# A Queuing Model for CPU Functional Unit and Issue Queue Configuration 

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#### Abstract

In a superscalar processor, instructions of various types flow through an execution pipeline, traversing hardware resources which are mostly shared among many different instruction types. A notable exception to shared pipeline resources is the collection of functional units, the hardware that performs specific computations. In a trade-off of cost versus performance, a pipeline designer must decide how many of each type of functional unit to place in a processor's pipeline. In this paper, we model a superscalar processor's issue queue and functional units as a novel queuing network. We treat the issue queue as a finite-sized waiting area and the functional units as servers. In addition to common queuing problems, customers of the network share the queue but wait for specific servers to become ready (e.g., addition instructions wait for adders). Furthermore, the customers in this queue are not necessary ready for service, since instructions may be waiting for operands. In this paper we model a novel queuing network that provides a solution to the expected queue length of each type of instruction. This network and its solution can also be generalized to other problems, notably other resource-allocation issues that arise in superscalar pipelines. Keywords: Modeling of computer architecture, processor architectures, hardware architecture


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## 1. Introduction

In a superscalar processor, instructions of various types flow in parallel through a pipeline of several stages, such as the simple pipeline shown in Figure 1. Instructions start in memory, from where they are fetched, and traverse the pipeline until their operations are completed and their results finalized. Instructions come in various types, such as integer addition, integer multiplication, floating-point addition, load, store, etc. In most stages of the pipeline, all types of instructions share a common space and do not travel along a type-specific data path.

However, as observable in Figure 1, there is one portion of the pipeline in which each specific type of instruction must travel a type-specific path into their respective functional units (FU), where the operation (addition, multiplication, etc.) is completed. At this stage of the pipeline, an issue queue (IQ) acts as a buffer and holds instructions until they can be issued to an FU. To be ready for issuing, an instruction must meet the following criteria:

1. each of this instruction's operands is ready;
2. there is an available FU of this instruction's type.

If either of these criteria is not met in a particular clock cycle, the instruction shall remain in the IQ indefinitely. Given that modern CPUs exploit instructionlevel parallelism, there are typically several of each type of FU to serve multiple instructions of a single type at a time. Clearly, in a system with bandwidth $B$ between the IQ and the FUs, it would be optimal to have the number of each type of FU equal to $B$ to fulfill its potential. In such a configuration, criteria (2) is always satisfied and we should intuitively expect higher throughput than a configuration with any fewer FUs. However, FUs are expensive in many dimensions, including economic cost, chip space, and idle power consumption. So by configuring a system to have ample FUs of each type, we optimize system performance with respect to this parameter but accept a serious tradeoff in cost. On the other hand, using just one FU for each instruction type to minimize costs instead may easily waste some instruction-level parallelism.


Figure 1: A typical 6-stage RISC instruction pipeline. Pipeline stages are numbered and shown with solid lines; data paths are dashed.

To overcome this problem, an FU configuration must be chosen to exploit instruction-level parallelism available while maintaining a reasonable cost. Choosing the best combination of FUs is inherently difficult due to the size of the respective state space. To analyze a possible FU configuration for a particular system, extensive simulation runs have to be performed to determine the performance implications of the chosen configuration. In this paper, we take a novel approach to analyzing an FU configuration by deriving a mathematical model to predict the expected number of instructions of each type that are queued in the pipeline stage immediately preceding the FUs.

Note that, in order to maximize program execution throughput, each type of instruction should maintain a similar flow through the IQ-FU stage. Instead, if one type of instruction is consumed by its FUs slower than others, disregarding the effect of data dependency among instructions, it most likely results from the insufficient number of FUs for this type of instruction. Consequently, more instructions of this type will tend to overwhelm and clog the IQ, and subsequently slow down the processing of other instructions as well due to data dependency among them. That is, the best FU configuration is the one that minimizes the total cost of the FU configuration and the queue length of each instruction type.

The goal of this paper is to model the IQ-FU stage as a novel queuing network and determine, given an FU configuration, derive the expected queue length of each instruction type. Under the general hypothesis of queuing theory,

(a) Standard IQ

(b) Abstracted IQ

Figure 2: Abstraction of an IQ, from the standard representation to a logical rearrangement of slots and bandwidth
the number of instructions of a given type sitting in IQ can be determined if the following parameters are given:

- the number of FUs for this type (and its execution/input latency),
- the prevalence of this type of instructions (i.e. its occurrence frequency), and
- how likely an instruction of this type in IQ is operand-ready to be issued.

Clearly the first parameter is readily available, given as an input parameter for the network. Each of the other two instead is real-time program execution dependent, acquisition of which can be either empirically derived or inferred based on system configuration.

As an illustrating example, a system of two instruction types is first considered, addition and multiplication instruction types. Such an IQ may at any time have any combination of addition and multiplication instructions in it awaiting issue. Of course, the total number of instructions inside the IQ cannot exceed the IQ's capacity, which we shall denote as $N$. One example situation of the aforementioned IQ is shown in Figure 2a,

This illustrates a situation in which there are four addition and two multiplication instructions awaiting issue in the IQ. Instructions arrive along the incoming bus, which has bandwidth $B_{\text {in }}$, and are issued out of the IQ along the


Figure 3: Fully generalizing an IQ into a framework suitable for a queuing model
outgoing bus, which has bandwidth $B_{\text {out }}$. Note that we can abstract this IQ into a logically-equivalent model as shown in Figure 2b,

Here, we logically group similar instructions and depict the bandwidth as being split among the instruction types. If we apply to Figure 2b the constraints that $A d d_{\text {in }}+$ Multiply $_{\text {in }} \leq B_{\text {in }}$ and $A d d_{\text {out }}+$ Multiply $_{\text {out }} \leq B_{\text {out }}$ in any particular clock cycle, we derive an equivalent model as in Figure 2a,

Abstracting the model even further, we may derive a queuing network as shown in Figure 3a Here, we have a queuing network with dynamically-sized queues with the restriction that their total size is less than $N$ (the physical IQ size) and that the bandwidth does not exceed that of the physical IQ. In keeping with queuing-theory convention, we label the distributions of incoming instructions as $\lambda_{t}$ and FUs (service units) as $\mu_{t}$ for each unique instruction type $t$.

Finally, we can generalize the IQ into the queuing network shown in Figure 3b. This generalization depicts a queuing network for an IQ in a system that supports $T$ unique instruction types, hence we have $T$ unique queuing lanes. To keep the model consistent, we must apply to Figure 3b the constraints that each logical queue can be up to length $N$, but also that the sum of all queues cannot exceed $N$.

Given that we are able to model an IQ as a queuing network, we can model its transitions as a Markov chain. A Markov chain describes a sequence of events in which the probability of the next state is determined solely by the current
state. In the case of an IQ, we may consider a Markov chain whose state is the combination of instructions currently in the IQ. For example, we might say that the IQ shown in Figures 2a and 2b is a Markov chain in state $\langle 4,2\rangle$ since there are four addition and two multiplication instructions in the IQ.

A compact way to represent a Markov Chain is with a transition matrix $|P|$ with dimension $|S| \times|S|$ where $S$ is the Markov Chain's state space, and each entry $P_{i, j}$ represents the transition probability of the Markov Chain from state $i$ to state $j$ in one time step. E.g., for a Markov chain with $|S|=3$, we may have

$$
P=\left[\begin{array}{lll}
0.25 & 0.75 & 0.00  \tag{1}\\
0.30 & 0.60 & 0.10 \\
0.00 & 0.80 & 0.20
\end{array}\right]
$$

In addition to being concise, we can extract from a transition matrix important properties of the underlying Markov chain. Of interest to us in this paper is the steady-state distribution, which, for some transition matrix $P$, is a row vector $\pi$ which exhibits the property

$$
\begin{equation*}
\pi P=\pi \tag{2}
\end{equation*}
$$

and, more importantly, each element $\pi_{i}$ holds the steady-state probability of state $i$, i.e., the percentage of time that the Markov chain will be in state $i$ as it transitions indefinitely. From 2, it is clear that $\boldsymbol{\pi}$ is obtained by finding the eigenvector of matrix $P$ whose eigenvalue is equal to 1.0 . Finding the steadystate distribution for $P$ from Equation 1 leads to

$$
\boldsymbol{\pi}=\left[\begin{array}{lll}
0.262 & 0.656 & 0.082 \tag{3}
\end{array}\right]
$$

According to this distribution, this Markov chain will spend $26.2 \%$ of its time in state $1,65.6 \%$ of its time in state 2 , and $8.2 \%$ of its time in state 3 . Intuitively, we may examine $P$ and see that it appears this Markov chain is usually in state 2 , sometimes in state 1 , and rarely in state 3 , which is in agreement with $\boldsymbol{\pi}$. If the Markov chain as in Figure 1 is used to represent the number of instructions
waiting in a queue for a single-instruction-type system, then the steady state result in Equation 3 leads to another measurement of significance, the expected queue length $L$, which is

$$
L=\sum_{i} i \cdot \pi_{i}=0.82
$$

Given that we may derive a transition matrix $P$ for an IQ modelled as a queuing network and then extract the steady-state distribution $\boldsymbol{\pi}$, we then know how often the IQ will be in each state. Recall that a state of the IQ is a combination of instructions of each type. Thus, extracting $\boldsymbol{\pi}$ tells us how often there will be, e.g., 4 addition and 2 multiplication instructions in the IQ, as in Figure 2a. Given this distribution, we can examine how often the IQ will be in an undesirable state, e.g., when it is inundated with instructions of a single type and likely creating a bottleneck in the pipeline. Such a situation may imply that the FU configuration is unbalanced. Additionally, if the steady-state distribution reveals that the IQ is in a full state at a higher than desirable rate, it would be clear that there are not enough FUs to service the instructions. In this paper we develop the theory to model a CPU's FU-IQ configuration as a novel queuing network and extract the steady state distribution.

This paper is structured as follows. Section 2 discusses some work related to this area, including FU configuration optimization and queuing networks. Section 3 describes the general algorithm which will be used in this paper to model an IQ as a Markov chain. Section4details the algorithm while considering a processor which supports a single instruction type, and shows a small example as we proceed through the section. Section 5 details the a processor which supports an arbitrary number of instruction types, and thus the ultimate goal of this paper. Section 6 shows an example of modeling a small IQ with the queuing network and algorithms described in this paper. Section 7 compares the algorithm and model to a simulation of a benchmark program to validate the model. Section 8 uses the presented theory to develop an optimization problem that shows how the proposed model can be applied.

## 2. Related Work

As discussed in Section 1 one could solve the problem of deriving an appropriate FU configuration by simply having many of each type of FU. However, the economic drawbacks of such a scheme typically outweigh the performance gain. One such economic problem is the power consumption of excess FUs, which must consume power even when not in use via static power dissipation. This problem was approached by Rele et al. 1] who described dynamic techniques to shut down FUs by up to $90 \%$ of the time with minimal performance degradation.

Previous research has been done to empirically derive optimal FU configurations via simulation [2]. Such approaches typically only examine the overall throughput of the processor to determine how well a particular configuration fits a processor, whereas our goal is to examine the pipeline effects of each configuration.

Reconfigurable computing $[3,4,[5,6,6,4,9,10,11]$ aims to do away with preconfigured FU pools. In such a scheme, an FPGA-like architecture is customized to the application at hand, and can be altered whenever the instructions to be executed change. For example, if a reconfigurable computing scheme was used to run a program consisting of primarily integer instructions, the majority of the FU-dedicated hardware can be configured to perform integer operations, while only a small portion be configured for floating-point operation. However, if this same architecture were used for a floating-point-intensive program, it can be configured in the opposite manner. Reconfigurable computing has been shown to have great increases in performance and efficiency. However, in such a system, there may not be a priori information ready for reconfiguration in real time and the process overhead of reconfiguration may easily outweigh the gain in performance thus acquired. In addition, the extra hardware overhead required for the system to be reconfigurable may easily diminish its intended benefit in cost efficiency, and the system thus configured may not be at all as efficient compared to an ordinary system.

None of these approaches mentioned above attempts to derive the mathematical abstraction of the FU configuration. Having a way to model the IQ abstractly will allow the derivation of results which are not affected by the parameters or limitations of a simulator. Having a way to choose an optimal FU configuration may be able to optimize cost compared to simpler approaches, and may be able to optimize performance compared to approaches such as reconfigurable computing.

Related work in queuing theory describes the solution of many types of queues [12]. Existing queues and their solutions include many network configurations, including

1. Single-queue, single-server
2. Single-queue, multiple-servers
3. Multiple-queues, multiple-servers
and also can include various constraints such as finite or infinite queuing space, or even no waiting area at all (if no server is ready, an arriving customer leaves immediately). Impatient customers are also considered, which models customers that leave a queue after a certain wait time. Networks of queues, such as a sequence of queues that a customer may traverse, are also common problems and have many applications in computer networks including job scheduling and thread I/O 13]. Typical parameters to current queuing networks include the input process (arrival of customers), the service process (service time distribution), and the number of servers [12]. The typical objectives of a queuing network problem is to determine the wait times for customers and the expected queue length.

However, no documented queuing network is applicable to our goal of modeling a superscalar processor's IQ. Our network must model a single, finite waiting area. Some unique attributes also apply to our network:

1. Customers in the waiting area wait for a specific type of server to be ready
2. Customers in the waiting area are not necessarily ready for service
3. The arrival and service of customers occur during discrete, alternating time periods

The first unique constraint considers that an operation of a certain type must wait for a FU of that type to be ready. The second unique constraint models the fact that instructions can be dispatched to the queue before their operands are ready, rendering them unready for computation despite occupying a slot in the finite waiting area. The third considers that a superscalar processor has discrete pipeline stages: the IQ is partially emptied as instructions are sent to the FUs, then afterwards the IQ is replenished with new instructions. This is in contrast to typical queues, where service events are independent of arrival events. These properties separate our work from existing queuing theory work.

## 3. Constructing the Model

In common queuing networks, a closed-form solution can usually be derived to describe the metrics of the network [12], including estimated wait time, estimated queue length, etc. However, a queuing network such as in Figure 3b has no known derivation. We aim to develop a model that takes as input two variables:

1. The distribution of incoming instruction rates, $\boldsymbol{\lambda}$
2. The probability of readiness for each instruction types, $\rho$
where elements $\lambda_{t} \in \boldsymbol{\lambda}$ and $\rho_{t} \in \boldsymbol{\rho}$ are the incoming distribution and readiness rate for instruction type $t$, respectively. The model is parameterized by the FUIQ configuration, and produces as output the steady state distribution. The model is further composed of two sub-models: a consumption model and an arrival model, with the queuing process modeling each of the CPU clock cycles as a combination of two discrete steps: instructions in IQ consumed by the FUs followed by new instructions arriving and dispatched into the IQ.

Our approach will follow three primary steps and then extract the target result in a fourth step:

1. Construct a transition matrix $C$ to model the consumption of instructions from the IQ to the FUs during the issue stage for each clock cycle
2. Construct a transition matrix $A$ to model the arrival of new instructions intro the IQ during the dispatch stage for each clock cycle
3. Show that the transition matrix on a per-clock-cycle basis is $P=C \times A$, or the matrix product of the arrival and consumption matrices
4. Extract from $P$ the steady-state distribution $\boldsymbol{\pi}$, which tells us the average queue length (IQ occupancy)

Moreover, we are interested in not just the overall IQ occupancy, but the length of each subqueue (see Figure 3b). We set up the model such that each state contains information about each type of instruction in the IQ, and therefore reveals the length of each subqueue for every instruction type.

For the sake of simplicity, we start by considering an IQ in a system which has just one instruction type and derive the transition matrix of the IQ. Building on the model of the single instruction type, we then derive a model for a system with an arbitrary number of instructions in the same manner. As it turns out, constructing a model for an arbitrary number of instructions can be done simply by using the joint probabilities that are derived for the model with just a single instruction type.

## 4. Modelling a Single Instruction Type

We begin by considering a system which has only one type of instruction and, therefore, one type of FU. To model this system as a Markov chain, we first describe that state space. Since the state space of an IQ Markov chain model is the set of all possible combinations of IQ occupancy, the state space for a system with a single instruction type can be described as

$$
\begin{equation*}
\mathcal{S}=\left\{s_{i} \mid i \in \mathbb{Z} \cap[0, N]\right\} \tag{4}
\end{equation*}
$$

where $N$ is the size of the IQ and $s_{i}$ represents the state when the IQ has exactly $i$ instructions currently residing in it. That is, the state space consists of one state for each possible number of instructions up to $N$, the size of the IQ.

To complete the Markov chain model, the state space must be augmented with transition probabilities between each pair of states. The transitions we are interested in are the changes in the number of instructions in the IQ after each clock cycle. In each clock cycle of the system, two events affect the number of instructions in the IQ:

1. Issuing to the FUs
2. Dispatching to the IQ

For each of these two events, we derive one transition matrix. We first derive a transition matrix $C$ which represents the consumption of instructions out of the IQ and into the FUs in any clock cycle. We then derive a transition matrix $A$ which represents the flow of instructions into the IQ in any clock cycle. We then show that by multiplying these matrices together, we may derive the complete transition matrix of the IQ; that is, $P=C \times A$.

### 4.1. Single-Type Consumption Model

In this section, we build the consumption matrix $C$ to represent the consumption of instructions from the IQ to the FUs. During the issue stage of the pipeline, instructions are sent from the IQ to appropriate FUs. There are two factors which limit the number of instructions issued:

1. The number of ready instructions in the IQ
2. The number of available FUs

Instructions may be dispatched into the IQ before their operands are ready, but instructions may not be issued to FUs until all operands are ready. During the time an instruction is dispatched with operands not ready, it cannot be considered ready for issue but will continue to occupy an IQ entry until it is. Therefore, in any state the number of instructions which are candidates for issue is probabilistically distributed based on how many we expect to have ready operands.

Consider transitioning from state $s_{i}$ to state $s_{j}$. First, if $j>i$ we should expect that the transition probability to be zero during the issue stage of the
pipeline since we cannot issue a negative number of instructions. Therefore, we have

$$
p_{s_{i}, s_{j}}=0 \quad \text { if } \quad i-j<0
$$

where $p_{s_{i}, s_{j}}$ denotes the corresponding transition probability.
Second, it is clear that we cannot issue more instructions that we have FUs available since each FU can process at most one instruction at a time. Let $\mathcal{F}$ denote the number of FUs available. Then we have

$$
p_{s_{i}, s_{j}}=0 \quad \text { if } \quad i-j>\mathcal{F}
$$

Third, we consider the case that $i-j=\mathcal{F}$, or that every FU gets issued an instruction. This is simply the case that at least $\mathcal{F}$ instructions in the IQ have their operands ready. Let $\rho$ represent the probability of an instruction being ready for issue. Given that there are currently $i$ instructions in the IQ, the transition probability under this situation is the probability when at least $\mathcal{F}$ are ready, which can be calculated as

$$
p_{s_{i}, s_{j}}=\sum_{k=\mathcal{F}}^{i}\binom{i}{k} \rho^{k}(1-\rho)^{i-k} \quad \text { if } \quad i-j=\mathcal{F}
$$

Fourth, and finally, we consider the case that $0 \leq i-j<\mathcal{F}$, or that fewer instructions were issued than there are FUs. This case reduces to the probability that exactly $i-j$ instructions in the IQ are ready for issue given that there are $i$ instructions in the IQ. Therefore we have the corresponding transition probability as

$$
p_{s_{i}, s_{j}}=\binom{i}{i-j} \rho^{i-j}(1-\rho)^{j} \quad \text { if } \quad 0 \leq i-j<\mathcal{F}
$$

Note that for $j=0$, i.e., when every instruction in the IQ was issued, this equation reduces to

$$
\begin{aligned}
p_{s_{i}, s_{j}} & =\binom{i}{i-j} \rho^{i-j}(1-\rho)^{j} \\
& =\binom{i}{i-0} \rho^{i-0}(1-\rho)^{0} \\
& =\rho^{i}
\end{aligned}
$$

which represents the probability of every instruction in the IQ being ready for issue.

To summarize the transition probabilities from state $s_{i}$ to state $s_{j}$ during issue, we have

$$
p_{s_{i}, s_{j}}= \begin{cases}0 & i-j<0  \tag{5}\\ \binom{i}{i-j} \rho^{i-j}(1-\rho)^{j} & 0 \leq i-j<\mathcal{F} \\ \sum_{k=\mathcal{F}}^{i}\binom{i}{k} \rho^{k}(1-\rho)^{i-k} & i-j=\mathcal{F} \\ 0 & i-j>\mathcal{F}\end{cases}
$$

We may then use the cases in Equation 5 to populate an issue-stage transition matrix (consumption matrix) for the issue Markov model of an IQ.

We conclude this subsection with a small example. Suppose that we have a system with an IQ of three entries and two FUs serving instruction of only one instruction type which has its readiness probability $\rho$ equal to 0.6. Again with the state space $\mathcal{S}=\left\{s_{0}, s_{1}, s_{2}, s_{3}\right\}$, the corresponding consumption matrix of dimension $|\mathcal{S}| \times|\mathcal{S}|$ can be constructed and populated with Equation 5.

$$
C=\left[\begin{array}{llll}
1.000 & 0.000 & 0.000 & 0.000  \tag{6}\\
0.600 & 0.400 & 0.000 & 0.000 \\
0.360 & 0.480 & 0.160 & 0.000 \\
0.000 & 0.648 & 0.288 & 0.064
\end{array}\right]
$$

in which $C_{i, j}$ denotes the transitioning probability from $i$ instructions to $j$ instructions in IQ. Note that this matrix is necessarily lower-triangular, since we are describing the probabilities of instructions being consumed and therefore we must transition to a state no greater than the current one. Also interesting to note is that in the lower-left triangle of this matrix we may also get zero-valued entries. These entries represent states which cannot be transitioned to due to insufficient functional units. For example, $C_{3,0}=0$ because $3-0>2$. That is, since we have only 2 FUs , it is impossible to issue three instructions in one clock cycle (and thus, it is impossible to transition from state $s_{3}$ to state $s_{0}$ in one issue cycle).

### 4.2. Single-Type Arrival Model

In this section, we derive the transition matrix $A$ to represent the arrival of new instructions into the IQ. During the dispatch stage of the pipeline, instructions are sent from the ROB into the IQ. The number of instructions which are dispatched depends primarily on two factors: the number of instructions in the ROB awaiting dispatch and the number of empty IQ entries. Suppose that the number of instructions in the ROB in a random clock cycle (i.e., the arrival rate of instructions) follows some probability mass function (pmf) $a(x)$. Neglecting the effect of limited bandwidth and given $a(x)$, we may build an instruction arrival model for any state of the IQ.

Consider a system in state $s_{i}$. For any state $s_{j}$, we shall build a Markov chain to model the probability of transitioning from state $s_{i}$ to state $s_{j}$ during the dispatch stage of the pipeline by partitioning the range of values that $i$ and $j$ may take on. First, we have that the IQ cannot lose instructions during the dispatch stage (at least 0 must arrive). Therefore we have

$$
p_{s_{i}, s_{j}}=0 \quad \text { if } \quad j<i
$$

Second, consider the case that $i \leq j<N$. This transition implies that $j-i$ instructions have entered the IQ, but that the IQ is not yet completely full. Therefore, in this case, the probability of transitioning to state $j$ is the case that exactly $j-i$ instructions are ready for dispatched. Since the arrival of instructions is modeled by $a(x)$, we have

$$
p_{s_{i}, s_{j}}=a(j-i) \quad \text { if } \quad i \leq j<N
$$

Third, and finally, we have the case that $j=N$. In this case, transitioning from state $s_{i}$ to state $s_{j}$ implies that enough instructions have arrived to fill up the IQ. Since this is the last case in a partition of the probability space, the likelihood of this case is the complement of the total likelihood of the previous cases. That is,

$$
\begin{equation*}
p_{s_{i}, s_{j}}=p_{s_{i}, s_{N}}=1-\sum_{\substack{s_{k} \in \mathcal{S} \\ k \neq N}} a(k-i) \tag{7}
\end{equation*}
$$



Figure 4: Example arrival distributions, starting from an empty IQ and a partially-occupied IQ, with 32 IQ entries
where $a(k-i)$ is defined by the previous cases.
To summarize the transition probabilities from state $s_{i}$ to state $s_{j}$ during dispatch, we have

$$
p_{s_{i}, s_{j}}= \begin{cases}0 & j<i  \tag{8}\\ a(j-i) & i \leq j<N \\ 1-\sum_{\substack{s_{k} \in \mathcal{S} \\ k \neq N}} a(k-i) & j=N\end{cases}
$$

Suppose that the instruction's incoming rate into the IQ follows the Poisson distribution with mean 1, i.e.,

$$
\begin{equation*}
a(k)=\frac{1^{k} e^{-1}}{k!}=\frac{1}{e k!} \tag{9}
\end{equation*}
$$

where $a(k)$ is the probability of $k$ instructions being ready for dispatch (arrival) to the IQ in any clock cycle. An example of this equation when applied to a queue of 32 entries is depicted in Figure 4a for $i=0$ (when the queue is initially empty). Another example in Figure 4b displays a case for $i=17$ (when the queue has initially 17 occupied slots). One can easily see the relationship between the function in Figure 4a (denoted as $a_{0}(j)$ ) and Figure 4b (denoted as $\left.a_{17}(j)\right)$, where the value of each point in the latter one is exactly shifted 17 positions to the right from the respective point in the former one; that is,

$$
a_{17}(j)=a_{0}(j-17) \quad \text { if } \quad j \geq 17
$$

except for the last point which corresponds to the summation of all the rest; that is,

$$
a_{17}(32)=\sum_{j=15}^{32} a_{0}(j)
$$

Or, in a generalized form,

$$
a_{i}(N)=\sum_{j=N-i}^{N} a_{0}(j)
$$

We may then use the cases in Equation 8 to populate a dispatch-stage transition matrix (arrival matrix) for the arrival Markov model of an IQ. A small example is adopted here to illustrate how this matrix is established. Assume an IQ with three entries and one instruction type, and the arrival rate function $a(x)$ follows Equation 9 Next, we consider the state space of this model: $\mathcal{S}=\{\langle 0\rangle,\langle 1\rangle,\langle 2\rangle,\langle 3\rangle\}$. The arrival matrix of a dimension $|\mathcal{S}| \times|\mathcal{S}|$ can then be constructed with one row and column per state. Following Equation 8 we derive the arrival matrix as

$$
A=\left[\begin{array}{llll}
0.368 & 0.368 & 0.184 & 0.080  \tag{10}\\
0.000 & 0.368 & 0.368 & 0.264 \\
0.000 & 0.000 & 0.368 & 0.632 \\
0.000 & 0.000 & 0.000 & 1.000
\end{array}\right]
$$

in which $A_{i, j}$ denotes the transition probability from $i$ instructions to $j$ instructions in IQ. From to the "shift nature" aforementioned, in the arrival matrix we see that the arrival distribution in each row is exactly one entry shifted right from the next row above, except for the last entry also including the last one from the row above. That is

$$
\begin{align*}
A_{i, j} & =A_{i-1, j-1} \forall i, j \text { s.t. } 1 \leq i \leq j \leq N-1  \tag{11}\\
A_{i, N} & =A_{i-1, N-1}+A_{i-1, N} \tag{12}
\end{align*}
$$

### 4.3. Single-Type Complete model

In Section 4.2 and 4.1 we partitioned the IQ's behavior during each clock cycle into two parts: a model $C$ to represent the consumption of instructions
from the IQ into the FUs and a model $A$ to represent the arrival of new instructions to the IQ. We now model the change in the IQ between clock cycles by combining these two models. That is, we use the arrival model and the consumption model to exhaustively describe the behavior of the IQ in a single Markov model. Consider an IQ in state $s_{i}$ at the beginning of a clock cycle, that is, there are currently $i$ instructions currently in the IQ. During the next clock cycle, the IQ undergoes two changes: the issue stage and, subsequently, the dispatch stage. We can therefore denote the change that the IQ undergoes in one complete clock cycle as

$$
\begin{equation*}
s_{i} \xrightarrow{C} \text { post-issue stage } \xrightarrow{A} \text { post-dispatch stage } \tag{13}
\end{equation*}
$$

where $\xrightarrow{C}$ and $\xrightarrow{A}$ represent the transitions of the issue (consumption of instructions) stage and dispatch (arrival of instructions) stage, respectively. Now consider some arbitrary end state $s_{j}$. To determine the probability of transitioning from state $s_{i}$ to state $s_{j}$ during one clock cycle, we must determine the likelihood that, after the dispatch stage, the IQ is in state $s_{j}$. Therefore, we can rewrite Equation 13 as

$$
\begin{equation*}
s_{i} \xrightarrow{C} \text { post-issue stage } \xrightarrow{A} s_{j} \tag{14}
\end{equation*}
$$

Furthermore, we note that the post-issue stage is an arbitrary state and denote it as $s_{m}$ and write

$$
\begin{equation*}
s_{i} \xrightarrow{C} s_{m} \xrightarrow{A} s_{j} \tag{15}
\end{equation*}
$$

and we have that the probability of transitioning from state $s_{i}$ to state $s_{j}$ in one clock cycle is the probability of Equation 15 for all possibilities of the arbitrary
$s_{m}$. Since each value of $s_{m} \in \mathcal{S}$ creates an independent event, we may say that

$$
\begin{align*}
p_{s_{i}, s_{j}} & =\sum_{s_{m} \in \mathcal{S}} p\left(s_{i} \xrightarrow{C} s_{m} \xrightarrow{A} s_{j}\right)  \tag{16}\\
& =\sum_{s_{m} \in \mathcal{S}} p\left(\left(s_{i} \xrightarrow{C} s_{m}\right) \wedge\left(s_{m} \xrightarrow{A} s_{j}\right)\right)  \tag{17}\\
& =\sum_{s_{m} \in \mathcal{S}} p\left(s_{i} \xrightarrow{C} s_{m}\right) \cdot p\left(s_{m} \xrightarrow{A} s_{j}\right)  \tag{18}\\
& =(C \times A)_{i, j}  \tag{19}\\
& =P_{i, j} \tag{20}
\end{align*}
$$

where $C$ is the consumption matrix derived in Section 4.1 and $A$ is the arrival matrix derived in Section [.2. Equation 17 is derived from Equation 16 by observing that the sequence of transitioning can be separated by the discrete nature of the pipeline. Then we infer Equation 18by noting that the arrival stage is independent of the consumption stage. Then we observe that by summing over all intermediate states $s_{m}$, we are computing the dot product of the $i^{t h}$ row of the consumption matrix and $j^{\text {th }}$ column of the arrival matrix, which corresponds to the value of $P_{i, j}$ where $P=C \times A$. Therefore, we can build the complete, per-clock-cycle transition matrix for the IQ by taking the matrix product of the issue and dispatch matrices.

Continuing with the example shown at the end of Subsections 4.2 and 4.1 with $\rho=0.6$ and $\lambda=\operatorname{Poi}(1)=\frac{1}{e k!}$, we conclude this subsection with construction of the per-clock-cycle transition matrix $P$ from the matrices derived in the preceding subsections. Thus

$$
P=C \times A=\left[\begin{array}{llll}
0.368 & 0.368 & 0.184 & 0.080  \tag{21}\\
0.221 & 0.368 & 0.258 & 0.154 \\
0.132 & 0.309 & 0.302 & 0.257 \\
0.000 & 0.238 & 0.344 & 0.417
\end{array}\right]
$$

Finally, extracting the steady-state distribution from $|P|$ gives us

$$
\boldsymbol{\pi}=\left[\begin{array}{llll}
0.171 & 0.323 & 0.278 & 0.231 \tag{22}
\end{array}\right]
$$

which indicated that this three-entry IQ would be empty during $17.1 \%$ of clock cycles and be full $23.1 \%$ of clock cycles, and the expected queue length is:

$$
L=\sum_{i} i \cdot \pi_{i}=1.572
$$

## 5. Modeling an Arbitrary Number of Instruction Types

In this section, we consider a system which has an arbitrarily-sized set of unique instruction types $\left\{I_{1}, I_{2}, \ldots I_{T}\right\}$, that is, a system with $T$ different types of instructions for some $T \in \mathbb{N}$, where each $I_{t}$ denotes a type of instruction. A system such as this one resembles a realistic CPU where instructions come in the form in integer addition, integer multiplication, floating-point addition, etc., and each instruction type must be issued to a corresponding type of FU. We take the simple probability models derived in Section 4 and show that by using joint probabilities of each instruction type we may develop a model of FU configuration for an arbitrary system.

### 5.1. State Space and State Labeling

In this subsection we describe the state space and assignment of a state to an IQ which may contain any number of arbitrary instruction types. For a system with $T$ instruction types, we can describe the IQ during an arbitrary clock cycle by the number of each type of instruction currently inside the IQ. Let $n_{t}$ denote the number of instructions of type $I_{t}$ currently residing in the queue. We may then view this enumeration as a set of $T$-tuples $\left\langle n_{1}, n_{2}, \ldots, n_{T}\right\rangle$ and use this $T$-tuple as a state. Collecting all possible $T$-tuples, we have that the state space of such a system is

$$
\mathcal{S}=\left\{\left\langle n_{1}, n_{2}, \ldots, n_{T}\right\rangle \mid n_{t} \in \mathbb{N}, \sum_{t} n_{t} \leq N\right\}
$$

where $T$ is the number of unique instruction types and $N$ is the size of the IQ. Lastly, we observe that, with a simple theory of permutation and combination,
the number of states for a system with $T$ instruction types and an IQ size of $N$ is

$$
|\mathcal{S}|=\frac{(T+N)!}{N!T!}
$$

which indicates that the state space of this problem is of a size of exponential of $T$ and $N$, a size simply too large for any attempt to perform exhaustive simulations to cover them all.

### 5.2. Matrix representation

To create a model for a multi-instruction-type system, we again will use transition matrices to describe the use of the IQ and its transitions during both the arrival and consumption stages of the pipeline. In the previous case of a single instruction type, we used matrices of size of $|\mathcal{S}| \times|\mathcal{S}|$, where $\mathcal{S}$ is the state space and $|\mathcal{S}|=N+1$. Each state $s$ was assigned its own row and column, and the entry $P_{i, j}$ was the probability of the transition $s_{i} \rightarrow s_{j}$. A similar approach is adopted here for the multi-instruction-type case, with the only exception being that states are $N$-tuples rather than integers. That is, for each model, we will use a transition matrix of size $|\mathcal{S}| \times|\mathcal{S}|$. Each state $\left\langle n_{1}, n_{2}, \ldots, n_{T}\right\rangle$ will be mapped to an index and assigned one row and one column in the matrix. Each element of the matrix $P_{\left\langle i_{1}, i_{2}, \ldots, i_{T}\right\rangle,\left\langle j_{1}, j_{2}, \ldots, j_{T}\right\rangle}$ holds the probability of the transition $\left\langle i_{1}, i_{2}, \ldots, i_{T}\right\rangle \rightarrow\left\langle j_{1}, j_{2}, \ldots, j_{T}\right\rangle$.

### 5.3. Multiple-Type Consumption model

In this section, we derive a model for the issue stage of the pipeline and build a transition matrix $C$ to describe the probabilities of IQ transition during the issue stage. In Section 4.1, we derived Equation 5 to compute the likelihood of transition of a single type of instruction in the IQ during the issue stage, i.e., the expected number of instructions that will be issued in one clock cycle based on the state of the IQ. We use this equation as a marginal probability and show that the transition probability in a system with an arbitrary number of instruction types is the joint probability of all instruction types as described in Equation 5

Suppose we have an IQ in some state $s_{i}=\left\langle i_{1}, i_{2}, \ldots, i_{T}\right\rangle$ and consider some arbitrary state $s_{j}=\left\langle j_{1}, j_{2}, \ldots, j_{T}\right\rangle$. For the transition $s_{i} \xrightarrow{C} s_{j}$ to occur during the issue stage, it must be the case that, for each instruction type $t$, we issued exactly $i_{t}-j_{t}$ instructions. This case is exactly the joint probability of the independent events governing the issue of each instruction type. Therefore, disregarding bandwidth constraints, we have that

$$
\begin{equation*}
p\left(s_{i} \xrightarrow{C} s_{j}\right)=\prod_{t=1}^{T} p_{i_{t}, j_{t}}^{[t]} \tag{23}
\end{equation*}
$$

where $p_{i_{t}, j_{t}}^{[t]}$ is defined by Equation 5 for the single-instruction-type consumption model for instruction type $I_{t}$. Note that each different instruction type may come with a different readiness parameter $\rho$ for its consumption model. For example, for a three-instruction-type system with instruction types from $\left\{I_{1}, I_{2}, I_{3}\right\}$, the probability of transitioning from state $\langle 4,5,3\rangle$ to state $\langle 3,1,2\rangle$ is

$$
p(\langle 4,5,3\rangle \xrightarrow{C}\langle 3,1,2\rangle)=p_{4,3}^{[1]} \cdot p_{5,1}^{[2]} \cdot p_{3,2}^{[3]}
$$

Represented is matrix form to derive the final consumption matrix $C$ for this three-instruction-type system, the entry at the row assigned for state $\langle 4,5,3\rangle$ and the column assigned for state $\langle 3,1,2\rangle$ can then be derived as

$$
C_{\langle 4,5,3\rangle,\langle 3,1,2\rangle}=C_{4,3}^{[1]} \cdot C_{5,1}^{[2]} \cdot C_{3,2}^{[3]}
$$

in which $C_{i_{t}, j_{t}}^{[t]}$ represents the respective consumption matrix for instruction type $I_{t}$. Thus

$$
\begin{equation*}
C_{s_{i}, s_{j}}=\prod_{t=1}^{T} C_{i_{t}, j_{t}}^{[t]} \tag{24}
\end{equation*}
$$

### 5.4. Multiple-Type Arrival model

In this section, an arrival model and matrix $A$ for the dispatch stage of the multi-instruction-type pipeline is to be derived by using the joint probability of the Equation 8 derived for the single-instruction-type system.

Similar to state space definition in the consumption model, for the transition $s_{i} \xrightarrow{A} s_{j}$ to occur during the dispatch stage, there are two conditions to consider:

1. $\sum_{t} j_{t}<N$
2. $\sum_{t} j_{t}=N$
where $N$ is the size of the IQ. The former case represents when there is still room in the IQ at the end of dispatch and the model is quite simple; the latter is a more special case where the IQ becomes full during dispatch and the model is more complicated, as we shall see.

### 5.4.1. Case 1: $\sum_{t} j_{t}<N$

In this case the IQ is not full at the end of the dispatch stage. This implies all arriving instructions (or all yet-to-dispatch instructions in the ROB) were able to be dispatched into the IQ, as evidenced by the leftover space after allocation. Therefore, in this case, the probability of the transition $s_{i} \xrightarrow{A} s_{j}$ is simply the joint probability that, for each instruction type $t$, exactly $j_{t}$ instructions were ready for dispatch. In other words, we have that for $\sum_{t} j_{t}<N$

$$
\begin{equation*}
p\left(s_{i} \xrightarrow{A} s_{j} \mid \sum_{t} j_{t}<N\right)=\prod_{t=1}^{T} p_{i_{t}, j_{t}}^{[t]} \tag{25}
\end{equation*}
$$

where $p_{i_{t}, j_{t}}^{[t]}$ is defined by Equation 8 for the arrival probability for the instruction type $I_{t}$, which is due to that each different instruction type may come with a different arrival distribution $(\lambda)$ for its arrival model. Similar to the consumption model derivation process which results in Equation 24 we have for all the entries of this arrival matrix $A$ that satisfy $\sum_{t} j_{t}<N$,

$$
\begin{equation*}
A_{s_{i}, s_{j}}=\prod_{t=1}^{T} A_{i_{t}, j_{t}}^{[t]} \tag{26}
\end{equation*}
$$

### 5.4.2. Case 2: $\sum_{t} j_{t}=N$

In this second case the IQ becomes full at the end of the dispatch stage. In the single-type case, this transition is from state $i$ to state $N$ where $N$ is the IQ size. Therefore, this was the probably that at least $N-i$ instructions were ready to arrive. However, in the case of multiple instruction types, there are multiple boundary states. For example, with a 3 -entry IQ and 2 instruction types, we
have instead a total of four boundary states $\langle 0,3\rangle,\langle 1,2\rangle,\langle 2,1\rangle$ and $\langle 3,0\rangle$. When the IQ becomes full, we must partition the probability between these boundary states.

Take the example of a transition from state $\langle 0,0\rangle$ to a boundary state $\langle 1,2\rangle$. The probability of this transition are from all events in which at least three instructions arrive, and that the combination of the first three instructions dispatched is one of type $I_{1}$ and two of type $I_{2}$.

We can compute the probability of at least three instructions arriving by subtracting from 1 the probability that fewer than three instructions arrive using previously-derived equations. To do so, we can simply iterate over the states whose IQ occupancy is less than $N$ (which is 3 in this case) and sum all their probabilities. If we denote the probability of at least three instructions arriving as $p\left(3^{+}\right)$and have $\mathcal{S}^{*}$ denote the set of all states that are not boundary states, we have that

$$
\begin{equation*}
p\left(3^{+}\right)=1-\sum_{s_{k} \in \mathcal{S}^{*}} p\left(\langle 0,0\rangle \xrightarrow{A} s_{k}\right) \tag{27}
\end{equation*}
$$

Note that each of the term $p\left(\langle 0,0\rangle \xrightarrow{A} s_{k}\right)$ in Equation 27 can be easily obtained by using the formula in Equation 25 since each state $s_{k}$ satisfy the condition of case 1.
$p\left(3^{+}\right)$thus derived includes all the probabilities reaching each of the four boundary states. Another probability needs to be factored in to produce the intended probability of reaching only state $\langle 1,2\rangle$. That is, one of the first three incoming instructions have to be type $I_{1}$, and the other two type $I_{2}$. Since order of the three does not matter, the total number of possible combination is $\frac{(1+2)!}{1!2!}=3$, each with a probability of

$$
p_{I_{1}} \cdot p_{I_{2}}^{2}
$$

where $p_{I_{1}}$ and $p_{I_{2}}$ each denotes the probability that an arbitrary incoming instruction is type $I_{1}$ and $I_{2}$, respectively. Note that these values can be easily
derived from the instruction-specific arrival distributions $\lambda_{1}$ and $\lambda_{2}$. Namely,

$$
p_{I_{1}}=\frac{\mu_{1}}{\mu_{1}+\mu_{2}}, \quad p_{I_{2}}=\frac{\mu_{2}}{\mu_{1}+\mu_{2}}
$$

where each $\mu_{i}$ represents the mean of the incoming distribution $\lambda_{i}$. This can be generalized for a $T$-instruction-type system, that is, the probability that the next incoming instruction being type $I_{t}$ is

$$
p_{I_{t}}=\frac{\mu_{t}}{\sum_{k=1}^{T} \mu_{k}}
$$

For this example, the transitioning probability from state $\langle 0,0\rangle$ to a boundary state $\langle 1,2\rangle$ is then

$$
\begin{equation*}
\left[1-\sum_{s_{x} \in \mathcal{S}^{*}} p\left(\langle 0,0\rangle \xrightarrow{A} s_{x}\right)\right] \frac{(1+2)!}{1!2!} p_{I_{1}}^{1} p_{I_{2}}^{2} \tag{28}
\end{equation*}
$$

Note that the term $\frac{(1+2)!}{1!2!}$ in this equation is the total number of permutations, which we also could denote more formally as the multinomial coefficient of a set of cardinality 3 with element multiplicities of 2 and 1 . When extended to a general case, the total number of permutations becomes $\left(\sum n_{t}\right)!/ \prod\left(n_{t}!\right)$ where $n_{t}$ represents the number of instructions that should come in during the current arrival stage.

We may then generalize this equation from some starting state $s_{i}=\left\langle i_{1}, i_{2}, \ldots, i_{T}\right\rangle$ to some boundary state $s_{j}=\left\langle j_{1}, j_{2}, \ldots, j_{T}\right\rangle$ as

$$
\begin{align*}
& p\left(s_{i} \xrightarrow{A} s_{j} \mid \sum_{t} j_{t}=N\right) \\
& =\left[1-\sum_{s_{x} \in \mathcal{S}^{*}} p\left(s_{i} \xrightarrow{A} s_{x}\right)\right] \frac{\left(\sum_{t} n_{t}\right)!}{\prod\left(n_{t}!\right)}\left[\prod_{t=1}^{T} p_{I_{t}}^{n_{t}}\right] \tag{29}
\end{align*}
$$

where $n_{t}=j_{t}-i_{t}$, that is, the number of instructions of type $I_{t}$ that should be dispatched, and $\mathcal{S}^{*}$ is the set of all non-boundary states, i.e.,

$$
\mathcal{S}^{*}=\mathcal{S} \backslash\left\{\left\langle n_{1}, n_{2}, \ldots, n_{T}\right\rangle \mid \sum_{i} n_{i}=N\right\}
$$

### 5.5. Multiple-Type Complete Model

Once the consumption model and arrive model are both derived, the complete model for multi-instruction-type case can be easily obtained following exactly the same process presented in Section 4.3 for the single-type-instruction case, namely, by multiplying the consumption and arrival matrices to produce one transition matrix which completely describes the IQ's behavior. That is, $P=C \times A$.

## 6. Example

In this section, we walk through an example implementation of the proposed algorithm using a simple, small system. Suppose that we have a system with a three-entry IQ $(N=3)$ and two instruction types $\left\{I_{1}, I_{2}\right\}(T=2)$. Furthermore, we may enumerate the state space as

$$
\mathcal{S}=\{\langle 0,0\rangle,\langle 0,1\rangle,\langle 0,2\rangle,\langle 0,3\rangle,\langle 1,0\rangle,\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,0\rangle,\langle 2,1\rangle,\langle 3,0\rangle\}
$$

where $\left\langle n_{1}, n_{2}\right\rangle$ represents the state during which the IQ holds $n_{1}$ and $n_{2}$ instructions of types $I_{1}$ and $I_{2}$, respectively.

As aforementioned, the proposed algorithm requires three inputs to produce a model with the following assumptions:

- the incoming rates of each instruction type: $\lambda_{1}=\operatorname{Poi}(1.5), \lambda_{2}=\operatorname{Poi}(1.0)$
- the probability of instructions being ready for issue: $\rho_{1}=0.75$ for $I_{1}$ and $\rho_{2}=0.8$ for $I_{2} ;$
- the FU configuration: two FUs for $I_{1}$ and one FU for $I_{2}$.

To simplify the state labeling, and enable the mapping of states to matrix entries, we define a bijective mapping from the states of the model to a subset of integers as

$$
M=\left(\begin{array}{ccc}
\langle 0,0\rangle & \langle 0,1\rangle & \langle 0,2\rangle \\
0 & \underset{2}{2} & \langle 0,3\rangle \\
3
\end{array} \underset{4}{\langle 1,0\rangle} \underset{5}{\langle 1,1\rangle} \underset{6}{\langle 1,2\rangle}\langle\underset{7}{\langle 2,0\rangle} \underset{8}{\langle 2,1\rangle} \underset{9}{\langle 3,0\rangle})\right.
$$


(a) Consumption

$$
\left[\begin{array}{llllllllll}
0.08 & 0.08 & 0.04 & 0.03 & 0.12 & 0.12 & 0.13 & 0.09 & 0.20 & 0.10 \\
0.00 & 0.08 & 0.08 & 0.11 & 0.00 & 0.12 & 0.34 & 0.00 & 0.26 & 0.00 \\
0.00 & 0.00 & 0.08 & 0.37 & 0.00 & 0.00 & 0.55 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.08 & 0.08 & 0.11 & 0.12 & 0.34 & 0.26 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.08 & 0.37 & 0.00 & 0.55 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.08 & 0.37 & 0.55 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00
\end{array}\right]
$$

(b) Arrival

Figure 5: Join consumption and arrival matrices

For each of subsequently-generated consumption, arrival and the final transition matrices, $C, A$, and $P$, it should be understood that each matrix is of size $10 \times 10$ with their row and column indices ranging from 0 to 9 . Specifically, the entry $P_{i, j}$ represents the transition probability from state $i$ to state $j$ as mapped under this enumeration. For example, the entry $P_{3,6}$ will hold the transition probability from state $\langle 0,3\rangle$ to state $\langle 1,2\rangle$.

The consumption matrix $C$ is built by first finding, for each instruction type $I_{t}$, the respective consumption matrix $C^{[t]}$, and then following Equation 23 and Equation 24 with a joint combination process. The joint consumption transition matrix derived is shown in Figure 5a.

As aforementioned, construction of the arrival matrix $A$ is divided to two steps. For all the entries that belong to Case 1 in which the end state is not one of boundary states, similar to the consumption model, it is derived by first finding, for each instruction type $I_{t}$, the respective consumption matrix $A^{[t]}$, and then following Equation 25 and Equation 26 with a joint combination process. For all the other entries (Case 2) that the end state is one of the boundary states, Equations 29 is used. Also note that the number of instructions for any instruction type cannot decrease, that is, the corresponding entry will be 0 :

$$
\exists k\left(i_{k}>j_{k}\right) \Rightarrow A_{s_{i}, s_{j}}=0
$$

where $s_{i}=\left\langle i_{1}, i_{2}, \ldots, i_{T}\right\rangle$ and $s_{j}=\left\langle j_{1}, j_{2}, \ldots, j_{T}\right\rangle$. The final arrival transition matrix $A$ is as shown in Figure 5b,

Multiplying the consumption and arrival matrices results in the per-clockcycle model $P$ as shown in Figure 6

$$
P=C \times A=\left[\begin{array}{cccccccccc}
0.08 & 0.08 & 0.04 & 0.03 & 0.12 & 0.12 & 0.13 & 0.09 & 0.20 & 0.10 \\
0.07 & 0.08 & 0.05 & 0.05 & 0.10 & 0.12 & 0.17 & 0.07 & 0.21 & 0.08 \\
0.00 & 0.08 & 0.08 & 0.12 & 0.00 & 0.12 & 0.35 & 0.00 & 0.25 & 0.00 \\
0.00 & 0.00 & 0.08 & 0.37 & 0.00 & 0.00 & 0.55 & 0.00 & 0.00 & 0.00 \\
0.06 & 0.06 & 0.03 & 0.02 & 0.11 & 0.11 & 0.13 & 0.10 & 0.23 & 0.14 \\
0.05 & 0.06 & 0.04 & 0.03 & 0.09 & 0.11 & 0.17 & 0.08 & 0.25 & 0.11 \\
0.00 & 0.06 & 0.06 & 0.09 & 0.00 & 0.11 & 0.36 & 0.00 & 0.32 & 0.00 \\
0.05 & 0.05 & 0.02 & 0.02 & 0.10 & 0.10 & 0.12 & 0.10 & 0.26 & 0.19 \\
0.04 & 0.05 & 0.03 & 0.03 & 0.08 & 0.10 & 0.16 & 0.08 & 0.29 & 0.15 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.07 & 0.07 & 0.10 & 0.12 & 0.34 & 0.31
\end{array}\right]
$$

Figure 6: Complete per-clock-cycle transition matrix $P$ for the example.

Finally, we extract the Left Perron vector $\boldsymbol{\pi}$ from $P$ to find that the steadystate distribution, expressed as a percentage of clock cycles, is

$$
\boldsymbol{\pi}=\left[\begin{array}{lllll}
.026 & .047 & .040 .066 & .058 & .096 \\
\text {. } 228 & .060 & .267 & .110
\end{array}\right]
$$

which in turn represents the probability that one of the ten possible states occurs. Re-inserting the state labels and displaying $\boldsymbol{\pi}$ as a mapping, we get

$$
\boldsymbol{\pi}=\left(\begin{array}{cccccccccc}
\langle 0,0\rangle & \langle 0,1\rangle & \langle 0,2\rangle\langle 0,3\rangle & \langle 1,0\rangle & \langle 1,1\rangle & \langle 1,2\rangle & \langle 2,0\rangle\langle 2,1\rangle & \langle 3,0\rangle \\
.026 & .047 & .040 & .066 & .058 & .096 & .228 & .060 & .267 & .110
\end{array}\right)
$$

If we neatly rearrange these probabilities into an organized state space we obtain the result as shown in Figure 7 On the horizontal axis is the number of instructions of type $I_{1}$, and on the depth axis is the number of instructions of type $I_{2}$. Vertically, we show the estimated percentage of time the IQ will spend in each state.

From the results of the Left Perron vector, we see that the IQ is full during approximately $67 \%$ of clock cycles, when $n_{1}+n_{2}=3$ for all states $\left\langle n_{1}, n_{2}\right\rangle$. One conclusion we can make is that adding more FUs to the system would alleviate the bottleneck in the IQ since it is full relatively often. In addition, the IQ usage is slightly dominated by instructions of type $I_{1}$, which is because the arrival rate of type $I_{1}\left(\mu_{1}=1.5\right)$ is higher than one of type $I_{2}\left(\mu_{2}=1\right)$, coupled with the


Figure 7: Graphical Representation of The Results
fact that the readiness parameter for type $I_{1}(\rho=0.75)$ is lower than type $I_{2}$ ( $\rho=0.8$ ). Furthermore, we may derive the average IQ usage of each instruction type by summing over the steady state. For some instruction type $I_{t}$, we have that $L_{t}$, the average number of instructions of type $t$ in the IQ, can be derived as

$$
\begin{equation*}
L_{t}=\sum_{s \in \mathcal{S}} n_{t} \boldsymbol{\pi}[s] \tag{30}
\end{equation*}
$$

where $n_{t}$ is the number of instruction of type $I_{t}$ in state $s$. That is, we sum the $t^{t h}$ index of each state $s$ multiplied by the probability of being in state $s$. We have then

$$
L_{1}=1.366, L_{2}=1.144
$$

The overall queue length can be derived using

$$
\begin{equation*}
L=\sum_{s \in \mathcal{S}}\left(\sum_{t=1}^{T} n_{t}\right) \boldsymbol{\pi}[s] \tag{31}
\end{equation*}
$$

or simply

$$
L=\sum_{t=1}^{T} L_{t}
$$

which leads to $L=2.51$, a relatively tight IQ utilization.
The most important analysis is to determine if the employment of various functional units leads to the best throughput. Adding more FUs for sure will
increase throughput, but it remains to be determined that which type of FU should be invested in order to obtain the most increase in throughput. Note that for all instructions of various types to flow through IQ in a "congruent" manner, that is, no instructions of any type is more "stagnant" than the others, which would subsequently cause instructions of other types to slow down as well due to data dependency among instructions across types. Such a congruency can be measured by how closely matched the ratios of arrival rate and expected queue length among all instruction types are. Let this "flow ratio" be denoted as $R_{t}$ for instruction type $I_{t}$, and

$$
R_{t}=\frac{\mu_{t}}{L_{t}}
$$

where $\mu_{t}$ is the mean of the incoming distribution for type $t$. The higher $R_{t}$ is, the better the instructions of type $I_{t}$ are flowing through the IQ. Thus we have

$$
R_{1}=1.098, R_{2}=0.874
$$

which shows that the FUs for type $I_{2}$ are more utilized than those for type $I_{1}$. This may call for an increase of the FU number for $I_{2}$ to see if the balance between $R_{1}$ and $R_{2}$ can be further improved. Or strictly theoretically speaking, the number of functional units for type $I_{2}$ should be increased by a factor of 1.256 (i.e., $R_{1} / R_{2}$ ) in order to match the flow ratio between the two types of instructions.

## 7. Model Validation

In this section we apply the proposed technique and validate its accuracy. That is, we intend to experiment with how well the proposed model predicts the IQ usage under a given configuration. We will experiment using the SimpleScalar [14] simulator with the SPEC CPU 2006 benchmark suite [15]. The benchmarks are compiled with the Alpha instruction set architecture. The simulator configuration used in shown in Table 1 .

Generally, queuing theory solutions are parameterized and leave it to the application to infer the parameters to use the model. Since the proposed model

Table 1: Parameters for the Simulation Environment Used in the Experiments

| Parameter | Value |
| :---: | :---: |
| Decode/Issue/Commit Width | $8 / 8 / 8$ |
| LSQ/ROB/IQ/RF/WB Size | $48 / 128 / 16 / 256 / 16$ |
| Mem. Ports/IALU/FP Mult./FP ALU | $2 / 4 / 1 / 1$ |

Predicted vs. Actual Queue lengths, Two Instruction Types


Figure 8: Modeled Predicted Queue Lengths vs. Actual Average Queue Lengths for Various SPEC CPU 2006 Floating-point Benchmarks
is a queuing theory model and therefore requires parameters a priori (the values for $\lambda$ and $\rho$ ), we validate the model by first running simulations and gathering statistics. We then use the statistics to infer the parameters to the model. We then query the model with the parameters to predict the queue lengths. We then compare the models prediction vs. the simulation results to validate how closely the model matches the IQ-FU instruction flow. We focus on the two most utilized instruction types from each benchmark. We find that in practice in this environment, an individual benchmark uses no more than two instruction types in meaningful quantities. Figure 8 shows the results from the simulations comparing the predicted queue lengths against the actual values. Overall, a very respectable average prediction error rate of $20.1 \%$ is observed. The discrepancies most likely are due to the parameters empirically acquired, namely the $\lambda$ 's and $\rho$ 's which may vary from benchmark to benchmark by a somewhat significant


Figure 9: Results of creating an optimization problem, where the cost of some FU configuration is the sum of the queue lengths and the cost of the FU configuration
margin.

## 8. As An Optimization Problem

In this section we use the proposed model to create an optimization problem that can be applied in practice. Suppose we have some FU configuration $\mathcal{F}$, where $\left|\mathcal{F}_{t}\right|$ represents the number of FUs for instruction type $t$. Using the proposed model, we can determine the mean length of the queue for each instruction type $t$ under the configuration $\mathcal{F}$ as $\ell(t, \mathcal{F})$ via Equation 30. Since a lower mean queue length indicates faster throughput, $\ell(t, \mathcal{F})$ becomes more costly as it rises.

Similarly, each FU of type $t$ costs $c\left(\mathcal{F}_{t}\right)$, and therefore having $\left|\mathcal{F}_{t}\right|$ FUs costs $\left|\mathcal{F}_{t}\right| \cdot c\left(\mathcal{F}_{t}\right)$. Therefore we can say that the total cost of configuration $\mathcal{F}$ is

$$
\begin{equation*}
C(\mathcal{F})=\sum_{t}\left(\ell(t, \mathcal{F})+c\left(\mathcal{F}_{t}\right) \cdot\left|\mathcal{F}_{t}\right|\right) \tag{32}
\end{equation*}
$$

For $T$ instruction types, equation 32 results in a convex, $T$-dimensional curve embedded in a $(T+1)$-dimensional space, and minimizing for some given cost parameters would result in an optimal $\mathcal{F}$.

For example, say we have $t_{0}$ as multiplication and $t_{1}$ as division. Since dividers require a more complicated circuit and more power than multiplication,
we can assign costs $c\left(\mathcal{F}_{0}\right)=1$ and $c\left(\mathcal{F}_{1}\right)=3$. If we have an IQ of size 16 , incoming distributions $\lambda_{0}=\operatorname{Poi}(2), \lambda_{1}=\operatorname{Poi}(1)$, and readiness rates $\rho_{0}=\rho_{1}=$ 0.8 , we can derive the cost curve across the state space of $\mathcal{F}$. Examining Figure 9a, we see that under these parameters the optimal FU configuration is $\left|\mathcal{F}_{0}\right|=$ $3,\left|\mathcal{F}_{1}\right|=2$ (i.e., 3 multipliers and 2 dividers) with a cost of 13.4 . If we double the costs of the FUs and fix the remaining parameters, we see in Figure 9b that the optimal FU configuration drops to $\left|\mathcal{F}_{0}\right|=2,\left|\mathcal{F}_{1}\right|=1$ for a cost of 21.5 , as we should expect since the only parameters to have changed are the costs of the FUs which are now more expensive relative to queue length.

Lastly, note that since 32 is convex, in practice we can use hill climbing to quickly find an optimal configuration starting from a configuration of 1 FU of each type and ascending through the state space.

## 9. Conclusion

In this paper we described a novel queuing network comprised of a shared waiting area with variably-sized pools of unique server types. Customers in the waiting area wait for availability of a specific type of server and cannot be served by any other type. Furthermore, a customer in the waiting area is not necessarily ready for service. We showed how to fit a superscalar processor's IQFU configuration to such a model with instructions as customers and functional units as servers. We showed how to solve for the expected queue length of each instruction type, revealing the efficiency of an arbitrary FU configuration and the IQ usage of every instruction type. We also showed how the proposed theory can be applied in practice as an optimization problem.

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