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This paper has been published in *Systems & Control Letters,* **61** (2012) 9991002

doi:10.1016/j.sysconle.2012.07.002

The published version can be downloaded from

http://dx.doi.org/10.1016/j.sysconle.2012.07.002

Abstract

We consider a model for the heat equation with memory, which has infinite propagation speed, like the standard heat equation. We prove that, in spite of this, for every $T > 0$ there exist square integrable initial data which cannot be steered to hit zero at time T, using square integrable controls.

We show that the counterexample we present complies with the restrictions imposed by the second principle of thermodynamics.

keyword Controllability, heat equation with memory, moment method

MSC 45K05, 93B03, 93B05, 93C22

*The research of the first author is partially supported by Romanian CNCS Grant PN-II-ID-PCE-2011-3-0211. The research of the second author fits the plans of INDAM-CNR and of the project "Groupement de Recherche en Contrôle des EDP entre la France et l'Italie (CONEDP)".

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1 Introduction

Heat equation with memory has a long history and it is now an active field of research (see the book [1]). Controllability problems for this equation have been studied in recent times by many authors, with different methods (see references below). In the linear approximation, the equation boils down to the following integrodifferential equation (see for example [6]):

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t} + \gamma u(x,t) + \int_{-\infty}^{t} M(t-s)u(x,s)\,\mathrm{d}s \\ &= k_0 \Delta u(x,t) + \int_{-\infty}^{t} N(t-s)\Delta u(x,s)\,\mathrm{d}s\,. \end{aligned} \tag{1}$$

Here $x \in \Omega$, a smooth region, and Δ denotes the laplacian. In order to have a well posed problem, we need initial conditions, which are

$$u(x,t) = \phi(x,t) \quad \text{for } t < 0, \quad u(x,0) = u_0(x) \tag{2}$$

and suitable boundary conditions (in this paper we consider Dirichlet boundary conditions).

Problem (1)-(2) with boundary conditions of Dirichlet type may or may not be well posed depending on the values of the constants γ and k_0 and the properties of the kernels $M(t)$ and $N(t)$ but, even if well posed, the equations might contrast with the principles of thermodynamics: conditions have to be imposed on the constants and the kernels not only for the well posedness of the equation but also in order to have a process which is acceptable from the point of view of thermodynamic. The condition $k_0 \geq 0$ is obvious, while the assumptions on the kernels required further investigation, based on the second principle of thermodynamics, see for example [10]. We recall these conditions in Section 3.

We mention now that Eq. (1) was first introduced to model heat diffusion in materials with memory (or in the extreme case of very low temperature) but it turned out to be important also to model nonfickian diffusion in materials with complex molecular structure, like polymers. See [7, 18] for overviews on these applications.

The dynamical behavior of Eq. (1) strongly depends on whether $k_0 = 0$ or $k_0 > 0$. The case $k_0 = 0$ was proposed by Gurtin and Pipkin in [11] (and in a special case by Cattaneo) in order to have a "hyperbolic" type of behavior (in particular, finite propagation speed. This requires smooth

kernels with $N(0) > 0$.) Instead, if $k_0 > 0$ we have infinite propagation speed, more similar to the standard (memoryless) heat equation. The case $k_0 > 0$ corresponds to a general model proposed in [4, 5] by Boltzmann (and in a special case by Maxwell.)

Now we discuss the results on controllability which are available up to now. In order to study controllability, we can assume $u = 0$ for $t < 0$ so that integrals are restricted to $[0, t]$ and only the initial condition $u(x, 0) = u_0(x)$ is required. In fact, the history up to time $t_0 = 0$, if nonzero, contributes a known additive term to the equation. When studying controllability, we can assume that this contribution is equal to zero and this does not change positive results on exact controllability or negative results concerning lack of controllability. So, from the point of view of controllability, we have to study the Volterra integrodifferential equation

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \gamma u(x,t) + \int_0^t M(t-s)u(x,s)\,\mathrm{d}s \\ &= k_0 \Delta u(x,t) + \int_0^t N(t-s)\Delta u(x,s)\,\mathrm{d}s \end{aligned} \tag{3}$$

with known initial condition $u(x, 0) = u_0(x)$.

Furthermore, the control can act either on $\partial\Omega$ or in a subregion of Ω. We shall study the case that the control acts in the boundary condition of Dirichlet type.

It is easy to guess that it is not possible to control the system to the rest, i.e. to have $u(x, t) = 0$ for every t large enough (unless $u_0 = 0$) a fact rigorously proved in [12]. So, the control problem to be studied is whether we can hit any prescribed target $\xi(x)$ at a certain time T, i.e. if we can achieve $u(x, T) = \xi(x)$, starting with a known initial condition. This problem has been studied for the Gurtin-Pipkin equation (i.e. the case $k_0 = 0$) in several papers with several different methods (see [3, 13, 14, 15, 16, 17] for boundary control and [9] for internal control.) The result is similar to the results which hold for the wave equation without memory: there exists a time T such that every square integrable target can be reached using square integrable controls.

When $k_0 > 0$ the dynamical behavior of the systems is more similar to that of the heat equation, and we can conjecture analogous controllability results: using square integrable controls, a dense subset of $L^2(\Omega)$ can be hit starting from zero (a fact proved in [3] for a special and important class of

kernels) and every initial condition u_0 can be steered to hit zero at a certain time T which, in the case of the heat equation, can be taken arbitrarily small. *Our goal here is to disprove this last conjecture.* We present a simple example which has the following property: for every $T > 0$ there exist square integrable initial conditions u_0 such that the condition $u(x,T) = 0$ is not achievable using square integrable controls. In order to see this, we present a counterexample and, furthermore, we shall see that this counterexample is not artificial, since this simple model is acceptable from the thermodynamics point of view.

2 The counterexample

The counterexample is the following simple model:

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + u(x,t) - \int_0^t e^{-(t-s)} u(x,s)\,\mathrm{d}s \\
&= \frac{\partial^2 u(x,t)}{\partial x^2} + \int_0^t e^{-(t-s)} \frac{\partial^2 u}{\partial x^2}(x,s)\,\mathrm{d}s \\
&u(x,0) = \xi(x) \quad x \in [0,\pi]\,, \quad u(\pi,t) = 0 \quad \forall t \geq 0\,.
\end{aligned} \tag{4}$$

The control acts at $x = 0$:

$$u(0,t) = \sqrt{\frac{\pi}{2}} g(t) \tag{5}$$

(the multiplicative constant is used to simplify the following computations.) Define

$$u_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi u(x,t) \sin nx\,\mathrm{d}x$$

Using (4) we see that

$$u_n' = -(n^2+1)u_n - (n^2-1)\int_0^t e^{-(t-s)} u_n(s)\,\mathrm{d}s + nf(t). \tag{6}$$

where $f(t)$ and $g(t)$ are related by

$$\begin{aligned}
f(t) &= g(t) + \int_0^t e^{-(t-s)} g(s)\,\mathrm{d}s\,, \\
g(t) &= f(t) - \int_0^t e^{-2(t-s)} f(s)\,\mathrm{d}s\,.
\end{aligned} \tag{7}$$

In order to solve (6), we first study the case that $f(t)$, hence $g(t)$, is of class C^1. So, we can differentiate both the sides and we obtain

$$u_n'' = -(n^2+1)u_n' - (n^2-1)u_n + (n^2-1)\int_0^t e^{-(t-s)}u_n(s)\,\mathrm{d}s + nf'(t).$$

The integral term is computed from (6) and we get

$$u_n'' = -(n^2+2)u_n' - 2n^2u_n + nf(t) + nf'(t) \tag{8}$$

with the initial conditions

$$\begin{cases} u_n(0) = \xi_n = \sqrt{2/\pi}\int_0^\pi \xi(x)\sin nx\,\mathrm{d}x\,, \\ u_n'(0) = -(n^2+1)u_n(0) + nf(0) = -(n^2+1)\xi_n + nf(0)\,. \end{cases} \tag{9}$$

The characteristic equation of Eq. (8) is $\lambda^2 + (n^2+2)\lambda + 2n^2 = 0$ and the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -n^2$. So, the solutions of Eq. (8) are

$$\begin{aligned} u_n(t) &= Ae^{-2t} + Be^{-n^2t} + \\ &+\frac{n}{2-n^2}\int_0^t \left[e^{-n^2(t-s)} - e^{-2(t-s)}\right]\left[f(s)+f'(s)\right]\,\mathrm{d}s = \\ &= e^{-2t}\left[A + \frac{n}{2-n^2}f(0)\right] + e^{-n^2t}\left[B - \frac{n}{2-n^2}f(0)\right] \\ &+\frac{n}{2-n^2}\int_0^t f(s)\left[(1-n^2)e^{-n^2(t-s)} + e^{-2(t-s)}\right]\,\mathrm{d}s \end{aligned}$$

Imposing the initial conditions (9) we see that

$$A = \frac{n}{n^2-2}f(0) - \frac{\xi_n}{n^2-2}\,, \qquad B = \frac{n^2-1}{n^2-2}\xi_n - \frac{n}{n^2-2}f(0)\,.$$

Hence

$$\begin{aligned} u_n(t) &= \left(\frac{1}{2-n^2}e^{-2t} + \frac{n^2-1}{n^2-2}e^{-n^2t}\right)\xi_n + \\ &+\frac{n}{2-n^2}\int_0^t \left[(1-n^2)e^{-n^2(t-s)} + e^{-2(t-s)}\right] f(s)\,\mathrm{d}s. \end{aligned} \tag{10}$$

Note that the transformation $f \to u_n$ is affine and continuous from $L^2(0,T)$ to $C(0,T)$ for every $T \geq 0$. Hence, (10) is the solution of Eq. (8) with initial condition $u_n(0) = \xi_n$ for every square integrable f.

If there exists $f \in L^2(0,T)$ such that $u(\cdot,T) = 0 \in L^2(0,\pi)$ then every component $u_n(T)$ has to be zero and this function f solves the moment problem

$$\begin{aligned} n\int_0^T \left[(1-n^2)e^{-n^2 s} + e^{-2s}\right] f(T-s)\,\mathrm{d}s = \\ = \left(-e^{-2T} + (1-n^2)e^{-n^2 T}\right)\xi_n. \end{aligned} \tag{11}$$

We replace $f(t)$ with its expressions given in (7). We get, when $n \geq 2$:

$$\begin{aligned} &\int_0^T f(T-s)e^{-n^2 s}\,\mathrm{d}s = \int_0^T g(T-s)e^{-n^2 s}\,\mathrm{d}s + \\ &+\int_0^T \left(\int_0^{T-s} e^{-(T-s-\tau)}g(\tau)\,\mathrm{d}\tau\right) e^{-n^2 s}\,\mathrm{d}s = \\ &= \int_0^T g(T-s)e^{-n^2 s}\,\mathrm{d}s \\ &+\int_0^T g(\tau)\left(\int_0^{T-\tau} e^{-(T-\tau)}e^{-(n^2-1)s}\,\mathrm{d}s\right)\,\mathrm{d}\tau = \\ &= \int_0^T g(T-\tau)e^{-n^2\tau}\,\mathrm{d}\tau + \\ &+\frac{1}{n^2-1}\left[\int_0^T g(T-\tau)e^{-\tau}\,\mathrm{d}\tau - \int_0^T g(T-\tau)e^{-n^2\tau}\,\mathrm{d}\tau\right]. \end{aligned}$$

and, analogously,

$$\begin{aligned} &\int_0^T f(T-s)e^{-2s}\,\mathrm{d}s = \int_0^T g(T-s)e^{-2s}\,\mathrm{d}s + \\ &+\int_0^T e^{-2s}\left(\int_0^{T-s} e^{-(T-s-\tau)}g(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s \\ &= \int_0^T g(\tau)e^{-(T-\tau)}\,\mathrm{d}\tau. \end{aligned}$$

Then, the moment problem (11) takes the form

$$\begin{aligned} n(n^2-2)\int_0^T g(T-\tau)e^{-n^2\tau}\,\mathrm{d}\tau = \\ = \left[e^{-2T} - (n^2-1)e^{-n^2 T}\right]\xi_n \qquad n \geq 2\,. \end{aligned} \tag{12}$$

Our goal now is the proof that this moment problem is not solvable. It is sufficient that we prove that the equations which correspond to every $n > N$, with N large enough, constitute a moment problem which is not solvable.

Let $T > 0$ be fixed and let $N > 0$. We consider the operator $\mathcal{T}_N$: $l^2 \mapsto l^2$ defined as follows: the k-entry of the sequence $(\mathcal{T}_N(\{\xi_n\}_{n>N}))$ is

$$(\mathcal{T}_N(\{\xi_n\}_{n>N}))_k = e^{-2T}\left[1 - (k^2-1)e^{-(k^2-2)T}\right]\xi_k\,.$$

Clearly, this operator is linear and continuous. Our starting point is the following simple observation:

Lemma 1 *Let $T > 0$ be fixed. There exists a number $N = N_T$ such that the operator $\mathcal{T}_N$ is boundedly invertible.*

<u>Proof</u>. For every $T > 0$ there exists N such that for every $k > N$ we have

$$0 < (k^2-1)e^{-(k^2-2)T} < \frac{1}{2}\,.$$

Then we have

$$\|\mathcal{T}_N(\{\xi_n\}_{n>N})\|_{l^2} \geq \frac{e^{-2T}}{2}\|\{\xi_n\}_{n>N}\|_{l^2}$$

from which the result follows. ∎

We examine now the moment problem (12) with any fixed $T > 0$ and we prove that it is not solvable. For this, it is sufficient to prove that the subset of the equations of index $n > N_T$ is not solvable (N_T is the number specified in Lemma 1).

Lemma 1 shows that we can replace (12) for $n > N_T$ with

$$n(n^2-2)\int_0^T e^{-n^2 s} g(T-s)ds = c_n, \quad n \geq N_T\,. \tag{13}$$

Here, $\{c_n\}_{n>N_T} \in l^2$ is arbitrary.

Now we need few definitions and properties of sequences $\{e_n\}$ in a Hilbert space H: 1) the sequence $\{e_n\}$ is *minimal* if we have, for every k, $e_k \notin \mathrm{cl\,span}\{e_n\,,\ n \neq k\}$; 2) a sequence $\{\psi_n\}$ is *biorthogonal* to $\{e_n\}$ if and only if $\langle e_n, \psi_j\rangle_H = \delta_{nj}$ (δ_{nj} is the Kronecker symbol); 3) the sequence $\{e_n\}$ admits biorthogonal sequences if and only if it is minimal and its admits a *unique* biorthogonal sequence if and only if it is (minimal and) complete in H; 4)

any minimal sequence $\{e_n\}$ always admits a *unique* "optimal" biorthogonal sequence, i.e. a sequence whose elements have minimal norm and this is the unique biorthogonal sequence whose elements belong to $\operatorname{cl}\operatorname{span}\{e_n\}$.

Now we invoke [2, Theorem I.2.1]: the moment problem (13) is solvable if and only if $\{n(n^2-2)e^{-n^2 s}\}_{n>N_T}$ is *-*uniformly minimal* i.e. it is minimal and has (at least) one biorthogonal sequence $\{\psi_n\}_{n>N_T}$ which is *bounded* (see [2, Definition I.1.15] and note that this terminology is not universally accepted.)

So, our goal is achieved if we prove:
zProof Let $T > 0$ be fixed, and let N_T be the number in Lemma 1. The sequence $\{n(n^2-2)e^{-n^2 t}\}_{n\geq N_T}$ $(t \in [0,T])$ has no bounded biorthogonal sequence in $L^2(0,T)$. ∎

Proof. It is sufficient to prove that the sequence $\{n(n^2-2)e^{-n^2 t}\}$ in $L^2(0,T)$ does not have a bounded biorthogonal sequence. We consider the sequence $\{e^{-n^2 t}\}$ first.

By the Theorem of Muntz (see [19]), the sequence $\{e^{-n^2 t}\}$ is minimal and not complete in $L^2(0,\infty)$. Denote by $E(\infty)$ the closed space generated by $\{e^{-n^2 t}\}$ in $L^2(0,\infty)$. The subspace $E(\infty)$ of $L^2(0,\infty)$ is known as a *Muntz space* (see [8, 19]). Denote by $E(T)$ the closed linear subspace of $L^2(0,T)$ generated by $\{e^{-n^2 t}\}$ and let $P_T : L^2(0,\infty) \to L^2(0,T)$ be the operator $P_T f = f|_{(0,T)}$. The operator P_T is linear and bounded and, by a theorem of Schwartz (see [19, formula (9.a) p. 55]), P_T is an isomorphism between $E(\infty)$ and $E(T)$.

We have to be precise on this point: below we shall consider P_T as an isomorphism between the two Hilbert spaces $E(\infty)$ and $E(T)$ so that its Hilbert space adjoint acts from the Hilbert space $E(T)$ to the Hilbert space $E(\infty)$ and it is an isomorphism too.

We introduce a convenient notation: we will denote by e_n the function $e^{-n^2 t}$ considered in $L^2(0,\infty)$ and by e_n^T its restriction to $(0,T)$.

Suppose that $\{\tilde{\psi}_n\}$ is any biorthogonal to $\{e_n^T\}$ in $L^2(0,T)$. Our goal is the proof that the sequence $\{\tilde{\psi}_n\}$ is not bounded (in fact, we shall see that it is exponentially unbounded.)

Let ψ_n be the orthogonal projection of $\tilde{\psi}_n$ on $E(T) = \operatorname{cl}\operatorname{span}\{e_n^T\}$. Then, $\{\psi_n\}$ is biorthogonal to $\{e_n^T\}$ too and

$$\|\psi_n\|_{L^2(0,T)} \leq \|\tilde{\psi}_n\|_{L^2(0,T)} .$$

We have

$$\delta_{jn} = (\psi_j, e_n^T)_{L^2(0,T)} = (\psi_j, e_n^T)_{E(T)} =$$
$$= (\psi_j, P_T e_n)_{E(T)} = (P_T^* \psi_j, e_n)_{E(\infty)}$$

and it follows that $\{P_T^* \psi_n\}$ is biorthogonal to $\{e_n\}$ and furthermore $\varphi_n = P_T^* \psi_n \in E(\infty)$ since $P_T \in \mathcal{L}(E(\infty), E(T))$. Hence, $\{\varphi_n\}$ is the "optimal" biorthogonal sequence of $\{e_n\}$.

In [8] the authors construct explicitly the "optimal" sequence $\{\varphi_n\}$ biorthogonal to $\{e_n\}$ and for this sequence $\{\varphi_n\}$ the following asymptotic relation is proved (see [8, Lemma 3.1]):

$$||\varphi_n||_{L^2(0,\infty)} = \frac{2}{n^2} e^{[\pi + O(1)]n}, \quad n \to \infty\,. \tag{14}$$

Since $P_T^* \in \mathcal{L}(E(T), E(\infty))$ is boundedly invertible, there exist *positive numbers* m and M such that for every n we have

$$m\|\psi_n\|_{L^2(0,T)} \leq \|P_T^* \psi_n\|_{L^2(0,+\infty)} \leq M\|\psi_n\|_{L^2(0,T)}$$

and we noted $P_T^* \psi_n = \varphi_n$. It follows that

$$\|\tilde{\psi}_n\|_{L^2(0,T)} \geq \|\psi_n\|_{L^2(0,T)} \geq \frac{1}{M}\|\varphi_n\|_{L^2(0,\infty)} \quad \forall n. \tag{15}$$

We recapitulate: we see from (14) that *any* biorthogonal sequence of $\{e^{-n^2 t}\}$ in $L^2(0, +\infty)$ is exponentially unbounded and from (15) we see that *any* biorthogonal sequence of $\{e^{-n^2 t}\}_{n \geq N_T}$ in $L^2(0, T)$ is exponentially unbounded too.

Let now $\{\hat{\psi}_n\}_{n > N_T}$ be a biorthogonal sequence of $\{n(n^2 - 2)e^{-n^2 t}\}_{n \geq N_T}$ in $L^2(0, T)$. The sequence of the vectors $\hat{\psi}_n / n(n^2 - 2)$ is a biorthogonal to $\{e^{-n^2 t}\}_{n \geq N_T}$ in $L^2(0, T)$ and we have seen that such biorthogonal sequence must be exponentially unbounded. So, the sequence $\{\hat{\psi}_n\}_{n > N_T}$ is exponentially unbounded too. ∎

In conclusion, any biorthogonal sequence of $\{n(n^2 - 2)e^{-n^2 s}\}_{n > N_T}$ in $L^2(0, T)$ is unbounded and so, from [2, Theorem I.2.1], the moment problem (13) is not solvable and we conclude:

Theorem 2 *Let $T > 0$ be arbitrary. There exist initial conditions $\xi(x) \in L^2(0, \pi)$ such that the condition $u(\cdot, T) = 0 \in L^2(0, \pi)$ cannot be achieved using $g \in L^2(0, T)$.*

So, system (4)-(5) is the counterexample to controllability we were looking for.

3 Thermodynamics conditions

Finally we show that our counterexample, i.e. Eq. (4) has been given for an equation which is physically significant, since it satisfies the conditions in [10, Sect. 2], derived from the constraints imposed by the second law of thermodynamics. In order to check this, we must recall the Boltzman model, which is as follows: the internal energy $\epsilon(x,t)$ and the heat flux $q(x,t)$ are related by

$$\epsilon(x,t) = \epsilon_0 + \alpha_0 u(x,t) + \int_{-\infty}^{t} \alpha'(t-s)u(s)\,\mathrm{d}s$$
$$q(x,t) = -k_0 \nabla u(x,t) - \int_{-\infty}^{t} k'(t-s)\nabla u(x,s)\,\mathrm{d}s$$

where u denotes temperature (the reason why the kernels are written as derivatives depends on the usual presentation of the Boltzman model, first written using Stieltjes integrals which are then integrated by parts. So, this notation is of no consequence here).

Equation (1) is obtained combining these relations with conservation of energy, i.e.

$$\frac{\mathrm{d}\epsilon(x,t)}{\mathrm{d}t} = -\nabla \cdot q(x,t)$$

which gives

$$\alpha_0 u_t + \alpha'(0)u(x,t) + \int_{-\infty}^{t} \alpha''(t-s)u(s)\,\mathrm{d}s =$$
$$= k_0 \Delta u(x,t) + \int_{-\infty}^{t} k'(t-s)\Delta u(x,s)\,\mathrm{d}s\,.$$

The conditions to be imposed to the constants and the kernels are:

- $\alpha_0 > 0$;
- $\alpha'(t) \in L^1(0,+\infty)$, $k'(t) \in L^1(0,+\infty)$ (vanishing memory);
- second law of of thermodynamics, which holds if and only if

$$\begin{cases} k_0 + \int_0^{+\infty} k'(s)\cos\omega s\,\mathrm{d}s > 0 & \forall \omega \in \mathbb{R}\,, \\ \omega \int_0^{+\infty} \alpha'(s)\sin\omega s\,\mathrm{d}s > 0 & \forall \omega \neq 0\,. \end{cases} \tag{16}$$

Comparing with (4), we see that

$$\alpha_0 = 1\,, \quad \alpha'(0) = 1\,, \quad \alpha''(t) = -e^{-t}\,, \quad k_0 = 1\,, \quad k'(t) = e^{-t}\,.$$

So, $k'(t)$ and $\alpha'(t) = e^{-t}$ belong to $L^1(0, +\infty)$. The cosine Fourier transform of $k'(t)$ and the sine Fourier transform of $\alpha'(t)$ are respectively $1/(1+\omega^2)$ and $\omega/(1+\omega^2)$ from which conditions (16) are easily seen.

Remark 3 *Essentially with the same computations as in Section 2, we can see that controllability does not hold for the equation*

$$u_t = u_{xx} - u + \int_0^t u_{xx}(s)\,\mathrm{d}s \quad \text{i.e.} \quad v_t = v_{xx} + \int_0^t e^{(t-s)} v_{xx}(s)\,\mathrm{d}s$$

(replace $v(x,t) = e^t u(x,t)$). We didn't present the computations for this equation which does not satisfies the physical constraints. The point is not so much that the kernel is not integrable on $[0, +\infty)$. In fact, when working on $[0, T]$ we can redefine the kernel to be equal to 0 for $t > T$. The point is that such a kernel would not satisfy the positivity constraints (16).

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