

# Stabilization of stochastic approximation by step size adaptation

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**Abstract:** A scheme for stabilizing stochastic approximation iterates by adaptively scaling the step sizes is proposed and analyzed. This scheme leads to the same limiting differential equation as the original scheme and therefore has the same limiting behavior, while avoiding the difficulties associated with projection schemes. The proof technique requires only that the limiting o.d.e. descend a certain Lyapunov function outside an arbitrarily large bounded set.

**Key words:** stochastic approximation, almost sure boundedness, step size adaptation, limiting o.d.e.

## 1 Introduction

Stochastic approximation was originally introduced in [15] as a scheme for finding zeros of a nonlinear function under noisy measurements. It has since become one of the main workhorses of statistical computation, signal processing, adaptive schemes in control engineering and artificial intelligence, economic models, and so on. See [4], [7], [9], [11], [13] for some recent texts that give an extensive account. One of the successful approaches for its convergence analysis has been the ‘o.d.e. approach’ of [10], [14] which treats it as

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a noisy discretization of an ordinary differential equation (o.d.e.) with slowly decreasing step sizes. The convergence analysis is usually of the form: if the iterates remain stable, i.e., a.s. bounded, then they converge a.s. to a set predicted by the o.d.e. analysis. Stability tests that establish a.s. boundedness are typically geared for specific applications and require stringent assumptions on the ‘drift’ term. See, e.g., [1], [8], [16] for some recent stability tests motivated by reinforcement learning applications, that crucially use resp. long term stability w.r.t. initial data, exact linear growth, or contraction-like properties for the drift. There does not seem to be a broad enough test to cover a reasonably generic class of stochastic approximation algorithms.

An alternative to establishing a priori stability is to force it by suitably modifying the algorithm, the most popular modification being to project it onto a bounded set every time it exits from the same [12], [9]. This, however, is not without its pitfalls. One major problem is that the projection operation can introduce spurious equilibria. Another is that the choice of the bounded set in question needs to be carefully done, in particular it should include the desired asymptotic limit (point or set) which is usually not known a priori.

Motivated by this, we propose and analyze a different scheme for stabilizing the iterates, viz., an adaptation of step sizes that controls the growth of the iterates without affecting their asymptotic behavior. This amounts to scaling the step sizes appropriately when the iterates are sufficiently far away from the origin. In fact, one can argue that at most a finite random number of steps differ from the original scheme.

Another offshoot of our analysis is that instead of requiring the o.d.e. to descend the Lyapunov function everywhere where the function isn’t at its minimum, we only require it to do so outside a sphere of arbitrarily large radius. While this is hardly surprising, the fact does not seem to have been formally recorded in literature.

## 2 Preliminaries

Throughout this article we allow the letter  $c$  to denote a possibly different constant in different places.

Consider the  $\mathbb{R}^d$ -valued stochastic approximation iterates

$$x_{n+1} = x_n + a(n)[h(x_n) + M_{n+1}], \quad (1)$$

and their ‘o.d.e.’ limit

$$\dot{x}(t) = h(x(t)). \quad (2)$$

Let  $W(\cdot) : \mathbb{R}^d \rightarrow [0, \infty)$  be a continuously differentiable Lyapunov function. We make the following assumptions regarding  $h(\cdot)$ ,  $a(n)$ ,  $M_{n+1}$ , and  $W(\cdot)$

**(A1)**  $h(\cdot)$  is locally Lipschitz.

**(A2)** Step size assumptions.

- (i)  $\sum_n a_n = \infty$ .
- (ii)  $\sum_n a_n^2 < \infty$ .

**(A3)** Martingale difference assumptions.

- (i)  $(M_n)$  is a martingale difference sequence w.r.t. the filtration  $(\mathcal{F}_n)$  where  $\mathcal{F}_n = \sigma(x_0, M_1, \dots, M_n)$ . Thus,  $E[M_{n+1}|\mathcal{F}_n] = 0$  a.s. for all  $n \geq 0$ .
- (ii)  $M_n$  is square integrable for all  $n \geq 0$  and there exists a locally bounded and measurable function  $f(\cdot) : \mathbb{R}^d \rightarrow [0, \infty)$  such that

$$E[\|M_{n+1}\|^2|\mathcal{F}_n] \leq f(x_n) \text{ a.s.}$$

**(A4)** Lyapunov function assumptions.

- (i)  $W(x) \geq 0$  for all  $x \in \mathbb{R}^d$  and  $W(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ .
- (ii) There exists a positive integer, say  $M$ , such that

$$h(x) \cdot \nabla W(x) < 0 \text{ whenever } W(x) \geq M.$$

We next define a generalization of the iteration scheme (1). First, choose a positive integer  $N$ , with  $M < N \leq \infty$ , such that there is a finite positive constant  $c_N$  satisfying

$$c_N > 1 \bigvee \left( \sup_{y \in \bar{H}^N \setminus H^M} \frac{\|h(y)\|^2 + f(y)}{W(y)} \right). \quad (3)$$

At least for finite  $N$ , assumptions **(A1)** and **(A3)**(ii) guarantee such a choice for  $c_N$ . Having chosen a suitable  $N$ , choose a locally bounded measurable function  $g(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$g(y) > 1 \bigvee \left( \mathbb{I} \{W(y) > N\} \sqrt{\frac{\|h(y)\|^2 + f(y)}{W(y)}} \right). \quad (4)$$

Again, assumptions **(A1)** and **(A3)**(ii) guarantee such a choice for  $g(\cdot)$ . We thus have, for some suitable  $N$ , possibly infinite, the following inequality

$$c_N W(y) > \frac{\|h(y)\|^2 + f(y)}{g(y)^2} \text{ if } W(y) \geq M. \quad (5)$$

Having chosen  $g(\cdot)$ , consider the iterates  $\{y_n\}$  generated by

$$y_{n+1} = y_n + a^\omega(n)[h(y_n) + M_{n+1}], \quad (6)$$

where

$$a^\omega(n) := a(n)/g(y_n). \quad (7)$$

This is a generalization of the original iteration scheme (1) since the step size  $a^\omega(n)$  is now an  $\mathcal{F}_n$ -measurable random step size. We note that by our choice

- $g(\cdot)$  is a locally bounded function, and
- $g(y) \geq 1$  for all  $y \in \mathbb{R}^d$ .

**Remark 1.** By choosing  $N$  large enough we can ensure  $g(y) = 1$  for  $y$  in an arbitrarily large sphere around the origin. If  $c_\infty < \infty$ , we can choose  $N = \infty$ , in which case  $g(y) = 1$  for all  $y \in \mathbb{R}^d$  and we recover the original scheme (1).

**Remark 2.** Since  $g(y) \geq 1$  for all  $y \in \mathbb{R}^d$ , it follows from assumption **(A2)**(ii) that the random step sizes satisfy

$$\sum_n a^\omega(n)^2 < \infty \text{ a.s.}$$

### 3 A test for stability

Let  $m$  be an arbitrary positive integer,  $m > M$ . Define the level set

$$H^m := \{x : W(x) < m\},$$

and let  $\bar{H}^m$  denote the closure of  $H^m$ . Since  $h(x) \cdot \nabla W(x) < 0$  whenever  $W(x) \geq M$ , we get

$$\dot{W}(x) := h(x) \cdot \nabla W(x) < 0 \text{ for } x \in \bar{H}^m \setminus H^M.$$

As  $\bar{H}^m \setminus H^M$  is a compact set, and  $\dot{W}(\cdot)$  is a continuous function, there must exist a negative constant  $c$  such that

$$\sup_{x \in \bar{H}^m \setminus H^M} \dot{W}(x) \leq c < 0. \quad (8)$$

Fix some  $T > 0$ . Note that **(A1)** and **(A4)** ensure the well-posedness of the o.d.e. given by (2) for  $t \geq 0$ . Let  $y^u(t)$  be the o.d.e. trajectory starting from  $u$ . Thus,  $\dot{y}^u(t) = h(y^u(t))$  for  $t \geq 0$ , and  $y^u(0) = u$ . Choose a positive but arbitrarily small  $\epsilon_m$  satisfying

$$\epsilon_m \leq 1 \wedge \inf \{ |W(u) - W(v)| : u, v \in \bar{H}^m \setminus H^M \text{ and } v = x^u(T) \}.$$

Note that  $\epsilon_m > 0$  is possible because of (8). Given  $\epsilon_m$ , choose a positive but arbitrarily small  $\delta_m$  such that:

$$\text{if } u, v \in \bar{H}^m \text{ and } \|u - v\| < \delta_m, \text{ then } |W(u) - W(v)| < \epsilon_m/2.$$

Note that  $\delta_m > 0$  is possible because  $W(\cdot)$  is a continuous function and  $\bar{H}^m$  is a compact set.

**Remark 3.** *The fact that both  $\epsilon_m$  and  $\delta_m$  can be chosen positive but arbitrarily small will prove crucial later.*

Let  $n_0 \geq 0$ . Given  $n_i(\omega)$ , define  $n_{i+1}(\omega)$  as

$$n_{i+1}(\omega) := \inf \left\{ n > n_i(\omega) : \sum_{n_i(\omega)}^n a^\omega(i) \geq T \right\}.$$

Consider the  $\delta_m$ -neighbourhood of  $H^m$ ,

$$N^{\delta_m}(H^m) := \left\{ x : \inf_{y \in H^m} \|x - y\| < \delta_m \right\}.$$

Note that  $\mathbb{I}\{y_n \in N^{\delta_m}(H^m)\} a^\omega(n) M_{n+1}$  is a martingale difference term. Since  $N^{\delta_m}(H^m)$  is a bounded set, and  $f(\cdot)$  is locally bounded, it follows from assumption **(A3)**(ii) and Remark 2 that

$$\begin{aligned} & \sum_n \mathbb{E} \left[ \left( \mathbb{I}\{y_n \in N^{\delta_m}(H^m)\} a^\omega(n) M_{n+1} \right)^2 \middle| \mathcal{F}_n \right] \\ & \leq \left( \sup_{y \in N^{\delta_m}(H^m)} f(\|y\|) \right) \times \sum_n a^\omega(n)^2 \\ & < \infty \text{ a.s.} \end{aligned} \tag{9}$$

This leads to:

**Lemma 4.** *Assume **(A2)**–**(A4)**. For any positive integer  $m > M$  we have:*

$$\sum_i \mathbb{I}\{y_n \in N^{\delta_m}(H^m)\} a^\omega(n) M_{n+1} \text{ converges a.s.}$$

*Proof.* This is immediate from (9) and the convergence theorem for square-integrable martingales, Theorem 3.3.4, p. 53, of [5].  $\square$

From Lemma 4 it follows that almost surely there exists an  $N(\omega, m)$  such that if  $n_0(\omega) \geq N(\omega, m)$  then

$$\sup_q \left\| \sum_{n_0(\omega)}^q \mathbb{I}\{y_n \in N^{\delta_m}(H^m)\} a^\omega(n) M_{n+1} \right\| < \frac{\delta_m}{2 \exp(KT)}. \tag{10}$$

**Remark 5.** *Note that Lemma 4 guarantees the convergence of the martingale  $\sum_i \mathbb{I}\{\dots\} a^\omega(n) M_{n+1}$  while not saying anything about  $\sum_n a^\omega(n)$ . Since the martingale converges, there must exist an  $N(\omega, m)$  satisfying (10) even if  $\sum_n a^\omega(n) < \infty$ . That  $\sum_n a^\omega(n) = \infty$  a.s. needs a proof. In what follows, we give a sufficient condition for stability and show that it is also sufficient for  $\sum_n a^\omega(n) = \infty$  a.s.*

Let  $K$  be the Lipschitz constant of  $h(\cdot)$  on  $N^{\delta_m}(H^m)$ . Without loss of generality we assume that  $N(\omega, m)$  is large enough that if  $n_0(\omega) \geq N(\omega, m)$ . Then

$$K \left( \sup_{y \in N^{\delta_m}(H^m)} \|h(y)\| \right) \left( \sum_{n_0(\omega)}^{\infty} a^\omega(n)^2 \right) < \frac{\delta_m}{2 \exp(KT)}. \quad (11)$$

**Lemma 6.** Assume **(A1)–(A4)**. Let  $m$  be a positive integer with  $m > M$ . Let  $n_0(\omega)$  satisfy  $n_0(\omega) \geq N(\omega, m)$ . Under this base condition, the following inductive step holds: if  $n_i(\omega) < \infty$  and  $y_{n_i}(\omega) \in H^m$ , then

1.  $y_j(\omega) \in N^{\delta_m}(H^m)$  a.s. for  $n_i(\omega) \leq j \leq n_{i+1}(\omega)$ ,
2.  $n_{i+1}(\omega) < \infty$  a.s., and
3. Almost surely, either
  - $W(y_{n_{i+1}}(\omega)) < W(y_{n_i}(\omega)) - \frac{\epsilon_m}{2}$ , or
  - $y_{n_{i+1}}(\omega) \in N^{\delta_m}(H^M)$ .

In particular, in either case,  $y_{n_{i+1}}(\omega) \in H^m$  a.s.

*Proof.* We first show by induction that  $y_j(\omega) \in N^{\delta_m}(H^m)$  for  $n_i(\omega) \leq j \leq n_{i+1}(\omega)$ . By assumption,  $y_{n_i}(\omega) \in H^m \subset N^{\delta_m}(H^m)$ . Fix  $j$  in the range  $n_i(\omega) < j \leq n_{i+1}(\omega)$ . Assume  $y_k(\omega) \in N^{\delta_m}(H^m)$  for  $n_i(\omega) \leq k \leq j-1$ . We need to show that  $y_j(\omega) \in N^{\delta_m}(H^m)$ . If

$$K \left( \sup_{n_i \leq k \leq j-1} \|h(y_k)\| \right) \left( \sum_{n_i(\omega)}^{j-1} a^\omega(n)^2 \right) < \frac{\delta_m}{2 \exp(KT)}, \quad (12)$$

and

$$\sup_{n_i \leq k \leq j-1} \left\| \sum_{n_i(\omega)}^k a^\omega(n) M_{n+1} \right\| < \frac{\delta_m}{2 \exp(KT)}, \quad (13)$$

then by a standard application of the Gronwall inequality (see, e.g., Lemma 2.1 in [7])  $y_j(\omega)$  will satisfy

$$\left\| y_j(\omega) - y^{y_{n_i}(\omega)} \left( \sum_{n_i(\omega)}^{j-1} a^\omega(n) \right) \right\| < \delta_m. \quad (14)$$

From the assumption that  $n_i(\omega) \geq N(\omega, m)$  it follows that (10) and (11) hold. These equations, coupled with the assumption that  $y_k(\omega) \in N^{\delta_m}(H^m)$  for  $n_i(\omega) \leq k \leq j-1$ , imply (12) and (13), which in turn imply (14). Since the o.d.e. trajectory will always be in  $H^m$  if it starts there, (14) implies

$$y_j(\omega) \in N^{\delta_m}(H^m).$$

Induction now proves the first claim.

For the second claim we give a proof by contradiction. Consequently, assume that  $n_{i+1}(\omega) = \infty$ . The first claim, which has already been proved, now gives  $y_j(\omega) \in N^{\delta_m}(H^m)$  a.s. for  $n_i(\omega) \leq j \leq \infty$ . Therefore, since  $g(\cdot)$  is a locally bounded function, we get  $\sup_{j \geq n_i(\omega)} g(y_j(\omega)) < \infty$ . By assumption **(A2)**(i) this gives

$$\sum_{j=n_i(\omega)}^{\infty} a^\omega(j) \geq \frac{\sum_{j=n_i(\omega)}^{\infty} a(j)}{\sup_{j \geq n_i(\omega)} g(y_j(\omega))} = \infty.$$

Since  $n_{i+1}(\omega) = \infty$  requires  $\sum_{j=n_i(\omega)}^{\infty} a^\omega(j) \leq T$ , we get the required contradiction. Thus  $n_{i+1}(\omega) < \infty$  a.s.

We turn to the final claim. Let  $z = y^{y_{n_i}(\omega)}(\sum_{n_i}^{n_{i+1}-1} a^\omega(n))$ , the o.d.e. trajectory after time  $\approx T$  starting from  $y_{n_i}(\omega)$ . Since the o.d.e. starts in  $H^m$ , it remains in  $H^m$ . There are two cases to consider.

- If  $z \in H^m \setminus H^M$ , the definition of  $\epsilon_m$  implies that  $W(z) \leq W(y_{n_i}(\omega)) - \epsilon_m$ . Since (14) holds for  $j = n_{i+1}(\omega)$ , we have  $\|y_{n_{i+1}}(\omega) - z\| < \delta_m$ . From the definition of  $\delta_m$  it follows that  $W(y_{n_{i+1}}(\omega)) < W(z) + \epsilon_m/2$ . We get  $W(y_{n_{i+1}}(\omega)) < W(y_{n_i}(\omega)) - \frac{\epsilon_m}{2}$ . In particular,  $y_{n_{i+1}}(\omega) \in H^m$ .
- If  $z \in H^M$ , then, since  $\|y_{n_{i+1}}(\omega) - z\| < \delta_m$ , we get  $y_{n_{i+1}}(\omega) \in N^{\delta_m}(H^M)$ . Since  $N^{\delta_m}(H^M) \subset H^{M+\frac{\epsilon_m}{2}}$  and  $m > M + 1/2$ , we get  $y_{n_{i+1}}(\omega) \in H^m$ .

The proof is complete.  $\square$

Define the stopping times

$$\tau_k^m(\omega) := \inf\{n \geq k : W(y_n(\omega)) < m\}.$$



The next result establishes the fact that if  $W(y_n(\omega)) < m$  for infinitely many  $n$ , then almost surely the iterates converge to  $\bar{H}^M$ .

**Proposition 7.** *Assume (A1)–(A4). For any arbitrary  $m > M$ , if  $\tau_k^m(\omega) < \infty$  for all  $k$ , then  $y_n(\omega) \rightarrow \bar{H}^M$  a.s.*

*Proof.* Assume  $\tau_k^m(\omega) < \infty$  for all  $k$ . From the definition of  $\tau_k^m$  this implies that given any  $k$  there exists an  $n$  with  $n \geq k$  such that  $y_n(\omega) \in H^m$ . In other words, the iterates are in  $H^m$  infinitely often. By Remark 5 there exists an  $N(\omega, m)$  satisfying (10). Since the iterates are in  $H^m$  infinitely often there exists an  $n_0 > N(\omega, m)$  such that  $y_{n_0} \in H^m$ . From Lemma 6 we know that if  $n_i(\omega) < \infty$  then almost surely  $n_{i+1}(\omega) < \infty$ . By induction it follows that  $n_i(\omega) < \infty$  a.s. for all  $i \in \mathbb{Z}^+$ . Invoking Lemma 6 again, we get that either  $W(y_{n_{i+1}}(\omega)) < W(y_{n_i}(\omega)) - \frac{\epsilon_m}{2}$ , or  $y_{n_{i+1}}(\omega) \in N^{\delta_m}(H^M)$ . Since  $W(\cdot)$  cannot keep decreasing by  $\epsilon_m/2$  forever, it follows that for some  $i$ ,  $y_{n_i}(\omega) \in N^{\delta_m}(H^M)$ . Note that  $N^{\delta_m}(H^M) \subset H^{M+\frac{\epsilon_m}{2}}$ . Consequently, if  $y_{n_i}(\omega) \in N^{\delta_m}(H^M)$  then  $W(y_{n_i}(\omega)) < M + \frac{\epsilon_m}{2}$  and so  $y_{n_{i+1}}(\omega) \in N^{\delta_m}(H^M)$ . It follows that the iterates  $y_{n_i}(\omega)$  will eventually get trapped in  $N^{\delta_m}(H^M)$ . Once the iterates  $y_{n_i}(\omega)$  are trapped in  $N^{\delta_m}(H^M) \subset H^{M+\frac{\epsilon_m}{2}}$ , the o.d.e. starting from  $y_{n_i}(\omega)$  will remain in  $H^{M+\frac{\epsilon_m}{2}}$ . It follows that once the iterates  $y_{n_i}(\omega)$  are trapped in  $N^{\delta_m}(H^M)$ , the intermediate iterates  $y_j(\omega)$ ,  $n_i(\omega) < j < n_{i+1}(\omega)$  will get trapped in  $N^{\delta_m}(H^{M+\frac{\epsilon_m}{2}})$ . Since both  $\epsilon_m$  and  $\delta_m$  were chosen arbitrarily small positive quantities (see Remark 3), the result follows.  $\square$

Consider two statements of stability: first

$$y_n(\omega) \rightarrow \bar{H}^M \text{ a.s.} \quad (15)$$

and second, for every positive integer  $k \geq 0$ ,

$$y_{n \wedge \tau_k^M}(\omega) \rightarrow \bar{H}^M \text{ a.s.} \quad (16)$$

The next result establishes the equivalence of the two stability statements.

**Lemma 8.** *Under assumptions (A1)–(A4), the two stability statements (15) and (16) are equivalent.*

*Proof.* Clearly (15) implies (16). For the converse, assume (16). We need to show that

$$\mathbb{P} \left[ y_{n \wedge \tau_k^M}(\omega) \rightarrow \bar{H}^M \ \forall k \text{ and } y_n(\omega) \not\rightarrow \bar{H}^M \right] = 0.$$

Fix an  $m > M$ . Let  $\omega$  be such that  $y_{n \wedge \tau_k^M}(\omega) \rightarrow \bar{H}^M \forall k$  and  $y_n(\omega) \not\rightarrow \bar{H}^M$ . Since  $y_n(\omega) \not\rightarrow \bar{H}^M$ , by Proposition 7 there exists a  $k$  such that  $\tau_k^m(\omega) = \infty$  a.s. For this choice of  $k$ , since  $m > M$ , it follows that  $\tau_k^M(\omega) = \infty$  a.s. Thus for this  $k$ ,  $y_{n \wedge \tau_k^M}(\omega) \rightarrow \bar{H}^M$  reduces to  $y_n(\omega) \rightarrow \bar{H}^M$  a.s. The result follows.  $\square$

On the basis of Lemma 8, we get the following test for stability: for every  $k$ , if  $y_{n \wedge \tau_k^M}(\omega) \rightarrow \bar{H}^M$  a.s. then  $\sup_n \|y_n\| < \infty$  a.s. Note that it does not require the o.d.e. to descend the Lyapunov function inside the arbitrarily large set  $H^M$ . In the next section we give a sufficient condition for this stability test.

## 4 A sufficient condition for stability

In this section we show that assumption **(A5)** below is sufficient for stability.

**(A5)** Let the  $W(\cdot)$  of **(A4)** be twice continuously differentiable such that all second order derivatives of  $W(\cdot)$  are bounded in absolute value by a constant.

We start with a few lemmas.

**Lemma 9.** Assume **(A1)**–**(A5)**. For any positive integer  $k$ , and for any  $\mathcal{F}_k$ -measurable set  $A$ , if  $\mathbb{E}[W(y_k(\omega); A)] < \infty$  then

$$\sup_{n \geq k} \mathbb{E} \left[ W \left( y_{n \wedge \tau_k^M}(\omega) \right); A \right] < \infty.$$

*Proof.* We have

$$y_{(n+1) \wedge \tau_k^M} = y_{n \wedge \tau_k^M} + a^\omega(n) \mathbb{I}\{\tau_k^M > n\} \left[ h \left( y_{n \wedge \tau_k^M} \right) + M_{n+1} \right].$$

Doing a Taylor expansion and using the fact that the second order space derivatives of  $W(\cdot)$  are bounded, we get

$$\begin{aligned} & W \left( y_{(n+1) \wedge \tau_k^M} \right) \\ \leq & W \left( y_{n \wedge \tau_k^M} \right) + a^\omega(n) \mathbb{I}\{\tau_k^M > n\} \nabla W \left( y_{n \wedge \tau_k^M} \right) \cdot \left[ h \left( y_{n \wedge \tau_k^M} \right) + M_{n+1} \right] \\ & + ca^\omega(n)^2 \mathbb{I}\{\tau_k^M > n\} \left\| h \left( y_{n \wedge \tau_k^M} \right) + M_{n+1} \right\|^2. \end{aligned}$$

Since  $\mathbb{I}\{\tau_k^M > n\} \nabla W(y_{n \wedge \tau_k^M}) \cdot h(y_{n \wedge \tau_k^M}) \leq 0$  and  $\mathbb{E}[h(y_{n \wedge \tau_k^M}) \cdot M_{n+1} | \mathcal{F}_n] = 0$ , we get

$$\begin{aligned} & \mathbb{E}\left[W(y_{(n+1) \wedge \tau_k^M}) | \mathcal{F}_n\right] \\ & \leq W(y_{n \wedge \tau_k^M}) + ca^\omega(n)^2 \left( \mathbb{E}\left[\mathbb{I}\{\tau_k^M > n\} \left( \|h(y_{n \wedge \tau_k^M})\|^2 + \|M_{n+1}\|^2 \right) | \mathcal{F}_n \right] \right). \end{aligned}$$

From (5) and the definition of  $a^\omega(n)$ , it follows that

$$\begin{aligned} E\left[W(y_{(n+1) \wedge \tau_k^M}) | \mathcal{F}_n\right] & \leq W(y_{n \wedge \tau_k^M}) + ca(n)^2 \cdot c_N W(y_{n \wedge \tau_k^M}) \\ & \leq (1 + ca(n)^2) W(y_{n \wedge \tau_k^M}) \\ & \leq \exp(ca(n)^2) W(y_{n \wedge \tau_k^M}) \end{aligned}$$

For  $n \geq k$ , integrating gives

$$E\left[W(y_{(n+1) \wedge \tau_k^M}); A\right] \leq \exp\left(c \sum_{i=k}^{\infty} a(i)^2\right) \mathbb{E}[W(y_k(\omega); A)] < \infty.$$

The result follows.  $\square$

The next lemma is independent of assumption **(A5)** and requires only assumptions **(A1)**–**(A4)** for its proof.

**Lemma 10.** *Assume **(A1)**–**(A4)**. Let  $k$  be an arbitrary positive integer. Let  $A$  be an arbitrary  $\mathcal{F}_k$ -measurable set. If*

$$\sup_{n \geq k} \mathbb{E}\left[W(y_{n \wedge \tau_k^M}(\omega)); A\right] < \infty$$

then

$$\mathbb{P}\left[A \cap \left(y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M\right)\right] = 0.$$

*Proof.* Assume  $y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M$ . Clearly, this implies that  $\tau_k^M(\omega) = \infty$  and so  $y_{n \wedge \tau_k^M}(\omega) = y_n(\omega)$ . It follows that  $y_n(\omega) \not\rightarrow \bar{H}^M$ . Now, let  $u$  be an arbitrary integer,  $u > M$ . By Proposition 7, since  $y_n(\omega) \not\rightarrow \bar{H}^M$ , there exists an integer  $l$ ,  $l \geq k$ , such that  $\tau_l^u(\omega) = \infty$  a.s. It follows that

$$\left\{A \cap \left(y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M\right)\right\} = \bigcup_l \left\{A \cap \left(y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \text{ and } \tau_l^u(\omega) = \infty\right)\right\}.$$

Since  $\{\tau_l^u(\omega) = \infty\} \subset \{\tau_{l+1}^u(\omega) = \infty\}$  for all  $l$ , it follows that there exists a positive integer  $L \geq k$  such that

$$\mathbb{P} \left[ A \cap \left( y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \text{ and } \tau_L^u(\omega) = \infty \right) \right] > \frac{1}{2} \times \mathbb{P} \left[ A \cap \left( y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \right) \right].$$

Combining everything we get the following inequalities

$$\begin{aligned} & \sup_{n \geq k} \mathbb{E} \left[ W(y_{n \wedge \tau_k^M}(\omega)); A \right] \\ & \geq \mathbb{E} \left[ W(y_{L \wedge \tau_k^M}(\omega)); A \right] \\ & \geq u \times \mathbb{P} \left[ A \cap \left( \tau_k^M(\omega) = \infty \text{ and } \tau_L^u(\omega) = \infty \right) \right] \\ & \geq u \times \mathbb{P} \left[ A \cap \left( y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \text{ and } \tau_L^u(\omega) = \infty \right) \right] \\ & \geq \frac{u}{2} \times \mathbb{P} \left[ A \cap \left( y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \right) \right]. \end{aligned}$$

Since  $u$  is arbitrary and  $\sup_{n \geq k} \mathbb{E} \left[ W(y_{n \wedge \tau_k^M}(\omega)); A \right] < \infty$ , the result follows.  $\square$

**Lemma 11.** *Assume (A1)–(A5). For  $k$  an arbitrary positive integer, we have*

$$\mathbb{P} \left[ y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \right] = 0.$$

*Proof.* Define  $A^l := \{\omega : W(y_k(\omega)) < l\}$ . Clearly  $A^l$  is  $\mathcal{F}_k$ -measurable and  $\mathbb{E}[W(y_k(\omega)); A^l] < l < \infty$ . It follows from Lemma 9 that

$$\sup_{n \geq k} \mathbb{E} \left[ W(y_{n \wedge \tau_k^M}(\omega)); A^l \right] < \infty.$$

Lemma 10 now gives us

$$\mathbb{P} \left[ A^l \cap \left( y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \right) \right] = 0.$$

Since  $\mathbb{P} \left[ \bigcup_l A^l \right] = 1$  it follows that  $\mathbb{P} \left[ y_{n \wedge \tau_k^M}(\omega) \not\rightarrow \bar{H}^M \right] = 0$   $\square$

We now give our main results and a couple of examples.

**Theorem 12.** *Under assumptions (A1)–(A5),*

$$y_n(\omega) \rightarrow \bar{H}^M \text{ a.s.}$$

*In particular,  $\sup_n \|y_n\| < \infty$  a.s.*

*Proof.* The result follows from Lemma 8 and Lemma 11.  $\square$

The next results establish that the iterates  $(y_n)$  indeed capture behaviour as time goes to infinity.

**Proposition 13.** *Under assumptions (A1)–(A5), almost surely  $a^\omega(n) = a(n)$  for all except finitely many  $n$ . In particular,*

$$\sum_n a^\omega(n) = \infty \text{ a.s.}$$

*Proof.* By Theorem 12,  $y_n(\omega) \rightarrow \bar{H}^M$  a.s. Since  $g(y) = 1$  for  $y \in H^N$ , and  $N > M$ , it follows that  $g(y_n) = 1$  for all except finitely many  $n$ .  $\square$

Finally, following [3] (see also [7], Chapter 2), we get

**Theorem 14.** *Under assumptions (A1)–(A5), the iterates  $(y_n)$  converge a.s. to an internally chain transitive set of the o.d.e.*

We also get a condition for the convergence of the iterates  $(x_n)$  obtained by the original iteration scheme as given by (1).

**Theorem 15.** *Under assumptions (A1)–(A5), if*

$$\sup_{x \in \mathbb{R}^d} \frac{\|h(x)\|^2 + f(x)}{1 \wedge W(x)} = c_\infty < \infty$$

*then the original iterates  $(x_n)$  converge a.s. to an internally chain transitive set of the o.d.e.*

*Proof.* By Remark 1 we can set  $N = \infty$  in (3). Now (4) gives  $g(x) = 1$  for all  $x \in \mathbb{R}^d$ . Equation (5) continues to hold with  $c_\infty$  in place of  $c_N$ . The choice of  $g(\cdot)$  gives  $a^\omega(n) = a(n)$  for all  $n$ , or  $x_n(\omega) = y_n(\omega)$  for all  $n$ . The result now follows from Theorem 14.  $\square$

**Example 16.** *Consider the scalar iteration*

$$x_{n+1} = x_n - a(n)x_n \exp(|x_n|)(1 + \xi_{n+1}),$$

*where  $\{\xi_n\}$  are i.i.d.  $N(0, 1)$  (say). Here  $W(x) = x^2$  and  $g(x) = O(\exp(|x|))$  will do.*

**Example 17.** Consider the scalar iteration (1) with bounded  $h(\cdot)$  satisfying

$$\lim_{x \uparrow \infty} h(x) = - \lim_{x \downarrow -\infty} h(x) = -1,$$

with  $\{M_n\}$  i.i.d. uniform on  $[-1, 1]$ . Then  $W(x) = x^2$  and  $g(x) \equiv 1$  will do. In particular, there is no need to adaptively scale the step sizes.

Note that neither of these two examples, even the apparently simple Example 17, is covered by the tests of [1], [8], [16].

**Acknowledgements:** The author would like to thank Prof. V. S. Borkar for suggesting this problem and for his comments on an earlier draft which included, in particular, the idea of using an adaptive scheme.

## References

- [1] ABOUNADI, J; BERTSEKAS, D. P.; AND BORKAR V. S. (2002) Stochastic approximation for nonexpansive maps: Application to Q-Learning algorithms, *SIAM J. Control Optim.* Volume 41 , Issue 1, pp. 1–22.
- [2] ANDRIEU, C.; MOULINES, É.; AND PRIOURET, P. (2005) Stability of stochastic approximation under verifiable conditions, *SIAM J. Control Optim.* Volume 44, Issue 1, pp. 283–312.
- [3] BENAÏM, M. (1999) Dynamics of stochastic approximation algorithms, *Séminaire de Probabilités XXXIII*, 1–68, Springer.
- [4] BENVENISTE, A.; METIVIER, M.; AND PRIOURET, P. (1990) *Adaptive Algorithms and Stochastic Approximation*, Springer, Berlin-Heidelberg.
- [5] BORKAR, V. S. (1995) *Probability Theory: An Advanced Course (Universitext)*, Springer-Verlag, New York.
- [6] BORKAR, V. S. (2002) On the lock-in probability of stochastic approximation, *Combinatorics, Probability and Computing* 11, 11–20.
- [7] BORKAR, V. S. (2008) *Stochastic Approximation: A Dynamical Systems Viewpoint*, Hindustan Publ. Co., New Delhi, and Cambridge University Press, Cambridge, UK.

- [8] BORKAR, V. S.; AND MEYN, S. P. (2000) The O.D.E. method for convergence of stochastic approximation and reinforcement learning, *SIAM J. Control Optim.* Volume 38, Issue 2, pp. 447–469.
- [9] CHEN, H.-F. (2002) *Stochastic Approximation and its Applications*, Kluwer Academic, Dordrecht.
- [10] DEREVITSKII, D. P.; AND FRADKOV, A. L. (1974) Two models for analyzing the dynamics of adaptation algorithms, *Automation and Remote Control* 35, 59–67.
- [11] DUFLO, M. (1997) *Random Iterative Models*, Springer, Berlin-Heidelberg.
- [12] KUSHNER, H. J.; AND CLARK, D. (1978) *Stochastic Approximation Algorithms for Constrained and Unconstrained Systems*, Springer New York.
- [13] KUSHNER, H. J.; AND YIN, G. (2003) *Stochastic Approximation and Recursive Algorithms and Applications* (2nd ed.), Springer, New York.
- [14] LJUNG, L. (1977) Analysis of recursive stochastic algorithms, *IEEE Trans. on Automatic Control* 22, 551–575.
- [15] ROBBINS, H.; AND MONRO, J. (1953) A stochastic approximation method, *Annals of Math. Stat.* 22, 400–407.
- [16] TSITSIKLIS, J. N. (1994) Asynchronous stochastic approximation and Q-learning, *Machine Learning* 16, 185–202.