

On q -ratio CMSV for sparse recovery

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Abstract

Sparse recovery aims to reconstruct an unknown sparse or approximately sparse signal from significantly few noisy incoherent linear measurements. As a kind of computable incoherence measure of the measurement matrix, q -ratio constrained minimal singular values (CMSV) was proposed in Zhou and Yu [1] to derive the performance bounds for sparse recovery. In this paper, we study the geometrical property of the q -ratio CMSV, based on which we establish new sufficient conditions for signal recovery involving both sparsity defect and measurement error. The ℓ_1 -truncated set q -width of the measurement matrix is developed as the geometrical characterization of q -ratio CMSV. In addition, we show that the q -ratio CMSVs of a class of structured random matrices are bounded away from zero with high probability as long as the number of measurements is large enough, therefore satisfy those established sufficient conditions. Overall, our results generalize the results in Zhang and Cheng [2] from $q = 2$ to any $q \in (1, \infty]$ and complement the arguments of q -ratio CMSV from a geometrical view.

Index Terms

Sparse recovery; q -ratio sparsity; q -ratio constrained minimal singular values; ℓ_1 -truncated set q -width.

I. INTRODUCTION

Sparse recovery (for instance, compressive sensing, introduced in [3], [4]) concerns the reconstruction of a sparse or compressible (approximately sparse) signal $x \in \mathbb{R}^N$ from noisy underdetermined linear measurements:

$$y = Ax + e, \quad (1)$$

where $y \in \mathbb{R}^m$ is the measurement vector with $m \ll N$, $A \in \mathbb{R}^{m \times N}$ is the measurement matrix, and $e \in \mathbb{R}^m$ is the noise vector. As is well-known that if the measurement matrix A satisfies the null space property (NSP) or restricted isometry property (RIP), then stable (w.r.t. sparsity defect) and robust (w.r.t.

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measurement error) recovery can be guaranteed via various algorithms, see [5], [6] for a comprehensive view.

However, due to the hardness of verifying NSP and computing restricted isometry constant (RIC) (see [7] and [8]), new kinds of computable quality measures of the measurement matrices are proposed. Specifically, the authors in [9] developed ℓ_1 -constrained minimal singular values (CMSV) $\rho_s(A) = \min_{z \neq 0, \|z\|_1^2 / \|z\|_2^2 \leq s} \frac{\|Az\|_2}{\|z\|_2}$ and derived the ℓ_2 -norm recovery error bounds. Its geometrical property was investigated in [2]. An equivalent quantity called strong restricted eigenvalue was also used in [10] and [11]. Meanwhile, in [12], the authors defined another similar quantity $\omega_\diamond(A, s) = \min_{z \neq 0, \|z\|_1 / \|z\|_\infty \leq s} \frac{\|Az\|_\diamond}{\|z\|_\infty}$ with $\|\cdot\|_\diamond$ denoting a general norm, and obtained the performance bounds on the ℓ_∞ norm of the reconstruction errors. Their corresponding generalizations to block sparse recovery and low-rank matrix recovery were done in [13] and [14]. Very recently, [1] generalized these two quantities to a more general quantity called q -ratio CMSV with any $q \in (1, \infty]$, and established the performance bounds for both ℓ_q norm and ℓ_1 norm of the reconstruction errors.

In this paper, we study the geometrical property of q -ratio CMSV and make the following contributions. First, we define the ℓ_1 -truncated set q -width and connect it with the q -ratio CMSV. Second, by using a newly developed geometrical tool, we establish the sufficient conditions and both ℓ_q and ℓ_1 norm error bounds for the recovery of both exactly sparse and compressible signals via Basis Pursuit (BP) [15] algorithm. Third, we show that a class of structured random matrices satisfy these sufficient conditions, since their q -ratio CMSVs are bounded away from zero with high probability as long as the number of measurements is large enough.

Throughout the paper, for any vector $z \in \mathbb{R}^N$ with entries z_i , we denote the ℓ_0 norm $\|z\|_0 = \sum_{i=1}^N 1\{z_i \neq 0\}$, the ℓ_∞ norm $\|z\|_\infty = \max_{1 \leq i \leq N} |z_i|$ and the ℓ_q norm $\|z\|_q = (\sum_{i=1}^N |z_i|^q)^{1/q}$ for $q \in (0, \infty)$. A signal x is called k -sparse if $\|x\|_0 \leq k$. We let $[N]$ be the set $\{1, 2, \dots, N\}$ and $|S|$ be the cardinality of a set S . We write S^c for the complement $[N] \setminus S$ of a set S in $[N]$. For a vector $x \in \mathbb{R}^N$ and a set $S \subset [N]$, we denote by x_S the vector coincides with x on the indices in S and is extended to zero outside S . For any matrix $A \in \mathbb{R}^{m \times N}$, $\text{Ker} A := \{z \in \mathbb{R}^N : Az = 0\}$. The q -radius of a set $T \subset \mathbb{R}^N$ is defined as $\text{rad}_q(T) = \sup_{t \in T} \|t\|_q$. We denote the unit ball of the ℓ_q -norm in \mathbb{R}^N by B_q^N and its unit sphere by S_q^{N-1} . $\langle \cdot, \cdot \rangle$ denotes the inner product function in \mathbb{R}^N .

II. q -RATIO SPARSITY, q -RATIO CMSV AND ℓ_1 -TRUNCATED SET q -WIDTH

We start with a kind of effective sparsity measure, called q -ratio sparsity, based on which the q -ratio CMSV is developed. It is entropy-based and has many nice properties, see [16], [17] for more detailed

arguments.

Definition 1: ([16], [17]) For any non-zero $z \in \mathbb{R}^N$ and non-negative $q \notin \{0, 1, \infty\}$, the q -ratio sparsity level of z is defined as

$$s_q(z) = \left(\frac{\|z\|_1}{\|z\|_q} \right)^{\frac{q}{q-1}}. \quad (2)$$

The cases of $q \in \{0, 1, \infty\}$ are evaluated as limits: $s_0(z) = \lim_{q \rightarrow 0} s_q(z) = \|z\|_0$, $s_1(z) = \lim_{q \rightarrow 1} s_q(z) = \exp(H_1(\pi(z)))$, $s_\infty(z) = \lim_{q \rightarrow \infty} s_q(z) = \frac{\|z\|_1}{\|z\|_\infty}$. Here $\pi(z) \in \mathbb{R}^N$ with entries $\pi_i(z) = |z_i|/\|z\|_1$ and H_1 is the ordinary Shannon entropy $H_1(\pi(z)) = -\sum_{i=1}^N \pi_i(z) \log \pi_i(z)$.

Now, we are ready to present the definition of q -ratio CMSV, which is a computable quality measure for the measurement matrix. The related theoretical sparse recovery results and computational aspects are referred to [1]. In this paper, we investigate it geometrically instead.

Definition 2: ([1]) For any real number $s \in [1, N]$, $q \in (1, \infty]$ and matrix $A \in \mathbb{R}^{m \times N}$, the q -ratio constrained minimal singular value (CMSV) of A is defined as

$$\rho_{q,s}(A) = \min_{z \neq 0, s_q(z) \leq s} \frac{\|Az\|_2}{\|z\|_q}. \quad (3)$$

To exploit the geometrical property of q -ratio CMSV, we define the ℓ_1 -truncated set q -width of a measurement matrix A as follows.

Definition 3: For any $r > 0$, $q \in (1, \infty]$ and matrix $A \in \mathbb{R}^{m \times N}$, we define the ℓ_1 -truncated set q -width of A as

$$R_{q,r}(A) = \min_{u \in B_1^N \cap rS_q^{N-1}} \|Au\|_2. \quad (4)$$

In what follows, the proposition gives a geometrical characterization of q -ratio CMSV.

Proposition 1: The following equation connects the q -ratio CMSV and the ℓ_1 -truncated set q -width:

$$\rho_{q,r^{-\frac{q}{q-1}}}(A) = r^{-1} \cdot R_{q,r}(A). \quad (5)$$

Proof. Based on the definitions, we have

$$\begin{aligned}
\rho_{q,r^{-\frac{q}{q-1}}}(A) &= \min_{z \neq 0, \|z\|_q \leq r^{-\frac{q}{q-1}}} \frac{\|Az\|_2}{\|z\|_q} \\
&= \min_{z \neq 0, \left\| \frac{z}{\|z\|_q} \right\|_1 \leq r^{-\frac{q}{q-1}}} \left\| A \frac{z}{\|z\|_q} \right\|_2 \\
&= \min_{\|v\|_q=1, \|v\|_1 \leq r^{-\frac{q}{q-1}}} \|Av\|_2 \\
&= \min_{\|u\|_q=r, \|u\|_1 \leq 1} r^{-1} \|Au\|_2 \\
&= r^{-1} \cdot R_{q,r}(A).
\end{aligned}$$

The following proposition is used to exploit the geometrical property of q -ratio CMSV, which is an extension of Lemma 1 in [2] and Proposition 5.1.2 in [18]. Their results correspond to the special case of $\alpha = 0$.

Proposition 2: Let T be a star body with respect to the origin that is a compact subset T of \mathbb{R}^N such that for any $t \in T$, the segment between 0 and t is contained in T , and for any $\alpha \geq 0$, $T_0 = \{h : \|\Phi h\|_2 \leq \alpha\}$, where Φ is an $m \times N$ matrix with row vectors denoted by Y_1, Y_2, \dots, Y_m . If

$$\min_{u \in T \cap rS_q^{N-1}} \sum_{i=1}^m \langle Y_i, u \rangle^2 > \alpha^2, \quad (6)$$

then $\text{rad}_q(T_0 \cap T) < r$.

Proof. Select any vector $t \in T$, and assume that $\|t\|_q \geq r$. Then $z = \frac{r}{\|t\|_q} t \in T \cap rS_q^{N-1}$, which implies that $\|\Phi z\|_2^2 = \sum_{i=1}^m \langle Y_i, z \rangle^2 > \alpha^2$. Therefore, $z \notin T_0 \Rightarrow t = \frac{\|t\|_q}{r} z \notin T_0$. Hence, by using the contrapositive conclusion, for any $t \in T_0 \cap T$, it holds that $\|t\|_q < r$. In other words, we have $\text{rad}_q(T_0 \cap T) = \sup_{t \in T_0 \cap T} \|t\|_q < r$, which completes the proof.

By using Proposition 2 and combining the facts that the unit ball of ℓ_1 norm B_1^N is a star body and that

$$R_{q,r}^2(A) = \min_{u \in B_1^N \cap rS_q^{N-1}} \|Au\|_2^2 = \min_{u \in B_1^N \cap rS_q^{N-1}} \sum_{i=1}^m \langle a_i, u \rangle^2,$$

we obtain the following proposition immediately. As shown later, it plays a crucial role in deriving the robust recovery results involving the measurement error via noisy BP algorithm, which was not considered in [2], [18].

Proposition 3: If $R_{q,r}(A) > \alpha$, then $\text{rad}_q(T_0 \cap B_1^N) < r$. In particular, taking $\alpha = 0$, we have if $R_{q,r}(A) > 0$, then $\text{rad}_q(\text{Ker}A \cap B_1^N) < r$.

III. SPARSE RECOVERY RESULTS

In this section, we first study the recovery conditions for the noise free BP algorithm:

$$\min_{z \in \mathbb{R}^N} \|z\|_1, \text{ s.t. } Az = y, \quad (7)$$

where $y = Ax$. First, it is well-known that when the true signal x is k -sparse, the sufficient and necessary condition for the exact recovery of the noise free BP problem is given by the NSP of order k :

$$\|h_S\|_1 < \|h_{S^c}\|_1, \forall h \in \text{Ker}A \setminus \{0\}, S \subset [N] \text{ with } |S| \leq k,$$

see Theorem 4.5 in [5]. Then, a sufficient condition for exact recovery of k -sparse signal via noise free BP (7), which generalizes Lemma 2 in [2] and Proposition 5.1.1 in [18] from $q = 2$ to any $q \in (1, \infty]$, goes as follows.

Proposition 4: If there exists some $q \in (1, \infty]$ such that $\text{rad}_q(\text{Ker}A \cap B_1^N) < r$ with $r \leq \frac{1}{2}k^{-\frac{q-1}{q}}$, then for every k -sparse signal x , the solution to the noise free BP algorithm (7) is unique and equal to x .

Proof. Let $h \in \text{Ker}A \setminus \{0\}$, and $S \subset [N]$ with $|S| \leq k$. By our assumption $\text{rad}_q(\text{Ker}A \cap B_1^N) < r \leq \frac{1}{2}k^{-\frac{q-1}{q}}$, we have

$$\|h\|_q < r\|h\|_1 \leq \frac{1}{2}k^{-\frac{q-1}{q}}\|h\|_1.$$

Thus,

$$\begin{aligned} \|h_S\|_1 &\leq k^{1-1/q}\|h_S\|_q \leq k^{1-1/q}\|h\|_q \\ &< k^{1-1/q} \cdot \frac{1}{2}k^{-\frac{q-1}{q}}\|h\|_1 = \frac{1}{2}\|h\|_1, \end{aligned}$$

which leads to $\|h_S\|_1 < \|h_{S^c}\|_1$. The proof is completed by using the NSP of order k .

Then a sufficient condition based on q -ratio CMSV is derived as follows.

Theorem 1: (Exactly sparse recovery) If there exists some $q \in (1, \infty]$ such that $\rho_{q,s}(A) > 0$ with $s \geq 2^{\frac{q}{q-1}}k$, then for every k -sparse signal x , the solution to the noise free BP algorithm (7) is unique

and equal to x .

Proof. By using Proposition 1 and the assumption, we have

$$\rho_{q,s}(A) = r^{-1} \cdot R_{q,r}(A) > 0, \quad (8)$$

and $s = r^{-\frac{q}{q-1}} \geq 2^{\frac{q}{q-1}}k$. Thus $R_{q,r}(A) > 0$ and $r \leq (2^{\frac{q}{q-1}}k)^{-\frac{q-1}{q}} = \frac{1}{2}k^{-\frac{q-1}{q}}$. By Proposition 3, we obtain that $\text{rad}_q(\text{Ker}A \cap B_1^N) < r$ with $r \leq \frac{1}{2}k^{-\frac{q-1}{q}}$. Then Proposition 4 implies the desired result.

Remark. The condition that $\rho_{q,s}(A) > 0$ with $s \geq 2^{\frac{q}{q-1}}k$ is equivalent to the condition $\rho_{q,2^{\frac{q}{q-1}}k}(A) > 0$ by using the non-increasing property of $\rho_{q,s}(A)$ with respect to s . Actually this sufficient condition can also obtained as a by product of the error bound results (11) and (12) in [1] for exactly sparse recovery via noisy BP.

Now, we are ready to establish the recovery result for compressible signals via noise free BP, with Proposition 5 in [2] corresponding to the special case of $q = 2$. In the following context, we let the ℓ_1 -error of best k -term approximation of x be $\sigma_k(x)_1 = \inf\{\|x - z\|_1, z \in \mathbb{R}^N \text{ is } k\text{-sparse}\}$ and assume that S is the index set that contains the largest k absolute entries of x so that $\sigma_k(x)_1 = \|x_{S^c}\|_1$.

Theorem 2: (Stable recovery) Let \hat{x} be a minimizer of the noise free BP (7). For any $1 < q \leq \infty$, assume that $\rho_{q,s}(A) > 0$ with $s \geq 4^{\frac{q}{q-1}}k$, then

$$\|\hat{x} - x\|_1 \leq 4\sigma_k(x)_1, \quad (9)$$

$$\|\hat{x} - x\|_q \leq k^{1/q-1}\sigma_k(x)_1. \quad (10)$$

Proof. As discussed in the Proof of Theorem 1, the assumption $\rho_{q,s}(A) > 0$ with $s \geq 4^{\frac{q}{q-1}}k$ implies that $\text{rad}_q(\text{Ker}A \cap B_1^N) < r$ with $r \leq \frac{1}{4}k^{-\frac{q-1}{q}}$. If $h \in \text{Ker}A \setminus \{0\}$, then we have

$$\begin{aligned} \|h_S\|_1 &\leq k^{1-1/q}\|h_S\|_q \leq k^{1-1/q}\|h\|_q \\ &< k^{1-1/q}r\|h\|_1 \leq \frac{1}{4}\|h\|_1, \end{aligned} \quad (11)$$

Therefore, it holds that $\|h_S\|_1 < \frac{1}{3}\|h_{S^c}\|_1$ for $h \in \text{Ker}A \setminus \{0\}$. Then obviously we can obtain

$$\|h_S\|_1 \leq \frac{1}{3}\|h_{S^c}\|_1, \text{ for all } h \in \text{Ker}A. \quad (12)$$

On the other hand, since \hat{x} is a minimizer of the noise free BP (7), by taking $h = \hat{x} - x$, we have

$$\begin{aligned}\|x\|_1 &\geq \|\hat{x}\|_1 = \|x + h\|_1 \\ &= \|x_S + h_S\|_1 + \|x_{S^c} + h_{S^c}\|_1 \\ &\geq \|x_S\|_1 - \|h_S\|_1 - \|x_{S^c}\|_1 + \|h_{S^c}\|_1,\end{aligned}$$

which implies that

$$\|h_{S^c}\|_1 \leq \|h_S\|_1 + 2\|x_{S^c}\|_1 \quad (13)$$

Since $h = \hat{x} - x \in \text{Ker} A$, then combining (12) and (13) leads to $\|h_{S^c}\|_1 \leq 3\|x_{S^c}\|_1$. As a consequence,

$$\begin{aligned}\|\hat{x} - x\|_1 &= \|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \\ &\leq (1 + \frac{1}{3})\|h_{S^c}\|_1 \\ &\leq 4\|x_{S^c}\|_1 = 4\sigma_k(x)_1.\end{aligned} \quad (14)$$

As for (10), if $h = \hat{x} - x \equiv 0$, then it holds trivially. And if $h \in \text{Ker} A \setminus \{0\}$, we have

$$\|h\|_q < r\|h\|_1 \leq \frac{1}{4}k^{-\frac{q-1}{q}} \cdot 4\sigma_k(x)_1 = k^{1/q-1}\sigma_k(x)_1. \quad (15)$$

The proof is completed.

Next, we extend the result to the noisy BP case, which has not been considered yet in the existing literature [2], [18]. That is, we focus on the following problem:

$$\min_{z \in \mathbb{R}^N} \|z\|_1, \quad \text{s.t. } \|Az - y\|_2 \leq \varepsilon, \quad (16)$$

where $y = Ax + e$ with $\|e\|_2 \leq \varepsilon$. The key gradient for the proof is the newly developed Proposition 3 in Section II.

Theorem 3: (Stable and robust recovery) Let \hat{x} be a minimizer of the noisy BP (16). For any $1 < q \leq \infty$, assume that $\rho_{q,s}(A) > 0$ with $s \geq 4^{\frac{q}{q-1}}k$, then

$$\|\hat{x} - x\|_1 \leq 4\sigma_k(x)_1 + \frac{8\varepsilon}{\rho_{q,s}(A)}k^{1-\frac{1}{q}}, \quad (17)$$

$$\|\hat{x} - x\|_q \leq k^{1/q-1}\sigma_k(x)_1 + \frac{2\varepsilon}{\rho_{q,s}(A)}. \quad (18)$$

Proof. We first show (17). We assume $h = \hat{x} - x \neq 0$ and $\|h\|_1 > \frac{8\varepsilon}{\rho_{q,s}(A)}k^{1-\frac{1}{q}}$ with $s \geq 4^{\frac{q}{q-1}}k$, otherwise (17) holds trivially. Then we have $\rho_{q,s}(A) > \frac{8\varepsilon k^{1-\frac{1}{q}}}{\|h\|_1}$. By using Proposition 1 and the assumption, we have $\rho_{q,s}(A) = r^{-1} \cdot R_{q,r}(A) > \frac{8\varepsilon k^{1-\frac{1}{q}}}{\|h\|_1}$ and $s = r^{-\frac{q}{q-1}} \geq 4^{\frac{q}{q-1}}k$. Thus $R_{q,r}(A) > \frac{8r\varepsilon k^{1-\frac{1}{q}}}{\|h\|_1}$ and

$r \leq (4^{\frac{q}{q-1}}k)^{-\frac{q-1}{q}} = \frac{1}{4}k^{\frac{1}{q}-1}$. As a consequence, Proposition 3 implies that $\text{rad}_q(T_0 \cap B_1^N) < r$ with $T_0 = \{z : \|Az\|_2 \leq \alpha = \frac{8r\varepsilon k^{1-\frac{1}{q}}}{\|h\|_1}\}$. And it is easy to verify that $\frac{4rk^{1-\frac{1}{q}}}{\|h\|_1}h \in T_0 \cap B_1^N$ by using the facts that $\|Ah\|_2 = \|(A\hat{x} - y) - (Ax - y)\|_2 \leq \|A\hat{x} - y\|_2 + \|e\|_2 \leq 2\varepsilon$ and $r \leq \frac{1}{4}k^{\frac{1}{q}-1}$. Therefore, we obtain

$$\left\| \frac{4rk^{1-\frac{1}{q}}}{\|h\|_1}h \right\|_q < r \Rightarrow \|h\|_q < \frac{1}{4}k^{\frac{1}{q}-1}\|h\|_1.$$

Furthermore, since \hat{x} is a minimizer of the noisy BP (16), by taking $h = \hat{x} - x$ and using the same argument as that used to obtain (13), we still have

$$\|h_{S^c}\|_1 \leq \|h_S\|_1 + 2\|x_{S^c}\|_1. \quad (19)$$

Hence,

$$\begin{aligned} \|\hat{x} - x\|_1 &= \|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \\ &\leq 2\|h_S\|_1 + 2\|x_{S^c}\|_1 \\ &\leq 2k^{1-\frac{1}{q}}\|h\|_q + 2\|x_{S^c}\|_1 \\ &< \frac{1}{2}\|h\|_1 + 2\|x_{S^c}\|_1 \\ &\Rightarrow \|h\|_1 < 4\|x_{S^c}\|_1 = 4\sigma_k(x)_1, \end{aligned} \quad (20)$$

which completes the proof of (17).

To show (18), we assume that $h = \hat{x} - x \neq 0$ and $\|h\|_q > \frac{2\varepsilon}{\rho_{q,s}(A)}$ with $s \geq 4^{\frac{q}{q-1}}k$, otherwise (18) holds trivially. We first show that under this assumption we have

$$\|h\|_q < \frac{1}{4}k^{\frac{1}{q}-1}\|h\|_1 \text{ for } h = \hat{x} - x \neq 0. \quad (21)$$

We prove it by contradiction. If for $h = \hat{x} - x \neq 0$, $\|h\|_q \geq \frac{1}{4}k^{\frac{1}{q}-1}\|h\|_1$, i.e., $s_q(h) \leq 4^{\frac{q}{q-1}}k$, then, based on the definition, we have

$$\begin{aligned} \rho_{q,s}(A) &= \min_{z \neq 0, s_q(z) \leq s} \frac{\|Az\|_2}{\|z\|_q} \leq \min_{z \neq 0, s_q(z) \leq 4^{\frac{q}{q-1}}k} \frac{\|Az\|_2}{\|z\|_q} \\ &\leq \frac{\|Ah\|_2}{\|h\|_q} \leq \frac{2\varepsilon}{\|h\|_q} \\ &\Rightarrow \|h\|_q \leq \frac{2\varepsilon}{\rho_{q,s}(A)}. \end{aligned} \quad (22)$$

This is a contradiction with our assumption. Thus we have $\|h\|_1 \geq 4k^{1-\frac{1}{q}}\|h\|_q > \frac{8\varepsilon}{\rho_{q,s}(A)}k^{1-\frac{1}{q}}$ with $s \geq 4^{\frac{q}{q-1}}k$. Finally, by adopting the result in proving (17), we can obtain that

$$\|h\|_q \leq \frac{1}{4}k^{\frac{1}{q}-1}\|h\|_1 < \frac{1}{4}k^{\frac{1}{q}-1} \cdot 4\sigma_k(x)_1 = k^{\frac{1}{q}-1}\sigma_k(x)_1, \quad (23)$$

which ends the proof of (18).

Remark. If we go back by combining (18) and (20), we can improve the error bound (17) to

$$\|\hat{x} - x\|_1 \leq 4\sigma_k(x)_1 + \frac{4\varepsilon}{\rho_{q,s}(A)} k^{1-\frac{1}{q}}. \quad (24)$$

Then (18) and (24) are exactly the results (17) and (18) given in [1], while we use a different proof procedure based on the geometrical characterization of q -ratio CMSV.

IV. STRUCTURED RANDOM MATRICES

The theoretical results of q -ratio CMSV for subgaussian random matrices were established in [1]. In this section, we further study the q -ratio CMSV for a class of structured random matrices. More specifically, given an $N \times N$ orthogonal matrix. The measurement matrix is generated by selecting m rows of this matrix at random. Consider a system of vectors ϕ_1, \dots, ϕ_n such that they are orthogonal, and for $1 \leq i \leq N$, $\|\phi_i\|_\infty \leq 1/\sqrt{N}$ and $\|\phi_i\|_2 = M$ where M is a fixed number. Define the random vector a as $a = \phi_i$ with probability $1/N$ and let a_1, \dots, a_N be independent copies of a . Then the $m \times N$ random measurement matrix A is constructed by taking its i -th row to be a_i . This kind of structured random matrices include the class of Fourier random matrices and the class of Hadamard random matrices. Next, we start with two key existing lemmas.

Lemma 1: (Proposition 6 in [2]) Let $m \times N$ random matrix A be defined as above with the number of measurements satisfying

$$m \geq \frac{9\delta^2 C^2 (\log m)^3 \log N}{M^2} \cdot k. \quad (25)$$

Then with probability greater than $1 - C \exp(-cM^2 ms^{-1})$, it has $\rho_{2,s}(A) > 0$ with $s \geq \delta^2 k$, and guarantees exactly sparse signals recovery with $\delta = 2$ and stable (and robust) recovery with $\delta = 4$, where C, c are some constants.

Lemma 2: (Proposition 2 in [1]) If $1 < q_2 \leq q_1 \leq \infty$, then for any real number $1 \leq s \leq N^{\frac{q_1(q_2-1)}{q_2(q_1-1)}}$, we have

$$\rho_{q_1,s}(A) \geq \rho_{q_2, s^{\frac{q_2(q_1-1)}{q_1(q_2-1)}}}(A) \geq s^{-\frac{q_2(q_1-1)}{q_1(q_2-1)}} \rho_{q_1, s^{\frac{q_2(q_1-1)}{q_1(q_2-1)}}}(A).$$

In particular, we have if $1 < q < 2$, then $\rho_{q,s}(A) \geq s^{-1} \rho_{2,s}(A)$, while if $2 \leq q \leq \infty$, then $\rho_{q,s}(A) \geq \rho_{2, s^{\frac{2(q-1)}{q}}}(A)$.

We see that the common assignment for successful recovery is to choose A such that $\rho_{q,s}(A) > 0$ with $s \geq \delta^{\frac{q}{q-1}}k$. Then the following probabilistic statements about q -ratio CMSV holds immediately by combining Lemma 1 and Lemma 2.

Theorem 4: Let $m \times N$ random matrix A be defined as before.

(a) When $1 < q < 2$, if the number of measurements satisfying

$$m \geq \frac{9\delta^{\frac{q}{q-1}}C^2(\log m)^3 \log N}{M^2} \cdot k, \quad (26)$$

then with probability greater than $1 - C \exp(-cM^2ms^{-1})$, it has $\rho_{q,s}(A) > 0$ with $s \geq \delta^{\frac{q}{q-1}}k$, and guarantees exactly sparse signals recovery with $\delta = 2$ and stable (and robust) recovery with $\delta = 4$, where C, c are some constants used in Lemma 1.

(b) When $2 \leq q \leq \infty$, if the number of measurements satisfying

$$m \geq \frac{9\delta^2C^2(\log m)^3 \log N}{M^2} \cdot k^{\frac{2(q-1)}{q}}, \quad (27)$$

then with probability greater than $1 - C \exp(-cM^2ms^{-1})$, it has $\rho_{q,s}(A) > 0$ with $s \geq \delta^{\frac{q}{q-1}}k$, and guarantees exactly sparse signals recovery with $\delta = 2$ and stable (and robust) recovery with $\delta = 4$, where C, c are some constants used in Lemma 1.

V. CONCLUSION

In this paper, we studied the q -ratio CMSV geometrically by introducing the ℓ_1 -truncated set q -width. The geometrical property of q -ratio CMSV was used to derive sufficient conditions and error bounds for both exactly sparse and compressible signal recovery. A class of structured random matrices is shown to satisfy these q -ratio CMSV based sufficient conditions.

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