# Explicit solutions for continuous time mean-variance portfolio selection with nonlinear wealth equations 

Shaolin Ji *<br>Xiaomin Shi ${ }^{\dagger}$

July 10, 2018


#### Abstract

This paper concerns the continuous time mean-variance portfolio selection problem with a special nonlinear wealth equation. This nonlinear wealth equation has a nonsmooth coefficient and the dual method developed in [6] does not work. We invoke the HJB equation of this problem and give an explicit viscosity solution of the HJB equation. Furthermore, via this explicit viscosity solution, we obtain explicitly the efficient portfolio strategy and efficient frontier for this problem. Finally, we show that our nonlinear wealth equation can cover three important cases.


Key words. mean-variance portfolio selection; nonlinear wealth equation; HJB equation; viscosity solution

Mathematics Subject Classification (2010) 60H10 93E20

## 1 Introduction

A mean-variance portfolio selection problem is to find the optimal portfolio strategy which minimizes the variance of its terminal wealth while its expected terminal wealth equals a prescribed level. Markowitz [12], [13] first studied this problem in the single-period setting. It's multi-period and continuous time counterparts have been studied extensively in the literature; see, e.g. [1], [7], [9], 10], 15] and the references therein. Most of the literature on mean-variance portfolio selection focuses on an investor with linear wealth equation. But in some cases, one need to consider nonlinear wealth equations. For example, a large investor's portfolio selection may affect the return of the stock's price which leads to a nonlinear wealth equation. When some taxes must be paid on the gains made on the stocks, we also have to deal with a nonlinear wealth equation.

As for the continuous time mean-variance portfolio selection problem with nonlinear wealth equation, Ji [6] obtained a necessary condition for the optimal terminal wealth when the coefficient of the wealth equation is smooth. [5] studied the continuous time mean-variance portfolio selection problem with higher borrowing

[^0]rate in which the wealth equation is nonlinear and the coefficient is not smooth. They employed the viscosity solution of the HJB equation to characterize the optimal portfolio strategy.

In this paper, the continuous time mean-variance portfolio selection problem with a special nonlinear wealth equation is studied. This nonlinear wealth equation has a nonsmooth coefficient and can cover the following three important models: the first model is proposed by Jouini and Kallal [8] and El Karoui et al [3] in which an investor has different expected returns for long and short position of the stock (see Example 4.1); the second one is given in section 4 of [2] for a large investor (see Example 4.2); the third one is introduced in [3] to study the wealth equation with taxes paid on the gains (see Example 4.3). We invoke the Hamilton-Jacobi-Bellman (HJB for short) equation of this problem and give an explicit viscosity solution of the HJB equation. Furthermore, via this explicit viscosity solution, we obtain explicitly the efficient portfolio strategy and efficient frontier for this problem.

The paper is organized as follows. In section 2, we formulate the problem. Our main results are given in section 3 . In section 4, we show that our wealth equation (2.1) can cover three important cases.

## 2 Formulation of the problem

Let $W$ be a standard 1-dimensional Brownian motion defined on a filtered complete probability space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}, P\right)$, where $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ denotes the natural filtration associated with the 1-dimensional Brownian motion $W$ and augmented. We denote by $M^{2}(0, T)$ the space of all $\mathcal{F}_{t}$-progressively measurable $\mathbb{R}$ valued processes $x$ such that $E \int_{0}^{T} x_{t}^{2} d t<\infty$.

We consider a financial market consisting of a riskless asset (the money market instrument or bond) whose price is $S^{0}$ and one risky security (the stock) whose price is $S^{1}$. An investor can decide at time $t \in[0, T]$ what amount $\pi_{t}$ of his wealth $X_{t}$ to invest in the stock. Of course, his decisions can only depend on the current information $\mathcal{F}_{t}$, that is, the portfolio $\pi$ is $\mathcal{F}_{t}$-adapted.

For given deterministic continuous functions $r_{t}, \underline{\theta}_{t}, \bar{\theta}_{t}, \sigma_{t}$ on $[0, T]$, consider the following nonlinear wealth equation:

$$
\left\{\begin{array}{l}
d X_{t}=\left(r_{t} X_{t}+\pi_{t}^{+} \sigma_{t} \underline{\theta}_{t}-\pi_{t}^{-} \sigma_{t} \bar{\theta}_{t}\right) d t+\pi_{t} \sigma_{t} d W_{t}  \tag{2.1}\\
X_{0}=x_{0}, t \in[0, T]
\end{array}\right.
$$

where the functions $x^{+}:=\left\{\begin{array}{l}x, \text { if } x \geq 0 ; \\ 0, \text { if } x<0,\end{array}\right.$ and $x^{-}:=\left\{\begin{array}{c}-x, \text { if } x \leq 0 ; \\ 0, \text { if } x>0 .\end{array}\right.$
We assume:
Assumption $2.1 \underline{\theta}_{t} \geq 0, \bar{\theta}_{t} \geq 0$, a.e. on $[0, T], \sigma_{t} \neq 0$, a.e. on $[0, T]$.

Remark 2.2 When $\underline{\theta}_{t}=\bar{\theta}_{t}$, a.e. on $[0, T]$, the wealth equation (2.1) reduces to the classical linear wealth equation.

For a given expectation level $K$, consider the following continuous time mean-variance portfolio selection
problem:

$$
\begin{align*}
& \text { Minimize } \operatorname{Var} X_{T}=E\left(X_{T}-K\right)^{2}, \\
& \text { s.t. }\left\{\begin{array}{l}
E X_{T}=K, \\
\pi \in M^{2}(0, T), \\
(X, \pi) \text { satisfies Eq.(2.1). }
\end{array}\right. \tag{2.2}
\end{align*}
$$

Throughout the paper, we assume that $K \geq x_{0} e^{\int_{0}^{T} r_{s} d s}$.
The optimal strategy $\pi^{*}$ is called an efficient strategy. Denote the optimal terminal value by $X_{T}^{*}$. Then, $\left(\operatorname{Var} X_{T}^{*}, K\right)$ is called an efficient point. The set of all efficient points $\left\{\left(\operatorname{Var} X_{T}^{*}, K\right) \mid K \in\left[x_{0} e^{\int_{0}^{T} r_{s} d s},+\infty\right)\right\}$ is called the efficient frontier.

Definition 2.3 $A$ portfolio $\pi$ is said to be admissible if $\pi \in M^{2}(0, T)$ and ( $\left.X, \pi\right)$ satisfies Eq.(2.1).
Denote by $\mathcal{A}\left(x_{0} ; 0, T\right)$ the set of portfolio $\pi$ admissible for the initial investment $x_{0}$. For simplicity, we set $\mathcal{A}\left(x_{0}\right):=\mathcal{A}\left(x_{0} ; 0, T\right)$.

## 3 Main results

To deal with the constraint $E X_{T}=K$, we introduce a Lagrange multiplier $-2 \lambda \in \mathbb{R}$ and get the following auxiliary optimal stochastic control problem:

$$
\begin{align*}
& \text { Minimize } E\left(X_{T}-K\right)^{2}-2 \lambda\left(E X_{T}-K\right)=E\left(X_{T}-d\right)^{2}-(d-K)^{2}:=\hat{J}(\pi, d), \\
& \text { s.t. }\left\{\begin{array}{l}
\pi \in M^{2}(0, T), \\
(X, \pi) \text { satisfies Eq.(2.1), }
\end{array}\right. \tag{3.1}
\end{align*}
$$

where $d:=K+\lambda$.

Remark 3.1 The link between problem (2.2) and (3.1) is provided by the Lagrange duality theorem (see Luenberger [11])

$$
\min _{\pi \in \mathcal{A}\left(x_{0}\right), E X_{T}=K} \operatorname{Var} X_{T}=\max _{d \in \mathbb{R}} \min _{\pi \in \mathcal{A}\left(x_{0}\right)} \hat{J}(\pi, d) .
$$

So the optimal problem (2.2) can be divided into two steps. The first step is to solve

$$
\begin{equation*}
\text { Minimize } E\left(X_{T}-d\right)^{2} \text {, s.t. } \pi \in \mathcal{A}\left(x_{0}\right) \text {, } \tag{3.2}
\end{equation*}
$$

for any fixed $d \in \mathbb{R}$. The second step is to find the Lagrange multiple which attains

$$
\max _{d \in \mathbb{R}} \min _{\pi \in \mathcal{A}\left(x_{0}\right)} \hat{J}(\pi, d)
$$

To solve the first step, we introduce the stochastic control problem

$$
\begin{equation*}
v(t, x ; d):=\inf _{\pi \in \mathcal{A}(x ; t, T)} E\left(X_{T}-d\right)^{2},(t, x) \in[0, T] \times \mathbb{R} \tag{3.3}
\end{equation*}
$$

on $[t, T]$, subject to

$$
\left\{\begin{array}{l}
d X_{s}=\left(r_{s} X_{s}+\pi_{s}^{+} \sigma_{s} \underline{\theta}_{s}-\pi_{s}^{-} \sigma_{s} \bar{\theta}_{s}\right) d s+\pi_{s} \sigma_{s} d W_{s}  \tag{3.4}\\
X_{t}=x
\end{array}\right.
$$

The value function $v(t, x ; d)$ is a viscosity solution of the following HJB equation (refer to [14]):

$$
\left\{\begin{array}{l}
\frac{\partial v}{\partial t}+\inf _{\pi \in \mathbb{R}}\left[\frac{\partial v}{\partial x}\left(r_{t} x+\pi^{+} \sigma_{t} \underline{\theta}_{t}-\pi^{-} \sigma_{t} \bar{\theta}_{t}\right)+\frac{1}{2} \frac{\partial^{2} v}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right]=0  \tag{3.5}\\
v(T, x ; d)=(x-d)^{2}
\end{array}\right.
$$

Note that the functions $x^{+}$and $x^{-}$are nonsmooth. Then (3.5) does not have a smooth solution. In the following theorem, we construct the viscosity solution of (3.5).

Theorem 3.2 Under Assumption [2.1, the viscosity solution of the above HJB equation (3.5) is given by

$$
v(t, x ; d)=\left\{\begin{array}{l}
e^{-\int_{t}^{T} \theta_{s}^{2} d s}\left(x e^{\int_{t}^{T} r_{s} d s}-d\right)^{2}, \text { if } x \leq d e^{-\int_{t}^{T} r_{s} d s} ;  \tag{3.6}\\
e^{-\int_{t}^{T} \bar{\theta}_{s}^{2} d s}\left(x e^{\int_{t}^{T} r_{s} d s}-d\right)^{2}, \text { if } x>d e^{-\int_{t}^{T} r_{s} d s}
\end{array}\right.
$$

and the associated optimal feedback control is given by

$$
\pi^{*}(t, x)=\left\{\begin{array}{l}
-\frac{\hat{\theta}_{t}}{\sigma_{t}}\left(x-d e^{-\int_{t}^{T} r_{s} d s}\right), \text { if } x \leq d e^{-\int_{t}^{T} r_{s} d s}  \tag{3.7}\\
-\frac{\bar{\theta}_{t}}{\sigma_{t}}\left(x-d e^{-\int_{t}^{T} r_{s} d s}\right), \text { if } x>d e^{-\int_{t}^{T} r_{s} d s}
\end{array}\right.
$$

Proof: Notice that the terminal condition of (3.5) $(x-d)^{2}$ is a convex function in $x$. We conjecture that $v(t, x ; d)$ is convex in $x$ on $[0, T]$.

Then,

$$
\begin{aligned}
& \inf _{\pi \in \mathbb{R}}\left[\frac{\partial v}{\partial x}\left(r_{t} x+\pi^{+} \sigma_{t} \underline{\theta}_{t}-\pi^{-} \sigma_{t} \bar{\theta}_{t}\right)+\frac{1}{2} \frac{\partial^{2} v}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right] \\
= & \left\{\begin{array}{l}
-\frac{1}{2} \frac{\left(\frac{\partial v}{\partial x}\right)^{2}}{\frac{\partial^{2}}{\partial x}} \underline{\theta}_{t}^{2}+\frac{\partial v}{\partial x} r_{t} x, \text { if } \frac{\partial v}{\partial x} \leq 0 ; \\
-\frac{1}{2} \frac{\left(\frac{\partial v}{\partial v}\right)^{2}}{\frac{\partial^{2} v}{\partial x^{2}}} \bar{\theta}_{t}^{2}+\frac{\partial v}{\partial x} r_{t} x, \text { if } \frac{\partial v}{\partial x}>0 .
\end{array}\right.
\end{aligned}
$$

And the infimum in the above formula is attained at

$$
\pi^{*}(t, x)= \begin{cases}-\frac{\underline{\theta}_{t} \frac{\partial v}{\partial x}}{\sigma_{t} \frac{\partial^{2} v}{\partial x}}, & \text { if } \frac{\partial v}{\partial x} \leq 0  \tag{3.8}\\ -\frac{\bar{\theta}_{t} \frac{\partial v}{\partial x}}{\sigma_{t} \frac{\partial v}{\partial x^{2}}}, & \text { if } \frac{\partial v}{\partial x}>0\end{cases}
$$

The HJB equation (3.5) becomes

$$
\left\{\begin{array}{l}
-\frac{\partial v}{\partial t}+\left[\frac{1}{2} \frac{\left(\frac{\partial v}{\partial x}\right)^{2}}{\frac{\partial^{2} v}{\partial x^{2}}} \underline{\theta}_{t}^{2}-\frac{\partial v}{\partial x} r_{t} x\right] I_{\left\{\frac{\partial v}{\partial x} \leq 0\right\}}+\left[\frac{1}{2} \frac{\left(\frac{\partial v}{\partial x}\right)^{2}}{\frac{\partial^{2} v}{\partial x^{2}}} \bar{\theta}_{t}^{2}-\frac{\partial v}{\partial x} r_{t} x\right] I_{\left\{\frac{\partial v}{\partial x}>0\right\}}=0,  \tag{3.9}\\
v(T, x ; d)=(x-d)^{2}
\end{array}\right.
$$

We divide $[0, T] \times \mathbb{R}$ into three disjoint regions

$$
\begin{aligned}
& \Gamma_{1}:=\left\{(t, x) \in[0, T] \times \mathbb{R} \mid x<d e^{-\int_{t}^{T} r_{s} d s}\right\} ; \\
& \Gamma_{2}:=\left\{(t, x) \in[0, T] \times \mathbb{R} \mid x>d e^{-\int_{t}^{T} r_{s} d s}\right\} ; \\
& \Gamma_{3}:=\left\{(t, x) \in[0, T] \times \mathbb{R} \mid x=d e^{-\int_{t}^{T} r_{s} d s}\right\} .
\end{aligned}
$$

It is easy to verify that $v(t, x ; d)$ defined in (3.6) is $C^{1,2}$ and satisfies (3.9) on $\Gamma_{1}$ and $\Gamma_{2}$.
On $\Gamma_{3}$,

$$
v(t, x ; d)=\frac{\partial v}{\partial t}(t, x ; d)=\frac{\partial v}{\partial x}(t, x ; d) \equiv 0
$$

Unfortunately, $\frac{\partial^{2} v}{\partial x^{2}}$ does not exist on $\Gamma_{3}$ since $e^{\int_{t}^{T}\left(r_{s}-\underline{\theta}_{s}^{2}\right) d s} \not \equiv e^{\int_{t}^{T}\left(r_{s}-\bar{\theta}_{s}^{2}\right) d s}$.
For any $\phi \in C^{\infty}([0, T] \times \mathbb{R})$, such that $(t, x) \in \Gamma_{3}$ is a minimum point of $\phi-v$, it's easy to verify that

$$
\frac{\partial \phi}{\partial t}(t, x)=\frac{\partial \phi}{\partial x}(t, x)=0
$$

and

$$
\frac{\partial^{2} \phi}{\partial x^{2}}(t, x) \geq \max \left\{2 e^{\int_{t}^{T}\left(r_{s}-\underline{\theta}_{s}^{2}\right) d s}, 2 e^{\int_{t}^{T}\left(r_{s}-\bar{\theta}_{s}^{2}\right) d s}\right\},(t, x) \in \Gamma_{3}
$$

Then for any $\phi \in C^{\infty}([0, T] \times \mathbb{R})$, such that $(t, x) \in \Gamma_{3}$ is a minimum point of $\phi-v$, we have

$$
\begin{aligned}
& \frac{\partial \phi}{\partial t}+\inf _{\pi \in \mathbb{R}}\left[\frac{\partial \phi}{\partial x}\left(r_{t} x+\pi^{+} \sigma_{t} \underline{\theta}_{t}-\pi^{-} \sigma_{t} \bar{\theta}_{t}\right)+\frac{1}{2} \frac{\partial^{2} \phi}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right] \\
= & \frac{1}{2} \inf _{\pi \in \mathbb{R}}\left[\frac{\partial^{2} \phi}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right] \\
\geq & \frac{1}{2} \inf _{\pi \in \mathbb{R}}\left[\max \left\{2 e^{\int_{t}^{T}\left(r_{s}-\underline{\theta}_{s}^{2}\right) d s}, 2 e^{\int_{t}^{T}\left(r_{s}-\bar{\theta}_{s}^{2}\right) d s}\right\} \sigma_{t}^{2} \pi^{2}\right] \\
= & 0 .
\end{aligned}
$$

Therefore, $v$ is a viscosity subsolution of the HJB equation (3.5).
Similarly, for any $\phi \in C^{\infty}([0, T] \times \mathbb{R})$, such that $(t, x) \in \Gamma_{3}$ is a maximum point of $\phi-v$, we have

$$
\frac{\partial \phi}{\partial t}(t, x)=\frac{\partial \phi}{\partial x}(t, x)=0 .
$$

In this case,

$$
\begin{aligned}
& \frac{\partial \phi}{\partial t}+\inf _{\pi \in \mathbb{R}}\left[\frac{\partial \phi}{\partial x}\left(r_{t} x+\pi^{+} \sigma_{t} \underline{\theta}_{t}-\pi^{-} \sigma_{t} \bar{\theta}_{t}\right)+\frac{1}{2} \frac{\partial^{2} \phi}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right] \\
= & \frac{1}{2} \inf _{\pi \in \mathbb{R}}\left[\frac{\partial^{2} \phi}{\partial x^{2}} \sigma_{t}^{2} \pi^{2}\right] \\
\leq & 0
\end{aligned}
$$

Therefore, $v$ is a viscosity supersolution of the HJB equation (3.5). Finally, the terminal condition $v(T, x ; d)=$ $(x-d)^{2}$ is satisfied. By the definition of viscosity solution, we know that $v(t, x ; d)$ defined in (3.6) is a viscosity solution of the HJB equation (3.5). By (3.8), it is easy to see that (3.7) holds.

This completes the proof.
Now we determine the Lagrange multiple $d^{*}$ which attains $\max _{d \in \mathbb{R}} \min _{\pi \in \mathcal{A}\left(x_{0}\right)} \hat{J}(\pi, d)$.
From (3.1),

$$
\begin{aligned}
& \min _{\pi \in \mathcal{A}\left(x_{0}\right)} \hat{J}(\pi, d) \\
= & v\left(0, x_{0} ; d\right)-(d-K)^{2} \\
= & \left\{\begin{array}{l}
e^{-\int_{0}^{T} \underline{\theta}_{s}^{2} d s}\left(x_{0} e^{\int_{0}^{T} r_{s} d s}-d\right)^{2}-(d-K)^{2}, \text { if } x_{0} \leq d e^{-\int_{0}^{T} r_{s} d s} ; \\
e^{-\int_{0}^{T} \bar{\theta}_{s}^{2} d s}\left(x_{0} e^{\int_{0}^{T} r_{s} d s}-d\right)^{2}-(d-K)^{2}, \text { if } x_{0}>d e^{-\int_{0}^{T} r_{s} d s .}
\end{array}\right.
\end{aligned}
$$

Set $d^{*}=\frac{K-x_{0} e_{0}^{T}\left(r_{s}-\underline{\theta}_{s}^{2}\right) d s}{1-e^{-\int_{0}^{T} \underline{\theta}_{s}^{2} d s}}$. We have

$$
\begin{align*}
& \max _{d \in \mathbb{R}} \min _{\pi \in \mathcal{A}\left(x_{0}\right)} \hat{J}(\pi, d) \\
= & \max _{d \in \mathbb{R}}\left[v\left(0, x_{0} ; d\right)-(d-K)^{2}\right] \\
= & v\left(0, x_{0} ; d^{*}\right)-\left(d^{*}-K\right)^{2} \\
= & \frac{1}{e^{\int_{0}^{T} \underline{\theta}_{s}^{2} d s}-1}\left(K-x_{0} e^{\int_{0}^{T} r_{s} d s}\right)^{2}, \tag{3.10}
\end{align*}
$$

Therefore, the Lagrange multiple $\lambda^{*}=d^{*}-K=\frac{K-x_{0} e^{T} r_{s} d s}{e^{\int_{0}^{T} \theta_{s}^{2} d s}-1} \geq 0$.
The above analysis boils down to the following theorem.
Theorem 3.3 The efficient strategy of the problem (2.2) can be written as a function of time $t$ and wealth $X$ :

$$
\pi^{*}(t, X)=\left\{\begin{array}{l}
-\frac{\hat{\theta}_{t}}{\sigma_{t}}\left(X-d^{*} e^{-\int_{t}^{T} r_{s} d s}\right), \text { if } X \leq d^{*} e^{-\int_{t}^{T} r_{s} d s} ;  \tag{3.11}\\
-\frac{\bar{\theta}_{t}}{\sigma_{t}}\left(X-d^{*} e^{-\int_{t}^{T} r_{s} d s}\right), \text { if } X>d^{*} e^{-\int_{t}^{T} r_{s} d s} .
\end{array}\right.
$$

Moreover, the efficient frontier is

$$
\operatorname{Var} X_{T}=\frac{1}{e^{\int_{0}^{T} \underline{\theta}_{s}^{2} d s}-1}\left(K-x_{0} e^{\int_{0}^{T} r_{s} d s}\right)^{2} \equiv \frac{1}{e^{\int_{0}^{T} \underline{\theta}_{s}^{2} d s}-1}\left(E X_{T}-x_{0} e^{\int_{0}^{T} r_{s} d s}\right)^{2}
$$

Remark 3.4 The efficient strategy (3.11) indicates that the investor should long the stock if his current wealth is less than $d^{*} e^{-\int_{t}^{T} r_{s} d s}$, otherwise he should take the short position.

## 4 Three examples

In this section, three examples are given to show the applications of our main results. The wealth equations in these examples are described by equation (2.1).
Example 4.1 Jouini and Kallal [8] and El Karoui et al [8] proposed the following model.
Under some circumstance, one has different expected returns for long and short position of the stock. In this case, the assets prices are given by

$$
\left\{\begin{array}{l}
d S_{t}^{0}=S_{t}^{0} r_{t} d t, S_{0}^{0}=s_{0} \\
d S_{t}^{1}=S_{t}^{1}\left[\left(\underline{b}_{t} I_{\left\{\pi_{t} \geq 0\right\}}+\bar{b}_{t} I_{\left\{\pi_{t}<0\right\}}\right) d t+\sigma_{t} d W_{t}\right], S_{0}^{1}=s_{1}>0
\end{array}\right.
$$

Then the wealth process $X \equiv X^{x, \pi}$ of the self-financed large investor who is endowed with initial wealth $x_{0}>0$ is governed by the following stochastic differential equation,

$$
\left\{\begin{aligned}
d X_{t} & =\pi_{t} \frac{d S_{t}^{1}}{S_{t}^{t}}+\left(X_{t}-\pi_{t}\right) \frac{d S_{t}^{0}}{S_{t}^{0}} \\
& =\left(r_{t} X_{t}+\pi_{t}^{+} \sigma_{t} \underline{\theta}_{t}-\pi_{t}^{-} \sigma_{t} \bar{\theta}_{t}\right) d t+\pi_{t} \sigma_{t} d W_{t} \\
X_{0} & =x_{0}
\end{aligned}\right.
$$

where $\underline{\theta}_{t}:=\frac{b_{t}-r_{t}}{\sigma_{t}}, \bar{\theta}_{t}:=\frac{\bar{b}_{t}-r_{t}}{\sigma_{t}}$ are risk premia for long and short positions.

Example 4.2 Cuoco and Cvitanic [2] gave the following large investor model.
The portfolio strategy of a large investor can infect the expected return of the stock.

$$
\left\{\begin{array}{l}
d S_{t}^{0}=S_{t}^{0} r_{t} d t, S_{0}^{0}=s_{0} \\
d S_{t}^{1}=S_{t}^{1}\left[\left(b_{t}-\varepsilon \operatorname{sgn}\left(\pi_{t}\right)\right) d t+\sigma_{t} d W_{t}\right], S_{0}^{1}=s_{1}>0
\end{array}\right.
$$

where $\varepsilon$ is a given small positive number, and

$$
\operatorname{sgn}(x)= \begin{cases}\frac{|x|}{x}, & \text { if } x \neq 0  \tag{4.1}\\ 0, & \text { otherwise }\end{cases}
$$

In this specific large investor model, longing the risky security depresses its expected return while shorting it increases its expected return as explained in [2].

The wealth equation can be written

$$
\left\{\begin{align*}
d X_{t} & =\left(r_{t} X_{t}+\left(b_{t}-r_{t}\right) \pi_{t}-\varepsilon\left|\pi_{t}\right|\right) d t+\pi_{t} \sigma_{t} d W_{t}  \tag{4.2}\\
& =\left(r_{t} X_{t}+\pi_{t}^{+} \sigma_{t} \underline{\theta}_{t}-\pi_{t}^{-} \sigma_{t} \bar{\theta}_{t}\right) d t+\pi_{t} \sigma_{t} d W_{t} \\
X_{0} & =x_{0}
\end{align*}\right.
$$

where $\underline{\theta}_{s}:=\frac{b_{s}-r_{s}-\varepsilon}{\sigma_{s}}$ and $\bar{\theta}_{s}:=\frac{b_{s}-r_{s}+\varepsilon}{\sigma_{s}}, s \in[0, T]$.
Example 4.3 El Karoui et al $[3]$ studied the following wealth equation with taxes.
We suppose the assets prices are given by

$$
\left\{\begin{array}{l}
d S_{t}^{0}=S_{t}^{0} r_{t} d t, S_{0}^{0}=s_{0} \\
d S_{t}^{1}=S_{t}^{1}\left(b_{t} d t+\sigma_{t} d W_{t}\right), S_{0}^{1}=s_{1}>0
\end{array}\right.
$$

And there are some taxes which must be paid on the gains made on the stock. In this case, the wealth equation satisfies

$$
\left\{\begin{align*}
d X_{t} & =\left(r_{t} X_{t}+\left(b_{t}-r_{t}\right) \pi_{t}-\alpha \pi^{+}\left(b_{t}-r_{t}\right)\right) d t+\pi_{t} \sigma_{t} d W_{t}  \tag{4.3}\\
& =\left(\left(r_{t} X_{t}+\pi_{t}^{+} \sigma_{t} \underline{\theta}_{t}-\pi_{t}^{-} \sigma_{t} \bar{\theta}_{t}\right) d t+\pi_{t} \sigma_{t} d W_{t}\right. \\
X_{0} & =x_{0}
\end{align*}\right.
$$

where $\underline{\theta}_{t}:=\frac{(1-\alpha)\left(b_{t}-r_{t}\right)}{\sigma_{t}}$ and $\bar{\theta}_{t}:=\frac{b_{t}-r_{t}}{\sigma_{t}}, \alpha \in[0,1)$ is a constant.

## References

[1] T. Bielecki, H. Jin, S. Pliska, X. Zhou. Continuous-time mean-variance portfolio selection with bankruptcy prohibition. Mathematical Finance, 15(2005), pp.213-244.
[2] D. Cuoco, J. Cvitanic. Optimal consumption choices for a 'large’ investor. Journal of Economic Dynamics and Control, 22(1998), pp. 401-436.
[3] N. El Karoui, S. Peng, M. Quenez, Backward stochastic differential equations in finance, Mathematical finance, 7(1997), pp. 1-71.
[4] N. El Karoui, S. Peng, M. Quenez, A dynamic maximum principle for the optimization of recursive utilities under constraints, The Annals of Applied Probability, 11(2001): pp. 664-693.
[5] C. Fu, L. Lavassani, X. Li, Dynamic mean-variance portfolio selection with borrowing constraint. European Journal of Operational Research, 200(2010), pp.312-319.
[6] S. Ji, Dual method for continuous-time Markowitz's problems with nonlinear wealth equations. Journal of Mathematical Analysis and Applications, 366(2010), pp.90-100.
[7] H. Jin, J. Yan, X, Zhou, Continuous-time mean-risk portfolio selection.Annales de l'Institut Henri Poincare Probabilites et Statistiques. 41(2005), pp. 559-580.
[8] E. Jouini, H. Kallal, Arbitrage in Securities Markets with Short-Sales Constraints. Mathematical Finance, 5(1995), pp. 197-232.
[9] D. Li, W. Ng, Optimal Dynamic Portfolio Selection: Multiperiod Mean-Variance Formulation. Mathematical Finance, 10(2000), pp.387-406.
[10] X. Li, X, Zhou, A. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints. SIAM Journal on Control and Optimization, 40(2002), pp. 1540-1555.
[11] D. Luenberger, Optimization by vector space methods. John Wiley and Sons, 1997.
[12] H. Markowitz, Portfolio selection. The journal of finance, 7(1952), pp. 77-91.
[13] H. Markowitz, Portfolio selection: efficient diversification of investments. Yale university press, 1968.
[14] J. Yong, X. Zhou, Stochastic controls: Hamiltonian systems and HJB equations. Springer-Verlag. New York, 1999.
[15] X. Zhou, D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework. Applied Mathematics and Optimization, 42(2000), pp. 19-33.


[^0]:    *Institute for Financial Studies, Shandong University, Jinan 250100, China and Institute of Mathematics, Shandong University, Jinan 250100, China, Email: jsl@sdu.edu.cn. This work was supported by National Natural Science Foundation of China (No. 11571203); Supported by the Programme of Introducing Talents of Discipline to Universities of China (No.B12023).
    ${ }^{\dagger}$ Corresponding author. Institute for Financial Studies, Shandong University, Jinan 250100, China. Email: shixm@mail.sdu.edu.cn, shixiaominhi@163.com. This work was supported by National Natural Science Foundation of China (No. 11401091).

