# Internal stabilization of the Oseen–Stokes equations by Stratonovich noise

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## Abstract

One designs a Stratonovich noise feedback controller with support in an arbitrary open subset $O_0$ of $O$ which exponentially stabilizes in probability, that is with probability one, the Oseen–Stokes systems in a domain $O \subset \mathbb{R}^d$, $d = 2, 3$. This completes the stabilization results from the author's work [6] which is concerned with design of an Ito noise stabilizing controller.

*Keywords:* Stratonovich noise, Navier-Stokes equation, eigenvalue, feedback controller
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## 1. Introduction and statement of the problem

This work is concerned with internal stabilization via Stratonovich noise feedback controller of Oseen–Stokes system

$$\begin{aligned}
&\frac{\partial X}{\partial t} - \nu \Delta X + (f \cdot \nabla)X + (X \cdot \nabla)g = \nabla p \ \text{ in } (0, \infty) \times O, \\
&\nabla \cdot X = 0 \quad \text{in } (0, \infty) \times O, \\
&X(0, \xi) = x(\xi), \ \xi \in O, \quad X = 0 \text{ on } (0, \infty) \times \partial O.
\end{aligned} \tag{1}$$

Here, $O$ is an open and bounded subset of $\mathbb{R}^d$, $d = 2, 3$, with smooth boundary $\partial O$ and $f, g \in C^2(\overline{O}; \mathbb{R}^d)$ are given functions. In the special case $g \equiv 0$, system (1) describes the dynamic of a fluid Stokes flow with partial inclusion of convection acceleration $(f \cdot \nabla)X$ ($X$ is the velocity field). The same equation describes the disturbance flow induced by a moving body with velocity $f$ through the fluid. Should we mention also that in the special case $f \equiv g \equiv X_e$, where $X_e$ is the equilibrium (steady-state) solution of the Navier–Stokes equation

$$\begin{aligned}
&\frac{\partial X}{\partial t} - \nu \Delta X + (X \cdot \nabla)X = \nabla p + f_e, \\
&\nabla \cdot X = 0, \quad X|_{\partial O} = 0,
\end{aligned} \tag{2}$$

and $f_e \in C(\overline{O}; \mathbb{R}^d)$, system (1) is the linearization of (2) around $X_e$. In this way, the stabilization of (1) can be interpreted as the first order stabilization procedure of steady-state Navier–Stokes flows.

Our aim here is to design a stochastic feedback controller of the form

$$u = \mathbf{1}_{O_0} \sum_{k=1}^{M} R_k(X) \circ \dot{\beta}_k, \ R_k \in L((L^2(O))^d), \tag{3}$$

which stabilizes in probability system (1) and has the support in an open subdomain $O_0$ of $O$.

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Here, $\{\beta_k\}_{k=1}^M$ is a system of mutually independent Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with filtration $\{\mathcal{F}_t\}_{t>0}$ while the corresponding closed loop system

$$\begin{aligned}
dX - &\nu \Delta X \, dt + (f \cdot \nabla)X \, dt + (X \cdot \nabla)g \, dt \\
&= \mathbf{1}_{O_0} \sum_{k=1}^{M} R_k(X) \circ d\beta_k + \nabla p \, dt
\end{aligned} \tag{4}$$

$$X(0) = x$$

is taken in Stratonovich sense (see, e.g., [1]) and this is the significance of the symbol $R_k(X) \circ \dot{\beta}_k$ in the expression of the noise controller (3). In the following, the terminology Stratonovich feedback controller or Ito feedback controller refer to the sense in which the corresponding stochastic equation (4) is considered: in the Stratonovich sense or, respectively, the Ito sense. We have denoted by $\mathbf{1}_{O_0}$ the characteristic function of the open set $O_0 \subset O$.

In [5], [6], [7], [8], the author has designed similar stabilizable Ito noise controllers for equation (1) and related Navier–Stokes equations. However, it should be said that, with respect to Ito noise controllers, the Stratonovich feedback controller (3) has the advantage to be stable with respect to smooth changes $\dot{\beta}_k^\varepsilon$ of the noise $\dot{\beta}_k$ and this fact is crucial not only from the conceptual point of view, but also for numerical simulations and practical implementation into system (1) of the random stabilizable feedback controller

$$u_\varepsilon(t) = \mathbf{1}_{O_0} \sum_{k=1}^{M} R_k(X(t)) \dot{\beta}_k^\varepsilon(t),$$

where $\dot{\beta}_k^\varepsilon$ is a smooth approximation of $\beta_k^\varepsilon$. If instead (3) we take $u$ to be an Ito stabilizable feedback controller, then the corresponding Ito stochastic closed loop equation is convergent for $\varepsilon \to 0$ to a Stratonovich equation of the form (4) which might be unstable because the stabilization effect of the noise controller is given by the Ito's formula which is valid for Ito stochastic equation only. Thus, for numerical implementations

of a stabilizanble noise feedback controller $u$ it is essential that it is of Stratonovich type. It should be said, however, that the option for an Ito stabilizable noise controller, like in [5]–[8], or a Stratonovich one, as in this work, is function of the specific techniques used to insert such a noise controller into system (1); by direct simulation or by numerical approximating procedure.

As regards the literature on stabilization of linear differential systems by Stratonovich noise, the pioneering works [2], [3] should be primarily cited. For linear PDEs, this procedure was developed in [9], [10] which are related to this work. For general results on internal stabilization of Navier–Stokes systems with deterministic feedback controllers, we refer to [4]. (See also [11] for noise stabilization effect into a different nonlinear PDE context.)

## 2. The noise stabilizing feedback controller

Consider the standard space of free divergence vectors $H = \{y \in (L^2(O))^d; \nabla \cdot y = 0 \text{ in } O, y \cdot n = 0 \text{ on } \partial O\}$ and denote by $\mathcal{A}_0 : D(\mathcal{A}_0) \subset H \to H$ the realization of the Oseen–Stokes operator in this space, that is,

$$\mathcal{A}_0 y = P(-\nu\Delta y + (f \cdot \nabla)y + (y \cdot \nabla)g), \quad y \in D(\mathcal{A}_0), \quad (5)$$

where $D(\mathcal{A}_0) = H \cap (H^2(O))^d \cap (H_0^1(O))^d$. Here, $P$ is the Leray projector on $H$ and $H^2(O)$, $H_0^1(O)$ are standard Sobolev spaces on $O$. In the following, it will be more convenient to represent equation (1) in the complex Hilbert space $\mathcal{H} = H + iH$ by extending $\mathcal{A}_0$ to $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ via standard procedure, $\mathcal{A}(x + iy) = \mathcal{A}_0 x + i\mathcal{A}_0 y$. The operator $\mathcal{A}$ has a countable set of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ (eventually complex) with the corresponding eigenvectors $\varphi_j$. Denote by $\mathcal{A}^*$ the adjoint of $\mathcal{A}$ with eigenvalues $\overline{\lambda}_j$ and eigenvectors $\varphi_j^*$. Each eigenvalue is repeated in the following according to its algebraic multiplicity $m_j$. Normalizing the system $\{\varphi_j\}_{j=1}^\infty$, we see that

$$|\nabla\varphi_j|_{\mathcal{H}}^2 = \lambda_j - \langle (f \cdot \nabla)\varphi_j + (\varphi_j \cdot \nabla)g, \varphi_j \rangle, \ \forall j,$$

and since, by the Fredholm–Riesz theory, $|\lambda_j| \to +\infty$ as $j \to \infty$, we infer that $\mathrm{Re}\ \lambda_j \to +\infty$ as $j \to \infty$. We denote by $N$ the minimal number of eigenvalues $\lambda_j$ for which

$$\mathrm{Re}\ \lambda_j > 0 \text{ for } j \geq N, \ \lambda_1 + \lambda_2 + \cdots + \lambda_N > 0. \quad (6)$$

(In the above sequence, each $\lambda_j$ is taken together its conjugate $\overline{\lambda}_j$ and, clearly, there is such a natural number $N$.)

Set $\mathcal{X}_u = $ lin span$\{\varphi_j\}_{j=1}^N$ and denote by $\mathcal{X}_s$ the algebraic complement of $\mathcal{X}_u$ in $\mathcal{X}$. It is well known that $\mathcal{X}_u$ and $\mathcal{X}_s$ are both invariant for $\mathcal{A}$ and, if we set

$$\mathcal{A}_u = \mathcal{A}|_{\mathcal{X}_u}, \quad \mathcal{A}_s = \mathcal{A}|_{\mathcal{X}_s},$$

we have for their spectra $\sigma(\mathcal{A}_u) = \{\lambda_j\}_{j=1}^N$, $\sigma(\mathcal{A}_s) = \{\lambda_j\}_{j=N+1}^\infty$ and, since $-\mathcal{A}_s$ is the generator of an analytic $C$-semigroup $e^{-\mathcal{A}_s t}$ in $\mathcal{X}_s$, we have

$$\|e^{-\mathcal{A}_s t}\|_{L(\mathcal{H})} \leq C \exp(-\mathrm{Re}\ \lambda_{N+1} t), \quad t \geq 0, \quad (7)$$

(see, e.g., [5], p. 14). In the following, we shall assume that

(i) *All the eigenvalues $\{\lambda_j\}_{j=1}^N$ are semisimple.*

This means that the algebraic multiplicity of each $\lambda_j$, $j = 1, ..., N$, coincides with its geometric multiplicity or, in other words, the finite-dimensional operator (matrix) $\mathcal{A}_u$ is diagonalizable. As we will see later on, this assumption is not essentially necessary but it simplifies however the construction of the stabilizing controller because it reduces the unstable part of the system to a diagonal finite-dimensional differential system. In particular, it follows by (i) that we can choose the dual systems $\{\varphi_j\}$ and $\{\varphi_j^*\}$ in such a way that

$$\langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}, \quad i, j = 1, ..., N. \quad (8)$$

(Here, and everywhere in the following, $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathcal{H}$ and $H$. By $|\cdot|_{\mathcal{H}}$ and $|\cdot|_H$ we denote the corresponding norms.)

We note that the uncontrolled Oseen–Stokes system (1) can be rewritten in the space $\mathcal{H}$ as

$$\frac{dX}{dt} + \mathcal{A}X = 0, \ t \geq 0, \quad X(0) = x, \quad (9)$$

and setting $X_u = \sum_{j=1}^N y_j \varphi_j$, $X_s = (I - P_N)X$, where $P_N$ is the algebraic projector on $\mathcal{X}_u$, we have

$$\frac{dX_u}{dt} + \mathcal{A}_u X_u = 0, \quad X_u(0) = P_N x, \quad (10)$$

$$\frac{dX_s}{dt} + \mathcal{A}_s X_s = 0, \quad X_s(0) = (I - P_N)x. \quad (11)$$

We set $A_u = \left\{\langle \mathcal{A}\varphi_j, \varphi_k^* \rangle\right\}_{j,k=1}^N = \mathrm{diag}\|\lambda_j\|_{j=1}^N$ and so, by (8), we may rewrite (10) in terms of $y = \left\{y_j = \langle X_u, \varphi_j^* \rangle\right\}_{j=1}^N$ as

$$\frac{dy}{dt} + A_u y = 0, \quad y(0) = \left\{\langle P_N x, \varphi_j^* \rangle\right\}_{j=1}^N. \quad (12)$$

Since $\mathrm{Tr}(-A_u) = -\lambda_1 - \lambda_2 - \cdots - \lambda_N < 0$, it follows by Theorem 2 in [3] that there is a sequence of skew-symmetric matrices $\{C^k\}_{k=1}^M$, where $M = N - 1$ such that the solution $y$ to the Stratonovich stochastic system

$$dy + A_u y\, dt = \sum_{k=1}^M C^k y \circ d\beta_k, \quad t \geq 0, \quad (13)$$

has the property

$$|y(t)| \leq C|y(0)|e^{-\gamma_0 t}, \ \mathbb{P}\text{-a.s.}, \ \forall t > 0, \quad (14)$$

where $\gamma_0 > 0$. The matrix $C^k$ is explicitly constructed in [3] and it will be used below to construct a stabilizable feedback controller of the form (3). Namely, we set in (3)

$$R_k(X) = \sum_{i,j=1}^N C_{ij}^k \langle X, \varphi_j^* \rangle \phi_i, \quad k = 1, ..., M, \quad (15)$$

where $\|C_{ij}^k\|_{i,j=1}^N = C^k$,

$$\phi_i = \sum_{\ell=1}^N \alpha_{i\ell} \varphi_\ell^*, \quad i = 1, ..., N, \quad (16)$$

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and $\alpha_{i\ell}$ are chosen in such a way that

$$\sum_{\ell=1}^{N} \alpha_{i\ell}\gamma_{\ell j} = \delta_{ij}, \quad i,j = 1,...,N. \tag{17}$$

Here, $\gamma_{\ell j} = \int_{O_0} \varphi_\ell^* \overline{\varphi}_j^* d\xi$ and since, by the unique continuation property (see [5], p. 157), the eigenfunction system $\{\varphi_j^*\}$ is linearly independent on $O_0$, we infer that the matrix $\|\gamma_{\ell j}\|_{\ell,j=1}^N$ is not singular and, therefore, there is a unique system $\{\alpha_{i\ell}\}$ which satisfies (17). Then, by (16), we see that

$$\left\langle \mathbf{1}_{O_0}\phi_i, \varphi_j^* \right\rangle = \delta_{ij}, \quad i,j = 1,...,N. \tag{18}$$

As mentioned earlier, $O_0$ is an arbitrary open subset of $O$.

Theorem 1 is the main result.

**Theorem 1.** *The solution $X$ to the closed loop system* (4)*, where $R_k$ are defined by* (15)*, is exponentially stable with probability one, that is,*

$$P\left[|X(t)|_{\mathcal{H}} \le Ce^{-\gamma t}|x|_H, \quad \forall t \ge 0\right] = 1, \tag{19}$$

*where $\gamma > 0$ and $C > 0$ are independent of $\omega \in \Omega$.*

*2.1. The proof of Theorem 1*

The idea of the proof, already used in stabilization theory of infinite-dimensional systems with finite-dimensional unstable subspaces, is to stabilize the finite-dimensional system (10) by a feedback controller of the form (3) and to reconstruct consequently the system via the infinite-dimensional stable complement (11).

Namely, taking into account (10), (11), we write the closed loop system (4), that is,

$$dX + \mathcal{A}X \, dt = P\left[\mathbf{1}_{O_0} \sum_{k=1}^{M} R_k(X) \circ d\beta_k\right] \tag{20}$$

as

$$dX_u + \mathcal{A}_u X_u dt = P_N \sum_{k=1}^{M} \sum_{i,j=1}^{N} C_{ij}^k \left\langle X_u, \varphi_j^* \right\rangle P(\phi_i) \circ d\beta_k, \tag{21}$$

$$dX_s + \mathcal{A}_s X_s dt = (I - P_N) \sum_{k=1}^{M} \sum_{i,j=1}^{N} C_{ij}^k \left\langle X_u, \varphi_j^* \right\rangle P(\phi_i) \circ d\beta_k, \tag{22}$$

where $X_u + X_s = X$.

Taking into account (18) and that $X_u = \sum_{j=1}^{N} y_j \varphi_j$, we may rewrite (21) as

$$dy_\ell + \lambda_\ell y_\ell dt = \sum_{k=1}^{M} C_{\ell j}^k y_j \circ d\beta_k, \quad \ell = 1,...,N, \tag{23}$$

and so, by (14), we have that

$$|y_\ell(t)| \le Ce^{-\gamma t}|y_\ell(0)|, \quad \mathbb{P}\text{-a.s.}, \quad \forall t > 0, \ \ell = 1,...,N. \tag{24}$$

(We have denoted by $C$ several positive constants independent of $t$ and $\omega \in \Omega$.)

As regards the existence of a solution $X_s$ to (22), this is standard and follows from the general theory of linear infinite-dimensional stochastic equations.

Now, in order to estimate $X_s$, it is convenient to replace (22) by its Ito formulation (see, e.g., [10])

$$\begin{aligned} dX_s + \mathcal{A}_s X_s \, dt &= \frac{1}{2} \sum_{k=1}^{M} [P(\mathbf{1}_{O_0}R_k)]^2 X_u dt \\ &+ \sum_{k=1}^{M} P(\mathbf{1}_{O_0}R_k(X_u)) d\beta_k. \end{aligned} \tag{25}$$

Taking into account that, by (7), $e^{-\mathcal{A}_s t}$ is an exponentially stable semigroup on $\mathcal{X}_s$, without loss of generality we may assume that $\mathrm{Re}\,\langle \mathcal{A}_s x, x \rangle \ge \gamma|x|_H^2$. (Otherwise, proceeding as in [6], we replace the scalar product $\langle x, y \rangle$ by $\langle Qx, y \rangle$ where $Q$ is the solution to the Lyapunov equation $\mathcal{A}_s Q + Q\mathcal{A}_s^* = \gamma I$.) Then, applying Ito's formula in (25), we see that

$$\begin{aligned} d|X_s(t)|_{\mathcal{H}}^2 + 2\gamma|X_s(t)|_{\mathcal{H}}^2 dt &= \left\langle X_s(t), \sum_{k=1}^{M}[P(\mathbf{1}_{O_0}R_k)]^2 X_u(t) \right\rangle dt \\ &+ \sum_{k=1}^{M}|P(\mathbf{1}_O R_k(X_u))|^2 dt + 2\sum_{k=1}^{M} P(\mathbf{1}_O R_k(X_u))X_s d\beta_k. \end{aligned} \tag{26}$$

Taking into account that, by (24), $|X_u(t)|_{\mathcal{H}} \le e^{-\gamma t}|X_u(0)|$, we infer by (26) that

$$E|X_s(t)|_{\mathcal{H}}^2 \le C|X_s(0)|^2 e^{-\gamma t}, \quad \forall t \ge 0.$$

Since $M(t) = \int_0^t \sum_{k=1}^{M} P(\mathbf{1}_{O_0}R_k(X_u))X_s \, d\beta_k$ is a local martingale, it follows by (26) and Lemma 3.1 in [6] that

$$|X_s(t)|_{\mathcal{H}} \le C|X_s(0)|e^{-\gamma t}, \quad \forall t \ge 0, \ \mathbb{P}\text{-a.s.},$$

which completes the proof.

## 3. The design of a real valued noise controller

A nice feature of the feedback controller given by Theorem 1 is its simple structure. Moreover, its computation does not rise any special problem because the stabilizing matrices $C^k$ for the diagonal system (23) can be explicitly expressed.

If all $\lambda_j$, $j = 1,...,N$, are real, then the feedback controller $\{R_k\}$ given by Theorem 1 as well as the closed loop system (4) are real, too. However, if some $\lambda_j$ are complex, then the above feedback controller does not stabilize system (1), but its complex realization in the space $\mathcal{H} = H + iH$. In this case, though the stabilizing feedback controller (3) is quite simple, its implementation into real systems rises some delicate problems because the feedback controller is in implicit form as function of $\mathrm{Re}\,X$, $\mathrm{Im}\,X$. In order to circumvent these inconvenience, we will derive below from the above construction a real valued noise feedback controller which has a stabilizing effect on the Oseen–Stokes system (1).

To this end, we denote by $\{\psi_j\}_{j=1}^N$ the orthogonalized system (via Schmidt procedure) $\{\mathrm{Re}\,\varphi_j, \mathrm{Im}\,\varphi_j, \ j = 1,...,N\}$. Taking

into account that each $\varphi_j$ in the above system arises together with its conjugate, the dimension of the system $\{\psi_j\}$ is again $N$. We also set

$$\mathcal{X}_u^{\mathrm{re}} = \lim \mathrm{span}\{\psi_j\}_{j=1}^N$$

and note that $\mathcal{X}_u^{\mathrm{re}} = \{\mathrm{Re}\, y;\ y \in \mathcal{X}_u\}$ and $H = \mathcal{X}_u^{\mathrm{re}} \oplus \mathcal{X}_s^{\mathrm{re}}$, where $X_s^{\mathrm{re}} = \{\mathrm{Re}\, y;\ y \in \mathcal{X}_s\}$. Moreover, the operator $\mathcal{A}_0$ leaves invariant spaces $\mathcal{X}_u^{\mathrm{re}}, \mathcal{X}_s^{\mathrm{re}}$ and if we set

$$\mathcal{A}_u^{\mathrm{re}} = \mathcal{A}_0|_{\mathcal{X}_u^{\mathrm{re}}}, \quad \mathcal{A}_s^{\mathrm{re}} = \mathcal{A}_0|_{\mathcal{X}_s^{\mathrm{re}}},$$

we have, of course,

$$\|e^{-\mathcal{A}_s^{\mathrm{re}} t}\|_{L(H)} \le C e^{-\gamma_0 t}, \quad \forall t > 0, \tag{27}$$

for some $\gamma_0 > 0$. It is also easily seen that

$$\mathrm{Tr}[-\mathcal{A}_u^{\mathrm{re}}] = \mathrm{Tr}[-\mathcal{A}_u] < 0.$$

Then, we define as in (15) the operators

$$\widetilde{R}_k(x) = \sum_{i,j=1}^N \widetilde{C}_{ij}^k \langle X, \varphi_j \rangle \widetilde{\phi}_i, \quad k = 1, ...., M, \tag{28}$$

where $\widetilde{C}^k = \|\widetilde{C}_{ij}^k\|_{i,j=1}^N$ is the matrix system which stabilizes with probability one, via Theorem 2 in [3], the finite-dimensional system

$$dy + A_u^{\mathrm{re}} y\, dt = \sum_{k=1}^M \widetilde{C}^k y \circ d\beta_k.$$

Here, $A_u^{\mathrm{re}} = \| \langle \mathcal{A}_u^{\mathrm{re}} \psi_i, \psi_j \rangle \|_{i,j=1}^N$ and $\widetilde{\phi}_i$ is given by

$$\widetilde{\phi}_i = \sum_{\ell=1}^N \widetilde{\alpha}_{i\ell} \psi_\ell, \quad i = 1, ...., N,$$

where $\widetilde{\alpha}_{i\ell}$ are chosen in such a way that

$$\sum_{\ell=1}^N \widetilde{\alpha}_{i\ell} \widetilde{\gamma}_{\ell j} = \delta_{ij}, \quad \widetilde{\gamma}_{\ell j} = \int_{O_0} \psi_\ell \psi_j d\xi.$$

As in the previous case, the system $\{\psi_j\}_{j=1}^N$ is still linearly independent on $O_0$ which implies that $\det \|\widetilde{\gamma}_{\ell j}\| \ne 0$ and the above system has a unique solution $\widetilde{\alpha}_{i\ell}$.

Then, arrived at this point, the proof of Theorem 1 applies neatly to conclude that the feedback noise controller (3) with $\widetilde{R}_k$ instead of $R_k$ is stabilizing in the probability system (9).

Namely, we have the following stabilization result.

**Theorem 2.** *The solution $X$ to the closed loop system* (4) *with $R_k = \widetilde{R}_k$ defined by* (28) *satisfies* (19) *in the real norm $|X(t)|_H$.*

**Remark 3.** It should be emphasized that there is a close connection between the unique continuation property of eigenfunctions $\varphi_j$ (or $\varphi_j^*$) of the Oseen–Stokes operator and the above construction of a stabilizing finite-dimensional feedback controller (3). In fact, as seen above, the design of $u$ in the form (15) or (28) is essentially based on this sharp property through existence in the algebraic system (17).

It should be mentioned also that, likewise the corresponding Ito noise controller (see [5]), the stabilizable feedback controller (3) is robust.

**Remark 4.** One might speculate that the noise feedback controller

$$u = \mathbf{1}_{O_0} \sum_{k=1}^M R_k(X - X_e) \circ d\beta_k \tag{29}$$

inserted in the right hand side of Navier–Stokes system (2) stabilizes exponentially in probability the equilibrium solution $X_e$ to (2). In general, this might not be true, but it happens for Ito noise of the form (29) for any $x$ in a sufficiently small neighborhood of $X_e$ (see [8]) and one might expect that the fixed point argument used there for the equivalent random system is still applicable in the present case. We expect to give details in a later work.

## 4. Conclusions

We have designed in this paper a Stratonovich stochastic feedback controller which exponentially stabilizes in probability a general Oseen–Stokes system from fluid dynamics. The controller has the support in an arbitrary open subset $O_0$ of the velocity field domain $O \subset \mathbb{R}^d$, $d = 2, 3$, and has a finite-dimensional linear structure which involves the dual eigenfunctions corresponding to unstable eigenvalues of the systems. The stabilization effect is independent of the Reynold number $1/\nu$ through the dimension $N$ of the stabilizing controller (3) might depend on $\nu$.

## References

[1] L. Arnold, *Stochastic Differential Equations; Theory and Applications*, Willey, 1974.

[2] L. Arnold, Stabilization by noise revisited, *Z. Angew. Math. Mech.* 70 (1990), 235-246.

[3] L. Arnold, H. Crauel, V. Wihstutz, Stabilization of linear systems by noise, *SIAM J. Contr. Opt.*, 21 (1983), 451-461.

[4] V. Barbu, R. Triggiani, Internal stabilization of Navier–Stokes equations with finite-dimensional controllers, *Indiana Univ. Math. J.*, vol. 53 (2004), 1443-1494.

[5] V. Barbu, *Stabilization of Navier–Stokes Flows*, Springer, London, 2010.

[6] V. Barbu, The internal stabilization by noise of the linearized Navier-Stokes equation, *ESAIM COCV*, vol. 41 (2011), 1-21.

[7] V. Barbu, Stabilization of a plane periodic channel flow by noise wall normal controllers, *System & Control Letters* 59 (2010), 608-614.

[8] V. Barbu, G. Da Prato, Internal stabilization of the Navier–Stokes equation, *SIAM J. Control Optimiz.*, 17 (2011), 117-130.

[9] T. Caraballo, J.A. Langa, Comparison of the long time-behavior of linear Ito and Stratonovich partial differential equations, *Stoch. Anal. Appl.*, 19 (2) (2001), 183-195.

[10] T. Caraballo, J.C. Robinson, Stabilization of linear PDEs by Stratonovich noise, *Syst. Control Lett.*, 53(2004), 41-50.

[11] T. Caraballo, H. Crauel, J.A. Langa and J.C. Robinson, The effect of noise of the Chafee-Infante equation; a nonlinear study case, *Proceedings Amer. Math. Soc.*, 135 (2007), 373-382.