Alternative tests for functional and pointwise output-controllability of linear time-invariant systems

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Abstract

This paper deals with the description of a new method for calculating the functional output-controllability. It is computed by means of the rank of a certain constant matrix which can be associated to the system. Moreover, a new method for the pointwise output-controllability determination by means of constructing the output-controllability matrix associated to the system using the residues of the given linear system is developed. Finally, a simple physical example is presented.

Keywords: Eigenvalues, Jordan normal form, Pointwise and Functional Output-controllability, Residues, Transfer function

1. Introduction

It is well known that many physical problems use state space representation for its description [1]

$$\left.\begin{array}{ll}\dot{X} & = AX + Bu \\ Y & = CX\end{array}\right\} \tag{1}$$

where $A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, and $C \in M_{p\times n}(\mathbb{C})$.

Although for its analysis and identification, it is usually defined by the transfer function obtained by applying Laplace transformation to equation (1). It is obtained in the following form

$$G(s) = C(sI - A)^{-1}B = \sum_{i=1}^{r}\sum_{j=1}^{m_i} \frac{R_{\lambda_i j}}{(s-\lambda_i)^j}. \tag{2}$$

The transfer function gives an input-output relationship of the system. The matrices $R_{\lambda_i,j}$ are called residues of the transfer functions and they are interesting because they describe the gain of the transfer function from input to output as well as describe which input-output pair have the largest influence on the desired eigenvalue of the state matrix (A), among other information.

Controllability of a dynamical system is largely studied by several authors and under many different points of view, (see [2, 3, 4, 5, 6] for example). Nevertheless, controllability for the output vector of a system has been less treated, (see [7, 8] for example). In this paper pointwise and functional output-controllability are analyzed and alternative tests to study these characteristics are presented.

In the literature (see [6, 9], for example), there are studies that in most of the cases, show how to obtain the residue for the case where the matrix A has simple eigenvalues, however, there are very few studies considering for the general case where the matrix A has eigenvalues of algebraic multiplicity greater than one. It is true that generically, the matrices have simple eigenvalues, but not all mathematical models representing physical problems are generic. In this paper, algorithms to obtain the residues for the general case are presented.

In [10], it is developed a calculation method for partial fraction expansion of transfer matrices which uses a Vandermonde matrix formed by the eigenvalues of the matrix of the system, however the method requires to calculate the powers of the matrix A.

The aim of this paper is to present new tests for pointwise and functional controllability. The first one is based on the residues of transfer matrix of the system. In order to do the first new test, it is presented a generalization of the residues calculation by considering not only the case of simple eigenvalues. The residues obtained are used to construct the pointwise output controllability matrix providing a new method to calculate it. From the functional output controllability matrix obtained in this paper, an efficient computational approach to analyze functional output controllability is shown.

This paper is organized as follows: Section II presents a generalized methodology to compute the residues. Alternative tests for calculation of pointwise and functional output controllability matrices are determined, and a new computational approach to analyze functional output controllability is also been proposed in Section III. Finally, an application example, where the general method is required, is shown in Section IV.

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2. Preliminaries

In this paper, it is considered the general state space system introduced in equation (1)

$$\left.\begin{array}{ll}\dot{X} & = AX + Bu \\ Y & = CX\end{array}\right\},$$

where X is the state vector, Y is the output vector,u is the input (or control) vector, $A \in M_n(\mathbb{C})$ is the state matrix, $B \in M_{n\times m}(\mathbb{C})$ is the input matrix, and $C \in M_{p\times n}(\mathbb{C})$ is the output matrix.

In the analysis of dynamic systems, it is often necessary to find the partial fraction expansion of the transfer functions defined in equation (2), in terms of individual modes.

Let λ_i, $i = 1, \ldots, r$ be the eigenvalues of A with multiplicities m_i respectively. Therefore, the dynamical state matrix $(sI-A)^{-1}$ in the transfer function can be expressed as

$$(sI - A)^{-1} = \frac{A_0 s^{n-1} + A_1 s^{n-2} + \ldots + A_{n-2}s + A_{n-1}}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\ldots(s-\lambda_r)^{m_r}}$$

being

$$\begin{array}{l} A_0 s^{n-1} + A_1 s^{n-2} + \ldots + A_{n-2}s + A_{n-1} = \operatorname{Adj}(sI - A) \\ (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\ldots(s-\lambda_r)^{m_r} = \det(sI - A). \end{array}$$

Then, it can be rearranged by the following expression: $(sI - A)^{-1} = \sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{K_{\lambda_i j}}{(s-\lambda_i)^j}$, where $K_{\lambda_i j}$ are the matrix of residues of the partial fraction expansion. From now onward, it will be written simply as K_{ij} in order to avoid confusion.

Then: $C(sI - A)^{-1}B = \sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R_{\lambda_i j}}{(s-\lambda_i)^j}$, where $R_{\lambda_i j}$ are the matrix residues which will be written R_{ij} from now onward. In the case where $m_i = 1$ for all i, it will be presented simply as R_i

The computation of residues can be reduced to the SISO systems partitioning matrices $C = \begin{pmatrix} C_1^t & \ldots & C_p^t\end{pmatrix}^t$ and $B = \begin{pmatrix} B_1 & \ldots & B_m\end{pmatrix}$ with $C_i \in M_{1\times n}(\mathbb{C})$ and $B_j \in M_{n\times 1}(\mathbb{C})$ for $i = 1, \ldots, p$ and $j = 1, \ldots, m$. A partition of the transfer matrix $G(s) = C(sI - A)^{-1}B$ is obtained in the following manner

$$(C_i)(sI - A)^{-1}(B_j) = (C_i(sI - A)^{-1}B_j) \tag{3}$$

and

$$C(sI - A)^{-1}B = \sum_{\ell=1}^{r}\sum_{k=1}^{m_i}\left(\frac{R_{i\lambda_\ell kj}}{(s-\lambda_\ell)^k}\right).$$

So, the study can be restricted to the SISO systems.

Also, the computation can be reduced to the normal canonical form since the transfer matrix is invariant under basis change in the state space on the system: $G(s) = C(sI - A)^{-1}B = CS(sI - J)^{-1}S^{-1}B$ where $S \in Gl(n; \mathbb{C})$ and $J = S^{-1}AS$ with J representing the Jordan canonical reduced form of the matrix (i.e. $J = \operatorname{diag}(J_1, \ldots, J_r)$, $J_i = \operatorname{diag}(J_{i_1}, \ldots, J_{i_{s_i}})$, $J_{i_j} = \lambda_i I + N$ where $N = \left(\begin{array}{c|c} 0 & I_{n_{i_j}-1} \\ 0 & 0\end{array}\right) \in M_{n_{i_j}}(\mathbb{C})$).

Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of the matrix A with multiplicities m_i respectively, where $n_{i_1} + \ldots + n_{i_{s_i}} = m_i$, $s_i = \dim \operatorname{Ker}(\lambda_i I - A)$.

Remark 1. In the case where $\lambda_i \neq \lambda_j$ for all $i \neq j$, the matrix J is obviously diagonal and it will be written as $J = D$.

In order to generalize the computation of residues, the system is divided into different cases by considering if the matrix is diagonal or not, if the eigenvalues are simple or not, and if the matrix is derogatory or not.

Case 1: matrix A is diagonalizable

Proposition 1. *a) If the matrix A has simple eigenvalues, the residue matrix R_i, which corresponds to the eigenvalue λ_i, is*

$$R_i = (R_{ki\ell}) = (C_k v_i u_i B_\ell).$$

where v_i is a right eigenvector (column vector) and u_i is a left eigenvector (row vector), both of them are chosen in such a way that $u_i v_i = 1$.
b) If the matrix A has multiple eigenvalues $\lambda_1, \ldots, \lambda_r$ with multiplicity m_i for all $i = 1, \ldots, r$. Then,

$$R_{i1} = (R_{ji1k}) = (\sum_{\ell=1}^{m_i} C_j v_\ell u_\ell B_k).$$

where $v_{i_1}, \ldots, v_{i_{m_i}}$ are the right eigenvalue and $u_{i_1}, \ldots, u_{i_{m_i}}$ are the corresponding left eigenvectors which are chosen such as

$$\begin{pmatrix} u_{i_1} \\ \vdots \\ u_{i_{m_i}}\end{pmatrix}\begin{pmatrix} v_{i_1} & \ldots & v_{i_{m_i}}\end{pmatrix} = I_r.$$

Case 2: non-derogatory matrices

a) with single eigenvalue

Suppose that matrix J has only one block, this means that the matrix A has a unique eigenvalue λ with a $\dim \operatorname{Ker}(A - \lambda I) = 1$. Then $J = S^{-1}AS$, with $J = \lambda I + N \in M_n(\mathbb{C})$.

Therefore,

$$(sI - J)^{-1} = \frac{1}{s-\lambda}I_n + \ldots + \frac{1}{(s-\lambda)^n}N^{n-1},$$

and the following result is obtained

Proposition 2.

$$\begin{array}{ll} R_{11} & = \sum_{i=1}^{n} Cv_i u_i B = \sum_{i=1}^{n} c_i b_i \\ R_{12} & = \sum_{i=1}^{n-1} Cv_i u_{i+1} B \\ \vdots & \\ R_{1n} & = Cv_1 u_n B. \end{array}$$

b) with r eigenvalues

Proposition 3.

$$R_{\lambda_i 1} = C\Pi_{m_i}B,\ 1 \le i \le n$$

where Π_{m_i} *is the projection to the* $Ker(A-\lambda_i)^{m_i}$.

Remark 2. In order to obtain the projection to the Ker, it suffices to obtain a sub-basis $(v_{i1},\dots,v_{im_i})$ (column vectors) corresponding to the eigenvalue block and the corresponding left sub-basis $(u_{i1},\dots,u_{im_i})$ (row vectors), as well as it has to verify the condition of normalization defined as,

$$\begin{pmatrix} v_{i1} & \dots & v_{im_i}\end{pmatrix}\begin{pmatrix} u_{i1}\\ \vdots \\ u_{im_i}\end{pmatrix} = I_{m_i},$$

Case 3: derogatory matrix
a) with single eigenvalue

In this case, suppose that the matrix A is equivalent to $J = \operatorname{diag}(J_1,\dots,J_s)$, with $J_i = \lambda I_i + N_i \in M_{n_i}$, $n_1+\dots+n_s = n$. Without loss of generality, it can be considered $n_1 \ge \dots \ge n_s$.

Calling $N = \operatorname{diag}(N_1,\dots,N_s)$, it can be obtained

$$(sI-J)^{-1} = \operatorname{diag}_i((sI_i-J_i)^{-1}) = \sum_1^s \frac{1}{(s-\lambda)^i}N^{i-1}$$

(observe that $N^i = 0$ for all $i \ge n_1$).

Therefore, the following result is developed

Proposition 4. $R_{11} = CB = c_1b_1+\dots+c_nb_n$.

b) derogatory matrix with multiple eigenvalues is a simple corollary.

Sometimes, a system is built by interconnecting different systems among them in engineering problems.

Let $\dot{X}_i = A_iX_i + B_iu_i$, $Y_i = C_iX_i$ for $i = 1,2$, be two systems which can be connected in different ways. The most common ones are considered in the following development.

i) Serie: serialized one after the other, so that the input information $u_2 = Y_1(t)$. Consequently

$$\left.\begin{aligned} \dot{X} &= \begin{pmatrix} A_1 & 0\\ B_2C_1 & A_2\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix} + \begin{pmatrix} B_1\\ 0\end{pmatrix}u\\ Y &= \begin{pmatrix} 0 & C_2\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix}\end{aligned}\right\}.$$

Calling (A,B,C) to this serial connected system, the following result is obtained.

Proposition 5. *Suppose that matrices* A_1 *and* A_2 *have simple eigenvalues* $\lambda_1^{(1)},\dots,\lambda_{n_1}^{(1)}$ *and* $\lambda_1^{(2)},\dots,\lambda_{n_2}^{(2)}$ *with* $\lambda_i^{(1)} \ne \lambda_j^{(2)}$ *for all* $1 \le i \le n_1$, $1 \le j \le n_2$. *Then, the residues of the serial connected system* (A,B,C) *can be computed as:*

$$R_{\lambda_i^{(1)}} = \frac{R_i^{(1)}R_j^{(2)}}{\lambda_i^{(1)}-\lambda_j^{(2)}},\quad R_{\lambda_j^{(2)}} = \frac{R_i^{(1)}R_j^{(2)}}{\lambda_j^{(2)}-\lambda_i^{(1)}}.$$

ii) Parallel: both systems receive the same input information and the outputs are added. Consequently

$$\left.\begin{aligned} \dot{X} &= \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix} + \begin{pmatrix} B_1\\ B_2\end{pmatrix}u\\ Y &= \begin{pmatrix} C_1 & C_2\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix}\end{aligned}\right\}.$$

Calling (A,B,C) to the parallel connected system, the following result is obtained.

Proposition 6. *Let* $\lambda_i^{(1)}$ *and* $\lambda_j^{(2)}$ *be the eigenvalues of the matrices* A_1 *and* A_2, *respectively. The residues of the parallel connected system* (A,B,C) *are calculated as:*

i) If $\lambda_i^{(1)} \ne \lambda_j^{(2)}$, $R_{\lambda_{i,j}^{(k)}} = R^{(k)}_{\lambda_{i,j}^{(k)}}$

ii) If $\lambda_i^{(1)} = \lambda_j^{(2)}$ *for some* i,j, $R_{\lambda_{i,j}^{(k)}}, = R^{(1)}_{\lambda_{i,j}^{(1)}} + R^{(2)}_{\lambda_{i,j}^{(2)}}$.

3. Pointwise and functional output-controllability

The output-controllability generally means, that the system can steer output of dynamical system independently of its state vector.

Definition 1. *Dynamical system* (1) *is said to be pointwise output-controllable if for every* $y(0)$ *and every vector* $y_1 \in \mathbb{R}^p$, *there exist a finite time* $t_1 > 0$ *and control* $u_1(t) \in \mathbb{R}^m$ *defined over* $[0,t_1]$, *that transfers the output from* $y(0)$ *to* $y_1 = y(t_1)$.

For a linear continuous-time system, like (1), described by matrices A, B, and C, the pointwise output-controllability matrix can be defined as

$$oC_{\mathrm{p}}(A,B,C) = \begin{pmatrix} CB & CAB & \dots & CA^{n-1}B\end{pmatrix} \quad (4)$$

and the following well-known result is obtained.

Theorem 1 ([4]). *Dynamical system* (1) *is pointwise output-controllable, if and only if* $\operatorname{rank} oC_{\mathrm{p}}(A,B,C) = p$.

Proposition 7. *The pointwise output-controllability is invariant under basis change in the state space on the system.*

Proof. Let $x = Sx_1$ be such that $J = S^{-1}AS$, and calling $C' = CS$ and $B' = S^{-1}B$, then

$$\operatorname{rank} oC_{\mathrm{p}}(A,B,C) = \operatorname{rank} oC_{\mathrm{p}}(J,B',C')$$

So, the matrix A in its Jordan reduced form can be considered.

$$J^i = \operatorname{diag}(J_1,\dots,J_r)^i = \operatorname{diag}((\lambda_1I_1+N_1)^i,\dots,(\lambda_rI_r+N_r)^i)$$

A more essentially restrictive condition is the functional output-controllable which is defined in the following manner

Definition 2. *A system is functional output-controllable if and only if its output can be steered along the arbitrary given curve over any interval of time. It means that if it is given any output $y_d(t)$, $t \geq 0$, there exists t_1 and a control u_t, $t \geq 0$, such that for any $t \geq t_1$, $y(t) = y_d(t)$.*

A necessary and sufficient condition for functional output-controllability is

Proposition 8. *[4, 11]*

$$\operatorname{rank}\begin{pmatrix} sI-A & B \\ C & 0 \end{pmatrix} = n+p, \ \forall s \in \mathbb{C}$$

It is easy to prove that pointwise and functional output-functional controllability are not equivalent.

Example 1. *Let (A,B,C) be a system with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, the transfer matrix is $\begin{pmatrix} \frac{1}{s}+\frac{1}{s^2} & \frac{1}{s}+\frac{1}{s^2} \\ \frac{1}{s^2} & \frac{1}{s^2} \end{pmatrix}$ whose row rank is 1 for $s=-1$, but the system is output controllable because of*

$$\operatorname{rank}\begin{pmatrix} CB & CAB \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 2.$$

Proposition 9. *The functional output-controllability character is invariant under strict equivalence of pencils.*

Proof.

$$\operatorname{rank}\begin{pmatrix} P^{-1} & W \\ 0 & S \end{pmatrix}\begin{pmatrix} sI-A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ V & R \end{pmatrix} = \operatorname{rank}\begin{pmatrix} sI-A & B \\ C & 0 \end{pmatrix}$$

3.1. Test for functional output-controllability

The functional output-controllability can be computed by means of the rank of a constant matrix in the following manner

Theorem 2. *The system (A,B,C) is functional output-controllable if and only if*

$$\begin{aligned}\operatorname{rank} oC_{\mathrm{f}}(A,B,C) &= \operatorname{rank}\begin{pmatrix} C & & & \\ CA & CB & & \\ CA^2 & CAB & CB & \\ \vdots & & & \ddots \\ CA^n & CA^{n-1}B & \dots & CAB \ CB \end{pmatrix} \\ &= (n+1)p\end{aligned}$$

The null terms are not written in the matrix.

Proof. Taking into account the proposition 9, the system in its Kronecker canonical reduced form can be considered (see [12], [13] for example): the system (A,B,C) can be reduced to (A_c,B_c,C_c) with

$$A_c = \begin{pmatrix} sI_1-N_1 & & & \\ & sI_2-N_2 & & \\ & & sI_3-N_3 & \\ & & & sI_4-J \end{pmatrix}, B_c = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_c = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

where $(C_1 \ \ 0 \ \ 0 \ \ 0) \in M_{p_1\times n}(\mathbb{C})$. It is important to highlight that not all the parts necessarily appear in reduced form.

So,

$$\begin{aligned}&\operatorname{rank}\begin{pmatrix} sI-A & B \\ C & 0 \end{pmatrix} = \\ &\operatorname{rank}\begin{pmatrix} sI_1-N_1 & B_1 \\ C_1 & 0 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} sI_2-N_2 & B_2 \end{pmatrix} \\ &\quad +\operatorname{rank}\begin{pmatrix} sI_3-N_3 \\ C_2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} sI_4-J \end{pmatrix} = \\ &n_1+p_1+n_2+n_3+n_4 = n+p_1,\end{aligned}$$

and the rank is $n+p$ if and only if $p=p_1$. Now it suffices to compute the row minimal indices of the pencil defined by the system.

Example 2. *Let (A,B,C) be a system with $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.*

It is easy to compute $\operatorname{rank} oC_{\mathrm{f}}(A,B,C) = 5 < 8$. Then the system it is not functional output-controllable.

But if we consider the system (A_1,B_1,C_1) with $A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & \varepsilon \\ 2 & 1 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, it can be obtained that $\operatorname{rank} oC_{\mathrm{f}}(A_1,B_1,C_1) = 8$. Then, the system it is functional output-controllable.

Remark 3.

i) If the system (A,B,C) is functional output-controllable, then $\operatorname{rank} C = p \leq n, m$.

ii) If

$$\operatorname{rank}\begin{pmatrix} C & & & \\ CA & CB & & \\ CA^2 & CAB & CB & \\ \vdots & & & \ddots \\ CA^{n-1} & CA^{n-2}B & \dots & CAB \ CB \end{pmatrix} = np$$

does not necessarily

$$\operatorname{rank}\begin{pmatrix} C & & & \\ CA & CB & & \\ CA^2 & CAB & CB & \\ \vdots & & & \ddots \\ CA^n & CA^{n-1}B & \dots & CAB \ CB \end{pmatrix} = (n+1)p$$

as it can be seen in the following example.

Example 3. *Let (A,B,C) with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.*

$$\operatorname{rank}\begin{pmatrix} C & \\ CA & CB \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 0 & \\ 0 & 1 & 0 \end{pmatrix} = 2 = np,$$

but

$$\operatorname{rank}\begin{pmatrix} C & & \\ CA & CB & \\ CA^2 & CAB & CB \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$2 < 3 = (n+1)p$.

Corollary 1. *The system (A, B, C) is functional output-controllable if and only if*

$$\operatorname{rank} C = p$$
$$\operatorname{rank} \begin{pmatrix} C & \\ CA & CB \end{pmatrix} = 2p$$
$$\vdots$$
$$\operatorname{rank} \begin{pmatrix} C & & & & \\ CA & CB & & & \\ CA^2 & CAB & CB & & \\ \vdots & & \ddots & & \\ CA^i & CA^{i-1}B & \dots & CAB & CB \end{pmatrix} = (i+1)p$$
$$\vdots$$

This corollary provides an iterative method to compute functional output-controllability. Calling oC_f the matrices in the corollary, it is shown an example of a flowchart in Figure 1.

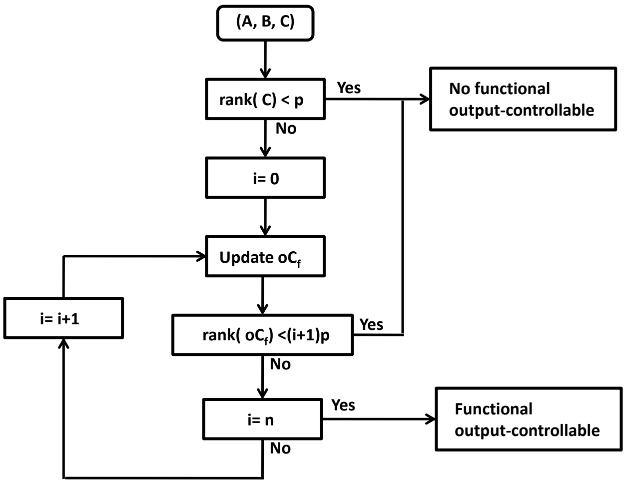

Figure 1: Flowchart showing the required iteration for functional output-controllability computation

Partitioning matrices $B = \begin{pmatrix} B_1 & \dots & B_m \end{pmatrix}$ and $C = \begin{pmatrix} C_1^t & \dots & C_p^t \end{pmatrix}^t$ by columns and rows respectively, it can be computed if there are any SISO subsystem, that are functional output-controllable, in the following manner

Corollary 2. *Let (A, B, C) be any system. The subsystem SISO system A, B_i, C_j for some $1 \leq i \leq m$, $1 \leq j \leq p$ is functional output-controllable, if and only if*

$$\operatorname{rank} \begin{pmatrix} C_j & & & & \\ C_jA & CB_i & & & \\ C_jA^2 & C_jAB_i & C_jB_i & & \\ \vdots & & \ddots & & \\ C_jA^n & C_jA^{n-1}B_i & \dots & C_jAB_i & C_jB_i \end{pmatrix} = (n+1).$$

From this result, it is easy to proof the following proposition.

Proposition 10. *Let (A, B, C) be a functional output-controllable system. Then, for all $1 \leq j \leq p$, there is at least one i, $1 \leq i \leq m$ such that the SISO system (A, B_i, C_j) is functional output-controllable.*

Remark 4. Notice that not necessarily all SISO subsystems are functional output-controllable, and in the case that all SISO subsystems are functional output-controllable, the complete system is not necessarily functional output-controllable.

3.2. Residues and pointwise output-controllability

Sometimes it is known the system through the transfer function, for that reason, it is interesting to analyze the pointwise output-controllability by means of their residues.

Proposition 11. *Let J be a non derogatory matrix with a only one eigenvalue. Then*

$$\operatorname{rank} oC = \operatorname{rank} \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \end{pmatrix}.$$

Proof. Matrix J is in the form $\lambda I + N$. Now, it suffices to observe that $CJ^\ell B = C(\sum_{j=0}^{\ell} \binom{\ell}{j} \lambda^{\ell-j} N^j)B$ and $C(sI - J)^{-1}B = \sum_{j=0}^{n} \dfrac{CN^jB}{(s-\lambda)^{j+1}}$.

Corollary 3. *Let A be a matrix with a single eigenvalue. Then*

$$\operatorname{rank} oC\mathrm{p}(A, B, C) = \operatorname{rank} \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \end{pmatrix}.$$

Proof. It suffices to apply 7 and 11.

Proposition 12. *Let A be a matrix having n simple eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\operatorname{rank} oC\mathrm{p}(A, B, C) = \\ \operatorname{rank} \begin{pmatrix} \sum_{i=1}^n R_{i1} & \sum_{i=1}^n \lambda_i R_{i1} & \dots & \sum_{i=1}^n \lambda_i^{n-1} R_{i1} \end{pmatrix}.$$

Proof. It is sufficient to consider the system in its diagonal reduced form.

A more general result is presented in the following.

Theorem 3. *Let $H(s)$ be a system with eigenvalues $\lambda_1, \dots, \lambda_r$. Then, the pointwise output-controllability matrix is*

$$oC\mathrm{p}(A, B, C) = \\ \begin{pmatrix} \sum_{i=1}^r R_{i1} & \sum_{i=1}^r \lambda_i R_{i1} + \sum_{i=1}^r R_{i2} & \dots & \sum_{i=1}^r \lambda_i^{n-1} R_{i1} + \sum_{i=1}^r (n-1)\lambda_i^{n-2} R_{i2} + \dots + \sum_{i=1}^r R_{in} \end{pmatrix}.$$

The proof is analogous to the particular cases presented.

Remark 5. Notice that proposition 11 is a direct corollary of this theorem.

4. Application example

In this section, it is presented an example of physical problems which highlights the need to know the residues of the transfer function corresponding to no simple eigenvalues.

This example is based on the exercise that can be found in [6]. It is presented a simplified synchronous machine against an infinite bus. The system scheme is shown in Figure 2. Applying Taylor's method to the mechanical equations, the linearized system equations can be described as follows:

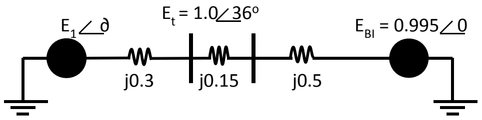

Figure 2: Synchronous machine infinite bus electrical scheme

$$\left.\begin{aligned} \dot{X} &= \begin{pmatrix} \Delta\dot{\omega}_r \\ \Delta\dot{\delta} \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{K_D}{2H} & -\frac{K_s}{2H} \\ \omega_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \Delta\omega_r \\ \Delta\delta \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{2H} \\ 0 \end{pmatrix}}_{B} \Delta T_m \\ Y &= X \end{aligned}\right\} \quad (5)$$

where $H = 3.5$, $K_s = \dfrac{E'E_B}{X_r}\cos\delta_0 = 0.757$, $\omega_0 = 120\pi$ and K_D is a constant parameter.

4.1. Eigenvalues and eigenvectors calculation

In order to obtain the eigenvalues of the matrix A, it can be compute the characteristic equation

$$\left.\begin{aligned} \lambda^2 + 0.143K_D\lambda + 40.79 = 0. \Leftrightarrow \\ \lambda^2 + 2\xi\cdot\omega_n\lambda + \omega_n^2 = 0. \end{aligned}\right\} \lambda = \xi\omega_n \pm j\omega_n\sqrt{\xi^2 - 1}$$

So, $\omega_n = \sqrt{40.79} = 6.387 rad/s = 1.0165 Hz$, $\xi = \dfrac{0.143K_D}{2\cdot 6.387} = 0.0112K_D$

In the case where $\xi = 1$, the matrix A has a double eigenvalue with a single eigenvector, in other words, the Jordan equivalent form and the generalized (right) eigenvectors are

$$J = \begin{pmatrix} -6.387 & 1 \\ 0 & -6.387 \end{pmatrix},\ S = \begin{pmatrix} -6.387 & 1 \\ 377 & 0 \end{pmatrix}.$$

Notice that the left generalized eigenvectors are compute as S^{-1}.

4.2. Residues calculation

$$R_{11} = \begin{pmatrix} \frac{1}{2H} \\ 0 \end{pmatrix},\ R_{12} = \begin{pmatrix} -\frac{6.387}{7} \\ \frac{377}{7} \end{pmatrix}.$$

4.3. Pointwise output-controllability demonstration

By using the general method, it can be demonstrated that a system is pointwise output-controllable if the rank is maximum

$$\operatorname{rank}(oC(A,B,C)) = \operatorname{rank}\begin{pmatrix} CB & CAB & \dots & CA^{n-1}B \end{pmatrix}$$

$$\Rightarrow \operatorname{rank}(oC(A,B,C)) = \operatorname{rank}\begin{pmatrix} \frac{1}{7} & \frac{-1}{0.5488} \\ 0 & \frac{377}{7} \end{pmatrix}$$

Therefore, it is output-observable. Now, it will be demonstrated by means of the new calculation method developed by using the residues.

$$\operatorname{rank} oC = \operatorname{rank}\begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \end{pmatrix}.$$

$$\Rightarrow \operatorname{rank}(oC) = \operatorname{rank}\begin{pmatrix} \frac{1}{7} & -\frac{6.387}{7} \\ 0 & \frac{377}{7} \end{pmatrix}$$

where, as it is expected, the system is pointwise output-controllable.

Notice that in this case, taking into account that $C = I$, the pointwise output-controllability coincides with the controllability of the system.

5. Conclusions

A new procedure to compute output-controllability (both pointwise and functional) has been presented in this paper. Also, a general approach for computing the residues of dynamical systems has been proposed. This method introduces the assumption of existence of non simple eigenvalues, as it does not occur up to now. Moreover, a relationship between the residues and the pointwise output-controllability concept is given. This relationship could simplify the computation of the pointwise output-controllability characteristic of the system. Also, a more computational efficient procedure to calculate functional output-controllability has been introduced. Finally, an application example is presented in order to show the real requirement of these general methods.

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