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Networked Control Systems for Multi-Input Plants Based on Polar Logarithmic Quantization*

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Abstract

This paper investigates the use of polar logarithmic quantization (PLQ) for multi-input systems, and the corresponding design issues for the underlying networked control system (NCS). It is shown that the PLQ induces sector bounded nonlinear uncertainties in multiplicative and relative forms for vector-valued analog signals, similar to those in the scalar case. For the two-input NCS, optimal quantization is obtained through minimization of the quantization error that are quantified explicitly. The results are extended to more than two-input NCSs with an upper bound derived for the quantization error. Feedback stabilization and control of the NCS are also investigated under the PLQ at the plant input under state feedback. The coarsest quantization density (CQD) is studied and obtained. Results in this paper are illustrated by a numerical example.

Keywords: Quadratic stability, quantization errors, robust control.

1. Introduction

This paper is motivated by logarithmic quantization proposed in [2] and studied in [3, 4, 5, 7, 10]. The CQD is obtained in [2] for single input systems under the state feedback control, and shown to be dictated by the Mahler measure [8]. By treating the quantization error as sector bounded nonlinear uncertainty [12], the CQD in [2] is re-derived in [5]. The sector bound approach to modeling the quantization error provides us a new insight and new design tool to tackle the information distortion caused by quantization for NCSs. For multi-input systems under state feedback control, logarithmic quantization can be applied to the vector input signal by quantizing each component of the vector independently, which is termed as *Cartesian logarithmic quantization* (CLQ). In [10], the largest quantization error under which stabilizability holds is obtained by optimal resource allocation

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and $\mathcal{H}_\infty$-based robust control, in spite of its being a μ synthesis problem. However because the CLQ for multi-input/multi-out (MIMO) systems leads to a more complex μ-type synthesis problem, generalization of the multi-input state feedback result in [10] to the case of MIMO output feedback control is extremely difficult, if it is not impossible.

It turns out that PLQ is more efficient than CLQ for vector signals [3, 4]. However very limited results, except a lower bound for the CQD in the 2-D case, are available in the existing literature. This paper will investigate several research issues for PLQ. The first is coding or quantization regarding how an analog vector in polar coordinates can be mapped to a digital one. This issue is closely related to the second one regarding quantification of the quantization error. Can the associated quantization error be modeled again by the sector bounded nonlinear uncertainty in form of multiplicative or relative errors? How to minimize such quantization errors? These two questions need to be answered together. After the first two issues are resolved, the next research issue is the stabilizability condition for the NCS when the PLQ is employed only at the plant input under state feedback. This condition can be converted to the problem of the largest quantization error under which the NCS is stabilizable. It is related to the notion of stability margin based on which the CQD can be obtained.

2. Problem Formulation

The multi-input feedback system under consideration is described by

$$x(k+1) = Ax(k) + Bu(k), \quad u(k) = \mathcal{Q}[Fx(k)], \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, and $F \in \mathbb{R}^{m\times n}$ is some state feedback gain. It follows that $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The nonlinear map $\mathcal{Q}[\cdot]$ represents the PLQ to be studied. Prior to our investigation, it is beneficial to review logarithmic quantization for the case $m = 1$, which is initially proposed in [2]. In this case, $\mathcal{Q}(v) = -\mathcal{Q}(-v)$ is defined by [5]

$$\mathcal{Q}(v) = u_{(i)} \ \ \forall\, v \in \left(\frac{u_{(i)}}{1+\delta},\ \frac{u_{(i)}}{1-\delta}\right] \tag{2}$$

where $u_{(i)} = \left(\frac{1-\delta}{1+\delta}\right)^i u_{(0)} > 0$, $i = 0, \pm1, \pm2, \cdots$, and $0 < \delta < 1$. In light of [5], the above logarithmic quantization induces a sector bounded memoryless nonlinearity in the form of

$$u(k) = [1 + \Delta]v(k), \quad \delta = \|\Delta\|_{i_E} := \sup_{v(k)\neq 0} \frac{|u(k) - v(k)|}{|v(k)|} < 1, \tag{3}$$

where $v(k) = Fx(k)$ and $\|\cdot\|_{i_E}$ is an induced norm. It is noted that the quantization error is captured by the memoryless nonlinear operator Δ that maps zero to zero, and the positive constant δ characterizes the quantization density defined in [2]. Assume that (A, B) is stabilizable. The supremum of δ, denoted by $\delta_{\max}$, over all quadratic stabilizing state feedback gains determines the CQD, and is known to be [2, 5]

$$\delta_{\max} = \frac{1}{M(A)}, \quad M(A) := \prod_{i=1}^{n} \max\{|\lambda_i(A)|, 1\}, \tag{4}$$

where $\lambda_i(\cdot)$ denotes the ith eigenvalue and $M(A)$ is called Mahler measure [8]. The quantity $\delta_{\max}$ is termed *stability margin* in the sense that stabilizing controllers exist for all uncertainties strictly bounded by $\delta_{\max}$, including sector bounded nonlinearities induced by quantization. This notion is borrowed from the robust control literature and has an important role to play in NCS stabilization.

A different logarithmic quantization is proposed in [10] by taking

$$\mathcal{Q}(v) = u_{(i)} \;\; \forall \; v \in \left(u_{(i)}(1-\delta), \; u_{(i)}(1+\delta)\right]. \tag{5}$$

The above $\mathcal{Q}[\,\cdot\,]$ used in (1) leads to the sector bounded memoryless nonlinearity in the relative form:

$$u(k) = [1+\Delta]^{-1}v(k), \quad \delta = \|\Delta\|_{i_E} := \sup_{v(k)\neq 0} \frac{|u(k)-v(k)|}{|u(k)|} < 1 \tag{6}$$

with again $v(k) = Fx(k)$. In this case, the stability margin $\delta_{\max}$, i.e., the supremum of δ over all quadratic stabilizing state feedback gains, has the same expression as in (4) which also provides the CQD for the corresponding NCS under logarithmic quantization.

Logarithmic quantization becomes more subtle for multi-input systems. A simple generalization is the so called CLQ by quantizing each component of the input vector independently. With $m > 1$ for the feedback system in (1), there holds either

$$\begin{array}{lcll} \text{(i)} \;\; u(k) & = & [I+\Delta]Fx(k) & \text{or} \\ \text{(ii)} \;\; u(k) & = & [I+\Delta]^{-1}Fx(k) & \end{array} \tag{7}$$

under CLQ, dependent on whether (2) or (5) is used for each component of $u(k)$. In either case,

$$\Delta = \text{diag}(\Delta_1, \Delta_2, \cdots, \Delta_m), \quad \delta_i = \|\Delta_i\|_{i_E} < 1. \tag{8}$$

The above Δ is a diagonal memoryless nonlinear operator with Δ_i a sector bounded nonlinear uncertainty. Thus CLQ induces diagonally *structured nonlinear uncertainty.* Feedback stabilizability in this case becomes a μ problem that is notoriously difficult. Indeed let us denote

$$D_\delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_m),$$

and $\mathcal{D}$ as the set of $m \times m$ diagonal matrices with positive diagonal entries. The subscript δ for D_δ indicates the determinant of D_δ, i.e., $\delta = \det(D_\delta)$. Let F be stabilizing and

$$T(z) = F(zI - A - BF)^{-1}B \tag{9}$$

be the closed-loop transfer matrix. Define

$$\|T\|_{\mathcal{H}_\infty} = \sup_{|z|>1} \overline{\sigma}[T(z)]$$

as the $\mathcal{H}_\infty$-norm of $T(z)$ with $\overline{\sigma}(\cdot)$ the maximum singular value. Then the quadratic stability of the closed-loop system in (1) is equivalent to [5, 9]

$$\inf_{D\in\mathcal{D}} \inf_{F} \|D^{-1}TDD_\delta\|_{\mathcal{H}_\infty} < 1.$$

Hence stabilizability of the closed-loop system in presence of CLQ involves $\mathcal{H}_\infty$ optimization over not only the stabilizing state feedback gain F, but also the scaling positive diagonal matrix D. Moreover the quantization error bounds $\{\delta_i\}_{i=1}^m$ also play a role. A useful result obtained in [10] is that the following inequality, required by feedback stability,

$$\inf_{\det(D_\delta)=\delta} \left\{ \inf_{F} \left[\inf_{D\in\mathcal{D}} \|D^{-1}TDD_\delta\|_{\mathcal{H}_\infty} \right] \right\} < 1 \tag{10}$$

holds for some stabilizing state feedback gain F, if and only if $\det(D_\delta) = \delta < M(A)^{-1}$. In fact the notion of resource allocation is introduced in [10] by treating $\frac{1}{\delta}$ as the total resource ($-\log\delta$ can be regarded as the total number of bits) available. As long as $\frac{1}{\delta} > M(A)$, a quadratically stabilizing feedback gain F exists and can be computed explicitly, $\{\delta_i\}_{i=1}^m$ exist with $0 < \delta_i < 1$ satisfying $\delta = \det(D_\delta)$, and a positive diagonal matrix D exists such that the inequality (10) holds.

To the authors of this paper, there are two major issues presented by PLQ employed in NCSs. These two issues are fundamental and present a significant challenge to design of NCSs that employ PLQ at the plant input or/and output. The first is that for a given analog vector v, how to obtain the quantized vector v_q by quantizing its polar coordinates such that either

$$\text{(a) } v_q = [I+\Delta]v \quad \text{or} \quad \text{(b) } v_q = [I+\Delta]^{-1}v \tag{11}$$

holds, and the quantization error bound $\tilde{\delta} := \|\Delta\|_{i_E}$ (that can be computed according to (3) or (6) by replacing the absolute value $|\cdot|$ by Euclidean norm $\|\cdot\|$) is minimized. Note that (a) and (b) in (11) are direct generalizations of (3) and (6), respectively. Different from CLQ, the uncertainty induced by the quantization error in (11) is *unstructured*, contrasting to that of CLQ, and can be regarded as generalized sector bounded uncertainty in either multiplicative or relative form. Hence it is much easier to synthesize the stabilizing feedback controller than the case of CLQ in [10]. The second is how to synthesize the stabilizing feedback controller when the input of the system involves unstructured multiplitive/relative uncertainty induced by the PLQ. Although quadratic stabilization becomes simpler compared with that for the structured uncertainty in (8), it is still an open problem to be investigated in this paper. These two problems will be tackled in the next two sections. Before concluding this section, it is important to note that the logarithmic quantization $\mathcal{Q}[\cdot]$ has an equivalent expression

$$u_q = \mathcal{Q}(v) = \left(\frac{1+\delta}{1-\delta}\right)^q \frac{v_0}{v_c}, \quad \text{if } \left(\frac{1+\delta}{1-\delta}\right)^q v_0 < v \leq \left(\frac{1+\delta}{1-\delta}\right)^{q+1} v_0 \tag{12}$$

by taking $u_{(0)} = v_0/(1+\delta)$ in (2) with $v_c = 1/(1+\delta)$ or $u_{(0)} = v_0/(1-\delta)$ in (5) with $v_c = (1-\delta)$, where $q = -i = 0, \pm 1, \pm 2, \cdots$. The above is more general and provides a parameter v_c. It covers (2) and (5) as a special case by taking v_c to some specific values. It also helps to derive the error bound of the unstructured time-varying uncertainty for the quantization error induced by the PLQ.

3. Polar Logarithmic Quantization

For a given analog m-D vector v, PLQ quantizes its polar coordinates. Let $v \in \mathbb{R}^m$ with v_i the ith entry of v. Its polar form is specified by [11] $v_1 = \rho \cos \phi_1$, $v_m = \rho \prod_{i=1}^{m-1} \sin \phi_i$, and

$$v_k = \rho \cos \phi_k \prod_{i=1}^{k-1} \sin \phi_i, \quad k = 2, \cdots, m-1,$$

where $\left(\rho, \{\phi_i\}_{i=1}^{m-1}\right)$ are its m coordinate variables. While $0 \leq \rho < \infty$ and $\phi_{m-1} \in [0,\ 2\pi)$, all other angle variables $\phi_i \in [0,\ \pi]$ [11]. There holds the identity

$$\cos^2\phi_1 + \prod_{i=1}^{m-1} \sin^2\phi_i + \sum_{k=2}^{m-1} \cos^2\phi_k \prod_{i=1}^{k-1} \sin^2\phi_i = 1. \tag{13}$$

By setting $\varphi_i = \frac{\pi}{2} - \phi_i$ for $1 \leq i < m-1$, $\varphi_i \in [-\frac{\pi}{2},\ \frac{\pi}{2}]$ is true for $1 \leq i < m-1$. In addition we replace ϕ_{m-1} by φ_{m-1} with its range $[-\pi,\ \pi)$. As a result,

$$\begin{array}{rclcrcl} v_1 &=& \rho \sin \varphi_1, & \vdots & v_{m-1} &=& \rho \sin \varphi_{m-1} \prod_{i=1}^{m-2} \cos \varphi_i, \\ v_2 &=& \rho \sin \varphi_2 \cos \varphi_1, & \vdots & v_m &=& \rho \prod_{i=1}^{m-1} \cos \varphi_i. \end{array}$$

The PLQ quantizes ρ by mapping the positive real axis to a countable set $\{\rho_q\}_{q=-\infty}^{\infty}$ defined by

$$\rho_q = \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)^q \frac{\rho_o}{\rho_c}, \quad \text{if } \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)^q \leq \frac{\rho}{\rho_o} < \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)^{q+1} \tag{14}$$

that is the same as (12) where $0 < \delta_\rho < 1$. The constant $\rho_o > 0$ divides the positive real axis into two parts with $\{\rho_q\}$ (more and more) densely distributed below ρ_o, and (more and more) scarcely distributed above ρ_o. The constant $\rho_c > 0$ is a parameter to be determined through minimization of the quantization error, and leads to unstructured multiplicative or relative nonlinear uncertainties.

In practice, the set of $\{\rho_q\}$ ranges over a finite interval with ρ_o adjusted to cover the dynamic range of the analog vector length ρ by the finite set $\{\rho_q\}$.

The PLQ quantizes $\{\varphi_i\}$ uniformly. Let $\kappa > 1$ be integer. The interval $[-\frac{\pi}{2},\ \frac{\pi}{2}]$ can be divided into κ uniform intervals with interval length 2θ and $\theta = \frac{\pi}{2\kappa}$. Each analog value of the argument φ_i with $0 < i < m-1$ is quantized uniformly according to

$$\varphi_{i_q} = 2\ell\theta \quad \forall\ \varphi_i\ \in [(2\ell-1)\theta,\ (2\ell+1)\theta) \tag{15}$$

for $\ell = 0, \pm 1, \cdots, \pm\lfloor \kappa/2 \rfloor$ where $\lfloor x \rfloor$ takes the integer part of x. The right end of the interval in (15) needs to be closed when $\ell = \lfloor \kappa/2 \rfloor$ and κ is odd. For quantization of φ_{m-1}, (15) can also be used but with $\ell = 0, \pm 1, \cdots, \pm\kappa$ so that it covers the interval $[-\pi,\ \pi)$.

The parameter ρ_c plays an important role for PLQ in deriving the error bound of two different forms. In order to facilitate the development, 2-D vectors are studied first. Consider

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \rho \begin{bmatrix} \sin(\varphi) \\ \cos(\varphi) \end{bmatrix}. \tag{16}$$

The uniform quantization on the argument φ, and logarithm quantization on the vector length ρ lead to the fan shaped tiles covering the two-dimensional plane. Fig. 1 shows one of the fan shaped regions, marked with A-B-E-D-A in solid line, where the vertical axis stands for v_1 and horizontal axis for v_2.

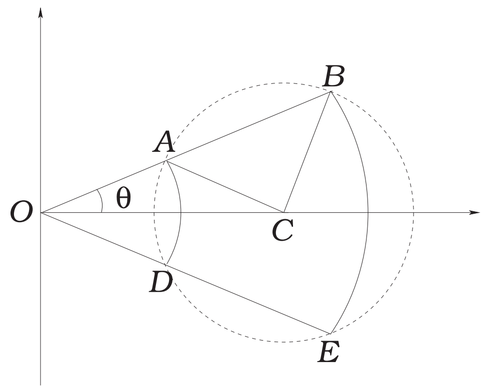

Fig. 1. Fan region with C the center point.

Similar to the scalar case, the logarithmic quantization (14) for the length and uniform quantization for the argument of the vector in (15) lead to the representation:

$$v = (I+\Delta)v_q, \qquad v_q = \rho_q \begin{bmatrix} \sin(\varphi_q) \\ \cos(\varphi_q) \end{bmatrix}, \tag{17}$$

where v_q is the quantized vector. All vectors lying in the same fan region are quantized to the same vector v_q. Hence Δ is clearly a nonlinear function of v and v_q. A significant problem is quantification and minimization of $\|\Delta\|_{i_E}$.

Consider the fan region in Fig. 1. For convenience define $\mathcal{S}_{\rm fan}$ as the collection of all vectors inside the fan shaped region A-B-E-D-A. Quantization error in relative form is studied first. For each $v \in \mathcal{S}_{\rm fan}$ with the same quantized vector v_q, the quantization error bound for (b) in (11) can be expressed as

$$\tilde{\delta} := \sup\left\{\frac{\|v - v_q\|}{\|v_q\|} \;:\; v \in \mathcal{S}_{\rm fan}\right\}.$$

The above is dependent on v_q, i.e., $\tilde{\delta} = \tilde{\delta}(v_q)$. A natural question arises: how to choose v_q such that $\tilde{\delta}(v_q)$ is the smallest possible? This leads to

$$\bar{\delta} := \inf_{v_q}\sup\left\{\frac{\|v - v_q\|}{\|v_q\|} \;:\; v \in \mathcal{S}_{\rm fan}\right\}. \tag{18}$$

Recall that $\tilde{\delta}$ represents the sector uncertainty bound. Its minimization will provide the correct value for ρ_c in (14) in order to have the smallest relative error in (17), and resolve the first major issue formulated in the previous section.

Since $\|\cdot\|$ is invariant under multiplication by orthogonal matrices, there is no loss of generality to rotate any fan shaped region to the position shown in Fig. 1. The point C represents the quantized vector, i.e., all vectors inside the fan region are quantized to the same vector v_q at point C. It is customary to define

$$\text{radius}(\mathcal{S}_{\rm fan}) := \min_{v_q}\left\{\max_{v\in\mathcal{S}_{\rm fan}} \|v - v_q\|\right\} \tag{19}$$

as the radius of the smallest ball containing $\mathcal{S}_{\rm fan}$.

Let $v_q^T = \begin{bmatrix} 0 & r_C \end{bmatrix}$ with v_q^T transpose of v_q. Then r_C is the length of the line OC. Let r_A and r_B be the length of the line OA and OB, respectively. Then by (14),

$$r_A = \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)^k \rho_{\rm o}, \quad r_B = \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)^{k+1} \rho_{\rm o}. \tag{20}$$

The point C lies at the center of the fan region in Fig. 1, if and only if the length of the line AC is the same as the length of the line BC that is equal to radius$(\mathcal{S}_{\rm fan})$. The simple geometric relation in Fig. 1 yields

$$(r_C - r_A\cos\theta)^2 + (r_A\sin\theta)^2 = (r_C - r_B\cos\theta)^2 + (r_B\sin\theta)^2 .$$

After rearrangement and noticing the logarithmic quantization in the radial direction, there hold

$$r_C = \frac{r_A + r_B}{2\cos\theta}, \quad r_B = \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right) r_A . \tag{21}$$

The second relation in (21) can be used to deduce

$$r_B - r_A = \frac{2\delta_\rho r_A}{1-\delta_\rho}, \qquad r_B + r_A = \frac{2r_A}{1-\delta_\rho}. \tag{22}$$

It follows from the first equation in (21) and the second equation in (22) that

$$r_A = (1-\delta_\rho) r_C \cos\theta \quad \Longrightarrow \quad \rho_c = (1-\delta_\rho)\cos\theta \tag{23}$$

by the fact that $r_C = \rho_q$ is some quantized value in (14) for some q where $\Longrightarrow$ stands for "implying". With $v_q = \begin{bmatrix} 0 & r_C \end{bmatrix}^T$ at the center of $\mathcal{S}_{\text{fan}}$, it can be verified that $\overline{\delta}$ defined in (18) is obtained as

$$\overline{\delta} = \frac{1}{r_C}\sqrt{(r_C - r_A\cos\theta)^2 + (r_A \sin\theta)^2} = \sqrt{\delta_\rho^2 + (1-\delta_\rho^2)\sin^2\theta} > \delta_\rho. \tag{24}$$

It is important to indicate that the expression in (24) holds for any vector inside the dashed circle covering the fan region.

Motivated by [10], a different representation from (17) can be derived that has form of

$$v = (I+\Delta)^{-1} v_q, \qquad v_q = \rho_q \begin{bmatrix} \sin(\varphi_q) \\ \cos(\varphi_q) \end{bmatrix}. \tag{25}$$

All vectors in the fan shaped region in Fig. 1 are quantized to a single vector v_q. Similar problems arise: what is v_q and what is the smallest uncertainty bound for the corresponding quantization error? The following is similar to (18):

$$\overline{\delta} := \inf_{v_q} \sup_{v \in \mathcal{S}_{\text{fan}}} \frac{\|v - v_q\|}{\|v\|} = \inf_{r_C} \sup_{|\varphi|\le\theta,\ r_A \le r < r_B} \left| e^{j\varphi} - \frac{r_C}{r} \right| \tag{26}$$

after replacing the two-dimensional plane by the complex plane, and using $e^{j\varphi}$ for $\frac{v}{\|v\|}$ and $re^{j\varphi}$ for v. Minimization in (26) leads to a different value of ρ_c for logarithmic quantization in (14) that also specifies the new center point.

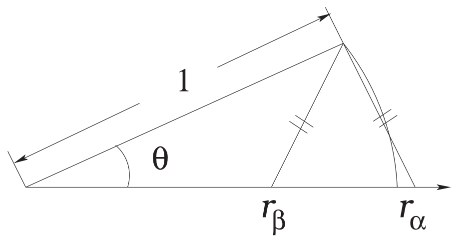

Fig. 2. Illustration of $|e^{j\theta} - r_\alpha| = |e^{j\theta} - r_\beta|$ in $\mathbb{C}$.

Specifically the second relation in (21) is useful in finding the new center point C represented by r_C, and the corresponding induced norm in (26). Indeed,

$$r_\alpha = \frac{r_C}{r_A}, \qquad r_\beta = \frac{r_C}{r_B} \quad \Longrightarrow \quad \frac{r_\alpha}{r_\beta} = \frac{1+\delta_\rho}{1-\delta_\rho}. \tag{27}$$

The center point r_C on the real axis can now be determined from the relation $|e^{j\theta} - r_\alpha| = |e^{j\theta} - r_\beta|$. See the illustration in Fig. 2.

The equidistance relation leads to

$$(r_\alpha - \cos\theta)^2 + \sin^2\theta = (r_\beta - \cos\theta)^2 + \sin^2\theta.$$

Using (27), the above yields

$$\cos\theta = \frac{1}{2}(r_\alpha + r_\beta) = \frac{r_\alpha}{1+\delta_\rho} = \frac{r_\beta}{1-\delta_\rho} \tag{28}$$

from which the center point at the complex plane is specified by

$$\begin{aligned} r_C &= r_A(1+\delta_\rho)\cos\theta = r_B(1-\delta_\rho)\cos\theta \\ &\Longrightarrow \ \rho_c = \frac{1}{(1+\delta_\rho)\cos\theta}. \end{aligned} \tag{29}$$

In addition the quantization error bound in (26) is obtained as

$$\begin{aligned} \overline{\delta} &= |e^{j\theta} - r_\alpha| = \sqrt{r_\alpha^2 - 2r_\alpha\cos\theta + 1} \\ &= \sqrt{1-(1-\delta_\rho^2)\cos^2\theta} \\ &= \sqrt{\delta_\rho^2 + (1-\delta_\rho^2)\sin^2\theta} > \delta_\rho \end{aligned} \tag{30}$$

that agrees with (24) in the case of $m = 2$. The result of PLQ derived for two-dimensional vectors is instrumental to its generalization to multi-dimensional vectors. The next theorem solves the first major issue posed in the previous section.

Theorem 3.1 *Let $\delta \in (0,\ 1)$ be real, $\kappa > 1$ be integer, and $v \in \mathbb{R}^m$. Consider the* PLQ *in* (14) *and* (15) *for some real $\rho_o > 0$ and $\theta = \frac{\pi}{\kappa}$. The following hold.*
(i) *If $\rho_c = (1-\delta_\rho)\cos^{m-1}\theta$, then each real vector $v \in \mathbb{R}^m$ and its quantization v_q satisfy $v = (I+\Delta)v_q$ for some sector bounded nonlinearity Δ that has the least induced norm bound*

$$\overline{\delta} = \sqrt{1-(1-\delta_\rho^2)\cos^{2(m-1)}\theta} > \delta. \tag{31}$$

(ii) *If $\rho_c = [(1+\delta_\rho)\cos^{m-1}\theta]^{-1}$, then each real vector $v \in \mathbb{R}^m$ and its quantization v_q satisfy $v = (I+\Delta)^{-1}v_q$ for some sector bounded nonlinearity Δ that has the same least induced norm bound as in* (31)*. That is, $\overline{\delta} = \sqrt{1-(1-\delta_\rho^2)\cos^{2(m-1)}\theta} > \delta_\rho$ for $m \geq 2$.*

Proof: Following the same development in the case of $m = 2$, relative quantization error is considered first. In this case, each $v \in \mathbb{R}^m$ is quantized to v_q and there holds $v = (I+\Delta)v_q$. The problem is what value of ρ_c in (14) minimizes the quantization error $\|\Delta\|_{i_E}$. Although $m > 2$,

Fig. 1 can still be served as reference with r_A and r_B the same as in (20). By denoting $\mathcal{S}_{\text{fan}}$ as the set of the vectors inside the hyper-fan volume symmetric with respect to the last axis of the Cartesian coordinate system, the center point of $\mathcal{S}_{\text{fan}}$ represented by vector $\begin{bmatrix} 0 & \cdots & 0 & r_C \end{bmatrix}^T$ can be determined by the following two vectors:

$$v_C - v_A = -\begin{bmatrix} r_A \sin\theta \\ r_A \cos\theta \sin\theta \\ \vdots \\ r_A \cos^{m-2}\theta \sin\theta \\ r_A \cos^{m-1}\theta - r_C \end{bmatrix}, \quad v_C - v_B = -\begin{bmatrix} r_B \sin\theta \\ r_B \cos\theta \sin\theta \\ \vdots \\ r_B \cos^{m-2}\theta \sin\theta \\ r_B \cos^{m-1}\theta - r_C \end{bmatrix}.$$

Using the same deduction for the case $m = 2$, the center point satisfies $\|v_C - v_A\| = \|v_C - v_B\|$. This equality and the identity (13) implies that

$$r_A^2 - 2r_A r_C \cos^{m-1}\theta = r_B^2 - 2r_B r_C \cos^{m-1}\theta.$$

After rearrangement and noticing the logarithmic quantization in the radial direction, there hold

$$r_C = \frac{r_A + r_B}{2\cos^{m-1}\theta}, \qquad r_B = \left(\frac{1+\delta_\rho}{1-\delta_\rho}\right) r_A. \tag{32}$$

Clearly the relation in (22) still holds, but (23) is now replaced by

$$r_A = (1-\delta_\rho) r_C \cos^{m-1}\theta \quad \Longrightarrow \quad \rho_c = (1-\delta_\rho)\cos^{m-1}\theta. \tag{33}$$

Recall that each m-D vector $v \in \mathbb{R}^m$ can be written as $v = (I_m + \Delta)v_q$ for some quantized vector v_q. The above ρ_c minimizes the relative error bound $\|\Delta\|_{i_E}$ when it is used in (14) for m-D logarithmic quantization in conjunction with the same uniform quantization for each of its arguments. With the center vector specified by (32) and by the identity (13), it can be shown that

$$\begin{aligned} \|\Delta\|_{i_E}^2 &= \frac{1}{r_C^2}\left(r_C^2 + r_A^2 - 2r_A r_C \cos^{m-1}\theta\right) \\ &= 1 + (1-\delta_\rho)^2 \cos^{2(m-1)}\theta - 2(1-\delta_\rho)\cos^{2(m-1)}\theta \\ &= 1 - (1-\delta_\rho^2)\cos^{2(m-1)}\theta > \delta_\rho^2 \end{aligned} \tag{34}$$

by $0 < \theta < \pi/2$. In the case when $m = 2$, the above equality is the same as (24).

For quantization in form of $v = (I + \Delta)^{-1} v_q$, the value of ρ_c in (14) needs to be synthesized differently in order to minimize the quantization error $\|\Delta\|_{i_E}$. Following the same steps as in the case of $m = 2$, the following can be proven:

$$\|\Delta\|_{i_E} = \sup_{v \in \mathcal{S}_{\rm fan}} \frac{\|v - v_q\|}{\|v\|} = \sup_{|\varphi_i| \le \theta,\ r_A \le r < r_B} \left\| \begin{bmatrix} \sin\varphi_1 \\ \sin\varphi_2 \cos\varphi_1 \\ \vdots \\ \sin\varphi_{m-1} \prod_{i=1}^{m-2} \cos\varphi_i \\ \prod_{i=1}^{m-1} \cos\varphi_i \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \frac{r_C}{r} \end{bmatrix} \right\|$$

that is similar to (26). The equidistance condition and the identity in (13) lead to

$$\|\Delta\|_{i_E}^2 = 1 - 2r_\alpha \cos^{m-1}\theta + r_\alpha^2 = 1 - 2r_\beta \cos^{m-1}\theta + r_\beta^2 \tag{35}$$

where r_α and r_β satisfy (27). A similar derivation to the case of $m = 2$ shows that

$$\cos^{m-1}\theta = \frac{r_\alpha + r_\beta}{2} = \frac{r_\alpha}{1+\delta_\rho} = \frac{r_\beta}{1-\delta_\rho}.$$

In light of the relations in (20) and (27), there holds

$$r_C = r_A(1+\delta_\rho)\cos^{m-1}\theta = r_B(1-\delta_\rho)\cos^{m-1}\theta \quad \Longrightarrow \quad \rho_c = \frac{1}{(1+\delta_\rho)\cos^{m-1}\theta}. \tag{36}$$

Substituting the above into the first equality of (35) and making use of the relation in (20) yield

$$\overline{\delta}^2 = 1 - 2(1+\delta_\rho)\cos^{2(m-1)}\theta + (1+\delta_\rho)^2\cos^{2(m-1)}\theta = 1 - (1-\delta_\rho^2)\cos^{2(m-1)}\theta$$

that is identical to the expression in (34). The proof can be completed if it can be shown that for a given ρ_q, the corresponding hyper-fan volume with the center vector $v_q^{(0)} = \begin{bmatrix} 0 & \cdots & 0 & \rho_q \end{bmatrix}^T$, denoted by $V_q^{(0)}$, covers every other hyper-fan volume corresponding to the same ρ_q with a different center vector. Note that $V_q^{(0)}$ involves angles $\{\varphi_i\}$ ranging $[-\theta,\ \theta)$ for $1 \le i < m$, while all others, denoted by $V_q^{(\ell)}$ with integer $\ell > 1$, involve different angle range for at least one i. According to [11], the "surface volume" generalized from the surface area of a sphere for a given ρ is given by

$$S_{V_q^{(0)}} = g_m(\rho) \int_{-\theta}^{\theta} \cdots \int_{-\theta}^{\theta} \left(\prod_{i=1}^{m-2} \cos^{m-i-1}\varphi_i \right) d\varphi_1 \cdots d\varphi_{m-1} \ge S_{V_q^{(\ell)}}$$

if $m > 2$ where $g_m(\rho)$ is a polynomial function of ρ. The reason lies in the fact that $\cos\varphi_i$ achieves the maximum at 0 and monotonically decreases as $|\varphi_i|$ increases to $\frac{\pi}{2}$ for $1 \le i < m-1$. Note that the integrand of $S_{V_q^{(\ell)}}$ for each ℓ does not involve φ_{m-1}, and $g_m(\rho) > 0$ is a fixed value for each fixed ρ. In addition the upper and lower integral limits involved in computing $S_{V_q^{(\ell)}}$ can all be translated to from $-\theta$ to θ. As a result the integrand of $S_{V_q^{(\ell)}}$ for $\ell \neq 0$ is no greater than the corresponding integrand of $S_{V_q^{(0)}}$ at each value of $\{\varphi_i\}_{i=1}^{m-2}$ over the product of intervals $[-\theta,\ \theta)^m$. This implies that each point inside the surface of $S_{V_q^{(\ell)}}$ is covered by the set of points inside the surface of $S_{V_q^{(0)}}$ after it is rotated to the same position as that of $S_{V_q^{(0)}}$. Consequently the hyper-fan volume $v_q^{(0)}$ covers every other hyper-fan volume $v_q^{(\ell)}$. The upper bound for $\|\Delta\|_{i_E}$ thus holds for every other hyper fan volume, which completes the proof. □

It is interesting to observe that the above result reduces to the case of $m = 2$ derived earlier. In summary, Theorem 3.1 provides the answer to our first major issue for PLQ.

4. Coarsest Quantization Density

The uncertainty bound $\overline{\delta}$ in Theorem 3.1 may not yield the CQD when PLQ is employed to quantize the input or/and output signals of the system. The reason lies in the fact that $\overline{\delta}$ is a function of both $\delta_\rho \in (0,\ 1)$ and integer $\kappa > 1$. On the other hand, the CQD is also, albeit a different, function of δ_ρ and κ. At this point, it is appropriate to introduce the quantization density defined in [2, 3]. Denote $\mathcal{B} \in \mathbb{R}^m$ as the unit ball in $\mathbb{R}^m$, and let $N(\epsilon)$ be the number of hyper-fan volumes inside $\frac{1}{\epsilon}\mathcal{B}\backslash\epsilon\mathcal{B}$ where $0 < \epsilon < 1$. It can be easily verified that

$$N(\epsilon) = 2^m \kappa^{m-1} \ln\left(\frac{1}{\epsilon}\right)\left[\ln\left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)\right]^{-1}.$$

The length in log-scale for $[a,\ b]$ is defined as $\ln\left(\left|\frac{b}{a}\right|\right)$. Thus the length of $[\epsilon,\ \epsilon^{-1}]$ in log-scale is $2\ln(\epsilon^{-1})$. The quantization density, denoted by $\eta_m(\delta_\rho, \kappa)$ for m-D vectors, can be defined as

$$\eta_m(\delta_\rho, \kappa) := \limsup_{\epsilon \to 0} \frac{N(\epsilon)}{2\ln(\epsilon^{-1})} = (2\kappa)^{m-1}\left[\ln\left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)\right]^{-1} \tag{37}$$

that is the number of hyper-fan volumes per unit length in log-scale over the ray ρ, which is finite by the fact $0 < \delta_\rho < \overline{\delta} < 1$.

Remark 4.1 The CQD is the infimum of $\eta_m(\delta_\rho, \kappa)$ over (δ_ρ, κ) subject to feedback stability. However the CQD cannot be computed directly that is why a lower bound is derived in [3, 4] for 2-D state feedback controllers. For this reason, the stability margin $\overline{\delta}_{\max}$ needs to be computed first with $\overline{\delta}_{\max}$ as the supremum of $\overline{\delta}$ over all quadratic stabilizing controllers. After $\overline{\delta}_{\max}$ is available, then minimization of $\eta_m(\delta_\rho, \kappa)$ can be carried out. To be specific, consider the limiting case of $\overline{\delta} = \overline{\delta}_{\max}$. Since $\overline{\delta}$ is a function of δ_ρ by Theorem 3.1, $\overline{\delta} = \overline{\delta}_{\max} = f(\delta_\rho)$ for some function $f(\cdot)$. The elementary form of $f(\cdot)$ leads to the inverse function

$$\delta_\rho = f^{-1}(\overline{\delta}_{\max}) = \sqrt{1 - (1 - \overline{\delta}_{\max}^2)/\cos^{2(m-1)}[\pi/(2\kappa)]} =: \delta_{\rho,\kappa}. \tag{38}$$

It follows that $\delta_{\rho,\kappa}$ is a function of κ by the fact that $\overline{\delta}_{\max}$ depends only on the plant model. Define the subset $\mathcal{S} := \left\{\kappa \in \{2, 3, 4 \cdots\} : \ 0 < \delta_{\rho,\kappa} < \overline{\delta}_{\max}\right\}$. Then the CQD can be obtained through minimization of $\eta_m(\delta_{\rho,\kappa}, \kappa)$ over $\kappa \in \mathcal{S}$ that is a 1-D search over $\mathcal{S}$, i.e., CQD $= \min_{\kappa \in \mathcal{S}} \eta_m(\delta_{\rho,\kappa}, \kappa)$. □

The following result provides upper and lower bounds for $\overline{\delta}_{\max}$ in the case of multi-input systems under state feedback control when the PLQ is employed to quantize the control input.

Lemma 4.2 *Consider feedback system in* (1) *where* $u(k) \in \mathbb{R}^m$ *and* PLQ *is employed for* $\mathcal{Q}[u(k)]$ *for each* k. *Assume that* (A, B) *is stabilizable,*

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \qquad B = \begin{bmatrix} B_s \\ B_u \end{bmatrix}$$

with compatible dimensions, A_s *is Schur stable, and none of the eigenvalues of* A_u *is stable. If* B_u *has rank* $m_u \le m$, *then the stability margin satisfies the inequality of* $M(A)^{-1} \le \overline{\delta}_{\max} \le M(A)^{-\frac{1}{m_u}}$.

Proof: In the case when $m_u < m$, $B_u = \tilde{B}_u V$ for some V satisfying $VV^T = I$, and $(A_u, \tilde{B}_u)$ remains to be stabilizable. Because we are concerned with only stabilization, $T(z)$ in (9) can be replaced by

$$T_u(s) = \tilde{F}_u(sI - A_u - \tilde{B}_u\tilde{F}_u)^{-1}\tilde{B}_u$$

by setting $F = \begin{bmatrix} F_s & F_u \end{bmatrix}$ with compatible partition to that of A and B, $F_s = 0$, and $F_u = V^T\tilde{F}_u$. The $\mathcal{H}_\infty$ norm property implies that $\|T\|_{\mathcal{H}_\infty} = \|T_u\|_{\mathcal{H}_\infty}$. In light of [5, 10], the stability margin for the underlying NCS with PLQ is now given by

$$\overline{\delta}_{\max} = \left(\gamma_{\text{opt}} := \inf_{\tilde{F}_u} \|T_u\|_{\mathcal{H}_\infty}\right)^{-1}.$$

By [2, 5], $\gamma_{\text{opt}} \le M(A)$. See also Lemma 2 in [10]. On the other hand the inequality $\gamma_{\text{opt}} \ge M(A)^{\frac{1}{m_u}}$ follows from the proof of Theorem 1 in [10]. We can thus conclude $M(A)^{-1} \le \overline{\delta}_{\max} \le M(A)^{-\frac{1}{m_u}}$. □

The next result generalizes the lower bound in [3, 4] to m-dimensional input signals.

Corollary 4.3 *Under the hypothesis of* Lemma 4.2 *and* $m_u = m$, *and subject to feedback stability, there holds*

$$\inf_{\delta_\rho, \kappa} \frac{\eta_m(\delta_\rho, \kappa)}{(2\kappa)^{m-1}} > \left[\ln\left(\frac{M(A)^{\frac{1}{m}} + 1}{M(A)^{\frac{1}{m}} - 1}\right)\right]^{-1}.$$

The result in Corollary 4.3 agrees with the lower bound for CQD in [3, 4], derived for the case $m = 2$. The proof is straightforward due to PLQ in Theorem 3.1 and the result in Lemma 4.2. Indeed there holds

$$\delta_\rho < \overline{\delta} < \overline{\delta}_{\max} \le M(A)^{-\frac{1}{m}}$$

in order for the quadratic stabilizing state feedback gain F to exist. Consequently

$$\begin{aligned}\inf_{\delta_\rho,\kappa} \frac{\eta_m(\delta_\rho,\kappa)}{(2\kappa)^{m-1}} &= \inf_{\delta,\kappa} \left[\ln\left(\frac{1+\delta_\rho}{1-\delta_\rho}\right)\right]^{-1} \\ &> \left[\ln\left(\frac{M(A)^{\frac{1}{m}}+1}{M(A)^{\frac{1}{m}}-1}\right)\right]^{-1}\end{aligned}$$

by $\delta_\rho < M(A)^{-\frac{1}{m}}$. Hence the proof is skipped.

While Corollary 4.3 provides a lower bound estimate, Theorem 3.1 and Lemma 4.2 can be used to compute the exact CQD as discussed in Remark 4.1. The following example is illustrative.

Example 1 Consider a 2-input system under state feedback specified by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{39}$$

Eigenvalues of A are $1 \pm j$ and -2, and thus $M(A) = 4$. Under the PLQ at the control input, the CQD is tied to the stability margin that can be computed using the proof of Lemma 4.2. Using the result in [10], $\gamma_{\text{opt}} = \sqrt{8} = 2.8284$ can be obtained which is between $\sqrt{M(A)} = 2$ and $M(A) = 4$, and thus $\overline{\delta}_{\max} = \gamma_{\text{opt}}^{-1} = 0.3536$ that is between 0.25 and 0.5. The numerical results are consistent with those in Lemma 4.2. To determine the values of δ and κ for PLQ in accordance with Remark 4.1, $\overline{\delta} = \overline{\delta}_{\max}$ is taken for the limiting case. By the expression in (38) for the case of $m = 2$,

$$\delta_\rho = \delta_{\rho,\kappa} = \sqrt{1 - (1 - \overline{\delta}^2_{\max})/\cos^2[\pi/(2\kappa)]}$$

that is a function of κ. Recall the subset $\mathcal{S}$ that is the collection of those integers $\kappa > 1$ such that $0 < \delta_{\rho,\kappa} < \overline{\delta}_{\max}$. The expression of the quantization density in (37) yields the CQD as

$$\text{CQD} = \min_{\kappa\in\mathcal{S}} \eta_m(\delta_{\rho,\kappa}, \kappa) = 23.5479,$$

and the CQD takes place at $\kappa = 6$ and $\delta_\rho = 0.2494$. That is, the optimal PLQ yields the fan shaped tile with angle expansion of $2\theta = 30^o$ and the ratio of $(1 + \delta_\rho)/(1 - \delta_\rho) = 1.6645$. □

5. Conclusion

This paper investigates logarithmic quantization in terms of how it can be used for multi-input systems under state feedback, what the corresponding feedback stabilizability condition is, and how the stabilizing feedback controllers should be synthesized. This problem is studied in the past

[3, 4, 7]. The results in this paper provide new insight to NCSs when PLQ is employed. It has the potential to address not only the issue of feedback stability but also of feedback performance. More interesting part is the optimal PLQ in terms of minimization of the multiplicative and relative quantization errors which are unstructured contrasting to those under CLQ. The issue of the CQD is also investigated which is hinged on the stability margin of the corresponding NCS, and depends on two parameters in quantizing the length and orientation of the signal vectors. Although this paper did not consider the design issue for NCSs employing PLQ at both input and output of the MIMO plants, the results in this paper can be used to tackle more general design issues. However explicit stabilizability condition and the CQD for such design problems will be difficult to obtain, which are under our investigation.

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