On the structure of uniformly hyperbolic chain control sets

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Abstract

We prove the following theorem: Let Q be an isolated chain control set of a control-affine system on a smooth compact manifold M. If Q is uniformly hyperbolic without center bundle, then the lift of Q to the extended state space $\mathcal{U} \times M$, where $\mathcal{U}$ is the space of control functions, is a graph over $\mathcal{U}$. In other words, for every control $u \in \mathcal{U}$ there is a unique $x \in Q$ such that the corresponding state trajectory $\varphi(t, x, u)$ evolves in Q.

Keywords: Nonlinear control; control-affine system; chain control set; uniform hyperbolicity

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1 Introduction

The notion of a uniformly hyperbolic set, which axiomatizes the geometric picture behind the "horseshoe", a general mechanism for producing complicated dynamics, was introduced by Smale in the 1960s. A uniformly hyperbolic set of a diffeomorphism $g : M \to M$ on a compact Riemannian manifold M is a closed invariant set Λ such that the tangent bundle over Λ splits into two subbundles, $T\Lambda = E^s \oplus E^u$, invariant under the differential dg with uniform exponential contraction (expansion) on E^s (E^u). For a flow $(\phi_t)_{t\in\mathbb{R}}$, generated by an ordinary differential equation $\dot{x} = f(x)$, a uniformly hyperbolic set is defined differently, because for any trajectory bounded away from equilibria, the vector $f(x) \in T_xM$ is neither contracted nor expanded exponentially. In this case, a uniformly hyperbolic set is a closed invariant set Λ such that $T\Lambda = E^s \oplus E^c \oplus E^u$ with three invariant subbundles, where additionally to the contracting and expanding bundles the one-dimensional center bundle E^c corresponds to the flow direction. Without the center bundle E^c in this definition, a flow could only have trivial uniformly hyperbolic sets, consisting of finitely many equilibria.

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The situation looks different for systems generated by equations with explicitly time-dependent right-hand sides. General models for such systems are skew-products, which are dynamical systems of the form $\Phi : \mathbb{T} \times B \times M \to B \times M$, $\Phi_t(b,x) = (\theta_t b, \varphi(t,x,b))$, with a time set $\mathbb{T} \in \{\mathbb{Z},\mathbb{R}\}$. The solutions of the equation are incorporated in the map φ, while θ is a 'driving system' on a base space B that models the time-dependency of the equation. Every non-autonomous difference equation $x_{t+1} = f(t,x_t)$ or differential equation $\dot{x} = f(t,x)$ with unique and globally defined solutions gives rise to a skew-product, where $B = \mathbb{T}$ and $\theta_t(s) = t + s$. Other examples with less trivial base dynamics are random dynamical systems and control-affine systems. If B is a compact space, M a smooth manifold and Φ respects these structures, a uniformly hyperbolic set can be defined as a compact Φ-invariant set $\Lambda \subset B \times M$ such that for every $(b,x) \in \Lambda$ the tangent space $T_x M$ splits into subspaces $E^s_{b,x} \oplus E^u_{b,x}$ depending on b and x. The invariance of the splitting now means that $\mathrm{d}\varphi_{t,b}(x)E^{s/u}_{b,x} = E^{s/u}_{\Phi_t(b,x)}$, and contraction (expansion) rates should be uniformly bounded in b and x. One major difference to the autonomous situation is that there can exist non-trivial uniformly hyperbolic sets (whose projection to M has nonempty interior) in the continuous-time case without the existence of a one-dimensional center subbundle. This, for instance, happens in random dynamical systems that arise as small time-dependent perturbations of a flow around a hyperbolic equilibrium (cf. [7] for the discrete-time case).

In this paper, we consider a special type of skew-product flow, namely the control flow generated by a control-affine system, i.e., a control system governed by differential equations of the form

$$\Sigma : \ \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad u \in \mathcal{U}.$$

The set $\mathcal{U}$ of admissible control functions consists of all measurable $u : \mathbb{R} \to \mathbb{R}^m$ with values in a compact and convex set $U \subset \mathbb{R}^m$, and $f_0, f_1, \ldots, f_m$ are $\mathcal{C}^1$-vector fields on a smooth manifold M. The set $\mathcal{U}$, endowed with the weak*-topology of $L^\infty(\mathbb{R},\mathbb{R}^m) = L^1(\mathbb{R},\mathbb{R}^m)^*$, is a compact metrizable space. For each $u \in \mathcal{U}$ and $x \in M$ a unique solution to the corresponding equation exists with initial value x at time $t = 0$. Writing $\varphi(\cdot,x,u)$ for this solution and assuming that all such solutions exist on $\mathbb{R}$, one obtains a continuous skew-product flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u,x) = (\theta_t u, \varphi(t,x,u)),$$

where $\theta_t u(s) = u(t+s)$ is the shift flow on $\mathcal{U}$. There are remarkable relations between dynamical properties of Φ and control-theoretic properties of Σ, a comprehensive study of which can be found in [2]. In particular, the notions of control and chain control sets are to mention here. Control sets are the maximal subsets of M on which complete approximate controllability holds. Their lifts to $\mathcal{U} \times M$ are maximal topologically transivite sets of Φ. In contrast, chain control sets are the subsets of M whose lifts are the maximal invariant chain transitive sets of Φ, and they can be seen as an outer approximation of the control sets, since under mild assumptions a control set is contained in a chain control set.

The purpose of this paper is to prove a theorem about the structure of a chain control set Q with a uniformly hyperbolic structure without center bundle. We show that the lift of such Q, defined by

$$\mathcal{Q} := \{(u,x) \in \mathcal{U} \times M \ : \ \varphi(\mathbb{R},x,u) \subset Q\},$$

has the property that each fiber $\{x \in M : (u,x) \in \mathcal{Q}\}$ is a singleton. In other words, $\mathcal{Q}$ is the graph of a (necessarily continuous) function $\mathcal{U} \to Q$. This simple structure can be seen as an analogue to the fact that a connected uniformly hyperbolic set of a flow without center bundle consists of a single equilibrium. Nevertheless, from the control-theoretic viewpoint uniformly hyperbolic chain control sets are not trivial, since they can have nonempty interior and in this case are the closures of control sets (cf. [1, 3]).

The paper is organized as follows. In Section 2 we review the shadowing lemma proved in [8] for uniformly hyperbolic sets of general skew-product maps. This is the main tool for the proof of our theorem, which is carried out in Section 3. The final Section 4 contains an application to invariance entropy.

2 A shadowing lemma for skew-product maps

In this section, we explain the contents of the shadowing lemma for skew-product maps proved in [8] by Meyer and Zhang. Let M be a Riemannian manifold (with metric $d(\cdot,\cdot)$) and B a compact metric space. Suppose that

$$\Phi : B \times M \to B \times M, \quad \Phi(b,x) = (\theta(b), \varphi(b,x)),$$

is a homeomorphism such that also $\theta : B \to B$ is a homeomorphism.[1] For fixed $b \in B$ assume that $\varphi_b := \varphi(b,\cdot) : M \to M$ is a diffeomorphism whose derivative depends continuously on (b,x). The orbit through (b,x) is the set $\mathrm{O}(b,x) = \left\{\Phi^k(b,x) : k \in \mathbb{Z}\right\}$. We write $\varphi(k,x,b)$ for the second component of $\Phi^k(b,x)$, i.e., $\Phi^k(b,x) = (\theta^k(b), \varphi(k,x,b))$. A sequence $(b_k,x_k)_{k\in\mathbb{Z}}$ in $B \times M$ is an α-pseudo-orbit if

$$b_{k+1} = \theta(b_k) \quad \text{and} \quad d(\varphi(b_k,x_k),x_{k+1}) < \alpha \quad \text{for all } k \in \mathbb{Z}.$$

A pseudo-orbit $(b_k,x_k)_{k\in\mathbb{Z}}$ is β-shadowed by an orbit $\mathrm{O}(b,x)$ if

$$b = b_0 \quad \text{and} \quad d(\varphi(k,x,b),x_k) < \beta \quad \text{for all } k \in \mathbb{Z}.$$

A set $\Lambda \subset M \times B$ is *invariant* if $\Phi(\Lambda) = \Lambda$. A closed invariant set Λ is *isolated* if there exists a neighborhood U of Λ such that $\Phi^k(b,x) \in \mathrm{cl}\, U$ for all $k \in \mathbb{Z}$ implies

[1] In [8], θ is assumed to be almost periodic. However, this is not used for the proof of the shadowing lemma.

$(b,x) \in \Lambda$. A closed invariant set Λ is *uniformly hyperbolic* if there are constants $C > 0$, $0 < \mu < 1$ and a continuous map $(b,x) \mapsto P(b,x) \in P(T_xM, T_xM)$, defined on Λ, where $P(T_xM, T_xM)$ denotes the space of all linear projections on T_xM, such that

(i) $P(\Phi(b,x))\mathrm{d}\varphi_b(x) = \mathrm{d}\varphi_b(x)P(b,x)$.

(ii) $\|\mathrm{d}\varphi_{k,b}(x)P(b,x)\| \leq C\mu^k$ for all $(b,x) \in \Lambda$, $k \geq 0$.

(iii) $\|\mathrm{d}\varphi_{k,b}(x)(I - P(b,x))\| \leq C\mu^{-k}$ for all $(b,x) \in \Lambda$, $k \leq 0$.

Here $\varphi_{k,b} = \varphi(k,\cdot,b)$. A reduced version of the shadowing lemma [8, Lem. 2.11] reads as follows.

2.1 Lemma: *Let $\Lambda \subset B \times M$ be a compact invariant uniformly hyperbolic set. Then there is a neighborhood U of Λ such that the following holds:*

(i) For any $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudo-orbit $(b_k, x_k)_{k\in\mathbb{Z}}$ in U is β-shadowed by an orbit $\{\Phi^k(b_0, y) : k \in \mathbb{Z}\}$.

(ii) There is $\beta_0 > 0$ such that $0 < \beta < \beta_0$ implies that the shadowing orbit in (i) is uniquely determined by the pseudo-orbit.

(iii) If Λ is an isolated invariant set of Φ, then the shadowing orbit is in Λ.

3 The main result

3.1 Preliminaries and assumptions

Consider a control-affine system

$$\Sigma: \ \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t)f_i(x(t)), \quad u \in \mathcal{U} = L^\infty(\mathbb{R}, U),$$

on a compact Riemannian manifold M with distance $d(\cdot,\cdot)$. The vector fields $f_0, f_1, \ldots, f_m$ are assumed to be of class $\mathcal{C}^1$ and the control range $U \subset \mathbb{R}^m$ is compact and convex. The set $\mathcal{U}$ of admissible control functions is endowed with the weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = L^1(\mathbb{R}, \mathbb{R}^m)^*$. We write

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u,x) = (\theta_t u, \varphi(t,x,u)),$$

for the associated control flow and $\varphi_{t,u} = \varphi(t,\cdot,u)$. A chain control set is a set $Q \subset M$ with the following properties:

(i) For every $x \in Q$ there exists $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset Q$.

(ii) For each two $x, y \in Q$ and all $\varepsilon, T > 0$ there are $n \in \mathbb{N}$, controls $u_0, \ldots, u_{n-1} \in \mathcal{U}$, states $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ and times $t_0, \ldots, t_{n-1} \geq T$ such that

$$d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon, \quad i = 0, 1, \ldots, n-1.$$

(iii) Q is maximal with (i) and (ii) in the sense of set inclusion.

Throughout the paper, we fix a chain control set Q and write

$$\mathcal{Q} = \{(u,x) \in \mathcal{U} \times M \ : \ \varphi(\mathbb{R}, x, u) \subset Q\}$$

for its *full-time lift*, which is a chain recurrent component of the control flow Φ on $\mathcal{U} \times M$ (cf. [2, Thm. 4.1.4]). We further assume that $\mathcal{Q}$ is an isolated invariant set for Φ, i.e., there exists a neighborhood $N \subset \mathcal{U} \times M$ of $\mathcal{Q}$ such that $\Phi(\mathbb{R}, u, x) \subset N$ implies $(u,x) \in \mathcal{Q}$. This, e.g., is the case if there are only finitely many chain control sets on M, because then the chain recurrent components are the elements of a finest Morse decomposition. We further assume that

$$T_x M = E^+_{u,x} \oplus E^-_{u,x}, \quad \forall (u,x) \in \mathcal{Q},$$

with subspaces $E^{\pm}_{u,x}$ satisfying

(H1) $\mathrm{d}\varphi_{t,u}(x) E^{\pm}_{u,x} = E^{\pm}_{\Phi_t(u,x)}$ for all $(u,x) \in \mathcal{Q}$ and $t \in \mathbb{R}$.

(H2) There exist constants $0 < c \leq 1$ and $\lambda > 0$ such that for all $(u,x) \in \mathcal{Q}$,

$$|\mathrm{d}\varphi_{t,u}(x) v| \leq c^{-1} \mathrm{e}^{-\lambda t} |v| \quad \text{for all } t \geq 0, \ v \in E^-_{u,x},$$

and

$$|\mathrm{d}\varphi_{t,u}(x) v| \geq c \mathrm{e}^{\lambda t} |v| \quad \text{for all } t \geq 0, \ v \in E^+_{u,x}.$$

From (H1) and (H2) it follows that $E^{\pm}_{u,x}$ depend continuously on (u,x) (cf. [6, Lem. 6.4]). We define the u-fiber of $\mathcal{Q}$ by

$$Q(u) := \{x \in Q \ : \ (u,x) \in \mathcal{Q}\}.$$

On $\mathcal{U}$ we fix a metric, compatible with the weak*-topology, of the form

$$d_{\mathcal{U}}(u,v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \mathrm{d}t|}{1 + |\int_{\mathbb{R}} \langle u(t) - v(t), x_n(t) \rangle \mathrm{d}t|},$$

where $\{x_n : n \in \mathbb{N}\}$ is a dense and countable subset of $L^1(\mathbb{R}, \mathbb{R}^m)$ and $\langle \cdot, \cdot \rangle$ is a fixed inner product on $\mathbb{R}^m$ (cf. [2, Lem. 4.2.1]).

3.2 Statement of results and proofs

We observe that the time-1-map $\Phi_1 : \mathcal{U}\times M \to \mathcal{U}\times M$, $(u,x) \mapsto (\theta_1 u, \varphi(1,x,u))$, of the control flow is a skew-product map and $\mathcal{Q}$ is a uniformly hyperbolic set for Φ_1 in the sense of Section 2. Moreover, $\mathcal{Q}$ is isolated for Φ_1, which easily follows from our assumption that $\mathcal{Q}$ is an isolated invariant set of the control flow. Continuous dependence of the derivative $\mathrm{d}\varphi_{1,u}(x)$ on (u,x) is proved in [6, Thm. 1.1].

3.1 Proposition: *For any $u,v \in \mathcal{U}$ the fibers $Q(u)$ and $Q(v)$ are homeomorphic.*

Proof: The proof is subdivided into three steps.

Step 1. We claim that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $u,v \in \mathcal{U}$,

$$\|u - v\|_\infty < \delta \quad \Rightarrow \quad d_{\mathcal{U}}(\theta_t u, \theta_t v) < \varepsilon \quad \text{for all } t \in \mathbb{R}, \tag{1}$$

where $\|\cdot\|_\infty$ is the L^∞-norm. To prove this, choose $N = N(\varepsilon) \in \mathbb{N}$ with

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Then put

$$c(\varepsilon) := \max_{1\leq n\leq N} \|x_n\|_1, \quad \delta := \frac{\varepsilon}{2c(\varepsilon)},$$

where $\|\cdot\|_1$ is the L^1-norm. Then, for every $t \in \mathbb{R}$, $\|u - v\|_\infty < \delta$ implies

$$\begin{aligned}
d_{\mathcal{U}}(\theta_t u, \theta_t v) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\int_{\mathbb{R}} \langle u(t+s) - v(t+s), x_n(s)\rangle \mathrm{d}s|}{1 + |\int_{\mathbb{R}} \langle u(t+s) - v(t+s), x_n(s)\rangle \mathrm{d}s|} \\
&< \sum_{n=1}^{N} \frac{1}{2^n} \left| \int_{\mathbb{R}} \langle u(t+s) - v(t+s), x_n(s)\rangle \mathrm{d}s \right| + \frac{\varepsilon}{2} \\
&\leq \sum_{n=1}^{N} \frac{1}{2^n} \int_{\mathbb{R}} |u(t+s) - v(t+s)| \cdot |x_n(s)| \mathrm{d}s + \frac{\varepsilon}{2} \\
&\leq \delta \sum_{n=1}^{N} \frac{1}{2^n} \max_{1\leq n\leq N} \int_{\mathbb{R}} |x_n(s)| \mathrm{d}s + \frac{\varepsilon}{2} < \frac{\varepsilon}{2c(\varepsilon)} c(\varepsilon) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Step 2. Consider the time-1-map Φ_1 of the control flow with the uniformly hyperbolic set $\mathcal{Q}$. Let $\beta > 0$ be given and choose $\alpha = \alpha(\beta)$ according to the shadowing lemma 2.1. Then choose $\varepsilon = \varepsilon(\alpha)$ such that

$$d_{\mathcal{U}}(u,v) < \varepsilon \quad \Rightarrow \quad d(\varphi(1,x,u), \varphi(1,x,v)) < \alpha, \tag{2}$$

whenever $u, v \in \mathcal{U}$ and $x \in Q$. This is possible by uniform continuity of $\varphi(1, \cdot, \cdot)$ on the compact set $Q \times \mathcal{U}$. We claim that for all sufficiently small β,

$$\sup_{t \in \mathbb{R}} d_{\mathcal{U}}(\theta_t u, \theta_t v) < \varepsilon \quad \Rightarrow \quad Q(u) \text{ and } Q(v) \text{ are homeomorphic.} \tag{3}$$

If $Q(u)$ and $Q(v)$ are both empty, there is nothing to show. Otherwise, we may assume that $Q(u) \neq \emptyset$. Then choose $x \in Q(u)$ arbitrarily and consider the doubly infinite sequence $x_n := \varphi(n, x, u)$, $n \in \mathbb{Z}$, which is completely contained in Q and (by (2)) satisfies

$$d(\varphi(1, x_n, \theta_n v), x_{n+1}) = d(\varphi(1, x_n, \theta_n v), \varphi(1, x_n, \theta_n u)) < \alpha$$

for all $n \in \mathbb{Z}$. Hence, $(x_n, \theta_n v)_{n \in \mathbb{Z}}$ is an α-pseudo-orbit for Φ_1. By the shadowing lemma there exists $y \in M$ with

$$d(\varphi(n, y, v), \varphi(n, x, u)) < \beta \quad \text{for all } n \in \mathbb{Z}.$$

Since $\mathcal{Q}$ is isolated, Lemma 2.1(iii) implies $(v, y) \in \mathcal{Q}$, i.e., $y \in Q(v)$. We claim that the map

$$h_{uv} : Q(u) \to Q(v), \quad x \mapsto y,$$

defined in this way, is a homeomorphism. If β is small enough, the shadowing orbit is unique by Lemma 2.1(ii), and hence h_{uv} is uniquely defined. Since the β-shadowing relation between u-orbits and v-orbits is symmetric, one can equivalently define a map $h_{vu} : Q(v) \to Q(u)$, which by uniqueness must be the inverse of h_{uv}. The proof for the continuity of h_{uv} is standard and will be omitted. (Continuity will also follow trivially from Proposition 3.3).

Step 3. From convexity of U it follows that for arbitrary $u, v \in \mathcal{U}$ the curve

$$w : [0,1] \to \mathcal{U}, \quad \tau \mapsto w_\tau, \quad w_\tau(t) :\equiv (1-\tau)u(t) + \tau v(t),$$

is well-defined. It is continuous w.r.t. the L^∞-topology on $\mathcal{U}$, since

$$\begin{aligned}
\|w_{\tau_1} - w_{\tau_2}\|_\infty &= \operatorname*{ess\,sup}_{t \in \mathbb{R}} |(1-\tau_1)u(t) + \tau_1 v(t) - (1-\tau_2)u(t) - \tau_2 v(t)| \\
&= \operatorname*{ess\,sup}_{t \in \mathbb{R}} |(\tau_2 - \tau_1)u(t) - (\tau_2 - \tau_1)v(t)| \\
&= \operatorname*{ess\,sup}_{t \in \mathbb{R}} |\tau_2 - \tau_1| \cdot |u(t) - v(t)| \leq |\tau_2 - \tau_1| \operatorname{diam} U.
\end{aligned}$$

Hence, for each $\tau \in [0,1]$ we can pick a (relatively) open subinterval $I_\tau \subset [0,1]$ containing τ such that $\|w_s - w_r\|_\infty$ is smaller than a given constant for all $s, r \in I_\tau$. By Step 1 this implies that $\sup_{t \in \mathbb{R}} d_{\mathcal{U}}(\theta_t w_s, \theta_t w_r) < \varepsilon$ for all $s, r \in I_\tau$ with a given $\varepsilon > 0$. Choose ε according to Step 2 so that $Q(w_s)$ and $Q(w_r)$ are homeomorphic for any two $s, r \in I_\tau$. By compactness, finitely many such intervals $I_{\tau_1}, \ldots, I_{\tau_l}$ are sufficient to cover $[0,1]$. We may assume that $0 = \inf I_{\tau_1} < \inf I_{\tau_2} < \cdots < \inf I_{\tau_l} < \sup I_{\tau_l} = 1$. To show that $Q(u)$ and $Q(v)$ are homeomorphic, we put $t_0 := 0$, $t_l := 1$ and pick $t_i \in I_{\tau_i} \cap I_{\tau_{i+1}} \neq \emptyset$ for $i = 1, \ldots, l-1$. Then there exist homeomorphisms $h_{i,i+1} : Q(w_{t_i}) \to Q(w_{t_{i+1}})$ for $0 \leq i \leq l-1$. The composition of these homeomorphisms gives a homeomorphism from $Q(u) = Q(w_{t_0})$ to $Q(v) = Q(w_{t_l})$. $\square$

3.2 Corollary: *If $Q(u)$ is a singleton for one $u \in \mathcal{U}$, then $\mathcal{Q}$ is the graph of a continuous map from $\mathcal{U}$ to Q.*

Proof: By the proposition, $Q(u)$ is a singleton for every $u \in \mathcal{U}$, say $Q(u) = \{x(u)\}$. Consider the map $u \mapsto (u, x(u))$, $\mathcal{U} \to \mathcal{Q}$. This is an invertible map between compact metric spaces with (obviously) continuous inverse. Hence, it is a homeomorphism. This implies that the function $u \mapsto x(u)$, $\mathcal{U} \to Q$, is continuous and $\mathcal{Q}$ is its graph. □

The next proposition shows that the fibers $Q(u)$ are finite.

3.3 Proposition: *If u is a constant control function, then $Q(u)$ consists of finitely many equilibria. Hence, there exists $n \in \mathbb{N}$ such that $Q(u)$ has precisely n elements for every $u \in \mathcal{U}$.*

Proof: Let $u \in \mathcal{U}$ be a constant control function. Observe that $Q(u)$ is a uniformly hyperbolic set for the diffeomorphism $g := \varphi_{1,u} : M \to M$. It is well-known that a diffeomorphism is expansive on a uniformly hyperbolic set, i.e., there is $\varepsilon > 0$ such that $d(g^k(x), g^k(y)) < \varepsilon$ for all $k \in \mathbb{Z}$ and $x, y \in Q(u)$ implies $x = y$ (see [5, Cor. 6.4.10]). If $x \in Q(u)$ and $w := f_0(x) + \sum_{i=1}^m u_i f_i(x) \neq 0$, then the Lyapunov exponent $l(w) := \limsup_{t\to\infty}(1/t)\log|\mathrm{d}\varphi_{t,u}(x)w|$ vanishes if the trajectory $\varphi(t, x, u)$ is bounded away from equilibria. A zero Lyapunov exponent, however, contradicts the existence of the uniformly hyperbolic splitting on $\mathcal{Q}$. Since the right-hand side of the system is bounded on compact sets, $l(w) < 0$ follows, implying $w \in E^-_{u,x}$. Writing $f := f_0 + \sum_{i=1}^m u_i f_i$, this yields $|\mathrm{d}\varphi_{t,u}(x)w| = |f(\varphi(t, x, u))| \leq c^{-1}\mathrm{e}^{-\lambda t}$ for $t \geq 0$. Because of the uniform hyperbolicity, there can be at most finitely many equilibria in the compact set $Q(u)$, and hence $\varphi(t, x, u) \to z_+$ for some equilibrium z_+. The same argumentation for the backward flow yields $\varphi(t, x, u) \to z_-$ for an equilibrium z_-. Choose $t_0 > 0$ large enough so that

$$d(\varphi(t, x, u), z_\pm) < \frac{\varepsilon}{2} \quad \text{if } |t| \geq \frac{t_0}{2}. \tag{4}$$

Then choose $\delta > 0$ small enough so that

$$d(x, y) < \delta \quad \Rightarrow \quad d(\varphi(t, x, u), \varphi(t, y, u)) < \varepsilon \quad \text{for all } |t| \leq t_0. \tag{5}$$

Finally, let $\tau \in (0, t_0/2)$ be chosen so that

$$d(x, \varphi(\tau, x, u)) < \delta. \tag{6}$$

We let $y := \varphi(\tau, x, u)$ and claim that $d(\varphi(t, x, u), \varphi(t, y, u)) < \varepsilon$ for all $t \in \mathbb{R}$, implying $x = y$. Indeed, by (6) and (5) we have

$$d(\varphi(t, x, u), \varphi(t, y, u)) < \varepsilon \quad \text{for all } |t| \leq t_0.$$

Now assume that $t \geq t_0$. Then (4) yields

$$d(\varphi(t, x, u), \varphi(t, y, u)) \leq d(\varphi(t, x, u), z_+) + d(z_+, \varphi(t + \tau, x, u)) < \varepsilon.$$

If $t < -t_0$, we obtain $t+\tau < -t_0+\tau < -t_0+t_0/2 = -t_0/2$, and hence (4) gives

$$d(\varphi(t,x,u),\varphi(t,y,u)) \leq d(\varphi(t,x,u),z_-) + d(z_-,\varphi(t+\tau,x,u)) < \varepsilon.$$

In particular, $d(g^k(x),g^k(y)) < \varepsilon$ for all $k \in \mathbb{Z}$, and hence $x = y = z_+ = z_-$. Consequently, $Q(u)$ consists of finitely many equilibria. $\square$

The next theorem is our main result.

3.4 Theorem: *Consider the control-affine system Σ with the uniformly hyperbolic chain control set Q with isolated lift $\mathcal{Q}$. Assume that $\operatorname{int} U \neq \emptyset$ and let u_0 be a constant control function with value in $\operatorname{int} U$. Additionally suppose that the following hypotheses are satisfied:*

- *(i) The vector fields $f_0, f_1, \ldots, f_m$ are of class $\mathcal{C}^\infty$ and the Lie algebra generated by them has full rank at each point of Q.*
- *(ii) For each $x \in Q(u_0)$ and each $\rho \in (0,1]$ it holds that $x \in \operatorname{int} \mathcal{O}^+_\rho(x)$, where $\mathcal{O}^+_\rho(x) = \{\varphi(t,x,u) : t \geq 0,\ u \in \mathcal{U}^\rho\}$ with*

$$\mathcal{U}^\rho = \{u \in \mathcal{U} \ : \ u(t) \in u_0 + \rho(U - u_0) \text{ a.e.}\}.$$

Then $\mathcal{Q}$ is a graph of a continuous function $\mathcal{U} \to Q$.

3.5 Remark: Before proving the theorem, we note that assumption (ii) is in particular satisfied if the system is locally controllable at (u_0,x) for each $x \in Q(u_0)$ (using arbitrarily small control ranges around u_0). A sufficient condition for this to hold, which is independent of ρ, is the controllability of the linearization around (u_0,x).

Proof: Without loss of generality, we assume that $u_0(t) \equiv 0$. Let $Q(u_0) = \{x_1,\ldots,x_n\}$. We consider for each $\rho \in (0,1]$ the control-affine system

$$\Sigma^\rho : \ \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad u \in \mathcal{U}^\rho.$$

From the assumptions (i) and (ii) it follows by [2, Cor. 4.1.7] that each x_i is contained in the interior of a control set D_i^ρ of Σ^ρ. Each D_i^ρ is contained in a unique chain control set E_i^ρ of Σ^ρ (cf. [2, Cor. 4.3.12]). By [2, Cor. 3.1.14] the chain control sets depend upper semicontinuously on ρ, hence $E_i^\rho \subset Q = E_i^1$ for every $1 \leq i \leq n$ and $\rho \in (0,1]$. This implies that each E_i^ρ is uniformly hyperbolic. By [1, Thm. 3] it follows that $E_i^\rho = \operatorname{cl} D_i^\rho$. If C_i denotes the chain recurrent component of the uncontrolled system $\dot{x} = f_0(x)$ which contains the equilibrium x_i, then $C_i \subset E_i^\rho$ for each ρ, because otherwise $E_i^\rho \cup C_i$ would satisfy the first two properties of chain control sets, contradicting maximality of E_i^ρ. Since each chain recurrent component is connected and $C_i \subset Q(u_0)$, we have $C_i = \{x_i\}$. By [2, Cor. 3.4.10], the chain control set E_i^ρ shrinks to $\{x_i\}$ as $\rho \searrow 0$.

Hence, for small ρ, the sets E_i^ρ are pairwisely disjoint. Since $E_i^1 = Q$ for each i, at some point the chain control sets have to merge as ρ increases. Since, by [2, Thm. 3.1.12], the control sets D_i^ρ depend lower semicontinuously on ρ, this is a contradiction if $n > 1$. It follows that $n = 1$ and Corollary 3.2 yields the assertion. □

3.6 Remark: Of course, in many cases it will be easier to check directly that $Q(u)$ is a single equilibrium for some constant control function u than verifying the conditions of the preceding theorem. We also note that the fact that $\mathcal{Q}$ is a graph over $\mathcal{U}$ implies the existence of a topological conjugacy between the shift flow θ on $\mathcal{U}$ and the restriction of the control flow to $\mathcal{Q}$ (cf. [3]).

4 Application to invariance entropy

The invariance entropy of a controlled invariant subset Q of M measures the complexity of the control task of keeping the state inside Q. In general, it is defined as follows. A pair (K, Q) of subsets of M is called admissible if K is compact and for every $x \in K$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, u) \subset Q$. In particular, if $K = Q$, this means that Q is a compact and controlled invariant set. For $\tau > 0$, a set $\mathcal{S} \subset \mathcal{U}$ is (τ, K, Q)-spanning if for every $x \in K$ there is $u \in \mathcal{S}$ with $\varphi([0,\tau], x, u) \subset Q$. Then $r_{\mathrm{inv}}(\tau, K, Q)$ denotes the number of elements in a minimal such set and we put $r_{\mathrm{inv}}(\tau, K, Q) := \infty$ if no finite (τ, K, Q)-spanning set exists. The *invariance entropy* of (K, Q) is

$$h_{\mathrm{inv}}(K, Q) := \limsup_{\tau \to \infty} \frac{1}{\tau} \log r_{\mathrm{inv}}(\tau, K, Q),$$

where log is the natural logarithm. From [3, Thm. 5.4] we can conclude the following result on the invariance entropy of admissible pairs (K, Q), where Q is a uniformly hyperbolic chain control set. The difference to [3, Thm. 5.4] is that we do not have to assume explicitly anymore that $\mathcal{Q}$ is a graph over $\mathcal{U}$.

4.1 Theorem: *Consider the control-affine system Σ with the uniformly hyperbolic chain control set Q with isolated lift $\mathcal{Q}$. Let the assumptions (i) and (ii) of Theorem 3.4 be satisfied, or alternatively, assume that $Q(u)$ is a singleton for some $u \in \mathcal{U}$. Then Q is the closure of a control set D and for every compact set $K \subset D$ of positive volume the pair (K, Q) is admissible and its invariance entropy satisfies*

$$h_{\mathrm{inv}}(K, Q) = \inf_{(u,x) \in \mathcal{Q}} \limsup_{\tau \to \infty} \frac{1}{\tau} \log \left| \det(\mathrm{d}\varphi_{\tau,u})|_{E^+_{u,x}} : E^+_{u,x} \to E^+_{\Phi_\tau(u,x)} \right|.$$

4.2 Remark: The paper [4] provides a rich class of examples for uniformly hyperbolic chain control sets that arise on the flag manifolds of a semisimple Lie group. The control-affine system in this case is induced by a right-invariant system on the group.

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