$ are bounded away from 0.

We next give a sufficient condition for $T_n^+ \rightarrow \infty$ in terms of the ρ_n in (15).

Lemma 3.1. *Under condition (15), $T_n^+ \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. Write $\rho^* \triangleq \limsup_{n \rightarrow \infty} \rho_n$. Pick $\varepsilon > 0$ such that $\rho^* + \varepsilon < 1$, and let N be such that $\rho_n < \rho^* + \varepsilon$ for $n > N$. Then

$$T_n^+ = \frac{1}{\lambda_n} + \frac{T_{n-1}^+}{\rho_n} > \frac{T_{n-1}^+}{\rho^* + \varepsilon}, \quad n > N,$$

from which it follows that $T_n^+ \rightarrow \infty$ as $n \rightarrow \infty$. \square

The following recurrences are also easily obtained:

$$\mu_n T_n^- - \lambda_n T_{n+1}^- = 1, \quad n \in \mathbb{N}, \quad (27)$$

which is given in [37, Eq. (1.10)], and its counterpart for costs,

$$\mu_n H_n^- - \lambda_n H_{n+1}^- = c_n, \quad n \in \mathbb{N}. \quad (28)$$

Yet, unlike recurrences (21) and (22), which determine the T_n^+ and H_n^+ , (27) and (28) do not determine the T_n^- and H_n^- , as they lack boundary conditions.

These are provided by the case $n = 0$ in the following result, which gives analogous relations to (23) and (24) linking T_{n+1}^- to \bar{P}_n and H_{n+1}^- to \bar{C}_n . Note that the identity in Lemma 3.2(a) is given in [37, Eq. (1.11)]. Yet, the argument outlined there bypasses a critical detail, as it entails summation of an infinite telescoping series, which requires showing that $\lambda_n p_n T_{n+1}^- \rightarrow 0$ as $n \rightarrow \infty$. We avoid this in the proof below by using instead an induction argument, drawing on the theory of renewal reward processes (see, e.g., [1, Ch. 7]).

Lemma 3.2. *For $n \in \mathbb{N}_0$,*

$$(a) \quad T_{n+1}^- = \frac{\bar{P}_n}{\lambda_n p_n};$$

$$(b) \quad H_{n+1}^- = \frac{\bar{C}_n}{\lambda_n p_n}.$$

Proof. (a) We use induction. For $n = 0$, consider the regenerative cycle starting from state 0 until the first return to 0, and denote by T_{00} its mean duration. By the *renewal reward theorem*, and using that $T_{00} = 1/\lambda_0 + T_1^-$, we have

$$p_0 = \frac{\mathbb{E}[\text{cost during a cycle}]}{\mathbb{E}[\text{duration of a cycle}]} = \frac{1/\lambda_0}{1/\lambda_0 + T_1^-},$$

and hence $T_1^- = \bar{P}_0/\lambda_0 p_0$. Suppose now that the result holds for $n - 1$, so that $T_n^- = \bar{P}_{n-1}/\lambda_{n-1} p_{n-1}$. Then, using this and (27) we obtain

$$\lambda_n p_n T_{n+1}^- = \mu_n p_n T_n^- - p_n = \lambda_{n-1} p_{n-1} T_n^- - p_n = \bar{P}_{n-1} - p_n = \bar{P}_n,$$

which completes the induction.

(b) For $n = 0$, let H_{00} be the mean cost accrued over the cycle in part (a). By the renewal reward theorem, and since $H_{00} = c_0/\lambda_0 + H_1^-$, we have

$$\zeta = \frac{\mathbb{E}[\text{cost during a cycle}]}{\mathbb{E}[\text{duration of a cycle}]} = \frac{c_0/\lambda_0 + H_1^-}{1/\lambda_0 + T_1^-},$$

whence $H_1^- = \bar{C}_0/\lambda_0 p_0$. Suppose now the result holds for $n - 1$, so that $H_n^- = \bar{C}_{n-1}/\lambda_{n-1} p_{n-1}$. Then, using this and (28) gives, as required.

$$\lambda_n p_n H_{n+1}^- = \mu_n p_n H_n^- - c_n p_n = \lambda_{n-1} p_{n-1} H_n^- - c_n p_n = \bar{C}_{n-1} - c_n p_n = \bar{C}_n.$$

\square

Remark 3.1. (a) From (23), (24) and Lemma 3.2 we obtain

$$P_n T_{n+1}^- = \bar{P}_n T_n^+ \quad \text{and} \quad C_n H_{n+1}^- = \bar{C}_n H_n^+. \quad (29)$$

(b) It follows from (23) and (24) that the Z_n in (17) and Z_n^+ in (20) satisfy

$$Z_n^+ = Z_n. \quad (30)$$

(c) As for the Z_{n+1}^- in (20), from Lemma 3.2 and (67) we can write (see (19))

$$Z_{n+1}^- = \frac{\bar{C}_n}{P_n} = \frac{\zeta - C_n}{1 - P_n} = Z_n + \frac{\bar{Z}_n}{P_n} = \zeta + P_n \frac{\bar{Z}_n}{P_n}. \quad (31)$$

Thus, $Z_{n+1}^- - \zeta$ is asymptotically equivalent to the tail ratio \bar{Z}_n/\bar{P}_n . Note that, typically, Z_{n+1}^- will *not* converge to ζ as $n \rightarrow \infty$.

Now, the aforementioned result that $\lambda_n p_n T_{n+1}^- \rightarrow 0$ as $n \rightarrow \infty$, and the corresponding result for H_{n+1}^- , follow immediately from Lemma 3.2 and (18).

Corollary 3.3. *As $n \rightarrow \infty$,*

(a) $\lambda_n p_n T_{n+1}^- \rightarrow 0$;

(b) $\lambda_n p_n H_{n+1}^- \rightarrow 0$.

We will also consider $T_{0n} \triangleq \mathbb{E}_0[\tau_n]$, the *mean first-passage time from 0 to n*, $T_{n0} \triangleq \mathbb{E}_n[\tau_0]$, the *mean first-passage time from n to 0*, and the corresponding mean costs $H_{0n} \triangleq \mathbb{E}_0[\int_0^{\tau_n} c_{X(s)} ds]$ and $H_{n0} \triangleq \mathbb{E}_n[\int_0^{\tau_0} c_{X(s)} ds]$. Note that

$$T_{0n} = \sum_{j=0}^{n-1} T_j^+ = \sum_{j=0}^{n-1} \frac{P_j}{\lambda_j p_j} \quad \text{and} \quad H_{0n} = \sum_{j=0}^{n-1} H_j^+ = \sum_{j=0}^{n-1} \frac{C_j}{\lambda_j p_j}, \quad (32)$$

whereas

$$T_{n0} = \sum_{j=0}^{n-1} T_{j+1}^- = \sum_{j=0}^{n-1} \frac{\bar{P}_j}{\lambda_j p_j} \quad \text{and} \quad H_{n0} = \sum_{j=0}^{n-1} H_{j+1}^- = \sum_{j=0}^{n-1} \frac{\bar{C}_j}{\lambda_j p_j}. \quad (33)$$

Remark 3.2. (a) From (23) and (32) we obtain

$$\lim_{n \rightarrow \infty} T_{0n} = \sum_{j=0}^{\infty} T_j^+ = \sum_{j=0}^{\infty} \frac{P_j}{\lambda_j p_j} = \infty,$$

since the last series diverges by the limit comparison test (since $P_j \rightarrow 1$ as $j \rightarrow \infty$) using that $\sum_{j=0}^{\infty} 1/\lambda_j p_j = \infty$ by (14).

(b) We will refer to $T_{\infty 0} \triangleq \lim_{n \rightarrow \infty} T_{n0} = \sum_{j=1}^{\infty} T_j^-$, which is given by

$$T_{\infty 0} = \sum_{n=0}^{\infty} \frac{\bar{P}_n}{\lambda_n p_n}. \quad (34)$$

In Feller's [38] classic boundary classification for birth–death processes, if $T_{\infty 0} < \infty$ the process is said to have an *entrance boundary at infinity*, and is then exponentially ergodic. See [39, Theorem 8.1]. Otherwise, the process is said to have a *natural boundary at infinity*. From [40, Eq. (6.5) and Theorem 6.4(ii)] it follows that, under condition (15),

$$T_{\infty 0} < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{\mu_n} < \infty. \quad (35)$$

Note that, in most applied models, the rightmost series in (35) diverges.

(c) For some results we will refer to the condition

$$T_{p0} < \infty, \quad (36)$$

where $T_{p0} \triangleq \mathbb{E}_p[\tau_0] = \sum_{n=1}^{\infty} p_n T_{n0}$ is the *mean first-passage time to 0 starting from steady state*. Note that from the identities for T_{n0} in (33) we have

$$T_{p0} = \sum_{n=0}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n}. \quad (37)$$

See also [41, Eq. (6.6)] and [36, Theorem 4 and Eq. (3.6)]. Furthermore, [41, Theorem 6.1] shows that satisfaction of (36) is equivalent to existence of the *deviation matrix* $D = (d_{mn})$ of the Markov chain, defined by

$$d_{mn} \triangleq \int_0^{\infty} (p_{mn}(t) - p_n) dt,$$

where $p_{mn}(t) \triangleq \mathbb{P}_m\{X(t) = n\}$ and \mathbb{P}_m is the probability starting from m . Note that the bias is given in terms of D by $\beta_m = \sum_n d_{mn} c_n$, i.e., $\beta = Dc$.

(d) The quantities $T_{\infty 0}$ and T_{p0} are related by

$$T_{\infty 0} = \sum_{n=0}^{\infty} \bar{P}_n T_n^+ + T_{p0}, \quad (38)$$

which readily follows from (34), (37), (23) and $P_n - \bar{P}_n = P_n^2 - \bar{P}_n^2$. In light of (38), we define

$$T_{\infty p} \triangleq \sum_{n=0}^{\infty} \bar{P}_n T_n^+, \quad (39)$$

which represents the *mean first passage-time to steady state starting at ∞* .

The following result shows that the ratios T_{n0}/T_{0n} strictly decrease to 0.

Lemma 3.4.

- (a) $\frac{\bar{P}_n}{P_n} < \frac{T_{n+1,0}}{T_{0,n+1}} < \frac{T_{n0}}{T_{0n}}$, for $n \in \mathbb{N}$;
- (b) $\frac{T_{n0}}{T_{0n}} \searrow 0$ as $n \rightarrow \infty$.

Proof. (a) Using (32), (33) and the mediant inequality (see (68)) we obtain

$$\frac{\bar{P}_n}{P_n} < \frac{T_{n+1,0}}{T_{0,n+1}} < \frac{T_{n0}}{T_{0n}} \text{ if and only if } \frac{\bar{P}_n}{P_n} < \frac{T_{n0}}{T_{0n}}. \quad (40)$$

We next show by induction that $\bar{P}_n/P_n < T_{n+1,0}/T_{0,n+1}$ for $n \geq 1$. For $n = 1$,

$$\frac{T_{10}}{T_{01}} = \frac{\bar{P}_0}{P_0} > \frac{\bar{P}_1}{P_1},$$

whence the result follows by (40). Suppose now the result holds for $n - 1$. Then,

$$\frac{T_{n0}}{T_{0n}} > \frac{\bar{P}_{n-1}}{P_{n-1}} > \frac{\bar{P}_n}{P_n}, \quad (41)$$

which, using (40), gives the result for n . This completes the induction. Now, part (a) follows since, by (40), it is a consequence of (41).

(b) Since

$$\frac{T_{n0}}{T_{0n}} = \frac{\sum_{j=0}^{n-1} \frac{\bar{P}_j}{\lambda_j p_j}}{\sum_{j=0}^{n-1} \frac{P_j}{\lambda_j p_j}},$$

with T_{0n} strictly increasing and divergent and $\bar{P}_{n-1}/P_{n-1} \rightarrow 0$, the *Stolz–Cesàro theorem* (see [42, Theorem 1.22]) gives that $T_{n0}/T_{0n} \rightarrow 0$. \square

By analogy with expression (30) for Z_n , we further define the cost ratios

$$Z_{0n} \triangleq \frac{H_{0n}}{T_{0n}} \quad \text{and} \quad Z_{n0} \triangleq \frac{H_{n0}}{T_{n0}}, \quad n \in \mathbb{N}. \quad (42)$$

Since $Z_n \rightarrow \zeta$ as $n \rightarrow \infty$, this raises the question of whether Z_{0n} and Z_{n0} converge to ζ . We will settle this in the affirmative for Z_{0n} . We need a preliminary result on invariance relations for certain ratios of mean costs to mean times.

Lemma 3.5.

- (a) $\frac{H_n^+ + H_{n+1}^-}{T_n^+ + T_{n+1}^-} = \zeta$, for $n \in \mathbb{N}_0$;
- (b) $\frac{H_{0n} + H_{n0}}{T_{0n} + T_{n0}} = \zeta$, for $n \in \mathbb{N}$.

Proof. (a) Using (23), (24) and Lemma 3.2, we have

$$\frac{H_n^+ + H_{n+1}^-}{T_n^+ + T_{n+1}^-} = \frac{\frac{C_n}{\lambda_n p_n} + \frac{\bar{C}_n}{\lambda_n p_n}}{\frac{P_n}{\lambda_n p_n} + \frac{\bar{P}_n}{\lambda_n p_n}} = \zeta.$$

(b) Using (32) and (33) we can write

$$\frac{H_{0n} + H_{n0}}{T_{0n} + T_{n0}} = \frac{\sum_{j=0}^{n-1} \frac{C_j}{\lambda_j p_j} + \sum_{j=0}^{n-1} \frac{\bar{C}_j}{\lambda_j p_j}}{\sum_{j=0}^{n-1} \frac{P_j}{\lambda_j p_j} + \sum_{j=0}^{n-1} \frac{\bar{P}_j}{\lambda_j p_j}} = \frac{\sum_{j=0}^{n-1} \frac{\zeta}{\lambda_j p_j}}{\sum_{j=0}^{n-1} \frac{1}{\lambda_j p_j}} = \zeta.$$

\square

We can now prove the following result on the convergence of Z_{0n} .

Lemma 3.6. $Z_{0n} \rightarrow \zeta$ as $n \rightarrow \infty$.

Proof. Since

$$Z_{0n} = \frac{H_{0n}}{T_{0n}} = \frac{H_0^+ + \dots + H_{n-1}^+}{T_0^+ + \dots + T_{n-1}^+},$$

with T_{0n} strictly increasing and divergent and $H_{n-1}^+/T_{n-1}^+ = Z_n \rightarrow \zeta$, the *Stolz–Cesàro theorem* (see [42, Theorem 1.22]) gives that $Z_{0n} \rightarrow \zeta$. \square

Remark 3.3. From Lemma 3.5(b) and (66) we can write

$$\zeta = \frac{H_{n0} + H_{0n}}{T_{n0} + T_{0n}} = Z_{n0} + \frac{T_{0n}}{T_{n0} + T_{0n}}(Z_{0n} - Z_{n0}),$$

hence

$$Z_{n0} = \zeta - \frac{T_{0n}}{T_{n0}}(Z_{0n} - \zeta). \quad (43)$$

Thus, the asymptotic behavior of Z_{n0} depends on that of the scaled tail of Z_{0n} in (43). See Lemmas 3.4(b) and 3.6. Typically, Z_{n0} will *not* converge to ζ .

4 Exact solution: explicit expressions and properties

4.1 Explicit expressions for the solution to Poisson's equation

We next derive explicit expressions for the exact solution to Poisson's equation (3). We start by obtaining expressions formulated in terms of the mean *upward* first-passage times and costs considered in §3.

Let $b(z; a)$ be the solution to (3) for given z when $b_0 = a$. To analyze $b(z; a)$, consider the reformulation (9) of (3). Note that

$$b_n = b_0 + \sum_{j=0}^{n-1} f_j, \quad n \in \mathbb{N}. \quad (44)$$

Let $f(z)$ be the solution to (9), which is constructed by the forward recurrence

$$f_0(z) = \frac{z - c_0}{\lambda_0}, \quad f_n(z) = \frac{z - c_n}{\lambda_n} + \frac{\mu_n}{\lambda_n} f_{n-1}(z), \quad n = 1, 2, \dots \quad (45)$$

The following result represents $f_n(z)$ and $b_n(z; a)$ as affine functions of z .

Proposition 4.1. *For $n \in \mathbb{N}_0$,*

- (a) $f_n(z) = T_n^+ z - H_n^+ = T_n^+(z - Z_n)$;
- (b) $b_n(z; a) = a + T_{0n}z - H_{0n} = a + T_{0n}(z - Z_{0n})$.

Proof. (a) Since T^+ and H^+ satisfy (21) and (22), $f = T^+z - H^+$ satisfies (45), whence $f(z) = T^+z - H^+ = T^+(z - Z)$, where we have also used (30).

(b) We have $b_0(z; a) = a$. For $n \geq 1$, from (44), part (a), (32) and (42),

$$b_n(z; a) = a + \sum_{j=0}^{n-1} f_j(z) = a + \sum_{j=0}^{n-1} (T_j^+ z - H_j^+) = a + T_{0n}z - H_{0n} = a + T_{0n}(z - Z_{0n}).$$

□

From Proposition 4.1 we immediately obtain the following result, which gives expressions for the $\varphi_n = f_n(\zeta)$ and $b_n = b_n(\zeta; a)$ solving (8) and (2).

Corollary 4.2. *For $n \in \mathbb{N}_0$,*

- (a) $\varphi_n = T_n^+(\zeta - Z_n)$;
- (b) $b_n = b_0 + T_{0n}(\zeta - Z_{0n})$.

Remark 4.1. (a) The identity in Corollary 4.2(a) was given in [15, Eq. (6)] for a particular queueing model, the M/M/m queue with $c_n = n$.

(b) Corollary 4.2(b) is consistent with known results for discrete-time Markov chains. See, e.g., [43, Prop. A.3.1(ii)] and [20, Theorem 1].

The following result gives alternate expressions for the exact solution to Poisson's equations (8) and (2), which are formulated in terms of the mean *downward* first-passage times and costs considered in §3.

Proposition 4.3.

- (a) $\varphi_n = T_{n+1}^-(Z_{n+1}^- - \zeta)$, for $n \in \mathbb{N}_0$;
- (b) $b_n = b_0 + T_{n0}(Z_{n0} - \zeta)$, for $n \in \mathbb{N}$.

Proof. (a) From Corollary 4.2(a), (30), Lemma 3.5(a) and (31), we obtain

$$\varphi_n = T_n^+(\zeta - Z_n) = T_n^+\zeta - H_n^+ = H_{n+1}^- - \zeta T_{n+1}^- = T_{n+1}^-(Z_{n+1}^- - \zeta).$$

(b) From Corollary 4.2(b), (42), and Lemma 3.5(b), we obtain

$$b_n - b_0 = T_{0n}(\zeta - Z_{0n}) = \zeta T_{0n} - H_{0n} = H_{n0} - \zeta T_{n0} = T_{n0}(Z_{n0} - \zeta).$$

□

4.2 Convexity of relative cost function

We next draw on the exact expressions derived above to obtain the practically relevant structural result of solutions to Poisson's equation (2) stated in Theorem 1.2. The short proof given next draws on substantial preliminary groundwork, which is laid in Appendix A.

Proof of Theorem 1.2. From Corollary 4.2(a), we immediately obtain

$$\Delta\varphi_n = \zeta\Delta T_n^+ - \Delta H_n^+,$$

and hence, since $\Delta T_n^+ > 0$ by Lemma A.3,

$$\Delta\varphi_n \geq 0 \quad \text{if and only if} \quad \frac{\Delta H_n^+}{\Delta T_n^+} \leq \zeta. \quad (46)$$

Now, Lemmas A.6 and A.7 ensure that $\Delta H_n^+/\Delta T_n^+$ grows to ζ as $n \rightarrow \infty$. In light of (46), this completes the proof. \square

4.3 Bias and asymptotic variance: exact expressions in terms of φ

We next turn to exact evaluation of the bias β and the asymptotic variance σ^2 . The following result gives new expressions in terms of the marginal relative cost φ . We assume that β is characterized by (6), and that σ^2 is well defined and finite, being given by (12). To ensure the validity of interchanging the order of summation in certain series arising in the proofs, we further assume that cost rates c_n are nonnegative and nondecreasing, so that Assumption 1.1(ii.a) holds.

Proposition 4.4. *Suppose that c satisfies Assumption 1.1(ii.a). Then*

$$\begin{aligned} \text{(a)} \quad & \beta_0 = -\sum_{j=0}^{\infty} \bar{P}_j \varphi_j, \quad \beta_n = \beta_0 + \sum_{j=0}^{n-1} \varphi_j, \quad n \in \mathbb{N}; \\ \text{(b)} \quad & \sigma^2 = 2 \sum_{n=0}^{\infty} \lambda_n p_n \varphi_n^2. \end{aligned}$$

Proof. (a) From (44) we have $\beta_n = \beta_0 + \sum_{j=0}^{n-1} \varphi_j$ and, using (6), we can write

$$0 = \sum_{n=0}^{\infty} \beta_n p_n = \sum_{n=0}^{\infty} \left(\beta_0 + \sum_{j=0}^{n-1} \varphi_j \right) p_n = \beta_0 + \sum_{j=0}^{\infty} \varphi_j \sum_{n=j+1}^{\infty} p_n = \beta_0 + \sum_{j=0}^{\infty} \bar{P}_j \varphi_j,$$

where the interchange in the order of summation is justified by Tonelli's theorem, since $\varphi \geq 0$ by Lemma A.1(b). Therefore,

$$\beta_0 = -\sum_{j=0}^{\infty} \bar{P}_j \varphi_j. \quad (47)$$

(b) We can write

$$\begin{aligned} \sigma^2 &= 2 \sum_{n=0}^{\infty} \beta_n c_n p_n = 2 \sum_{n=0}^{\infty} \left(\beta_0 + \sum_{j=0}^{n-1} \varphi_j \right) c_n p_n \\ &= 2\beta_0 \zeta + 2 \sum_{j=0}^{\infty} \varphi_j \sum_{n=j+1}^{\infty} c_n p_n = 2\beta_0 \zeta + 2 \sum_{j=0}^{\infty} \varphi_j \bar{C}_j \\ &= 2 \sum_{n=0}^{\infty} (\bar{C}_n - \zeta \bar{P}_n) \varphi_n = 2 \sum_{n=0}^{\infty} \lambda_n p_n (H_{n+1}^- - T_{n+1}^-) \varphi_n = 2 \sum_{n=0}^{\infty} \lambda_n p_n \varphi_n^2, \end{aligned}$$

using in turn (12), part (a), (18), Lemma 3.2 and Proposition 4.3(a), and the interchange in the order of summation is justified as in part (a). \square

5 Approximate numerical solution by forward recurrence

This section presents an error analysis of standard forward recurrence for the numerical solution of Poisson's equation. It further addresses how the resulting approximation errors in the computed solution affect the accuracy of computed approximations to the bias β and the asymptotic variance σ^2 .

Given an approximation \hat{x} to a number x , we denote by $E_{\text{abs}}(\hat{x}) \triangleq \hat{x} - x$ and $E_{\text{rel}}(\hat{x}) \triangleq E_{\text{abs}}(\hat{x})/x$ the corresponding approximation errors in absolute and relative terms, respectively, provided that $x \neq 0$ in the latter case.

Note that, typically, \hat{x} will be a floating-point approximation to x , and hence its relative error will be bounded as (see, e.g., [12, Ch. 2])

$$|E_{\text{rel}}(\hat{x})| < u, \quad (48)$$

where u is the *unit roundoff*. Thus, in IEEE standard arithmetic, $u = 2^{-24} \approx 5.96 \times 10^{-8}$ for single precision, and $u = 2^{-53} \approx 1.11 \times 10^{-16}$ for double precision.

5.1 Approximate numerical evaluation of φ and b : Error amplification factors

We start by addressing the approximate numerical evaluation of relative costs b_n and marginal relative costs φ_n , using an approximate input $z = \hat{\zeta} \neq \zeta$ instead of $z = \zeta$ in (3) and (9), respectively. We will assume that to be the only source of error, so the computed approximation to φ is $\hat{\varphi} = f(\hat{\zeta})$. See (45).

We next address how the approximation errors in the input $\hat{\zeta}$ are amplified to corresponding errors in the computed outputs $\hat{\varphi}_n$ and \hat{b}_n , where the latter are computed from $\hat{\varphi}$ through (44) with $\hat{b}_0 = b_0 = a$. The following result identifies the corresponding *error amplification factors*.

Note that, as is standard in error analysis of numerical algorithms (see, e.g., [26]), we call A_n the absolute (resp. relative) *error amplification factor* of a computed quantity, such as $\hat{\varphi}_n$, with respect to the error in the approximate input, which is $\hat{\zeta}$ in our case, if $E_{\text{abs}}(\hat{\varphi}_n) = A_n E_{\text{abs}}(\hat{\zeta})$ (resp. $E_{\text{rel}}(\hat{\varphi}_n) = A_n E_{\text{rel}}(\hat{\zeta})$).

We assume henceforth that relative errors are well defined, i.e., $\zeta, \varphi_n, b_n \neq 0$.

Proposition 5.1 (Error amplification factors).

- (a) $E_{\text{abs}}(\hat{\varphi}_n) = T_n^+ E_{\text{abs}}(\hat{\zeta})$, for $n \in \mathbb{N}_0$;
- (b) $E_{\text{rel}}(\hat{\varphi}_n) = \zeta \frac{T_n^+}{\varphi_n} E_{\text{rel}}(\hat{\zeta}) = \frac{\zeta}{\zeta - Z_n} E_{\text{rel}}(\hat{\zeta})$, for $n \in \mathbb{N}_0$;
- (c) $E_{\text{abs}}(\hat{b}_n) = T_{0n} E_{\text{abs}}(\hat{\zeta})$, for $n \in \mathbb{N}$;
- (d) $E_{\text{rel}}(\hat{b}_n) = \zeta \frac{T_{0n}}{b_n} E_{\text{rel}}(\hat{\zeta}) = \frac{\zeta}{a/T_{0n} + \zeta - Z_{0n}} E_{\text{rel}}(\hat{\zeta})$, for $n \in \mathbb{N}$.

Proof. All parts follow straightforwardly from Proposition 4.1. Thus, e.g., for part (a), taking $z = \hat{\zeta}$ and $z = \zeta$ in Proposition 4.1(a) gives $\hat{\varphi}_n = T_n^+(\hat{\zeta} - Z_n)$ and $\varphi_n = T_n^+(\zeta - Z_n)$, hence $\hat{\varphi}_n - \varphi_n = T_n^+(\hat{\zeta} - \zeta)$. \square

The following result elucidates the numerical instability phenomenon (cf. Table 1 in §1), by clarifying the asymptotic behavior of the error amplification factors in Proposition 5.1. Note that $\text{sgn}(x) \in \{-1, 0, 1\}$ denotes the sign of x .

Corollary 5.2. As $n \rightarrow \infty$,

- (a.1) $E_{\text{abs}}(\hat{\varphi}_n) \rightarrow \text{sgn}(E_{\text{abs}}(\hat{\zeta})) \cdot \infty$ if and only if $\lambda_n p_n \rightarrow 0$;
- (a.2) $\sup_n E_{\text{abs}}(\hat{\varphi}_n) = \infty$ if and only if $\inf_n \lambda_n p_n = 0$;
- (b) $|E_{\text{rel}}(\hat{\varphi}_n)| \rightarrow \infty$;
- (c) $E_{\text{abs}}(\hat{b}_n) \rightarrow \text{sgn}(E_{\text{abs}}(\hat{\zeta})) \cdot \infty$;

(d) $|E_{\text{rel}}(\hat{b}_n)| \rightarrow \infty$.

Proof. The results follow from (25), (26), Proposition 5.1, Lemma 3.1, (18), Remark 3.2(a) and Lemma 3.6. \square

Remark 5.1.

- (a) Corollary 5.2(b, d) shows that forward recurrence is inherently numerically unstable, in that the magnitudes of *relative errors* in the computed approximations $\hat{\varphi}_n$ and \hat{b}_n to φ_n and b_n diverge to infinity as $n \rightarrow \infty$, as they are inversely proportional to the asymptotically vanishing *tails* $\zeta - Z_n$ and $\zeta - Z_{n0}$ (approximately so for \hat{b}_n when $a \neq 0$), respectively. Corollary 5.2(c) shows that such is also the case for the approximation error of \hat{b}_n .
- (b) It is of interest to consider how the relative approximation errors of $\hat{\varphi}_n$ and \hat{b}_n grow asymptotically in relation to $1/p_n$. From Proposition 5.1(b, d) it follows that $p_n E_{\text{rel}}(\hat{\varphi}_n)$ and $p_n E_{\text{rel}}(\hat{b}_n)$ are asymptotically proportional to $p_n/(\zeta - Z_n)$ and $p_n/(\zeta - Z_{n0})$, respectively. Note that the latter ratios might, e.g., converge to a finite limit or diverge to infinity.
- (c) The approximation error of $\hat{\varphi}_n$ may be bounded or unbounded. Corollary 5.2(a.2) shows that it is bounded if and only if $\lambda_n p_n$ is bounded away from 0. As an example where $E_{\text{abs}}(\hat{\varphi}_n)$ is bounded, take $\lambda_n \triangleq \lambda \rho^{-n-1}$ and $\mu_n \triangleq \mu \rho^{-n-1}$, with $\rho \triangleq \lambda/\mu < 1$. Then, $p_n = (1 - \rho)\rho^n$, so $\lambda_n p_n \equiv \mu - \lambda$.
- (d) If arrival rates λ_n are bounded above, so that $\bar{\lambda} \triangleq \sup_n \lambda_n < \infty$, as is often the case in applied models, Proposition 5.1(a) and (21) give that, for large n , $|E_{\text{abs}}(\hat{\varphi}_n)| \approx |E_{\text{abs}}(\hat{\zeta})|/(\lambda_n p_n) \geq |E_{\text{abs}}(\hat{\zeta})|/(\bar{\lambda} p_n)$. In such a case, $\hat{\varphi}_n$ will substantially deviate from φ_n for unlikely states since, asymptotically, $|E_{\text{abs}}(\hat{\varphi}_n)|$ will grow at least inversely proportional to p_n .
- (e) Corollary 5.2(a.1) explains the opposite signs in the behavior for large n of the computed approximations $\hat{\varphi}_n$ and $\tilde{\varphi}_n$ in Table 1 (see §1).

The following result highlights the strikingly different asymptotic behavior of the computed $\hat{\varphi}_n$ and \hat{b}_n compared to the exact φ_n and b_n . Note that the notation $y_n = \Theta(x_n)$ means that y_n is *asymptotically proportional* to x_n .

Corollary 5.3. *Let $\hat{\zeta} \neq \zeta \neq 0$. As $n \rightarrow \infty$,*

- (a) *If $\lambda_n p_n \rightarrow 0$, $\hat{\varphi}_n = \Theta(T_n^+)$ and $\varphi_n = o(T_n^+)$, and hence $\varphi_n = o(\hat{\varphi}_n)$;*
- (b) *$\hat{b}_n = \Theta(T_{0n})$ and $b_n = o(T_{0n})$, and hence $b_n = o(\hat{b}_n)$.*

Proof. (a) By Proposition 4.1(a), $\hat{\varphi}_n = T_n^+(\hat{\zeta} - Z_n)$ and $\varphi_n = T_n^+(\zeta - Z_n)$, which gives the result using that $Z_n \rightarrow \zeta$ and $T_n^+ \rightarrow \infty$ (see (18) and Lemma 3.1), since $\hat{\varphi}_n/T_n^+ \rightarrow \hat{\zeta} - \zeta \neq 0$ and $\varphi_n/T_n^+ \rightarrow 0$ as $n \rightarrow \infty$.

(b) The result follows similarly as part (a) using Proposition 4.1(b) and that $Z_{0,n+1} \rightarrow \zeta$ and $T_{0,n+1} \rightarrow \infty$ (see Lemma 3.6 and Remark 3.2(a)). \square

5.2 Error analysis of bias and asymptotic variance computation

We next address how the approximation errors in the computed $\hat{\varphi}_n$ resulting from errors in input $\hat{\zeta}$, as characterized in Proposition 5.1, affect the accuracy of the computed bias $\hat{\beta}$ and asymptotic variance $\hat{\sigma}^2$.

Specifically, we discuss next whether the expressions given in Proposition 4.4 for β_n and σ^2 are also valid to approximately evaluate such quantities, substituting $\hat{\varphi}$ for φ . We would thus obtain approximations $\hat{\beta}$ and $\hat{\sigma}^2$ given by

$$\begin{aligned} \hat{\beta}_0 &= - \sum_{j=0}^{\infty} \bar{P}_j \hat{\varphi}_j, & \hat{\beta}_n &= \hat{\beta}_0 + \sum_{j=0}^{n-1} \hat{\varphi}_j, & n \in \mathbb{N}, \\ \hat{\sigma}^2 &= 2 \sum_{n=0}^{\infty} \lambda_n p_n \hat{\varphi}_n^2. \end{aligned} \tag{49}$$

Yet, Proposition 5.5 below shows that the expressions in (49) will typically be invalid, due to divergence of the infinite series involved. We need a preliminary result. See Remark 3.2 and the expressions for $T_{\infty 0}$ and $T_{\infty p}$ in (34) and (39).

Lemma 5.4.

- (a) $\sum_{n=0}^{\infty} P_n T_n^+ = \infty$;
- (b) $T_{\infty p} = \infty$ if and only if $T_{\infty 0} = \infty$;
- (c) $\frac{\sum_{n=0}^N P_n \varphi_n}{\sum_{n=0}^N P_n T_n^+} \rightarrow 0$ as $N \rightarrow \infty$.

Proof. (a) This part follows from $\sum_{n=0}^{\infty} T_n^+ = \infty$ (see Remark 3.2(a)) and the limit comparison test, using that $P_n \rightarrow 1$ as $n \rightarrow \infty$.

(b) By the limit comparison test, $T_{\infty p} = \sum_{n=0}^{\infty} \bar{P}_n T_n^+ = \infty$ if and only if $T_{\infty 0} = \sum_{n=0}^{\infty} \bar{P}_n / \lambda_n p_n = \infty$, using that, by (23), $\lambda_n p_n T_n^+ = P_n \rightarrow 1$ as $n \rightarrow \infty$.

(c) This part follows from the Stolz–Cesàro theorem (see [42, Theorem 1.22]), since $\sum_{n=0}^N P_n T_n^+ \nearrow \infty$ strictly and $\varphi_n / T_n^+ = \zeta - Z_n \rightarrow 0$ as $n \rightarrow \infty$. \square

In practice, one would approximate the infinite series in Proposition 4.4 by truncation. Thus, let $\beta_{0,N} \triangleq -\sum_{n=0}^N \bar{P}_n \varphi_n$ and $\sigma_N^2 \triangleq 2 \sum_{n=0}^N \lambda_n p_n \varphi_n^2$, which converge to β_0 and σ^2 as $N \rightarrow \infty$. Consider also the computed approximations $\hat{\beta}_{0,N} \triangleq -\sum_{n=0}^N \bar{P}_n \hat{\varphi}_n$ and $\hat{\sigma}_N^2 \triangleq 2 \sum_{n=0}^N \lambda_n p_n \hat{\varphi}_n^2$. The following result gives expressions for the latter quantities and elucidates their asymptotic behavior.

Proposition 5.5. For $N \in \mathbb{N}_0$,

- (a) $\hat{\beta}_{0,N} = \beta_{0,N} - E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N \bar{P}_n T_n^+$;
- (b) if $T_{\infty 0} = \infty$ then $|\hat{\beta}_{0,N}| \rightarrow \infty$; otherwise, $\hat{\beta}_{0,N} \rightarrow \beta_0 - T_{\infty p} E_{\text{abs}}(\hat{\zeta})$;
- (c) $\hat{\sigma}_N^2 = \sigma_N^2 + 4E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N P_n \varphi_n + 2E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=0}^N P_n T_n^+$;
- (d) $\hat{\sigma}_N^2 \rightarrow \infty$ as $N \rightarrow \infty$.

Proof. (a) From Proposition 5.1(a), i.e., $\hat{\varphi}_n = \varphi_n + T_n^+ E_{\text{abs}}(\hat{\zeta})$, we have

$$\hat{\beta}_{0,N} = -\sum_{n=0}^N \bar{P}_n \hat{\varphi}_n = -\sum_{n=0}^N \bar{P}_n \varphi_n - E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N \bar{P}_n T_n^+ = \beta_{0,N} - E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N \bar{P}_n T_n^+.$$

(b) The result follows by letting $N \rightarrow \infty$ in part (a), using Lemma 5.4(b).

(c) Using Proposition 5.1(a) and (23), we obtain

$$\begin{aligned} \hat{\sigma}_N^2 &= 2 \sum_{n=0}^N \lambda_n p_n \hat{\varphi}_n^2 = 2 \sum_{n=0}^N \lambda_n p_n (\varphi_n + E_{\text{abs}}(\hat{\zeta}) T_n^+)^2 \\ &= \sigma_N^2 + 4E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N \lambda_n p_n T_n^+ \varphi_n + 2E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=0}^N \lambda_n p_n (T_n^+)^2 \\ &= \sigma_N^2 + 4E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^N P_n \varphi_n + 2E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=0}^N P_n T_n^+. \end{aligned} \tag{50}$$

(d) The result follows from part (c) and Lemma 5.4(c). \square

6 Approximate numerical solution by backward recurrence

Instead of computing the approximate solution to Poisson's equation through standard forward recurrence, we now consider using *backward recurrence*, which to the author's knowledge has not been previously explored for this model.

This approach is based on the observation that the $f_n(z)$ in §4.1 satisfy the backward recurrence (cf. (45))

$$f_{n-1}(z) = \frac{c_n - z}{\mu_n} + \frac{\lambda_n}{\mu_n} f_n(z), \quad n \in \mathbb{N}. \quad (51)$$

Given an approximation $\hat{\zeta} \neq \zeta$, fix a large integer N and set $\hat{\varphi}_N^N$ to an approximation to φ_N , which we require to satisfy the asymptotic condition

$$\lambda_N p_N \hat{\varphi}_N^N \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (52)$$

One could use an asymptotic expansion to φ_N , or just set $\hat{\varphi}_N^N$ to an arbitrary value, e.g., 0. Note that φ_N does satisfy (52), since, by Corollary 4.2(a), (23) and (18), $\lambda_N p_N \varphi_N = P_N(\zeta - Z_N) \rightarrow 0$. Hence, condition (52) is equivalent to

$$\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (53)$$

Then, compute $\hat{\varphi}_{N-1}^N, \dots, \hat{\varphi}_0^N$ by the backward recurrence (cf. (51))

$$\hat{\varphi}_{n-1}^N = \frac{c_n - \hat{\zeta}}{\mu_n} + \frac{\lambda_n}{\mu_n} \hat{\varphi}_n^N, \quad n = N, N-1, \dots, 1. \quad (54)$$

Note that, setting $z = \zeta$ in (51), the φ_n satisfy

$$\varphi_{n-1} = \frac{c_n - \zeta}{\mu_n} + \frac{\lambda_n}{\mu_n} \varphi_n, \quad n \in \mathbb{N}. \quad (55)$$

As for the approximate relative costs \hat{b}_n^N , they are computed by (cf. (44))

$$\hat{b}_0^N = b_0, \quad \hat{b}_n^N = b_0 + \sum_{j=0}^{n-1} \hat{\varphi}_j^N, \quad n = 1, \dots, N. \quad (56)$$

6.1 Error analysis of backward recurrence scheme

We next turn to analyzing the approximation errors of the computed $\hat{\varphi}_n^N$ and \hat{b}_n^N . We need the following preliminary result.

Lemma 6.1. *For $n = 0, 1, \dots, N$,*

$$\lambda_n p_n f_n(z) = \sum_{j=n+1}^N c_j p_j - z \sum_{j=n+1}^N p_j + \lambda_N p_N f_N(z). \quad (57)$$

Proof. We use backward induction on n . The case $n = N$ follows trivially. Suppose (57) holds for some $1 \leq n \leq N$. Then

$$\begin{aligned} \lambda_{n-1} p_{n-1} f_{n-1}(z) &= \lambda_{n-1} p_{n-1} \frac{c_n - z}{\mu_n} + \lambda_{n-1} p_{n-1} \frac{\lambda_n}{\mu_n} f_n(z) \\ &= p_n (c_n - z) + \lambda_n p_n f_n(z) \\ &= p_n (c_n - z) + \sum_{j=n+1}^N c_j p_j - z \sum_{j=n+1}^N p_j + \lambda_N p_N f_N(z) \\ &= \sum_{j=n}^N c_j p_j - z \sum_{j=n}^N p_j + \lambda_N p_N f_N(z), \end{aligned}$$

using in turn (51), $\lambda_{n-1} p_{n-1} = \mu_n p_n$ and (57), which completes the induction. \square

The next result follows readily from Lemma 6.1 by taking $z = \zeta$ and $z = \hat{\zeta}$.

Corollary 6.2. For $n = 0, 1, \dots, N-1$,

$$(a) \quad \lambda_n p_n \varphi_n = \sum_{j=n+1}^N c_j p_j - \zeta \sum_{j=n+1}^N p_j + \lambda_N p_N \varphi_N;$$

$$(b) \quad \lambda_n p_n \hat{\varphi}_n^N = \sum_{j=n+1}^N c_j p_j - \hat{\zeta} \sum_{j=n+1}^N p_j + \lambda_N p_N \hat{\varphi}_N^N.$$

The next result gives expressions for the errors $E_{\text{abs}}(\hat{\varphi}_n^N)$ and $E_{\text{abs}}(\hat{b}_n^N)$.

Lemma 6.3.

$$(a) \quad \lambda_n p_n E_{\text{abs}}(\hat{\varphi}_n^N) = \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) - (P_N - P_n) E_{\text{abs}}(\hat{\zeta}), \quad 0 \leq n < N;$$

$$(b) \quad E_{\text{abs}}(\hat{b}_n^N) = \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{j=0}^{n-1} \frac{1}{\lambda_j p_j} - E_{\text{abs}}(\hat{\zeta}) \sum_{j=0}^{n-1} \frac{P_N - P_j}{\lambda_j p_j}, \quad 1 \leq n \leq N.$$

Proof. (a) This part follows straightforwardly from Corollary 6.2, by subtracting the expressions given there for $\lambda_n p_n \hat{\varphi}_n^N$ and $\lambda_n p_n \varphi_n$.

(b) Using (44), (56) and part (a), we obtain

$$E_{\text{abs}}(\hat{b}_n^N) = \sum_{j=0}^{n-1} E_{\text{abs}}(\hat{\varphi}_j^N) = \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{j=0}^{n-1} \frac{1}{\lambda_j p_j} - E_{\text{abs}}(\hat{\zeta}) \sum_{j=0}^{n-1} \frac{P_N - P_j}{\lambda_j p_j}.$$

□

The following result is the counterpart of Proposition 5.1 for the backward recurrence scheme. It relates the approximation errors in the computed $\hat{\varphi}_n^N$ and \hat{b}_n^N to those in the input $\hat{\zeta}$, by identifying the corresponding *asymptotic error amplification factors* as $N \rightarrow \infty$.

Proposition 6.4 (Asymptotic error amplification). For each n , as $N \rightarrow \infty$,

$$(a) \quad E_{\text{abs}}(\hat{\varphi}_n^N) \rightarrow -T_{n+1}^- E_{\text{abs}}(\hat{\zeta});$$

$$(b) \quad E_{\text{rel}}(\hat{\varphi}_n^N) \rightarrow -\zeta \frac{T_{n+1}^-}{\varphi_n} E_{\text{rel}}(\hat{\zeta}) = \frac{\zeta}{Z_{n+1}^- - \zeta} E_{\text{rel}}(\hat{\zeta});$$

$$(c) \quad E_{\text{abs}}(\hat{b}_n^N) \rightarrow -T_{n0} E_{\text{abs}}(\hat{\zeta});$$

$$(d) \quad E_{\text{rel}}(\hat{b}_n^N) \rightarrow -\zeta \frac{T_{n0}}{b_n} E_{\text{rel}}(\hat{\zeta}) = -\frac{\zeta}{b_0/T_{n0} + Z_{n0} - \zeta} E_{\text{rel}}(\hat{\zeta}).$$

Proof. (a) From Lemma 6.3(a), (52), (53), Lemma 3.2(a) and (29), we obtain

$$\begin{aligned} E_{\text{abs}}(\hat{\varphi}_n^N) &= \frac{\lambda_N p_N}{\lambda_n p_n} E_{\text{abs}}(\hat{\varphi}_N^N) - \frac{P_N - P_n}{\lambda_n p_n} E_{\text{abs}}(\hat{\zeta}) \\ &\rightarrow -\frac{\bar{P}_n}{\lambda_n p_n} E_{\text{abs}}(\hat{\zeta}) = -T_{n+1}^- E_{\text{abs}}(\hat{\zeta}) \text{ as } N \rightarrow \infty. \end{aligned}$$

(b) This part follows from part (a) and Proposition 4.3(a).

(c) This part follows by letting $N \rightarrow \infty$ in Lemma 6.3(b) and using (33).

(d) The result follows using part (c) and Proposition 4.3(b).

□

Thus, Proposition 6.4(a, c) ensures that, for a fixed state n , the computed $\hat{\varphi}_n^N$ and \hat{b}_n^N are asymptotically related to the exact φ_n and b_n by

$$\hat{\varphi}_n^N \rightarrow \varphi_n - T_{n+1}^- E_{\text{abs}}(\hat{\zeta}) \quad \text{and} \quad \hat{b}_n^N \rightarrow b_n - T_{n0} E_{\text{abs}}(\hat{\zeta}) \quad \text{as } N \rightarrow \infty. \quad (58)$$

How does this compare to the quality of approximations $\hat{\varphi}_n$ and \hat{b}_n computed by forward recursion in §5? The next result shows that the backward recurrence approximations $\hat{\varphi}_n^N$ and \hat{b}_n^N are asymptotically much more accurate, in that $E_{\text{abs}}(\hat{\varphi}_n^N)/E_{\text{abs}}(\hat{\varphi}_n) \approx 0$ and $E_{\text{abs}}(\hat{b}_n^N)/E_{\text{abs}}(\hat{b}_n) \approx 0$ for large n and $N > n$.

Proposition 6.5 (Relative accuracy of backward versus forward recurrence).

$$(a) \quad \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} E_{\text{abs}}(\hat{\varphi}_n^N)/E_{\text{abs}}(\hat{\varphi}_n) = 0;$$

$$(b) \quad \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} E_{\text{abs}}(\hat{b}_n^N)/E_{\text{abs}}(\hat{b}_n) = 0.$$

Proof. (a) From Propositions 5.1(a) and 6.4(a), and the leftmost identity in (29) we have, for each n ,

$$\lim_{N \rightarrow \infty} \frac{E_{\text{abs}}(\hat{\varphi}_n^N)}{E_{\text{abs}}(\hat{\varphi}_n)} = -\frac{T_{n+1}^-}{T_n^+} = -\frac{\bar{P}_n}{P_n}.$$

The result now follows since $\bar{P}_n/P_n \searrow 0$ as $n \rightarrow \infty$.

(b) From Propositions 5.1(c) and 6.4(c), we have, for each n ,

$$\lim_{N \rightarrow \infty} \frac{E_{\text{abs}}(\hat{b}_n^N)}{E_{\text{abs}}(\hat{b}_n)} = -\frac{T_{n0}}{T_{0n}}.$$

The result now follows since $T_{n0}/T_{0n} \searrow 0$ as $n \rightarrow \infty$ by Lemma 3.4(b). \square

7 A mixed forward–backward recurrence scheme

The results in §6.1 show that backward recurrence gives more accurate computed solutions to Poisson’s equation than forward recurrence for large states, whereas it gives less accurate results for small states. This insight leads us to propose a novel mixed *forward–backward recurrence* scheme that keeps the pros and avoids the cons of the pure schemes.

In light of Propositions 5.1 and 6.4, let us define, for a given large integer N and for states $n < N$, the approximate marginal relative costs

$$\tilde{\varphi}_n^N \triangleq \begin{cases} \hat{\varphi}_n, & \text{if } P_n \leq \bar{P}_n \\ \hat{\varphi}_n^N, & \text{otherwise,} \end{cases}$$

noting that $P_n \leq \bar{P}_n$ if and only if $P_n \leq 1/2$, and the approximate relative costs

$$\tilde{b}_n^N \triangleq \begin{cases} \hat{b}_n, & \text{if } T_{0n} \leq T_{n0} \\ \hat{b}_n^N, & \text{otherwise.} \end{cases}$$

Remark 7.1. Since both \bar{P}_n/P_n and T_{n0}/T_{0n} strictly decrease to 0 as $n \rightarrow \infty$ (see Lemma 3.4), letting $m \triangleq \min\{n \geq 0 : \bar{P}_n/P_n < 1\}$ and $M \triangleq \min\{n \geq 1 : T_{n0}/T_{0n} < 1\}$, we have $P_n \leq \bar{P}_n$ if and only if $n < m$, and $T_{0n} \leq T_{n0}$ if and only if $n < M$. Note that it must be the case that $m \leq M$ by Lemma 3.4(a).

Now, letting m and M be as in Remark 7.1, define

$$A_n \triangleq \begin{cases} T_n^+, & \text{if } n < m \\ -T_{n+1}^-, & \text{otherwise,} \end{cases} \quad \text{and} \quad B_n \triangleq \begin{cases} T_{0n}, & \text{if } n < M \\ -T_{n0}, & \text{otherwise.} \end{cases}$$

In the following result, we assume that relative errors are well defined.

Proposition 7.1 (Asymptotic error amplification). *For each n , as $N \rightarrow \infty$,*

- (a) $E_{\text{abs}}(\tilde{\varphi}_n^N) \rightarrow A_n E_{\text{abs}}(\hat{\zeta});$
- (b) $E_{\text{rel}}(\tilde{\varphi}_n^N) \rightarrow \frac{\zeta A_n}{\varphi_n} E_{\text{rel}}(\hat{\zeta});$
- (c) $E_{\text{abs}}(\tilde{b}_n^N) \rightarrow B_n E_{\text{abs}}(\hat{\zeta});$
- (d) $E_{\text{rel}}(\tilde{b}_n^N) \rightarrow \frac{\zeta B_n}{b_n} E_{\text{rel}}(\hat{\zeta}).$

Proof. The result follows directly from Propositions 5.1 and 6.4. \square

The following result shows that forward-backward recurrence is asymptotically more accurate than either of the pure recurrence schemes.

Corollary 7.2. *For each n ,*

- (a) $\lim_{N \rightarrow \infty} |E_{\text{abs}}(\tilde{\varphi}_n^N)| = \min\{|E_{\text{abs}}(\hat{\varphi}_n)|, \lim_{N \rightarrow \infty} |E_{\text{abs}}(\hat{\varphi}_n^N)|\};$
- (b) $\lim_{N \rightarrow \infty} |E_{\text{abs}}(\tilde{b}_n^N)| = \min\{|E_{\text{abs}}(\hat{b}_n)|, \lim_{N \rightarrow \infty} |E_{\text{abs}}(\hat{b}_n^N)|\}.$

Proof. The result follows directly from Propositions 5.1, 6.4 and 7.1. \square

7.1 Error analysis of bias and asymptotic variance computation

Let $\beta_{0,N}$ and σ_N^2 be as in §5.2, which we assume satisfy $\beta_{0,N} \rightarrow \beta_0 = -\sum_{n=0}^{\infty} \bar{P}_n \varphi_n$ and $\sigma_N^2 \rightarrow \sigma^2 = 2 \sum_{n=0}^{\infty} \lambda_n p_n \varphi_n^2$ as $N \rightarrow \infty$ (cf. Proposition 4.4), and consider the computed approximations $\tilde{\beta}_{0,N} \triangleq -\sum_{n=0}^N \bar{P}_n \tilde{\varphi}_n^N$ and $\tilde{\sigma}_N^2 \triangleq 2 \sum_{n=0}^N \lambda_n p_n (\tilde{\varphi}_n^N)^2$. The next result shows that, unlike the $\hat{\beta}_{0,N}$ and $\hat{\sigma}_N^2$ in §5.2, $\tilde{\beta}_{0,N}$ and $\tilde{\sigma}_N^2$ have bounded approximation errors as $N \rightarrow \infty$, for which explicit expressions are given, provided that $T_{p0} < \infty$ (see Remark 3.2(c)) and $\hat{\varphi}_N^N$ satisfies more stringent asymptotic accuracy requirements than (53).

We consider the following conditions on the quality of approximation $\hat{\varphi}_N^N$:

$$\lambda_N p_N T_{N0} E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0 \text{ as } N \rightarrow \infty, \quad (59)$$

$$\lambda_N p_N H_{N0} E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0 \text{ as } N \rightarrow \infty, \quad (60)$$

and

$$\lambda_N p_N \sqrt{T_{0N}} E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (61)$$

Proposition 7.3. *Suppose that $T_{p0} < \infty$. Then,*

- (a) *under (59), $E_{\text{abs}}(\tilde{\beta}_{0,N}) \rightarrow (T_{p0} - T_{m0}) E_{\text{abs}}(\hat{\zeta})$ as $N \rightarrow \infty$;*
- (b) *under (59), (60) and (61),*

$$E_{\text{abs}}(\tilde{\sigma}_N^2) \rightarrow 4\beta_m E_{\text{abs}}(\hat{\zeta}) + 2(T_{p0} + T_{0m} - T_{m0}) E_{\text{abs}}^2(\hat{\zeta}) \text{ as } N \rightarrow \infty.$$

Proof. (a) From Proposition 5.1(a) and Lemma 6.3(a) we have, for $N \geq m+2$,

$$\begin{aligned} E_{\text{abs}}(\tilde{\beta}_{0,N}) &= -\sum_{n=0}^{N-1} \bar{P}_n E_{\text{abs}}(\tilde{\varphi}_n^N) = -\sum_{n=0}^{m-1} \bar{P}_n E_{\text{abs}}(\hat{\varphi}_n) - \sum_{n=m}^{N-1} \bar{P}_n E_{\text{abs}}(\hat{\varphi}_n^N) \\ &= -E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^{m-1} \bar{P}_n T_n^+ - \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{n=m}^{N-1} \frac{\bar{P}_n}{\lambda_n p_n} \\ &\quad + E_{\text{abs}}(\hat{\zeta}) \sum_{n=m}^{N-1} \frac{\bar{P}_n}{\lambda_n p_n} (P_N - P_n). \end{aligned}$$

Now, from the above identity we obtain, as $N \rightarrow \infty$,

$$\begin{aligned} E_{\text{abs}}(\tilde{\beta}_{0,N}) &\rightarrow \left(\sum_{n=m}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n} - \sum_{n=0}^{m-1} \frac{P_n \bar{P}_n}{\lambda_n p_n} \right) E_{\text{abs}}(\hat{\zeta}) \\ &= \left(\sum_{n=0}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n} - \sum_{n=0}^{m-1} \frac{\bar{P}_n}{\lambda_n p_n} \right) E_{\text{abs}}(\hat{\zeta}) = (T_{p0} - T_{m0}) E_{\text{abs}}(\hat{\zeta}), \end{aligned}$$

where we have used (23), (33), (59), and (37).

(b) We have

$$\tilde{\sigma}_N^2 = 2 \sum_{n=0}^{N-1} \lambda_n p_n (\tilde{\varphi}_n^N)^2 = 2 \sum_{n=0}^{m-1} \lambda_n p_n \hat{\varphi}_n^2 + 2 \sum_{n=m}^{N-1} \lambda_n p_n (\hat{\varphi}_n^N)^2.$$

Now, on the one hand, using Proposition 5.1(a) and (23) gives

$$\begin{aligned} \sum_{n=0}^{m-1} \lambda_n p_n \hat{\varphi}_n^2 &= \sum_{n=0}^{m-1} \lambda_n p_n (\varphi_n + T_n^+ E_{\text{abs}}(\hat{\zeta}))^2 \\ &= \sum_{n=0}^{m-1} \lambda_n p_n \varphi_n^2 + 2 E_{\text{abs}}(\hat{\zeta}) \sum_{n=0}^{m-1} P_n \varphi_n + E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=0}^{m-1} \frac{P_n^2}{\lambda_n p_n}. \end{aligned}$$

On the other hand, using Lemma 6.3(a), $\sum_{n=m}^{N-1} \lambda_n p_n (\hat{\varphi}_n^N)^2$ equals

$$\begin{aligned} &\sum_{n=m}^{N-1} \lambda_n p_n \left(\varphi_n + \frac{\lambda_N p_N}{\lambda_n p_n} E_{\text{abs}}(\hat{\varphi}_N^N) - \frac{P_N - P_n}{\lambda_n p_n} E_{\text{abs}}(\hat{\zeta}) \right)^2 \\ &= \sum_{n=m}^{N-1} \lambda_n p_n \varphi_n^2 + 2 \sum_{n=m}^{N-1} \varphi_n (\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) - E_{\text{abs}}(\hat{\zeta}) (P_N - P_n)) \\ &\quad + \sum_{n=m}^{N-1} \frac{1}{\lambda_n p_n} (\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) - (P_N - P_n) E_{\text{abs}}(\hat{\zeta}))^2 \\ &= \sum_{n=m}^{N-1} \lambda_n p_n \varphi_n^2 + 2 \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{n=m}^{N-1} \varphi_n \\ &\quad - 2 E_{\text{abs}}(\hat{\zeta}) \sum_{n=m}^{N-1} (P_N - P_n) \varphi_n + (\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N))^2 \sum_{n=m}^{N-1} \frac{1}{\lambda_n p_n} \\ &\quad - 2 E_{\text{abs}}(\hat{\zeta}) \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{n=m}^{N-1} \frac{P_N - P_n}{\lambda_n p_n} + E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=m}^{N-1} \frac{(P_N - P_n)^2}{\lambda_n p_n}, \end{aligned}$$

and hence we have, as $N \rightarrow \infty$,

$$\sum_{n=m}^{N-1} \lambda_n p_n (\hat{\varphi}_n^N)^2 \rightarrow \sum_{n=m}^{\infty} \lambda_n p_n \varphi_n^2 - 2 E_{\text{abs}}(\hat{\zeta}) \sum_{n=m}^{\infty} \bar{P}_n \varphi_n + E_{\text{abs}}^2(\hat{\zeta}) \sum_{n=m}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n},$$

as the other terms vanish under the assumptions. Thus,

$$\begin{aligned} \lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{n=m}^{N-1} \varphi_n &= \lambda_N p_N (\beta_N - \beta_m) E_{\text{abs}}(\hat{\varphi}_N^N) \\ &\approx \lambda_N p_N (H_{N0} - \zeta T_{N0}) E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0 \text{ as } N \rightarrow \infty, \end{aligned}$$

where we have used Propositions 4.4(a) and 4.3(b), and then (59) and (60).

Also, as $N \rightarrow \infty$,

$$\begin{aligned} (\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N))^2 \sum_{n=m}^{N-1} \frac{1}{\lambda_n p_n} &\approx (\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N))^2 (T_{0N} + T_{N0}) \\ &= \left(\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sqrt{T_{0N} + T_{N0}} \right)^2 \rightarrow 0, \end{aligned}$$

where we have used (32), (33), (59) and (61).

Further, from (33) and (59) we have, as $N \rightarrow \infty$,

$$\lambda_N p_N E_{\text{abs}}(\hat{\varphi}_N^N) \sum_{n=m}^{N-1} \frac{P_N - P_n}{\lambda_n p_n} \approx \lambda_N p_N T_{N0} E_{\text{abs}}(\hat{\varphi}_N^N) \rightarrow 0.$$

It follows from the above that, as $N \rightarrow \infty$, $E_{\text{abs}}(\tilde{\sigma}_N^2)$ converges to

$$\begin{aligned} &4 \left(\sum_{n=0}^{m-1} P_n \varphi_n - \sum_{n=m}^{\infty} \bar{P}_n \varphi_n \right) E_{\text{abs}}(\hat{\zeta}) + 2 \left(\sum_{n=0}^{m-1} \frac{P_n^2}{\lambda_n p_n} + \sum_{n=m}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n} \right) E_{\text{abs}}^2(\hat{\zeta}) \\ &= 4 \left(\sum_{n=0}^{m-1} \varphi_n - \sum_{n=0}^{\infty} \bar{P}_n \varphi_n \right) E_{\text{abs}}(\hat{\zeta}) + 2 \left(\sum_{n=0}^{m-1} \frac{P_n^2 - \bar{P}_n^2}{\lambda_n p_n} + \sum_{n=0}^{\infty} \frac{\bar{P}_n^2}{\lambda_n p_n} \right) E_{\text{abs}}^2(\hat{\zeta}) \\ &= 4\beta_m E_{\text{abs}}(\hat{\zeta}) + 2(T_{0m} - T_{m0} + T_{p0}) E_{\text{abs}}^2(\hat{\zeta}), \end{aligned}$$

using Proposition 4.4(a), (32), (33), $P_n^2 - \bar{P}_n^2 = P_n - \bar{P}_n$, and (37). \square

8 An example

This section illustrates the application of the above results to the M/M/1+M queueing model with deadlines to the end of service considered in §1.

For this model the steady-state probabilities are given by (see [44, p. 89])

$$p_n = \frac{e^{-\kappa}}{\mathcal{P}(\alpha, \kappa)} \frac{\kappa^{\alpha+n}}{\Gamma(\alpha+n+1)}, \quad n \in \mathbb{N}_0, \quad (62)$$

where $\alpha \triangleq \mu/\theta$ and $\kappa \triangleq \lambda/\theta$. Note that $\Gamma(a)$ is the gamma function, $\gamma(a, x) \triangleq \int_0^x t^{a-1} e^{-t} dt$ and $\Gamma(a, x) \triangleq \int_x^\infty t^{a-1} e^{-t} dt$ are the lower and upper incomplete gamma functions, and $\mathcal{P}(a, x) \triangleq \gamma(a, x)/\Gamma(a)$ and $\mathcal{Q}(a, x) \triangleq \Gamma(a, x)/\Gamma(a)$ are the lower and upper normalized gamma functions, respectively. See [45, §8.2].

The mean steady-state cost has the evaluation

$$\zeta = \lambda - \mu(1 - p_0) = \lambda - \mu \left(1 - \frac{\theta \kappa^\alpha e^{-\kappa}}{\mu \gamma(\alpha, \kappa)} \right) = \lambda - \mu + \frac{\theta \kappa^\alpha e^{-\kappa}}{\gamma(\alpha, \kappa)}, \quad (63)$$

and the cumulative steady-state probabilities are given by

$$P_n = 1 - \frac{\mathcal{P}(\alpha+n+1, \kappa)}{\mathcal{P}(\alpha, \kappa)}, \quad n \in \mathbb{N}_0. \quad (64)$$

From (64), expressions for the mean first-passage times and costs considered in the above analyses are readily obtained. From these, and using Corollary 4.2, the following analytical expression for the marginal relative cost is obtained:

$$\varphi_n = 1 - \frac{\gamma(\alpha+n+1, \kappa)}{\gamma(\alpha, \kappa) \kappa^{n+1}} = 1 - \frac{\kappa^{\alpha-1}}{\gamma(\alpha, \kappa)} \int_0^\kappa (t/\kappa)^{\alpha+n} e^{-t} dt, \quad n \in \mathbb{N}_0. \quad (65)$$

Now, it is evident from the rightmost expression in (65) that $\varphi_n \nearrow 1$ as $n \rightarrow \infty$.

Recall that Table 1 in §1 illustrated the explosive numerical instability of the approximate solution to Poisson's equation by standard forward recurrence. We now consider the same instance, but approximate the marginal costs φ_n by the $\tilde{\varphi}_n^N$ calculated by the forward-backward recurrence scheme presented in §6. To test the accuracy of results, the values obtained were compared to the computed values of the φ_n using (65), which we denote by $fl(\varphi_n)$, where $fl(x)$ denotes the floating-point approximation of a number x . Computations were done in Matlab with standard double-precision arithmetic, so the relative error of $fl(x)$ is bounded above by the unit roundoff $u = 2^{-53} \approx 1.1 \times 10^{-16}$. We thus consider that the approximation to ζ computed by Matlab is $\hat{\zeta} = fl(\zeta)$.

We found that taking $N = 42$ and $\hat{\varphi}_N^N = 0$ suffices to obtain extremely accurate approximations to φ_n for the values of n in Table 1 in which inaccuracies were evident, i.e., n larger than 11. Table 2 shows the results. The $\tilde{\varphi}_n^N$ column shows the evaluations of such quantities with 15 significant digits, which precisely match those of the $fl(\varphi_n)$. The column labeled $\zeta A_n/\varphi_n$ evaluates the theoretical asymptotic relative-error amplification factors in Proposition 7.1(b). The results there show that the relative error of $\tilde{\varphi}_n^N$ is, in theory, substantially reduced with respect to that of $\hat{\zeta}$. The next column, labeled $2^{53}|E_{\text{rel}}(\tilde{\varphi}_n^N)|$, evaluates the ratio of the absolute value of the actual relative error of $\tilde{\varphi}_n^N$ (evaluated as $|\tilde{\varphi}_n^N - fl(\varphi_n)|/fl(\varphi_n)$) to that of $\hat{\zeta}$, which is taken equal to the unit roundoff u . The results differ slightly from the theory, but they show that the relative error of $\tilde{\varphi}_n^N$ is near u in the worst case.

The values shown in the last two columns, labeled T_{n+1}^- and T_n^+ , explain the vastly improved accuracy of forward-backward recurrence with respect to forward recurrence. Recall from Proposition 7.1(a) that the absolute approximation error of $\tilde{\varphi}_n^N$, for large n and N , is approximately proportional to T_{n+1}^- , while Proposition 5.1(a) shows that the absolute approximation error of $\hat{\varphi}_n$ is proportional to T_n^+ . These columns show that the T_{n+1}^- are very small and vanish as n grows, while the T_n^+ quickly grow to infinity.

Table 2: Accurate numerical computation of φ_n by forward-backward recurrence.

n	$\tilde{\varphi}_n^N$	$\zeta A_n/\varphi_n$	$2^{53} E_{\text{rel}}(\tilde{\varphi}_n^N) $	T_{n+1}^-	T_n^+
12	0.925174342237504	6.47×10^{-2}	0	0.150	8.4×10^7
13	0.930359089413224	6.00×10^{-2}	0	0.140	7.0×10^8
14	0.934875921107126	5.57×10^{-2}	1.07	0.131	6.2×10^9
15	0.938845492334662	5.21×10^{-2}	0	0.123	5.9×10^{10}
16	0.942361160780650	4.89×10^{-2}	0	0.116	5.9×10^{11}
17	0.945496267896444	4.61×10^{-2}	1.06	0.109	6.2×10^{12}
18	0.948309214061184	4.36×10^{-2}	1.05	0.104	6.9×10^{13}
19	0.950847068147842	4.13×10^{-2}	1.05	0.099	8.4×10^{14}
20	0.953148181463212	3.93×10^{-2}	0	0.094	9.8×10^{15}
21	0.955244111686174	3.75×10^{-2}	0	0.090	1.3×10^{17}
22	0.957161059916347	3.58×10^{-2}	1.04	0.086	1.7×10^{18}
23	0.958920958494403	3.42×10^{-2}	1.04	0.082	2.3×10^{19}
24	0.960542304575400	3.28×10^{-2}	0	0.079	3.4×10^{20}
25	0.962040806065022	3.15×10^{-2}	1.04	0.076	5.0×10^{21}
26	0.963429887334373	3.03×10^{-2}	0	0.073	7.8×10^{22}
27	0.964721088932251	2.92×10^{-2}	1.04	0.071	1.3×10^{24}
28	0.965924386304869	2.82×10^{-2}	0	0.068	2.1×10^{25}
29	0.967048446017873	2.72×10^{-2}	0	0.066	3.6×10^{26}

We now turn to application of the results in §5.2. It is easily shown that the conditions (59)–(61) hold for this model. Using the same N as above we obtain $\tilde{\beta}_{0,N} \approx -0.417521221604055$, which coincides in all digits shown with $fl(\beta_0)$, evaluated as $-\sum_{n=0}^N \bar{P}_n fl(\varphi_n)$. Since $T_{p0} \approx 0.761$ and $T_{10} \approx 1.117$, $T_{p0} - T_{10} \approx -0.356$, and hence, from Proposition 7.3(a) one can argue heuristically that

$$|E_{\text{abs}}(\tilde{\beta}_{0,N})| \approx |(T_{p0} - T_{10})E_{\text{abs}}(\hat{\zeta})| \approx \hat{\zeta} |(T_{p0} - T_{10})E_{\text{rel}}(\hat{\zeta})| < 0.143 u,$$

and hence $|E_{\text{rel}}(\tilde{\beta}_{0,N})| \approx |E_{\text{abs}}(\tilde{\beta}_{0,N})/\tilde{\beta}_{0,N}| < 0.34 u$.

As for the asymptotic variance, we have $\tilde{\sigma}_N^2 \approx 0.589053281069282$, which matches $fl(\sigma^2)$, evaluated as

$2 \sum_{n=0}^N \lambda_n p_n fl(\varphi_n)^2$, in all 15 digits. Furthermore, using Proposition 7.3(b) and $\beta_1 \approx 0.025$, one can argue that

$$|E_{\text{abs}}(\tilde{\sigma}_N^2)| \approx 4|\beta_1 E_{\text{abs}}(\hat{\zeta})| \approx 4\hat{\zeta} |\beta_1 E_{\text{rel}}(\hat{\zeta})| < 0.04 u,$$

and hence $|E_{\text{rel}}(\tilde{\sigma}_N^2)| \approx |E_{\text{abs}}(\tilde{\sigma}_N^2)/\tilde{\sigma}_N^2| < 0.07 u$.

9 Conclusions

While there is extensive work on the numerical instability analysis of linear recurrences, to date there is a dearth of research on its application to recurrences arising in applied probability, such as the Poisson equation considered herein. In this paper, the rich structure of this equation has been exploited to develop an error analysis elucidating the instability phenomenon in its numerical solution, as well as a means of overcoming it. It would be interesting to extend these results beyond the present scope to general continuous-time Markov chains.

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A Groundwork for the proof of Theorem 1.2

This appendix lays the groundwork for the proof of Theorem 1.2 given in §4.2. Recall from §1 that we write $d_n \triangleq \mu_n - \lambda_n$ and $\Delta x_n \triangleq x_n - x_{n-1}$.

We use below the following identities: for $a, b, c, d \in \mathbb{R}$ with $b, d \neq 0$,

$$\frac{a+c}{b+d} = \frac{a}{b} + \frac{d}{b+d} \left(\frac{c}{d} - \frac{a}{b} \right) \quad (66)$$

and

$$\frac{c}{b} + \frac{b}{b-d} \left(\frac{a}{b} - \frac{c}{d} \right) = \frac{a-c}{b-d} = \frac{a}{b} + \frac{d}{b-d} \left(\frac{a}{b} - \frac{c}{d} \right). \quad (67)$$

We will further use the following inequalities: if $b, d > 0$,

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d} \quad \text{if and only if} \quad \frac{a}{b} \leq \frac{c}{d}, \quad (68)$$

which is the classic *mediant inequality*. Further, if $b > d > 0$,

$$\frac{a}{b} \leq \frac{a-c}{b-d} \quad \text{if and only if} \quad \frac{c}{d} \leq \frac{a}{b}. \quad (69)$$

Note that, in both (67) and (69), the leftmost inequalities are strict if and only if the rightmost inequalities are strict.

Lemma A.1. *Under Assumption 1.1(ii.a),*

- (a) Z_n is nondecreasing;
- (b) φ_n is nonnegative.

Proof. (a) Let $n \geq 1$. Then, using (30), (21), (22), and (68), we obtain

$$\Delta Z_n = \frac{\mu_n H_{n-1}^+ + c_n}{\mu_n T_{n-1}^+ + 1} - \frac{H_{n-1}^+}{T_{n-1}^+} = \frac{c_n - Z_{n-1}}{\mu_n T_{n-1}^+ + 1}. \quad (70)$$

It now follows that $\Delta Z_n \geq 0$ since, by (17), (30), and Assumption 1.1(ii.a),

$$Z_{n-1} = \frac{\sum_{j=0}^{n-1} c_j p_j}{\sum_{j=0}^{n-1} p_j} \leq c_n.$$

(b) The result follows by part (a), Corollary 4.2(a) and (18). \square

Lemma A.2.

(a)

$$\begin{aligned} \lambda_n \Delta H_n^+ &= \begin{cases} \Delta c_1 + H_0^+ \Delta d_1, & n = 1 \\ \Delta c_n + \mu_{n-1} \Delta H_{n-1}^+ + H_{n-1}^+ \Delta d_n, & n \geq 2 \end{cases} \\ &= \Delta c_n + (\mu_{n-1} + \Delta d_n) \Delta H_{n-1}^+ + H_{n-2}^+ \Delta d_n, \quad n \geq 2; \end{aligned}$$

(b)

$$\begin{aligned} \lambda_n \Delta T_n^+ &= \begin{cases} T_0 \Delta d_1, & n = 1 \\ \mu_{n-1} \Delta T_{n-1}^+ + T_{n-1}^+ \Delta d_n, & n \geq 2 \end{cases} \\ &= (\mu_{n-1} + \Delta d_n) \Delta T_{n-1}^+ + T_{n-2}^+ \Delta d_n, \quad n \geq 2. \end{aligned}$$

Proof. (a) From (22) we obtain

$$\lambda_n \Delta H_n^+ = c_n + d_n H_{n-1}^+, \quad n \geq 1, \quad (71)$$

For $n = 1$, using (71) and $c_0 + d_0 H_0^+ = 0$ (cf. (22)), we obtain

$$\lambda_1 \Delta H_1^+ = c_1 + d_1 H_0^+ = c_1 + H_0^+ \Delta d_1 + d_0 H_0^+ = \Delta c_1 + H_0^+ \Delta d_1.$$

For $n \geq 2$, using twice (71) yields

$$\begin{aligned} \lambda_n \Delta H_n^+ &= c_n + d_n H_{n-1}^+ = \Delta c_n + H_{n-1}^+ \Delta d_n + c_{n-1} + d_{n-1} H_{n-1}^+ \\ &= \Delta c_n + H_{n-1}^+ \Delta d_n + c_{n-1} + d_{n-1} H_{n-2}^+ + d_{n-1} \Delta H_{n-1}^+ \\ &= \Delta c_n + H_{n-1}^+ \Delta d_n + \lambda_{n-1} \Delta H_{n-1}^+ + d_{n-1} \Delta H_{n-1}^+ \\ &= \Delta c_n + \mu_{n-1} \Delta H_{n-1}^+ + H_{n-1}^+ \Delta d_n. \end{aligned}$$

(b) From (21) we readily obtain

$$\lambda_n \Delta T_n^+ = 1 + d_n T_{n-1}^+, \quad n \geq 1. \quad (72)$$

This part follows as part (a) using (72) and $1 + d_0 T_0 = 0$ (cf. (21)). \square

Lemma A.3. Under Assumption 1.1(i.a, ii.a),

(a) H_n^+ is nondecreasing;

(b) T_n^+ is increasing.

Proof. (a) This part follows immediately by induction using Lemma A.2(a) and Assumption 1.1(i.a, ii.a), which yields $H_{n-1}^+ \geq 0$ and $\Delta H_n^+ \geq 0$ for $n \geq 1$.

(b) This part follows similarly using Lemma A.2(b) and Assumption 1.1(i.a), which yields $\Delta T_n^+ > 0$ for $n \geq 1$. \square

Lemma A.4. Under Assumption 1.1(i.a, ii.a), $Z_n \leq \Delta H_n^+ / \Delta T_n^+$.

Proof. We have, using (67), (69), and Lemmas A.1 and A.3,

$$\frac{\Delta H_n^+}{\Delta T_n^+} = \frac{H_n^+}{T_n^+} + \frac{T_{n-1}^+}{\Delta T_n^+} \left(\frac{H_n^+}{T_n^+} - \frac{H_{n-1}^+}{T_{n-1}^+} \right) \geq \frac{H_n^+}{T_n^+} = Z_n. \quad (73)$$

□

Lemma A.5. Under Assumption 1.1(i, ii),

$$\left(\frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_{n-1}^+}{T_{n-1}^+} \right) \Delta d_n \leq \frac{\Delta c_n}{T_{n-1}^+}, \quad n \geq 1 \text{ (with equality for } n = 1.) \quad (74)$$

Proof. We prove the result by induction. For $n = 1$, using Lemmas A.2 and A.3(b), and Assumption 1.1(i.a), it follows that (74) holds with equality:

$$\frac{\Delta H_1^+}{\Delta T_1^+} = \frac{H_0^+ \Delta d_1 + \Delta c_1}{T_0 \Delta d_1} = \frac{H_0^+}{T_0} + \frac{\Delta c_1}{T_0 \Delta d_1}. \quad (75)$$

Now, suppose that (74) holds for some $n \geq 1$. If $\Delta d_{n+1} = 0$, then it trivially holds for $n + 1$, since $\Delta c_{n+1} \geq 0$ by Assumption 1.1(ii.a).

So consider the case $\Delta d_{n+1} \neq 0$. Then $0 < \Delta d_{n+1} \leq \Delta d_n$ by Assumption 1.1(i). Using Lemmas A.2 and A.3(b), and (66), we can write

$$\begin{aligned} \frac{\Delta H_{n+1}}{\Delta T_{n+1}} &= \frac{H_n^+ \Delta d_{n+1} + \mu_n \Delta H_n^+ + \Delta c_{n+1}}{T_n^+ \Delta d_{n+1} + \mu_n \Delta T_n^+} \\ &= \frac{H_n^+}{T_n^+} + \frac{\mu_n \Delta T_n^+}{\lambda_{n+1} \Delta T_{n+1}} \left(\frac{\mu_n \Delta H_n^+ + \Delta c_{n+1}}{\mu_n \Delta T_n^+} - \frac{H_n^+}{T_n^+} \right). \end{aligned}$$

Hence, we can reformulate the required result that (74) holds for $i + 1$ as

$$\frac{\mu_n \Delta T_n^+}{\lambda_{n+1} \Delta T_{n+1}} \left(\frac{\mu_n \Delta H_n^+ + \Delta c_{n+1}}{\mu_n \Delta T_n^+} - \frac{H_n^+}{T_n^+} \right) \leq \frac{\Delta c_{n+1}}{T_n^+ \Delta d_{n+1}}. \quad (76)$$

In turn, we can reformulate the latter inequality as follows:

$$\frac{T_n^+ \Delta d_{n+1}}{\lambda_{n+1} \Delta T_{n+1}} \left(\mu_n \Delta H_n^+ + \Delta c_{n+1} - \mu_n \Delta T_n^+ \frac{H_n^+}{T_n^+} \right) \leq \Delta c_{n+1},$$

i.e.,

$$\frac{T_n^+ \Delta d_{n+1}}{\lambda_{n+1} \Delta T_{n+1}} \left(\Delta c_{n+1} + \mu_n \Delta T_n^+ \left(\frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_n^+}{T_n^+} \right) \right) \leq \Delta c_{n+1},$$

i.e., using again Lemma A.2(b),

$$\frac{T_n^+ \Delta d_{n+1} \mu_n \Delta T_n^+}{\lambda_{n+1} \Delta T_{n+1}} \left(\frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_n^+}{T_n^+} \right) \leq \frac{\mu_n \Delta T_n^+}{\lambda_{n+1} \Delta T_{n+1}} \Delta c_{n+1},$$

i.e.,

$$\frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_n^+}{T_n^+} \leq \frac{\Delta c_{n+1}}{T_n^+ \Delta d_{n+1}}. \quad (77)$$

To prove (77), we write, using (66),

$$\begin{aligned} \frac{H_n^+}{T_n^+} + \frac{\Delta c_{n+1}}{T_n^+ \Delta d_{n+1}} &= \frac{H_n^+ \Delta d_{n+1} + \Delta c_{n+1}}{T_n^+ \Delta d_{n+1}} \\ &= \frac{(\Delta H_n^+) \Delta d_{n+1} + H_{n-1}^+ \Delta d_{n+1} + \Delta c_{n+1}}{(\Delta T_n^+) \Delta d_{n+1} + T_{n-1}^+ \Delta d_{n+1}} \\ &= \frac{\Delta H_n^+}{\Delta T_n^+} + \frac{T_{n-1}^+}{T_n^+} \left(\frac{H_{n-1}^+}{T_{n-1}^+} + \frac{\Delta c_{n+1}}{T_{n-1}^+ \Delta d_{n+1}} - \frac{\Delta H_n^+}{\Delta T_n^+} \right), \end{aligned}$$

whence

$$\begin{aligned} \frac{H_n^+}{T_n^+} + \frac{\Delta c_{n+1}}{T_n^+ \Delta d_{n+1}} - \frac{\Delta H_n^+}{\Delta T_n^+} &= \frac{T_{n-1}^+}{T_n^+} \left(\frac{H_{n-1}^+}{T_{n-1}^+} + \frac{\Delta c_{n+1}}{T_{n-1}^+ \Delta d_{n+1}} - \frac{\Delta H_n^+}{\Delta T_n^+} \right) \\ &\geq \frac{T_{n-1}^+}{T_n^+} \left(\frac{H_{n-1}^+}{T_{n-1}^+} + \frac{\Delta c_n}{T_{n-1}^+ \Delta d_n} - \frac{\Delta H_n^+}{\Delta T_n^+} \right) \geq 0, \end{aligned}$$

where the first and second inequalities follow by Assumption 1.1(i.b, ii.b) and the induction hypothesis (74), respectively. Therefore, (77) holds, and hence so does (76), which completes the induction proof. \square

Lemma A.6. Under Assumption 1.1(i, ii), $\frac{\Delta H_n^+}{\Delta T_n^+}$ is nondecreasing.

Proof. Fix $n \geq 1$. We can write, using Lemma A.2,

$$\frac{\Delta H_{n+1}}{\Delta T_{n+1}} = \frac{(\mu_n + \Delta d_{n+1})\Delta H_n^+ + \Delta c_{n+1} + H_{n-1}^+ \Delta d_{n+1}}{(\mu_n + \Delta d_{n+1})\Delta T_n^+ + T_{n-1}^+ \Delta d_{n+1}}. \quad (78)$$

We need to distinguish two cases. If $\Delta d_{n+1} = 0$, the latter identity gives

$$\frac{\Delta H_{n+1}}{\Delta T_{n+1}} = \frac{\mu_n \Delta H_n^+ + \Delta c_{n+1}}{\mu_n \Delta T_n^+} = \frac{\Delta H_n^+}{\Delta T_n^+} + \frac{\Delta c_{n+1}}{\mu_n \Delta T_n^+} \geq \frac{\Delta H_n^+}{\Delta T_n^+},$$

as required, using Lemma A.3(b) and Assumption 1.1(ii.a).

If $\Delta d_{n+1} \neq 0$, it must be $\Delta d_{n+1} > 0$ by Assumption 1.1(i.a). We have

$$\begin{aligned} \frac{\Delta H_{n+1}}{\Delta T_{n+1}} &= \frac{\Delta H_n^+}{\Delta T_n^+} + \frac{T_{n-1}^+ \Delta d_{n+1}}{\lambda_{n+1} \Delta T_{n+1}} \left(\frac{\Delta c_{n+1}}{T_{n-1}^+ \Delta d_{n+1}} - \left(\frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_{n-1}^+}{T_{n-1}^+} \right) \right) \\ &\geq \frac{\Delta H_n^+}{\Delta T_n^+} + \frac{T_{n-1}^+ \Delta d_{n+1}}{\lambda_{n+1} \Delta T_{n+1}} \left(\frac{\Delta c_{n+1}}{T_{n-1}^+ \Delta d_{n+1}} - \frac{\Delta c_n}{T_{n-1}^+ \Delta d_n} \right) \geq \frac{\Delta H_n^+}{\Delta T_n^+}, \end{aligned}$$

using (78), (66), Lemmas A.2(b), A.3(b) and A.5, and Assumption 1.1(i.b, ii.b). \square

Lemma A.7. $\frac{\Delta H_n^+}{\Delta T_n^+} \rightarrow \zeta$ as $n \rightarrow \infty$.

Proof. We can write

$$\begin{aligned} \frac{\Delta H_n^+}{\Delta T_n^+} - \frac{H_n^+}{T_n^+} &= \frac{T_{n-1}^+}{\Delta T_n^+} \Delta Z_n = \frac{T_{n-1}^+}{\Delta T_n^+} \frac{1}{1 + \mu_n T_{n-1}^+} (c_n - Z_{n-1}) \\ &= \frac{T_{n-1}^+}{\lambda_n \Delta T_n^+} \frac{c_n - Z_{n-1}}{T_n^+} = \frac{T_{n-1}^+}{1 + d_n T_{n-1}^+} \frac{c_n - Z_{n-1}}{T_n^+}, \end{aligned}$$

where we have used in turn (30), (73), (70), (21), and (72).

Now, we have

$$\frac{T_{n-1}^+}{1 + d_n T_{n-1}^+} = \frac{1}{1/T_{n-1}^+ + d_n} \rightarrow \frac{1}{1/T_\infty^+ + d_\infty} < \infty \text{ as } n \rightarrow \infty, \quad (79)$$

where $0 < T_\infty^+ \leq \infty$ and $0 < d_\infty \leq \infty$ are the limits, possibly infinite, of T_n^+ and d_n as $n \rightarrow \infty$. See Lemma A.3(b) and Assumption 1.1(i). Note that Assumption 1.1(i.a) and ergodicity ensure that $d_\infty > 0$. Otherwise, it would be $\mu_n \leq \lambda_n$ for all n , and the chain would not be ergodic.

Furthermore, using (70) and (18) gives

$$\frac{c_n - Z_{n-1}}{T_n^+} = \Delta Z_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore we obtain, as required,

$$\lim_{n \rightarrow \infty} \frac{\Delta H_n^+}{\Delta T_n^+} = \lim_{n \rightarrow \infty} \frac{H_n^+}{T_n^+} = \zeta.$$

□

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