

# Feedback boundary stabilization of wave equations with interior delay

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**Abstract.** In this paper we consider a boundary stabilization problem for the wave equation with interior delay. We prove an exponential stability result under some Lions geometric condition. The proof of the main result is based on an identity with multipliers that allows to obtain a uniform decay estimate for a suitable Lyapunov functional.

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## 1 Introduction

We study the boundary stabilization of a wave equation in an open bounded domain  $\Omega$  of  $\mathbb{R}^n$ ,  $n \geq 2$ . We denote by  $\partial\Omega$  the boundary of  $\Omega$  and we assume that  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0, \Gamma_1$  are closed subsets of  $\partial\Omega$  with  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Moreover we assume  $meas\Gamma_0 > 0$ . The system is given by :

$$u_{tt}(x, t) - \Delta u(x, t) + au_t(x, t - \tau) = 0, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

$$u(x, t) = 0, \quad x \in \Gamma_0, \quad t > 0 \quad (1.2)$$

$$\frac{\partial u}{\partial \nu}(x, t) = -ku_t(x, t), \quad x \in \Gamma_1, \quad t > 0 \quad (1.3)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.4)$$

$$u_t(x, t) = g(x, t), \quad x \in \Omega, \quad t \in (-\tau, 0), \quad (1.5)$$

where  $\nu$  stands for the unit normal vector of  $\partial\Omega$  pointing towards the exterior of  $\Omega$  and  $\frac{\partial u}{\partial \nu}$  is the normal derivative. Moreover, the constant  $\tau > 0$  is the time delay,  $a$  and  $k$  are two positive numbers and the initial data are taken in suitable spaces.

Denoting by  $m$  the standard multiplier, that is  $m(x) = x - x_0$ , we assume

$$m(x) \cdot \nu(x) \leq 0, \quad x \in \Gamma_0, \quad \text{and} \quad m(x) \cdot \nu(x) \geq \delta > 0, \quad x \in \Gamma_1. \quad (1.6)$$

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Delay effects arise in many applications and practical problems and it is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in absence of delay (see e.g. [5, 6, 3, 4]).

The exponential stability of (1.1)–(1.5) with  $a = 0$  has been studied in [2] where it has been shown that the system is exponentially stable if  $\Gamma_1$  satisfies some geometric condition (BLR). Moreover, if  $\tau = 0$ , that is in absence of delay, the above problem for any  $a > 0$  is exponentially stable even if  $k = 0$  (see e.g. [12], [7]). On the contrary, in presence of a delay term there are instability phenomena. In fact, as shown in [9], it is possible to find for the above problem in the case  $k = 0$  a sequence  $\{\tau_k\}_k$  of delays with  $\tau_k \rightarrow 0$  for which the corresponding solutions  $u_k$  have an increasing energy.

In [9] in order to contrast the destabilizing effect of the time delay a “good” (not delayed) damping term is introduced in (1.1). More precisely the problem considered in [9] is

$$u_{tt}(x, t) - \Delta u(x, t) + au_t(x, t - \tau) + a_0 u_t(x, t) = 0, \quad x \in \Omega, \quad t > 0, \quad (1.7)$$

$$u(x, t) = 0, \quad x \in \Gamma_0, \quad t > 0 \quad (1.8)$$

$$\frac{\partial u}{\partial \nu}(x, t) = 0, \quad x \in \Gamma_1, \quad t > 0. \quad (1.9)$$

with  $a_0, a > 0$  and initial data in suitable spaces. If  $a_0 > a$ , it was shown in [9] that system (1.7)–(1.9) is uniformly exponentially stable, see also [8, 1, 10] for related results.

In this paper the idea is to contrast the effect of the time delay by using the dissipative boundary feedback (1.3) (i.e., by giving the control in the feedback form  $-k u_t(x, t)$ ,  $x \in \Gamma_1$ ,  $t > 0$ ). We will show that if the condition (1.6) is satisfied (geometric Lions condition), then for any  $k > 0$  system (1.1)–(1.5) is exponentially stable for  $a$  sufficiently small.

Let  $A = -\Delta$  be the unbounded operator in  $H = L^2(\Omega)$  with domain

$$H_1 = \mathcal{D}(A) = \left\{ u \in H^2(\Omega), u|_{\Gamma_0} = 0, \frac{\partial u}{\partial \nu}|_{\Gamma_1} = 0 \right\}.$$

We define by

$$B \in \mathcal{L}(L^2(\Omega); L^2(\Omega)), \quad Bu = B^*u = \sqrt{a}u, \quad \forall u \in L^2(\Omega),$$

and

$$C \in \mathcal{L}(L^2(\Gamma_1); H_{-1}), \quad Cu = \sqrt{k} A_{-1} N u, \quad \forall u \in L^2(\Gamma_1),$$

$$C^*w = \sqrt{k} w|_{\Gamma_1}, \quad \forall w \in \mathcal{D}(A^{\frac{1}{2}}) = H_{\frac{1}{2}},$$

where  $H_{-1} = (\mathcal{D}(A))'$  (the duality is in the sense of  $H$ ),  $A_{-1}$  is the extension of  $A$  to  $H$ , namely for all  $h \in H$  and  $\varphi \in \mathcal{D}(A)$ ,  $A_{-1}h$  is the unique element in  $H_{-1}$  such that (see for instance [11])

$$\langle A_{-1}h; \varphi \rangle_{H_{-1}-H_1} = \int_{\Omega} h A \varphi \, dx.$$

Here and below  $N \in \mathcal{L}(L^2(\Gamma_1); L^2(\Omega))$ ,  $\forall v \in L^2(\Gamma_1)$ ,  $Nv$  is the unique solution (transposition solution) of

$$\Delta w = 0, \quad w|_{\Gamma_0} = 0, \quad \frac{\partial w}{\partial \nu}|_{\Gamma_1} = v.$$

Setting  $z(t, \theta) = \dot{u}(t + \theta)$ ,  $\theta \in (-\tau, 0)$ , the evolution equation (1.1)–(1.5) is equivalent to

$$\ddot{u}(t) + Au(t) + CC^*\dot{u}(t) + BB^*z(t, -\tau) = 0, \quad t \geq 0, \quad (1.10)$$

$$\dot{z}(t, \theta) - z_\theta(t, \theta) = 0, \quad \theta \in (-\tau, 0), \quad t \geq 0, \quad (1.11)$$

$$z(t, 0) = \dot{u}(t), \quad t \geq 0, \quad (1.12)$$

$$u(0) = u^0, \quad \dot{u}(0) = u^1, \quad z(0, \theta) = g(\theta), \quad \theta \in (-\tau, 0), \quad (1.13)$$

where  $z_\theta = \partial_\theta z$  and  $g \in L^2(-\tau, 0; H_{\frac{1}{2}})$ .

To study the well-posedness of the system (1.10)–(1.13), we write it as an abstract Cauchy problem in a product space, and use the semigroup approach. For this purpose, take the Hilbert space  $\mathcal{H} := H_{\frac{1}{2}} \times H \times L^2(-\tau, 0; H_{\frac{1}{2}})$  and the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset \mathcal{H} \longrightarrow \mathcal{H}, \quad \mathcal{A}_d \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix} = \begin{pmatrix} u_2 \\ -Au_1 - CC^*u_2 - BB^*z(-\tau) \\ \partial_\theta z \end{pmatrix}, \quad (1.14)$$

where

$$\begin{aligned} \mathcal{D}(\mathcal{A}_d) := & \left\{ (u_1, u_2, z) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \times H^1(-\tau, 0; H_{\frac{1}{2}}), \right. \\ & \left. Au_1 + CC^*u_2 + BB^*z(-\tau) \in H, z(0) = u_2 \right\}. \end{aligned} \quad (1.15)$$

**Proposition 1.1.** *1. The operator  $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$  generates a strongly continuous semigroup  $(\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{H}$ .*

*2. The system (1.1)–(1.5) is well-posed. More precisely, for every  $(u^0, u^1, g) \in \mathcal{H}$ ,*

*the function  $w$  given by the first component of  $\mathcal{T}(t) \begin{pmatrix} u^0 \\ u^1 \\ g \end{pmatrix}$  is the mild solution of (1.1)–(1.5). In particular, for  $(u^0, u^1, g) \in \mathcal{D}(\mathcal{A}_d)$ , the problem (1.10)–(1.13) admits a unique classical solution*

$$(u, v, z) \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), \mathcal{H}),$$

*and thus the problem (1.1)–(1.5) admits a unique classical solution*

$$u \in C^1([-\tau, +\infty), H_{\frac{1}{2}}) \cap C^2([0, +\infty), H).$$

For any regular solution of problem (1.1)–(1.5) we define the energy

$$E(t) := E_S(t) + \frac{\xi}{2} \int_{t-\tau}^t \int_{\Omega} u_t^2(x, s) dx ds$$

$$= \frac{1}{2} \int_{\Omega} \{|\nabla u(x, t)|^2 + u_t^2(x, t)\} dx + \frac{\xi}{2} \int_{t-\tau}^t \int_{\Omega} u_t^2(x, s) dx ds, \quad (1.16)$$

where  $\xi$  is a strictly positive real number.

The main result of this paper is the following.

**Theorem 1.2.** *For any  $k > 0$  there exist positive constants  $a_0, C_1, C_2$  such that*

$$E(t) \leq C_1 e^{-C_2 t} E(0), \quad (1.17)$$

*for any regular solution of problem (1.1)–(1.5) with  $0 \leq a < a_0$ . The constants  $a_0, C_1, C_2$  are independent of the initial data but they depend on  $k$  and on the geometry of  $\Omega$ .*

The opposite problem, that is to contrast the effect of a time delay in the boundary condition with a velocity term in the wave equation, is still, as far as we know, open and it seems to be much harder to deal with. This will be the object of a future research. However, there is a positive answer by Datko, Lagnese and Polis [4] in the one dimensional case for the problem

$$u_{tt}(x, t) - u_{xx}(x, t) + 2au_t(x, t) + a^2 u(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (1.18)$$

$$u(0, t) = 0, \quad t > 0 \quad (1.19)$$

$$u_x(1, t) = -ku_t(1, t - \tau), \quad t > 0; \quad (1.20)$$

$$(1.21)$$

with  $a, k$  positive real numbers. Indeed, through a careful spectral analysis, in [4] the authors have shown that, for any  $a > 0$ , if  $k$  satisfies

$$0 < k < \frac{1 - e^{-2a}}{1 + e^{-2a}}, \quad (1.22)$$

then the spectrum of the system (1.18)–(1.20) lies in  $\text{Re } \omega \leq -\beta$ , where  $\beta$  is a positive constant depending on the delay  $\tau$ .

The paper is organized as follows. The second section deals with the well-posedness of the problem while, in the last section, we prove the exponential stability of the delayed system (1.1)–(1.5) by introducing a suitable Lyapunov functional.

## 2 Wellposedness

For the well-posedness of the equivalent equations (1.1)–(1.5) and (1.10)–(1.13), we show that the operator  $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$  defined by (1.14)–(1.15) generates a contraction semigroup on the Hilbert space  $\mathcal{H} := H_{\frac{1}{2}} \times H \times L^2(-\tau, 0; H_{\frac{1}{2}})$ .

We introduce in  $\mathcal{H}$  the new inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \\ z_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ z_2 \end{pmatrix} \right\rangle = \langle u_1, v_1 \rangle_{H_{\frac{1}{2}}} + \langle u_2, v_2 \rangle_H + \xi \int_{-\tau}^0 \langle B^* z_1(\theta), B^* z_2(\theta) \rangle_{L^2(\Omega)} d\theta,$$

where  $\xi$  is a strictly positive constant.

It can be easily seen that  $\mathcal{H}$  endowed with this inner product is a Hilbert space, and its associated norm is equivalent to the canonical norm of  $\mathcal{H}$ .

*Proof.* (of Proposition 1.1)

We show that there exists a positive constant  $c$  such that  $\mathcal{A}_d - cI$  is dissipative. Let  $\begin{pmatrix} u \\ v \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}_d)$ , then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\left\langle \mathcal{A}_d \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} v \\ -Au - CC^*v - BB^*z(-\tau) \\ \partial_\theta z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle \\
&= -\langle C^*v, C^*v \rangle_{L^2(\Gamma_1)} - \langle B^*z(-\tau), B^*v \rangle_{L^2(\Omega)} \\
&\quad + \xi \int_{-\tau}^0 \langle B^* \partial_\theta z(\theta), B^*z(\theta) \rangle_{L^2(\Omega)} d\theta \\
&= -\langle B^*z(-\tau), B^*v \rangle_{L^2(\Omega)} - \|C^*v\|_{L^2(\Gamma_1)}^2 + \frac{\xi}{2} \|B^*v\|_{L^2(\Omega)}^2 \\
&\quad - \frac{\xi}{2} \|B^*z(-\tau)\|_{L^2(\Omega)}^2 \\
&\leq \left( \frac{1}{2\xi} + \frac{\xi}{2} \right) \|B^*v\|_{L^2(\Omega)}^2.
\end{aligned}$$

Since  $B^*$  is bounded from  $L^2(\Omega)$  into itself, we deduce that there exists  $c > 0$  such that

$$\left\langle \mathcal{A}_d \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle \leq c \|v\|_{L^2(\Omega)}^2.$$

This shows that  $\mathcal{A}_d - cI$  is dissipative.

Next, we show that  $(\lambda I - \mathcal{A}_d)$  is surjective for some  $\lambda > 0$ .

Given a vector  $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{H}$ , we need  $\begin{pmatrix} u \\ v \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}_d)$  such that

$$(\lambda I - \mathcal{A}_d) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.$$

This is equivalent to

$$\lambda u - v = f, \tag{2.1}$$

$$Au + (\lambda + CC^*)v + BB^*z(-\tau) = g, \tag{2.2}$$

$$\lambda z - \partial_\theta z = h. \tag{2.3}$$

The function  $z \in H^1(-\tau, 0; H_{\frac{1}{2}})$  given by

$$z(\theta) = e^{\lambda\theta}v + \int_{\theta}^0 e^{\lambda(\theta-\sigma)}h(\sigma)d\sigma$$

is a solution to the equation (3) and verifies  $z(0) = v$ . By replacing  $v$  from (2.1) and  $z$  in the equation (2.2) we are reduced to find  $u \in D(A)$  solution of

$$\left( \lambda^2 I + A + \lambda(CC^* + BB^* e^{-\lambda\tau}) \right) u = k \quad (2.4)$$

with

$$k = g + \left( \lambda + CC^* + e^{-\lambda\tau} BB^* \right) f + BB^* \int_{-\tau}^0 e^{-\lambda(\tau+\sigma)} h(\sigma) d\sigma.$$

We now solve the equation (2.4). Assuming that  $u \in D(A)$  exists and is a solution of (2.4), then we have

$$\left\langle (\lambda^2 I + A + \lambda(CC^* + BB^* e^{-\lambda\tau}))u, \zeta \right\rangle = \langle k, \zeta \rangle, \forall \zeta \in H_{\frac{1}{2}}$$

or equivalently

$$\Lambda(u, \zeta) = \langle k, \zeta \rangle, \forall \zeta \in H_{\frac{1}{2}}, \quad (2.5)$$

where

$$\Lambda(u, \zeta) := \lambda^2 \langle u, \zeta \rangle + \left\langle A^{\frac{1}{2}} u, A^{\frac{1}{2}} \zeta \right\rangle + \lambda(\langle C^* u, C^* \zeta \rangle + e^{-\lambda\tau} \langle B^* u, B^* \zeta \rangle).$$

Since  $\Lambda$  is a bilinear coercive form on  $H_{\frac{1}{2}}$ , the Lax-Milgram lemma leads to the existence and uniqueness of a solution  $u \in H_{\frac{1}{2}}$  to (2.5). Some integrations by parts allow to show that  $u \in D(A)$  and is indeed solution of equation (2.4). Consequently,  $(\lambda I - \mathcal{A}_d)$  is surjective and therefore  $(\lambda I - (\mathcal{A}_d - cI))$  is also surjective. The density of  $\mathcal{D}(\mathcal{A}_d)$  is clear. Finally, the Lumer-Phillips theorem leads to the fact that  $\mathcal{A}_d - cI$  generates a strongly continuous semigroup of contraction in  $\mathcal{H}$ , hence  $\mathcal{A}_d$  generates a strongly continuous semigroup in  $\mathcal{H}$ .  $\square$

It is easy to see that if  $u : [-\tau, \infty) \rightarrow H_{\frac{1}{2}}$  is a classical solution of (1.1)–(1.5), then  $(u, \dot{u}, \dot{u}(t+\cdot))$  is the classical solution of the equation (1.10)–(1.13). The first assertion of Proposition 1.1 provides the converse result and then the well-posedness of the evolution equation (1.1)–(1.5).

The well-posedness part follows from the first assertion of Proposition 1.1.

### 3 Proof of Theorem 1.2

**Proposition 3.1.** *For any solution of problem (1.1)–(1.5) the following estimate holds:*

$$E'(t) \leq \frac{a+\xi}{2} \int_{\Omega} u_t^2(x, t) dx + \frac{a-\xi}{2} \int_{\Omega} u_t^2(x, t-\tau) dx - k \int_{\Gamma_1} u_t^2(x, t) d\Gamma. \quad (3.1)$$

*Proof.* Differentiating (1.16) we obtain

$$E'(t) = \int_{\Omega} \{ \nabla u(t) \nabla u_t(t) + u_t(t) u_{tt}(t) \} dx + \frac{\xi}{2} \int_{\Omega} u_t^2(t) dx - \frac{\xi}{2} \int_{\Omega} u_t^2(t-\tau) dx,$$

and then, integrating by parts and using (1.1), (1.2), (1.3),

$$E'(t) = -a \int_{\Omega} u_t(t) u_t(t-\tau) dx - k \int_{\Gamma_1} u_t^2(t) d\Gamma + \frac{\xi}{2} \int_{\Omega} u_t^2(t) dx - \frac{\xi}{2} \int_{\Omega} u_t^2(t-\tau) dx. \quad (3.2)$$

Applying Cauchy-Schwarz's inequality in (3.2) we obtain (3.1).  $\square$

**Proposition 3.2.** *For any regular solution of problem (1.1) – (1.5) and for every  $\epsilon > 0$ , we have*

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &\leq - \int_{\Omega} u_t^2(t) dx - \left( 1 - \frac{\epsilon}{2} C(P) \right) \int_{\Omega} |\nabla u(t)|^2 dx \\ &\quad - 2a \int_{\Omega} (m \cdot \nabla u(t)) u_t(t-\tau) dx - (n-1)a \int_{\Omega} u(t) u_t(t-\tau) dx \\ &\quad + \left[ \left( \|m\|_{\infty}^2 \frac{2}{\delta} + \frac{(n-1)^2}{2\epsilon} \right) k^2 + \|m\|_{\infty} \right] \int_{\Gamma_1} u_t^2(t) d\Gamma, \end{aligned} \quad (3.3)$$

where  $C(P)$  is a sort of Poincaré constant, which is a positive constant depending on  $\Omega$  and independent of the solution  $u$ .

*Proof.* Differentiating and integrating by parts we obtain

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &= - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx - 2a \int_{\Omega} (m \cdot \nabla u) u_t(t-\tau) dx \\ &\quad - (n-1)a \int_{\Omega} u(t) u_t(t-\tau) dx + \int_{\Gamma} (m \cdot \nu) (u_t^2 - |\nabla u|^2) d\Gamma \\ &\quad + 2 \int_{\Gamma} (m \cdot \nabla u) \frac{\partial u}{\partial \nu} d\Gamma + (n-1) \int_{\Gamma} u \frac{\partial u}{\partial \nu} d\Gamma. \end{aligned} \quad (3.4)$$

Now, since  $u = 0$  on  $\Gamma_0$  and  $m \cdot \nu \leq 0$  on  $\Gamma_0$ , from (3.4) we deduce

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &\leq - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ &\quad - 2a \int_{\Omega} (m \cdot \nabla u) u_t(t-\tau) dx - (n-1)a \int_{\Omega} u(t) u_t(t-\tau) dx + \|m\|_{\infty} \int_{\Gamma_1} u_t^2 d\Gamma \\ &\quad - \delta \int_{\Gamma_1} |\nabla u|^2 d\Gamma + 2 \int_{\Gamma_1} (m \cdot \nabla u) \frac{\partial u}{\partial \nu} d\Gamma + (n-1) \int_{\Gamma_1} u \frac{\partial u}{\partial \nu} d\Gamma, \end{aligned} \quad (3.5)$$

where we have used also  $m \cdot \nu \geq \delta$  on  $\Gamma_1$ .

We can estimate

$$\begin{aligned} 2 \int_{\Gamma_1} (m \cdot \nabla u) \frac{\partial u}{\partial \nu} d\Gamma &\leq \frac{\delta}{2} \int_{\Gamma_1} |\nabla u|^2 d\Gamma + 2 \frac{\|m\|_{\infty}^2}{\delta} \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma \\ &= \frac{\delta}{2} \int_{\Gamma_1} |\nabla u|^2 d\Gamma + 2 \frac{\|m\|_{\infty}^2}{\delta} k^2 \int_{\Gamma_1} u_t^2(t) d\Gamma. \end{aligned} \quad (3.6)$$

Moreover,

$$\begin{aligned} (n-1) \int_{\Gamma_1} u \frac{\partial u}{\partial \nu} d\Gamma &\leq \frac{\epsilon}{2} \int_{\Gamma_1} u^2 d\Gamma + \frac{(n-1)^2}{2\epsilon} \int_{\Gamma_1} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma \\ &\leq \frac{\epsilon}{2} C(P) \int_{\Omega} |\nabla u|^2 dx + \frac{(n-1)^2}{2\epsilon} k^2 \int_{\Gamma_1} u_t^2(t) d\Gamma, \end{aligned} \quad (3.7)$$

where we have used trace inequality and Poincaré's theorem.

Substituting (3.6) and (3.7) in (3.5), we obtain the estimate (3.3).  $\square$

**Remark 3.3.** In the above proposition  $C(P)$  is the smallest positive constant such that

$$\int_{\Gamma_1} \varphi^2(x) d\Gamma \leq C(P) \int_{\Omega} |\nabla \varphi(x)|^2 dx, \quad \forall \varphi \in H_{\Gamma_0}^1(\Omega).$$

**Corollary 3.4.** For any regular solution of (1.1) – (1.5)

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &\leq - \int_{\Omega} u_t^2(t) dx \\ &\quad - \left( 1 - \frac{\epsilon}{2} C(P) - a \|m\|_{\infty}^2 - \frac{a}{2} (n-1)^2 C_0(P) \right) \int_{\Omega} |\nabla u(t)|^2 dx \\ &\quad + \frac{3}{2} a \int_{\Omega} u_t^2(t-\tau) dx + \left[ \left( \|m\|_{\infty}^2 \frac{2}{\delta} + \frac{(n-1)^2}{2\epsilon} \right) k^2 + \|m\|_{\infty} \right] \int_{\Gamma_1} u_t^2(t) d\Gamma, \end{aligned} \quad (3.8)$$

where  $(C_0(P))^{1/2}$  is the so-called Poincaré constant.

*Proof.* We apply Cauchy-Schwarz's inequality to the integral in the second line of (3.3).  $\square$

Now, let us introduce the functional  $\mathcal{S}(t) := \int_{\Omega} \int_{t-\tau}^t e^{s-t} u_t^2(x, s) ds dx$ .

We can easily estimate

$$\begin{aligned} \mathcal{S}'(t) &= \int_{\Omega} u_t^2(t) dx - \int_{\Omega} e^{-\tau} u_t^2(t-\tau) dx - \int_{\Omega} \int_{t-\tau}^t e^{s-t} u_t^2(x, s) ds dx \\ &\leq \int_{\Omega} u_t^2(t) dx - e^{-\tau} \int_{\Omega} u_t^2(t-\tau) dx - e^{-\tau} \int_{\Omega} \int_{t-\tau}^t u_t^2(x, s) ds dx. \end{aligned} \quad (3.9)$$

Let us introduce the Lyapunov functional

$$\mathcal{E}(t) := E(t) + \gamma_1 \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx + \gamma_2 \mathcal{S}(t), \quad (3.10)$$

where  $\gamma_1, \gamma_2$  are suitable positive small constants that will be precised later on.

Note that  $\mathcal{E}(t)$  is equivalent to the energy  $E(t)$  if  $\gamma_1$  is small enough. In particular, there exists a positive constant  $C_1$  and suitable positive constants  $\alpha_1, \alpha_2$  such that

$$\alpha_1 E(t) \leq \mathcal{E}(t) \leq \alpha_2 E(t), \quad \forall 0 < \gamma_1, \gamma_2 \leq C_1. \quad (3.11)$$



**Proposition 3.5.** *For every  $k > 0$  there exist  $a_0, c_1, c_2$  such that for any solution of problem (1.1) – (1.5) with  $0 \leq a < a_0$  we have*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 t} \mathcal{E}(0), \quad t > 0. \quad (3.12)$$

*The constants  $a_0, c_1, c_2$  are independent of the initial data but they depend on  $k$  and on the geometry of  $\Omega$ .*

*Proof.* Differentiating the Lyapunov functional  $\mathcal{E}$  and using the propositions above we deduce

$$\begin{aligned} \mathcal{E}'(t) \leq & \left( \frac{a+\xi}{2} - \gamma_1 + \gamma_2 \right) \int_{\Omega} u_t^2(x, t) dx - \gamma_2 e^{-\tau} \int_{\Omega} \int_{t-\tau}^t u_t^2(x, s) ds dx \\ & + \left( \frac{a-\xi}{2} + \frac{3}{2} a \gamma_1 - \gamma_2 e^{-\tau} \right) \int_{\Omega} u_t^2(x, t - \tau) dx \\ & - \gamma_1 \left( 1 - \frac{\epsilon}{2} C(P) - a \|m\|_{\infty}^2 - \frac{a}{2} (n-1)^2 C_0(P) \right) \int_{\Omega} |\nabla u(t)|^2 dx \\ & + \left\{ \gamma_1 k^2 \left( \|m\|_{\infty}^2 \frac{2}{\delta} + \frac{(n-1)^2}{2\epsilon} \right) + \gamma_1 \|m\|_{\infty} - k \right\} \int_{\Gamma_1} u_t^2(x, t) dx. \end{aligned} \quad (3.13)$$

For a fixed  $k > 0$  we want to chose  $\epsilon, \gamma_1, \gamma_2 < C_1$  and  $a$  sufficiently small in order to obtain

$$\mathcal{E}'(t) \leq -c \mathcal{E}(t). \quad (3.14)$$

Applying the second inequality of (3.11) estimate (3.12) easily follows.

To show that (3.13) implies (3.14) we simply need that

$$\begin{aligned} \frac{a+\xi}{2} - \gamma_1 + \gamma_2 &< 0, \\ \frac{a-\xi}{2} + \frac{3}{2} a \gamma_1 - \gamma_2 e^{-\tau} &\leq 0, \\ 1 - \frac{\epsilon}{2} C(P) - a \|m\|_{\infty}^2 - \frac{a}{2} (n-1)^2 C_0(P) &> 0, \\ \gamma_1 k^2 \left( \|m\|_{\infty}^2 \frac{2}{\delta} + \frac{(n-1)^2}{2\epsilon} \right) + \gamma_1 \|m\|_{\infty} - k &\leq 0. \end{aligned}$$

These conditions are equivalent to

$$\frac{a+\xi}{2} < \gamma_1 - \gamma_2, \quad (3.15)$$

$$a \left( \frac{1}{2} + \frac{3}{2} \gamma_1 \right) - \frac{\xi}{2} \leq \gamma_2 e^{-\tau}, \quad (3.16)$$

$$a (\|m\|_{\infty}^2 + \frac{1}{2} (n-1)^2 C_0(P)) < 1 - \frac{\epsilon}{2} C(P), \quad (3.17)$$

$$\gamma_1 \left[ k^2 \left( \|m\|_{\infty}^2 \frac{2}{\delta} + \frac{(n-1)^2}{2\epsilon} \right) + \|m\|_{\infty} \right] - k \leq 0. \quad (3.18)$$

For any  $k > 0$  this last condition is satisfied for  $\gamma_1$  sufficiently small (once  $\epsilon$  is fixed, see below). It then remains to the conditions (3.15) to (3.17). For the first one, we need to assume that  $\gamma_1 > \gamma_2$ , while for (3.17) we need to fix  $\epsilon$  small enough such that

$$1 - \frac{\epsilon}{2} C(P) > 0. \quad (3.19)$$

Then we now fix  $\gamma_1, \gamma_2$  and  $\epsilon$  fulfilling the above requirements and look at (3.15) to (3.17) as conditions on  $a$  and  $\xi$ . These conditions are simply linear constraints and a simple analysis shows that the set of pairs  $(a, \xi)$  fulfilling these constraints is not empty (see Figure 1).

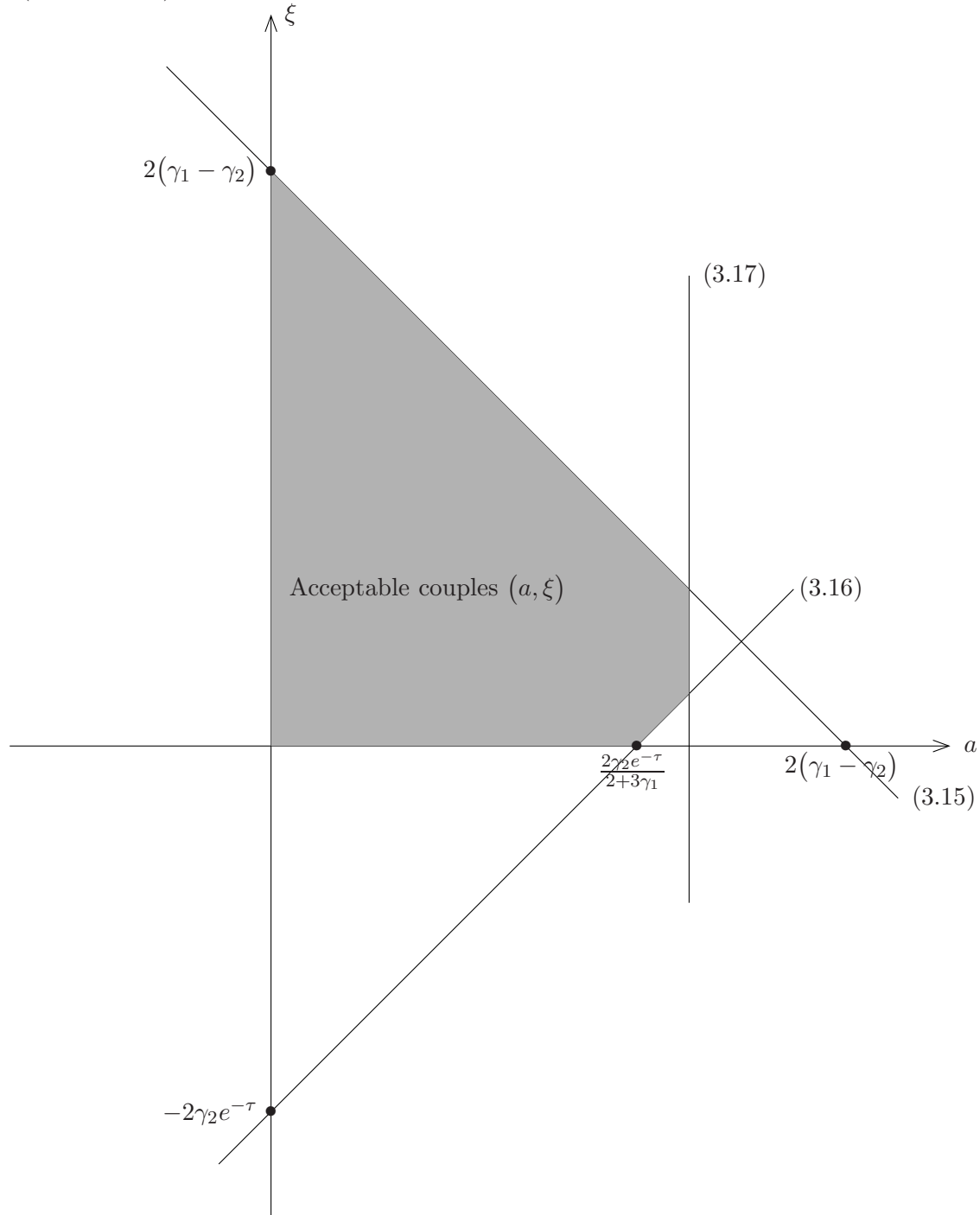


Figure 1

According to this figure, it is clear that for  $a$  and  $\xi$  small enough, (3.15) to (3.17)

are valid. Note further that due to (3.18) if  $k$  goes to  $\infty$  or to 0, then  $\gamma_1$  must tend to zero, and therefore  $\gamma_2$  as well and the maximal value  $a_0$  of  $a$  goes to zero.  $\square$

From Proposition 3.5 and the energy equivalence (3.11) we deduce estimate (1.17).

**Remark 3.6.** We can make explicit the relation between  $k$  and  $a_0$  by choosing the constants  $\xi, \gamma_1, \gamma_2$  in the definitions (1.16), (3.10) of the energy  $E(\cdot)$  and of the Lyapunov functional  $\mathcal{E}(\cdot)$  in such a way that conditions (3.15)–(3.18) are satisfied. Moreover, we need to fix  $\epsilon > 0$  in the estimate (3.3) such that (3.19) holds. For instance, fix

$$\epsilon = \frac{1}{C(P)}, \quad \xi = 2a.$$

Now, choose

$$\gamma_1 = \min \left\{ \frac{1}{3}, (2\|m\|_\infty + C_0(P) + 1)^{-1}, k \left( k^2 \left( \|m\|_\infty^2 \frac{2}{\delta} + \frac{(n-1)^2}{2} C(P) \right) + \|m\|_\infty \right)^{-1} \right\},$$

and  $\gamma_2 = \frac{\gamma_1}{2}$ .

The choice of  $\gamma_1 \leq (2\|m\|_\infty + C_0(P) + 1)^{-1}$  ensures the equivalence between the energy  $E(\cdot)$  and the Lyapunov functional  $\mathcal{E}(\cdot)$ . Moreover, with the above choices of  $\gamma_1$  and  $\gamma_2$  conditions (3.16) and (3.18) are satisfied for any  $a > 0$ .

The remaining conditions are satisfied for all  $0 \leq a < a_0$ , with

$$a_0 = \min \left\{ \frac{\gamma_1}{3}, \frac{1}{2} (\|m\|_\infty^2 + \frac{1}{2} (n-1)^2 C_0(P))^{-1} \right\},$$

that is

$$a_0 = \min \left\{ \frac{1}{9}, \frac{1}{3} (2\|m\|_\infty + C_0(P) + 1)^{-1}, \frac{k}{3} \left( k^2 \left( \|m\|_\infty^2 \frac{2}{\delta} + \frac{(n-1)^2}{2} C(P) \right) + \|m\|_\infty \right)^{-1}, \right. \\ \left. \frac{1}{2} (\|m\|_\infty^2 + \frac{1}{2} (n-1)^2 C_0(P))^{-1} \right\}$$

Note that  $a_0$  depends only on the geometry of the domain  $\Omega$  and on  $k$ . Moreover, observe that  $a_0 \rightarrow 0$  if  $k \rightarrow 0$  and, also, if  $k \rightarrow +\infty$ . This is in agreement with the result of [4] (cfr. (1.22)).

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