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Abstract—The controllability and observability of Boolean control network(BCN) are two fundamental properties. But the verification of latter is much harder than the former. This paper considers the observability of BCN via controllability. First, the set controllability is proposed, and the necessary and sufficient condition is obtained. Then a technique is developed to convert the observability into an equivalent set controllability problem. Using the result for set controllability, the necessary and sufficient condition is also obtained for the observability of BCN.

Index Terms-Boolean control network, set controllability, observability, semi-tensor product of matrices.

I. INTRODUCTION

BOOLEAN network was firstly proposed by Kauffman to describe gene regularity networks [1]. Since then it has attracted much attention from biologists, physicists, and system scientists [2–4].

Recently, a new matrix product, called the semi-tensor product (STP) of matrices was introduced. STP has then been successfully applied to modeling and controlling Boolean networks [5-7]. Inspired by STP, the theory of Boolean control networks (BCNs) as well as the control of general logical systems have been developed rapidly. A set of systematic results have been obtained. For instance, the controllability and observability of Boolean networks have been discussed in [8, 9]; the disturbance decoupling has been considered in [10, 11]; the optimal control has been investigated in [12, 13]; the stability and stabilization have been studied in [14, 15], just to mention a few.

Among them the controllability and observability of BCN are of particular importance. Particularly, the controllability via free control sequence is fundamental, and it has been solved elegantly by [16]. Unlike the controllability, the observability has also been discussed for long time and various kinds of observability have been proposed and investigated [8, 16-18]. A comparison for various kinds of observability has been presented in [19]. Moreover, [19] has also pointed out that one of them, which will be specified later, is the most sensitive observability. Here "most sensitive one" means all other kinds of observability implies this one. In addition, [19] has also provided necessary and sufficient conditions for various kinds of observability via finite automata approach. Motivated by the idea of [19], [20] proposed a numerical method to verify the (most sensitive) observability.

Since for BCN the controllability is much easier understandable and verifiable than the observability, this paper proposes a method to verify observability via controllability. In this paper the set controllability of BCN is proposed first. The idea comes from [9], where some states are forbidden and it is a special case of our set controllability. Hence the result about set controllability can be considered as a generalization of the corresponding result in [9]. Then a properly designed extended system of the original BCN is built. It is proved that the observability of the original BCN is equivalent to the set controllability of the extended system. Then the observability of BCN is converted into a set controllability problem and then is solved completely. In fact, the result is equivalent to the necessary and sufficient condition proposed in [20]. But the new result is concise and easily verifiable.

The rest of this paper is organized as follows: Section 2 describes the set controllability of BCN. The set controllability matrix is constructed. Using it an easily verifiable necessary and sufficient condition is obtained. As an application, the output controllability problem is also solved. Section 3 constructs an extended system and carefully designs the initial and destination sets. Then the observability of a BCN becomes the set controllability of its extended system. Some examples are presented to describe the design procedure. Section 4 is a concluding remark.

Before ending this section, a list of notations is presented:

- (1) $\mathcal{M}_{m \times n}$: the set of $m \times n$ real matrices.
- $\operatorname{Col}(M)$: the set of columns of a matrix M. $\operatorname{Col}_i(M)$: the i-th column of M.
- (3) $\mathcal{D} := \{0, 1\}.$
- (4) δ_n^i : the *i*-th column of the identity matrix I_n .
- (5) $\Delta_n := \left\{ \delta_n^i | i = 1, \cdots, n \right\}.$

(6)
$$\mathbf{1}_{\ell} = (1, 1, \cdots, 1)^{T}$$

- (7) A matrix $L \in \mathcal{M}_{m \times n}$ is called a logical matrix if the columns of L are of the form δ_m^k . That is, $\operatorname{Col}(L) \subset \Delta_m$. Denote by $\mathcal{L}_{m \times n}$ the set of $m \times n$ logical matrixes.
- (8) If $L \in \mathcal{L}_{n \times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted as $L = \delta_n[i_1, i_2, \cdots, i_r]$.
- (9) Denote by $\mathcal{B}_{m \times n}$ the set of $m \times n$ Boolean matrices.
- (10) Let $A, B \in \mathcal{B}_{m \times n}$. Then $A +_{\mathcal{B}} B$ is the Boolean addition (with respect to $+_{\mathcal{B}} = \vee$ and $\times_{\mathcal{B}} = \wedge$).
- (11) Let $A \in \mathcal{B}_{m \times n}$, $B \in \mathcal{B}_{p \times q}$. Then $A \ltimes_{\mathcal{B}} B$ is the Boolean (semi-tensor) product (with respect to $+_{\mathcal{B}} = \vee$ and $\times_{\mathcal{B}} = \wedge$).

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- (12) $A^{(k)} := \underbrace{A \ltimes_{\mathcal{B}} \cdots \ltimes_{\mathcal{B}} A}_{k}.$
- (13) $P^0 \subset 2^N$ is the set of initial sets, where N = $\{1,2,\cdots,n\}$ is the set of state nodes of a BCN, and 2^N is the power set of N.
- (14) $P^d \subset 2^N$ is the set of destination sets.

II. SET CONTROLLABILITY

A Markov-type Boolean control network with n nodes is described as [5]

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)) \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ y_j(t) = h_j(x_1(t), \cdots, x_n(t)), \quad j = 1, \cdots, p, \end{cases}$$
(1)

where $x_i \in \mathcal{D}, i = 1, \cdots, n$ are state variables; $u_i \in \mathcal{D},$ $i = 1, \dots, m$ are controls; $y_j \in \mathcal{D}, j = 1, \dots, p$ are outputs; $f_i: \mathcal{D}^{m+n} \to \mathcal{D}, i = 1, \cdots, n, \text{ and } h_j: \mathcal{D}^n \to \mathcal{D}, j =$ $1, \cdots, p$ are Boolean functions.

Definition II.1. The system (1) is

- 1) controllable from x_0 to x_d , if there are a T > 0 and a sequence of control $u(0), \dots, u(T-1)$, such that driven by these controls the trajectory can go from $x(0) = x_0$ to $x(T) = x_d$;
- 2) controllable at x_0 , if it is controllable from x_0 to destination $x_d = x$, $\forall x$;
- 3) controllable, if it is controllable at any x_0 .

Under the vector form expression:

$$1 \sim \delta_2^1, 0 \sim \delta_2^2,$$

we have $x_i, u_i, y_i \in \Delta_2$. Using Theorem A.5, (1) can be converted into its algebraic form as

$$\begin{cases} x(t+1) = Lu(t)x(t) \\ y(t) = Hx(t), \end{cases}$$
(2)

where $x(t) = \ltimes_{i=1}^{n} x_i(t), y(t) = \ltimes_{i=1}^{p} y_i(t), u(t) =$ $\ltimes_{j=1}^{m} u_j(t)$, and $L \in \mathcal{L}_{2^n \times 2^{n+m}}, H \in \mathcal{L}_{2^p \times 2^n}$.

Define

$$M := \sum_{\mathcal{B}} \sum_{j=1}^{2^m} L \delta_{2^m}^j, \tag{3}$$

and set

$$\mathcal{C} := \sum_{\mathcal{B}}^{2^n} M^{(i)},\tag{4}$$

which is called the controllability matrix. Then we have the following result:

Theorem II.2. [16] Consider the controllability of system (1) (by free control sequence). Assume its controllability matrix is $C = (c_{i,j})$, then we have the following results:

1) State x_i is controllable from x_j , if and only if, $c_{i,j} = 1$.

- 2) System (1) is controllable at x_i , if and only if, $\operatorname{Col}_i(\mathcal{C}) =$ $1_{2^{n}}$.
- 3) System (1) is controllable, if and only if, $C = \mathbf{1}_{2^n \times 2^n}$.

Denote by $N = \{1, 2, \dots, n\}$ the set of state nodes. Assume $s \in 2^N$, the index vector of s, denoted by $V(s) \in \mathbb{R}^n$, is defined as

$$\left(V(s)\right)_i = \begin{cases} 1, & i \in s \\ 0, & i \notin s. \end{cases}$$

Define the set of initial sets P^0 and the set of destination sets P^d respectively as follows:

$$P^{0} := \left\{ s_{1}^{0}, s_{2}^{0}, \cdots, s_{\alpha}^{0} \right\} \subset 2^{N}, P^{d} := \left\{ s_{1}^{d}, s_{2}^{d}, \cdots, s_{\beta}^{d} \right\} \subset 2^{N}.$$
(5)

Using initial sets and destination sets, the set controllability is defined as follows.

Definition II.3. Consider system (1) with a set of initial sets P^0 and a set of destination sets P^d . The system (1) is

- 1) set controllable from $s_i^0 \in P^0$ to $s_i^d \in P^d$, if there exist $x_0 \in s_i^0$ and $x_d \in s_i^d$, such that x_d is controllable from $x_0;$
- set controllable at s⁰_j, if for any s^d_i ∈ P^d, the system is controllable from s⁰_j to s^d_i;
 set controllable, if it is set controllable at any s⁰_j ∈ P⁰.

Using the set of initial sets and the set of destination sets defined in (5), we can define the initial index matrix J_0 and the destination index matrix J_d respectively as

$$J_0 := \begin{bmatrix} V(s_1^0) & V(s_2^0) & \cdots & V(s_{\alpha}^0) \end{bmatrix} \in \mathcal{B}_{2^n \times \alpha}; \\ J_d := \begin{bmatrix} V(s_1^d) & V(s_2^d) & \cdots & V(s_{\beta}^d) \end{bmatrix} \in \mathcal{B}_{2^n \times \beta}.$$
(6)

Using (6), we define a matrix, called the set controllability matrix, as

$$\mathcal{C}_S := J_d^T \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{B}} J_0 \in \mathcal{B}_{\beta \times \alpha}.$$
 (7)

Note that hereafter all the matrix products are assumed to be Boolean product $(\times_{\mathcal{B}})$. Hence the symbol $\times_{\mathcal{B}}$ is omitted.

According to the definition of set controllability, the following result is easily verifiable.

Theorem II.4. Consider system (1) with the set of initial sets P^0 and the set of destination sets P^d as defined in (5). Moreover, the corresponding set controllability matrix $C_S = (c_{ij})$ is defined in (7). Then

- 1) system (1) is set controllable from s_i^0 to s_i^d , if and only *if*, $c_{i,j} = 1$;
- 2) system (1) is set controllable at s_i^0 , if and only if $\operatorname{Col}_{i}(\mathcal{C}_{S}) = \mathbf{1}_{\beta};$
- 3) system (1) is set controllable, if and only if, $C_S = \mathbf{1}_{\beta \times \alpha}$.

Example II.5. Consider the following system [5]

$$\begin{cases} x_1(t+1) = (x_1(t) \leftrightarrow x_2(t)) \lor u_1(t) \\ x_2(t+1) = \neg x_1(t) \land u_2(t), \\ y(t) = x_1(t) \land x_2(t). \end{cases}$$
(8)

It is easy to calculate that

$$\mathcal{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

1) Assume

$$\begin{cases} P^{d} = \left\{ s_{1}^{d} = \left\{ \delta_{4}^{1}, \delta_{4}^{2} \right\}, & s_{2}^{d} = \left\{ \delta_{4}^{3}, \delta_{4}^{4} \right\} \right\}; \\ P^{0} = \left\{ s_{1}^{0} = \left\{ \delta_{4}^{1} \right\}, & s_{2}^{0} = \left\{ \delta_{4}^{2}, \delta_{4}^{3}, \delta_{4}^{4} \right\} \right\}. \end{cases}$$
(9)

Then we have

$$J_d = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$\mathcal{C}_S = J_d^T \mathcal{C} J_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence, the system (8) is set controllable with respect to the initial set P^0 and the destination set P^d defined by (9).

2) Assume

$$\begin{cases} P^{d} = \left\{ s_{1}^{d} = \left\{ \delta_{4}^{3} \right\} \right\}; \\ P^{0} = \left\{ s_{1}^{0} = \left\{ \delta_{4}^{1}, \delta_{4}^{2}, \delta_{4}^{3} \right\}, s_{2}^{0} = \left\{ \delta_{4}^{1}, \delta_{4}^{4} \right\} \right\}. \end{cases}$$
(10)

Then

$$J_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad J_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

And

$$\mathcal{C}_S = J_d^T \mathcal{C} J_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Hence, the system (8) is not set controllable with respect to the initial set P^0 and destination set P^d defined by (10).

As an application, we consider the output controllability [22].

Definition II.6. Consider system (1). It is said to be output controllable, if for any $x(0) = x_0$ and any y_d , there exist a T > 0 and a sequence of control $u(0), u(1), \dots, u(T-1)$ such that $y(T) = y_d$.

Definition II.7. Consider system (1).

1) A partition is called an output-based partition, if

$$s_j^d = \left\{ x \mid Hx = \delta_{2^p}^j \right\}, \quad j = 1, \cdots, 2^p.$$
 (11)

2) A partition is called a finest partition, if

$$s_i^0 = \{x_i\}, \quad i = 1, \cdots, 2^n.$$
 (12)

Using (11) and (12), we define

$$\begin{cases} P^{d} := \left\{ s_{j}^{d} \mid j = 1, \cdots, 2^{p} \right\}; \\ P^{0} := \left\{ s_{i}^{0} \mid i = 1, \cdots, 2^{n} \right\}. \end{cases}$$
(13)

Taking the construction of P^d and P^0 into consideration, the following result is an immediate consequence of the definition.

Theorem II.8. System (1) is output controllability, if and only if, it is set controllability with respect to the set pairs (P^d, P^0) , defined in (13).

Note that corresponding to P^0 , defined in (13), the initial index matrix is an identity matrix, and the destination index matrix is H^T . Hence for output controllability, denoting by C_Y the output controllability matrix, we have

$$\mathcal{C}_Y = \mathcal{C}_S = H\mathcal{C},\tag{14}$$

where C_S is the set controllability matrix with respect to the set pair (P^d, P^0) defined in (13).

The output controllability has been discussed in [23]. Comparing our result with the direct approach in [23], the advantage of set controllability approach is obvious.

Example II.9. Consider system (8) again. It is easy to figure out that

$$J_d = (\delta_2[1, 2, 2, 2])^T$$

$$\mathcal{C}_Y = J_d^T \mathcal{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} > 0.$$

Hence, system (8) is output controllable.

Then

III. OBSERVABILITY VIA SET CONTROLLABILITY APPROACH

As discussed in [19] the following one is the most sensitive observability among those in recent literature.

Definition III.1. [19] System (1) is observable, if for any two initial states $x_0 \neq z_0$, there exist an integer $T \geq 0$ and a control sequence $u = \{u(0), u(1), \dots, u(T-1)\}$, such that the corresponding output sequence $y(i) = y_i(x_0, u)$, $i = 0, 1, \dots, T$ is not equal to $\tilde{y}_i(z_0, u)$.

Next, we consider two kinds of state pairs.

Definition III.2. A pair $(x, z) \in \Delta_{2^n} \times \Delta_{2^n}$ is yindistinguishable if Hx = Hy. Otherwise, (x, z) is called y-distinguishable.

Following [20], we split the product state space $\Delta_{2^n} \times \Delta_{2^n}$ into a partition of three components as

$$D = \{ zx \mid z = x \},\tag{15}$$

$$\Theta = \{ zx \mid z \neq x \text{ and } Hz = Hx \}, \tag{16}$$

$$\Xi = \{ zx \mid Hz \neq Hx \}. \tag{17}$$

Using algebraic form (2), we construct a dual system as

$$\begin{cases} z(t+1) = Lu(t)z(t) \\ x(t+1) = Lu(t)x(t). \end{cases}$$
(18)

Then the observability problem of system (1) can be converted into a set controllability problem of the extended system (18). Construct the initial sets and the destination sets as follows:

$$P^0 := \bigcup_{zx \in \Theta} \{zx\}$$
(19)

and

$$P^d := \{\Xi\}.$$
 (20)

Note that (19) means that each $zx \in \Theta$ is an element of P^0 , while (20) means P^d has only one element, which is Ξ . Then we have the following result:

Theorem III.3. System (1) is observable, if and only if, system (18) is set controllable from P^0 to P^d , which are defined in (19) and (20) respectively.

Proof. (Necessary): Assume the system is observable. Then for any two initial points $z_0 \neq x_0$, there exists a control sequence $\{u(t) \mid t = 0, 1, \cdots\}$ such that the corresponding output sequences $\{y(t) \mid t = 0, 1, \cdots\}$ and $\{\tilde{y}(t) \mid t =$ $0, 1, \cdots\}$ are not the same. Let $T \geq 0$ be the smallest t such that $y(t) \neq \tilde{y}(t)$. If T = 0, $(z_0, x_0) \in \Xi^c$ is a distinguishable pair. Assume T > 0. Applying the sequence of controls to system (18), (z_0, x_0) can be driven to (z(T), x(T)). Since $Hz(T) = y(T) \neq \tilde{y}(T) = Hx(T)$, we have $(z(T), x(T)) \in$ Ξ . That is, system (18) is set controllable from P^0 to P^d .

(Sufficiency): Assume a pair $z_0 \neq x_0$ is given. If $(z_0, x_0) \in \Xi$, we are done. Otherwise, since the system (18) is set controllable from P^0 to $P^d = \{\Xi\}$, there exists a control sequence $\{u(t) \mid t = 0, 1, \cdots\}$ which drives (z_0, x_0) to $(z_T, x_T) \in \Xi$.

It is worth noting that system (18) is essentially a combination of two independent systems corresponding to z and x respectively. Only the same control sequence is applied to them. Hence we have z_T is on the trajectory of (2) with the above mentioned control sequence $\{u(t) \mid t = 0, 1, \dots\}$, that is, $z_T = x(z_0, u(0), u(1), \dots, u(T-1))$, and $x_T = x(x_0, u(0), u(1), \dots, u(T-1))$. Since $(z_T, x_T) \in \Xi$, which means that using this control sequence to system (2), it distinguishes z_0 and x_0 .

Example III.4. Consider the reduced model for the lac operon in the bacterium Escherichia coli [24]

$$\begin{cases} x_1(t+1) = \neg u_1(t) \land (x_2(t) \lor x_3(t)) \\ x_2(t+1) = \neg u_1(t) \land u_2(t) \land x_1(t) \\ x_3(t+1) = \neg u_1(t) \land (u_2(t) \lor (u_3(t) \land x_1(t))) , \end{cases}$$
(21)

where x_1, x_2 and x_3 represent the lac mRNA, the lactose in high and medium concentrations, respectively; u_1, u_2 and u_3 are the extracellular glucose, high and medium extracellular lactose, respectively.

1) Assume that the outputs are

$$\begin{cases} y_1(t) = x_1(t) \lor \neg x_2(t) \lor x_3(t) \\ y_2(t) = \neg x_1(t) \lor x_2(t) \land \neg x_3(t) \\ y_3(t) = \neg x_1(t) \land \neg x_2(t) \lor x_3(t). \end{cases}$$
(22)

Its algebraic form is

$$x(t+1) = Lu(t)x(t)$$

$$y(t) = Hx(t),$$
(23)

where

Construct the dual system as

$$\begin{cases} z(t+1) = Lu(t)z(t) \\ x(t+1) = Lu(t)x(t). \end{cases}$$
(24)

It is easy to figure out that

$$\begin{split} \Theta &= \left\{ \{\delta_8^2, \delta_8^4\}, \{\delta_8^2, \delta_8^6\}, \{\delta_8^2, \delta_8^8\}, \{\delta_8^4, \delta_8^6\}, \\ &\quad \{\delta_8^4, \delta_8^8\}, \{\delta_8^6, \delta_8^8\} \right\} \\ &\sim \left\{ \delta_{64}^{12}, \delta_{64}^{14}, \delta_{64}^{16}, \delta_{64}^{30}, \delta_{64}^{32}, \delta_{64}^{48} \right\} \\ &\coloneqq \left\{ \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \right\}; \end{split}$$

and

$$\begin{split} \Xi &= \left\{ \{\delta_8^1, \delta_8^2\}, \{\delta_8^1, \delta_8^3\}, \{\delta_8^1, \delta_8^4\}, \{\delta_8^1, \delta_8^5\}, \{\delta_8^1, \delta_8^6\}, \\ &\{\delta_8^1, \delta_8^7\}, \{\delta_8^1, \delta_8^8\}, \{\delta_8^2, \delta_8^3\}, \{\delta_8^2, \delta_8^5\}, \{\delta_8^2, \delta_8^7\}, \\ &\{\delta_8^4, \delta_8^3\}, \{\delta_8^4, \delta_8^5\}, \{\delta_8^4, \delta_8^7\}, \{\delta_8^6, \delta_8^3\}, \{\delta_8^6, \delta_8^5\}, \\ &\{\delta_8^6, \delta_8^7\}, \{\delta_8^8, \delta_8^3\}, \{\delta_8^8, \delta_8^5\}, \{\delta_8^8, \delta_8^7\}, \{\delta_8^3, \delta_8^5\}, \\ &\{\delta_8^2, \delta_{64}^7\}, \{\delta_{55}^8, \delta_{7}^8\} \right\} \\ &\sim \left\{ \delta_{64}^2, \delta_{64}^3, \delta_{64}^4, \delta_{64}^5, \delta_{64}^6, \delta_{64}^7, \delta_{64}^8, \delta_{64}^{11}, \delta_{64}^{13}, \delta_{64}^{15}, \delta_{64}^{21}, \\ &\delta_{64}^{23}, \delta_{64}^{27}, \delta_{64}^{29}, \delta_{64}^{31}, \delta_{64}^{33}, \delta_{64}^{43}, \delta_{64}^{45}, \delta_{64}^{47}, \delta_{64}^{69}, \delta_{64}^{63}, \delta_{64}^{64} \right\} \end{split}$$

Set w(t) = z(t)x(t), then (24) can be expressed as

$$z(t+1) = L \left(I_{64} \otimes \mathbf{1}_8^T \right) u(t)w(t)$$

$$x(t+1) = L \left(\mathbf{1}_8^T \otimes I_8 \right) u(t)w(t).$$

Finally, we have

$$w(t+1) = Mu(t)w(t),$$
 (25)

where

$$M = \delta_{64}[64, 64, 64, \dots, 60, 60, 60, 64] \in \mathcal{L}_{64 \times 512}$$

:= [M₁, M₂, M₃, M₄, M₅, M₆, M₇, M₈].

Then the controllability matrix of (24) can be calculated by

$$\mathcal{C} := \sum_{\mathcal{B}}^{64} \left(\sum_{j=1}^{8} M_i \right)^{(j)} \in \mathcal{B}_{64 \times 64}$$

Finally, we consider the set controllability of (24). Using the initial set $P^0 = \{\theta \in \Theta\} = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and the destination set $P^d = \Xi$, we have

$$J_d = \sum_{\delta_{64}^i \in \Xi} \delta_{64}^i;$$

and

$$J_0 = \delta_{64}[12, 14, 16, 30, 32, 48]$$

It follows that

$$C_S = J_d^T C J_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} > 0.$$

According to Theorem III.3, system (21) with outputs (23) is observable.

2) Assume the measured outputs of system (21) are

$$\begin{cases} y_1(t) = x_1(t) \\ y_2(t) = x_2(t). \end{cases}$$
(26)

Its algebraic form is

$$y(t) = Hx(t), \tag{27}$$

where

$$H = \delta_4[1, 1, 2, 2, 3, 3, 4, 4].$$

It is easy to figure out that

$$\begin{split} \Theta &= \left\{ \{\delta_8^1, \delta_8^2\}, \{\delta_8^3, \delta_8^4\}, \{\delta_8^5, \delta_8^6\}, \{\delta_8^7, \delta_8^8\} \right\} \\ &\sim \left\{ \delta_{64}^2, \delta_{64}^{20}, \delta_{64}^{38}, \delta_{64}^{56} \right\} \\ &:= \left\{ \theta_1, \theta_2, \theta_3, \theta_4 \right\}; \end{split}$$

and

$$\begin{split} \Xi &= \left\{ \{\delta_8^1, \delta_8^3\}, \{\delta_8^1, \delta_8^4\}, \{\delta_8^1, \delta_8^3\}, \{\delta_8^1, \delta_8^6\}, \{\delta_8^1, \delta_8^4\}, \\ \{\delta_8^1, \delta_8^8\}, \{\delta_8^2, \delta_8^3\}, \{\delta_8^2, \delta_8^4\}, \{\delta_8^2, \delta_8^2\}, \{\delta_8^2, \delta_8^2\}, \\ \{\delta_8^2, \delta_8^7\}, \{\delta_8^2, \delta_8^8\}, \{\delta_8^3, \delta_8^5\}, \{\delta_8^3, \delta_8^6\}, \{\delta_8^3, \delta_8^7\}, \\ \{\delta_8^3, \delta_8^8\}, \{\delta_8^4, \delta_8^5\}, \{\delta_8^4, \delta_8^6\}, \{\delta_8^4, \delta_8^6\}, \{\delta_8^4, \delta_8^6\}, \\ \{\delta_8^5, \delta_8^7\}, \{\delta_8^5, \delta_8^8\}, \{\delta_8^6, \delta_8^7\}, \{\delta_8^6, \delta_8^8\} \right\} \\ &\sim \left\{ \delta_{64}^3, \delta_{64}^4, \delta_{64}^5, \delta_{64}^6, \delta_{64}^7, \delta_{64}^8, \delta_{61}^{14}, \delta_{64}^{12}, \delta_{64}^{13}, \delta_{64}^{14}, \delta_{64}^{15}, \\ \delta_{16}^{16}, \delta_{64}^{21}, \delta_{22}^{22}, \delta_{23}^{22}, \delta_{24}^{22}, \delta_{29}^{30}, \delta_{31}^{31}, \delta_{32}^{32}, \delta_{39}^{39}, \delta_{64}^{40}, \\ \delta_{64}^{47}, \delta_{64}^{48} \right\}. \end{split}$$

Using the initial set $P^0 = \{\theta \in \Theta\} = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and the destination set $P^d = \Xi$, we have

$$J_d = \sum_{\delta_{64}^i \in \Xi} \delta_{64}^i;$$

and

$$J_0 = \delta_{64}[2, 20, 38, 56].$$

It follows that

$$\mathcal{C}_S = J_d^T \mathcal{C} J_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}.$$

According to Theorem III.3, system (21) with outputs (26) is not observable.

IV. CONCLUSION

In this paper the set controllability of BCN is proposed, and necessary and sufficient condition is obtained. As an application, the output controllability is converted into a set controllability problem and is solved easily. Then an extended system is constructed for a given BCN. It has been proved that the observability of the given BCN is equivalent to the set controllability of the extended system. Then the observability of a BCN is verified via the set controllability of the extended system by providing a concise and easily verifiable necessary and sufficient condition. A numerical example has been presented to demonstrate the theoretical result. The method reveals a relationship between controllability and observability of BCN.

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APPENDIX

SEMI-TENSOR PRODUCT OF MATRICES

Semi-tensor product of matrices was proposed by us. It is convenient in dealing with logical functions. We refer to [5, 6] and the references therein for details. In the follows we give a very brief survey.

Definition A.1. Let $A \in \mathcal{M}_{m \times n}$ and $B \in \mathcal{M}_{p \times q}$. Denote by $t := \operatorname{lcm}(n, p)$ the least common multiple of n and p. Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := (A \otimes I_{t/n}) (B \otimes I_{t/p}) \in \mathcal{M}_{(mt/n) \times (qt/p)}.$$
 (28)

- **Remark A.2.** When n = p, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
 - When n = rp, denote it by A ≻_r B; when rn = p, denote it by A ≺_r B. These two cases are called the multi-dimensional case, which is particularly important in applications.
 - STP keeps almost all the major properties of the conventional matrix product unchanged.

We cite some basic properties which are used in this note.

Proposition A.3. 1) (Associative Low)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C.$$
⁽²⁹⁾

2) (Distributive Low)

$$(A+B) \ltimes C = A \ltimes C + B \ltimes C.$$

$$A \ltimes (B+C) = A \ltimes B + A \ltimes C.$$
(30)

3)

$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{31}$$

4) Assume A and B are invertible, then

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}.$$
 (32)

Proposition A.4. Let $X \in \mathbb{R}^t$ be a column vector. Then for a matrix M

$$X \ltimes M = (I_t \otimes M) \ltimes X. \tag{33}$$

Finally, we consider how to express a Boolean function into an algebraic form.

Theorem A.5. Let $f : \mathcal{D}^n \to \mathcal{D}$ be a Boolean function expressed as

$$y = f(x_1, \cdots, x_n). \tag{34}$$

Identifying

$$1 \sim \delta_2^1, \quad 0 \sim \delta_2^2. \tag{35}$$

Then there exists a unique logical matrix $M_f \in \mathcal{L}_{2 \times 2^n}$, called the structure matrix of f, such that under vector form, by using (35), (34) can be expressed as

$$y = M_f \ltimes_{i=1}^n x_i, \tag{36}$$

which is called the algebraic form of (34).