

ON COALGEBRA BASED ON CLASSES

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ABSTRACT. The category **Class** of classes and functions is proved to have a number of properties suitable for algebra and coalgebra: every endofunctor has an initial algebra and a terminal coalgebra, the categories of algebras and coalgebras are complete and cocomplete, and every endofunctor generates a free completely iterative monad. A description of a terminal coalgebra for the power-set functor is provided.

1. INTRODUCTION: DON'T BE AFRAID OF CLASSES

This paper does not, despite its title, concern the foundations. It concerns coalgebra in a surprisingly coalgebra-friendly category

Class

of classes and functions — and the main message is that one almost does not need foundations for that, or just the reasonable minimum of foundations. What is the definition of reasonable minimum? In category theory one always works with “large” and “small” — and this is all one needs. Thus, “large” refers to, say, set theory which is a model of ZFC (Zermelo-Fraenkel axioms including the Axiom of Choice). And “small” means that a universe (of small sets) is once for all chosen within the given universe (of all sets). This is all one needs: the chosen universe of all small sets is itself a large set, and we denote by

\aleph_∞

its cardinality. This means that the category **Set** of all small sets is, obviously, equivalent to the category $\mathbf{SET}_{<\aleph_\infty}$ of all large sets of cardinality smaller than \aleph_∞ . And if one forms, analogously, the category $\mathbf{SET}_{\leq\aleph_\infty}$ of all large sets of cardinality less or equal to \aleph_∞ , then **Class** is equivalent to it. Thus, one can think of the difference between **Set** and **Class** as of the difference between being smaller than, or smaller or equal to, \aleph_∞ . The cardinal \aleph_∞ is strongly inaccessible (i.e., for every cardinal $\alpha < \aleph_\infty$ we have $2^\alpha < \aleph_\infty$), and conversely, for every choice of a strongly inaccessible uncountable cardinal \aleph_∞ there is a universe of small sets with $\mathbf{Set} \simeq \mathbf{SET}_{<\aleph_\infty}$.

In what follows we work with the category of all sets of cardinality less than \aleph_∞ as **Set** and with the category of all sets of cardinality at most \aleph_∞ as the category **Class**.

2. ALL ENDOFUNCTORS ARE SET-BASED

The concept of a set-based endofunctor of **Class** has been introduced by Peter Aczel and Nax Mendler [AM] in order to prove their “Final Coalgebra Theorem”, see the next section. An endofunctor $F : \mathbf{Class} \rightarrow \mathbf{Class}$ is called *set-based* provided that for every class X and every element of FX there exists a small subset $m : M \hookrightarrow X$ such that that element lies in the image of Fm . It turns out that every endofunctor has this property. The following proof, based on ideas of Václav Koubek [Ko], uses a classical set-theoretical result of Alfred Tarski; see [T]:

Theorem 2.1. *For every infinite cardinal λ there exists, on a set X of cardinality λ , an almost disjoint collection of subsets $X_i \subseteq X$, $i \in I$, i.e., a collection satisfying $\text{card}(X_i \cap X_j) < \lambda$ for all $i \neq j$ in I , such that I has more than λ elements.*

Theorem 2.2. *Every endofunctor of **Class** is set-based.*

Remark. We prove a more general statement: given an infinite regular ¹ cardinal λ , then every endofunctor of $\mathbf{SET}_{\leq\lambda}$ (the category of all sets of cardinality at most λ) is λ -accessible. Recall from [AP₁] that for an endofunctor F of $\mathbf{SET}_{\leq\lambda}$ the following conditions are equivalent:

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¹Recall that λ is regular if it is not a sum of less than λ smaller cardinals.

- (1) F is λ -accessible, i.e., preserves λ -filtered colimits.
- (2) For every set X in $\mathbf{SET}_{\leq \lambda}$ and every element x of FX there exists a subset $m : M \hookrightarrow X$ of cardinality less than λ such that x lies in the image of Fm .
- (3) F is a quotient of a λ -ary polynomial functor.

(Given a signature Σ , i.e., a set of operation symbols σ with prescribed arities $\text{ar}(\sigma)$, finite or infinite, then the corresponding functor

$$H_{\Sigma} : X \mapsto \prod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

is called *polynomial*. It is λ -ary if $\text{ar}(\sigma) < \lambda$ for all $\sigma \in \Sigma$.)

In particular, set-based and \aleph_{∞} -accessible are equivalent.

Proof of Theorem and Remark. Let λ be an infinite cardinal. Given $F : \mathbf{SET}_{\leq \lambda} \rightarrow \mathbf{SET}_{\leq \lambda}$ and a set X in $\mathbf{SET}_{\leq \lambda}$, then for every element $x \in FX$ we are to find a subset $m : M \hookrightarrow X$ with $\text{card } M < \lambda$ and $x \in Fm[FM]$. If $\text{card } X < \lambda$ there is nothing to prove, assume $\text{card } X = \lambda$. We can further assume, without loss of generality, that F preserves finite intersections. In fact, by a theorem of Věra Trnková, see, e.g., Theorem III.4.5 of [AT], there exists a functor F' preserving finite intersections and such that the restrictions of F and F' to the full subcategory $\mathbf{Set}_{\leq \lambda}$ of all nonempty sets are naturally isomorphic. Since F is λ -accessible iff F' is, we can assume $F' = F$. By Theorem 2.1 there exists an almost disjoint collection of subsets $v_i : X_i \hookrightarrow X$, $i \in I$, with $\text{card } I > \lambda$. Since the collection of all subsets of X of cardinality less than λ has cardinality λ (due to the regularity of λ), we can suppose without loss of generality that each X_i has cardinality λ — in fact, by discarding all X_i of cardinalities less than λ we still obtain an almost disjoint collection of more than λ members. For each $i \in I$ thus there exists an isomorphism

$$w_i : X \rightarrow X_i$$

and we put

$$y_i = F(v_i w_i)(x) \in FX.$$

Since F is an endofunctor of $\mathbf{SET}_{\leq \lambda}$, the set FX has cardinality smaller than that of I , consequently, the elements y_i are not pairwise disjoint. Choose $i \neq j$ in I with

$$y_i = y_j$$

and form a pullback (an intersection of v_i and v_j):

$$\begin{array}{ccc} & M & \\ u_i \swarrow & & \searrow u_j \\ X_i & & X_j \\ v_i \searrow & & \swarrow v_j \\ & X & \end{array}$$

Since F preserves this pullback and $Fv_i(Fw_i(x)) = Fv_j(Fw_j(x))$, there exists

$$y \in FY \quad \text{with} \quad Fu_i(y) = Fw_i(x).$$

For the subobject $m = w_i^{-1}u_i : M \hookrightarrow X$ this implies

$$Fm(y) = x$$

and this concludes the proof. \square

Remark 2.3. Denote by $J : \mathbf{Set} \rightarrow \mathbf{Class}$ the inclusion functor. That a functor $F : \mathbf{Class} \rightarrow \mathbf{Class}$ is set-based can be equivalently restated as being naturally isomorphic to a left Kan extension $\text{Lan}_J K$ for some functor $K : \mathbf{Set} \rightarrow \mathbf{Class}$.

Thus, Theorem 2.2 says that restriction along J , i.e., the functor

$$_{\perp} \phi J : [\mathbf{Class}, \mathbf{Class}] \rightarrow [\mathbf{Set}, \mathbf{Class}]$$

is an equivalence of categories. We denote the pseudoinverse of this restriction by $(-)^\sharp$.

In fact, $J : \mathbf{Set} \rightarrow \mathbf{Class}$ is a free cocompletion of \mathbf{Set} under colimits of transfinite chains, i.e., colimits of chains indexed by the set of all small ordinals (see [AMV]). Thus, every functor of the form F^\sharp (i.e., every endofunctor of \mathbf{Class}) preserves colimits of transfinite chains.

Notation 2.4. For an endofunctor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ we denote by

$$F^\infty : \mathbf{Class} \longrightarrow \mathbf{Class}$$

the extension $(J\emptyset F)^\sharp$ of the composite $J\emptyset F : \mathbf{Set} \longrightarrow \mathbf{Class}$.

3. ALL FUNCTORS ARE VARIETORS AND COVARIETORS

In the present section we show that endofunctors H of \mathbf{Class} have a surprisingly simple structure, and they admit free H -algebras (i.e., are variators) and cofree H -coalgebras (i.e., are covariators) — moreover, these algebras and coalgebras can be explicitly described.

3.1. Polynomial Endofunctors. Classical Universal Algebra deals with Σ -algebras in the category \mathbf{Set} , where Σ is a (small) signature, i.e., a small set of operation symbols σ with prescribed arities $\text{ar}(\sigma)$ which are (in general, infinite) small cardinal numbers. Thus, if

\mathbf{Card}

denotes the class of all small cardinal numbers, then a small signature is a small set Σ equipped with a function $\text{ar} : \Sigma \longrightarrow \mathbf{Card}$. And Σ -algebras are just algebras over the polynomial endofunctor H_Σ of \mathbf{Set} given on objects, X , by

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

Quite analogously, in \mathbf{Class} we work with (large) *signatures* as classes Σ equipped with a function $\text{ar} : \Sigma \longrightarrow \mathbf{Card}$ (thus, largeness refers to the possibility of having a proper class of operations, arities are small). Here, again, we obtain a *polynomial endofunctor* H_Σ defined on classes X by

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

and analogously on morphisms.

Proposition 3.2. *Every endofunctor H of \mathbf{Class} is a quotient of a polynomial functor. That is, there exists a natural epitransformation $\varepsilon : H_\Sigma \longrightarrow H$ for some signature Σ .*

Proof. Let Σ be the signature which, for every small cardinal n has as n -ary symbols precisely the elements of Hn . Then the function

$$\varepsilon_X : \coprod_{n \in \mathbf{Card}} \coprod_{\sigma \in Hn} X^n \longrightarrow HX$$

which to every $f : n \longrightarrow X$ in the σ -th summand X^n assigns $Hf(\sigma)$ in HX is a component of a natural transformation (due to Yoneda lemma). And ε is pointwise surjective: for a small set M put $n = \text{card } M$ and choose an isomorphism $f : n \longrightarrow M$. Then every element of HM has the form $Hf(\sigma)$ for a unique $\sigma \in \Sigma$. Thus, ε_M is surjective. For a general X use the fact that H is set-based (Theorem 2.2), thus for an element $x \in HX$ there exists a small subset $m : M \hookrightarrow X$ and $y \in HM$ such that $x = Hm(y)$. Since ε_M is surjective, there exists $z \in H_\Sigma M$ such that $\varepsilon_M(z) = y$. Define $t = H_\Sigma m(z) \in H_\Sigma X$. Due to naturality of ε it follows that $\varepsilon_X(t) = x$. \square

Example 3.3. The power-set functor $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ extends uniquely to $\mathcal{P}^\infty : \mathbf{Class} \longrightarrow \mathbf{Class}$, see Remark 2.3. The functor \mathcal{P}^∞ assigns to every class X the class of all small subsets of X . We can represent \mathcal{P}^∞ as a quotient of H_{Σ^0} where Σ^0 is the signature which possesses, for every cardinal $n \in \mathbf{Card}$, a unique operation σ_n : here

$$\varepsilon_X : H_{\Sigma^0} = \coprod_{n \in \mathbf{Card}} X^n \longrightarrow \mathcal{P}^\infty X$$

assigns to every $f : n \longrightarrow X$ the image $f[n] \subseteq X$.

3.4. Algebras. Recall that for an endofunctor H of \mathbf{Class} an *H-algebra* is a class A together with a function $\alpha : HA \longrightarrow A$. Given another algebra $\beta : HB \longrightarrow B$, a *homomorphism* from A to B is a function $f : A \longrightarrow B$ such that the following square

$$\begin{array}{ccc} HA & \xrightarrow{\alpha} & A \\ Hf \downarrow & & \downarrow f \\ HB & \xrightarrow{\beta} & B \end{array}$$

commutes. The category of all H -algebras and homomorphisms is denoted by

$$\mathbf{Alg} H$$

Examples 3.5.

- (1) $\mathbf{Alg} H_\Sigma$ is the category of Σ -algebras (i.e., classes A endowed, for every n -ary symbol σ , with an n -ary operation on A) which, except for the “size” of underlying sets, is just the classical category of Universal Algebra.
- (2) $\mathbf{Alg} \mathcal{P}^\infty$ has as objects classes A together with a function $\alpha : \mathcal{P}^\infty A \rightarrow A$. This can be equivalently considered as a variety of Σ^0 -algebras as follows: let E be the class of all equations

$$\sigma_n(x_i)_{i < n} \approx \sigma_m(y_j)_{j < m}$$

where n and m are small cardinals and the variables x_i and y_j are such that the sets $\{x_i \mid i < n\}$ and $\{y_j \mid j < m\}$ are equal. Then $\mathbf{Alg} \mathcal{P}^\infty$ is isomorphic to the variety of all Σ^0 -algebras satisfying the above equations.

Remark 3.6. Recall from [AT] that a *basic equation* is an equation between two flat terms, i.e., terms of the form $\sigma(x_i)_{i < n}$ where σ is an n -ary operation symbol and x_i are (not necessarily distinct) variables. The example $\mathbf{Alg} \mathcal{P}^\infty$ above is quite typical: every category $\mathbf{Alg} H$ is a variety presented by basic equations (and vice versa) — this has been shown for **Set** in [AT], let us recall it and extend to the present ambient:

Given a functor H represented as in Proposition 3.2, consider all the basic equations

$$\sigma(x_i)_{i < n} \approx \rho(y_j)_{j < m}$$

where $\sigma, \rho \in \Sigma$ and for the set $V = \{x_i \mid i < n\} \cup \{y_j \mid j < m\}$ of variables we have:

ε_V merges the n -tuple $(x_i)_{i < n}$ in the σ -summand of $H_\Sigma V$ with the m -tuple $(y_j)_{j < m}$ in the ρ -summand of $H_\Sigma V$.

Then $\mathbf{Alg} H$ is equivalent to the variety of all Σ -algebras presented by the above equations.

Conversely, given a class E of basic equations in signature Σ , there is a quotient H of H_Σ such that the variety of Σ -algebras presented by E is isomorphic to $\mathbf{Alg} H$.

Corollary 3.7. *For every endofunctor H of **Class** an initial H -algebra exists.*

In fact, we can describe an initial H -algebra, I , in two substantially different ways:

- (1) I is a quotient of the initial Σ -algebra modulo the congruence generated by the given basic equations.

Recall here the description of initial Σ -algebras well-known in Universal Algebra: it is the algebra of all well-founded Σ -trees. This remains unchanged in case of large signatures, the only difference is that all Σ -trees do not form a small set (but each Σ -tree is small, by definition). That is, by a Σ -tree we mean an ordered, labelled tree on a small set of nodes, where labels are operation symbols, and every node labelled by an n -ary symbol has precisely n children. The algebra

$$I_\Sigma$$

of all *well-founded* Σ -trees, i.e., Σ -trees in which every branch is finite, has operations given by tree-tupling. This is an initial algebra in $\mathbf{Alg} H_\Sigma$.

Given a quotient $\varepsilon : H_\Sigma \rightarrow H$, form the smallest congruence \sim on I_Σ which is generated by all the basic equations corresponding to ε . Then I_Σ / \sim is an initial algebra of $\mathbf{Alg} H$.

- (2) I is a colimit of the transfinite chain $W : \mathbf{Ord} \rightarrow \mathbf{Class}$ (where \mathbf{Ord} is the chain of all small ordinals) given by iterating H on the initial object \emptyset of **Class**:

$$W_i = H^i(\emptyset)$$

and

$$I = \operatorname{colim}_{i \in \mathbf{Ord}} W_i.$$

More precisely, there is a unique chain W for which we have

First step: $W_0 = \emptyset$, $W_1 = H(\emptyset)$ and $W_{0,1} : \emptyset \rightarrow H(\emptyset)$ unique.

Isolated step: $W_{i+1} = H(W_i)$ and $W_{i+1,j+1} = H(W_{i,j})$.

Limit step: $W_j = \operatorname{colim}_{i < j} W_i$ with colimit cocone $(W_{i,j})_{i < j}$.

A colimit of this chain exists (see Observation 3.9 below) and is preserved by H , see 2.3, therefore, if $I = \text{colim } W_i$ then

$$HI \cong \text{colim}_{i \in \text{Ord}} H(W_i) = \text{colim}_{i \in \text{Ord}} W_{i+1} \cong I$$

and the canonical isomorphism $HI \longrightarrow I$ defines an initial H -algebra, see [A1].

Example 3.8. An initial \mathcal{P}^∞ -algebra. Whereas the power-set functor \mathcal{P} has no initial algebra, \mathcal{P}^∞ does (since every endofunctor of **Class** does). The above chain W_i coincides with the chain of sets defined by the cumulative hierarchy:

$$\begin{aligned} W_0 &= \emptyset \\ W_{i+1} &= \exp W_i \end{aligned}$$

and

$$W_j = \bigcup_{i < j} W_i \text{ for limit ordinals } j.$$

Consequently, we can describe an initial \mathcal{P}^∞ -algebra as

$$I = \mathbf{Set} = \text{the class of all small sets}$$

with the structure map $\mathcal{P}^\infty I \longrightarrow I$ given by the union. This has been first observed by Jan Rutten and Danielle Turi [RT].

The other option of describing I is also interesting: let us first form the initial Σ^0 -algebra. Since operations of any arity are unique, we can first forget the labelling, thus

$$I_{\Sigma^0} = \text{the algebra of all well-founded trees}$$

To every well-founded tree t let us assign the corresponding non-ordered tree (obtained by forgetting the linear ordering of children of any node) and recall that a non-ordered tree is called *extensional* provided that every pair of distinct siblings defines a pair of non-isomorphic subtrees. For every $t \in I_{\Sigma^0}$ denote by $[t]$ the *extensional quotient* of the (non-ordered version of) t ; that is, the extensional tree obtained from t by iteratively merging any pair of siblings defining isomorphic subtrees. Then an initial \mathcal{P}^∞ -algebra can be described as

$$I_{\Sigma^0} / \sim \quad \text{where } t \sim t' \text{ iff } [t] = [t']$$

This follows from the above result of [RT] due to the Axiom of Extensionality for **Set**.

Observation 3.9. The category **Class** has all small limits and all class-indexed colimits. That is, given a functor $D : \mathcal{D} \longrightarrow \mathbf{Class}$ then

- (a) if \mathcal{D} is small then $\lim D$ exists

and

- (b) if \mathcal{D} has only a class of morphisms then $\text{colim } D$ exists.

In fact, the strongly inaccessible cardinal \aleph_∞ which is the cardinality of all classes, satisfies

$$(\aleph_\infty)^n = \aleph_\infty \quad \text{for all } n \in \mathbf{Card}$$

Consequently, a cartesian product of a small collection of classes is a class — thus, **Class** has small products. And since small limits are always subobjects of small products, it follows that **Class** has all small limits. Analogously, since

$$\aleph_\infty \otimes \aleph_\infty = \aleph_\infty$$

it follows that **Class** has all class-indexed coproducts: a disjoint union of a class of classes is a class. Since class-indexed colimits are always quotients of class-indexed coproducts, it follows that **Class** has class-indexed colimits.

Example 3.10. Class-indexed limits do not exist, in general. For example, if I is a proper class then 2^I (a cartesian product of I copies of the two-element set $2 = \{0, 1\}$) is not a class, having cardinality $2^{\aleph_\infty} > \aleph_\infty$. It follows that a product of I copies of 2 does not exist in the category **Class**: it is trivial that if $(\pi_i : L \longrightarrow 2)_{i \in I}$ were such a product, then for every subclass $J \subseteq I$ we have the unique $u_J : 1 \longrightarrow L$ with $\pi_i \circ u_J$ given by 0 for $i \in J$ and 1 for $j \in I \setminus J$. Then the u_J 's are pairwise distinct, thus, $\text{card } L > \aleph_\infty$, a contradiction.

Recall from [AT] that an endofunctor H of **Class** is called a *variator* provided that every object of **Class** generates a free H -algebra. Equivalently, if the forgetful functor $\mathbf{Alg} H \rightarrow \mathbf{Class}$ has a left adjoint.

And an initial algebra of the functor

$$H(-) + A$$

is precisely a free H -algebra on H . The former exists by Corollary 3.7, thus we obtain

Corollary 3.11. *Every endofunctor of **Class** is a variator.*

Remark 3.12. The category $\mathbf{Alg} H$ has all small limits and all class-indexed colimits for every endofunctor H of **Class**: the limits are (obviously) created by the forgetful functor. The existence of colimits follows from the fact that (Epi, Mono) is a factorization system in **Class** and every endofunctor H preserves epimorphisms (since they split). Since H is a variator, $\mathbf{Alg} H$ has all colimits which **Class** has, see [Ke], Theorem 16.5 (where $\mathbf{Alg} H$ is shown to be reflective in the category H/\mathbf{Class} having all colimits that **Class** has).

3.13. Coalgebras of an endofunctor H of **Class** are classes A together with a function $\alpha : A \rightarrow HA$. Given another algebra $\beta : B \rightarrow HB$, a *homomorphism* from A to B is a function $f : A \rightarrow B$ such that the following square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & HA \\ f \downarrow & & \downarrow Hf \\ B & \xrightarrow{\beta} & HB \end{array}$$

commutes. The category of all H -coalgebras and homomorphisms is denoted by

$$\mathbf{Coalg} H$$

Examples 3.14.

- (1) Coalgebras over polynomial functors describe deterministic dynamic systems, see [R]. For example, if Σ consists of a binary symbol and a nullary one, then a coalgebra

$$A \rightarrow A \times A + 1$$

describes a system with the state-set A and two deterministic inputs (0, 1, say) with exceptions: to every state a the pair (a_0, a_1) of states is assigned, representing the reaction of a to 0 and 1, respectively — unless a is an exception, mapped to the unique element of 1.

- (2) \mathcal{P}^∞ -coalgebras can be identified with large small-branching graphs, i.e., classes A endowed with a binary relation (represented by the function $A \rightarrow \mathcal{P}^\infty A$ assigning to every node the small set of its descendants).

Theorem 3.15. *Every endofunctor H of **Class** has a terminal coalgebra.*

Several proofs of this theorem are known. The first one is due to Peter Aczel and Nax Mendler [AM]. Their Final Coalgebra Theorem states that every set-based endofunctor has a terminal coalgebra — but we know from Section 2 that all endofunctors are set-based. Another proof follows, as Michael Barr has noticed in [B], from the theory of accessible categories in the monograph [MP]. A third proof can be derived from the result of James Worell [W] that every λ -accessible endofunctor H of **Set** has a terminal coalgebra obtained by 2λ steps of the dual chain of 3.7(2). That is, define a chain V of (in general, large) sets as follows:

First step: $V_0 = 1$, $V_1 = H(1)$ and $V_{0,1} : H(1) \rightarrow 1$ unique.

Isolated step: $V_{i+1} = H(V_i)$ and $V_{i+1,j+1} = H(V_{i,j})$.

and

Limit step: $V_j = \lim_{i < j} V_i$ with limit cone $(V_{i,j})_{i < j}$

Then

$$V_{2\lambda} \text{ is a terminal coalgebra of } H.$$

By applying this to $\lambda = \aleph_\infty$ we “almost” obtain a construction of terminal coalgebras of endofunctors of **Class** (first, one has to extend the endofunctor to the category of all large sets but this brings no difficulty). There is a catch here: although the resulting limit $V_{2\aleph_\infty}$ is indeed a class (which follows from Worell’s result), the intermediate step V_{\aleph_∞} can “slip” outside the scope of classes:

Example 3.16. A terminal coalgebra T of \mathcal{P}^∞ has, in the non-well-founded set theory of Peter Aczel, see [A] or [BM], a beautiful description: T is the class of all non-well-founded sets — see [RT]. However, we work here in the well-founded set theory ZFC. An explicit (but certainly not very beautiful) description of T is presented in Section 5 below.

Here we just observe that the chain V above “jumps” out of the realm of classes: if we put $V_0 = 1$, $V_{i+1} = \exp V_i$ and $V_j = \lim_{i < j} V_i$ for all ordinals in \mathbf{Ord} , then we cannot form

$$V_{\aleph_\infty} = \lim_{i \in \mathbf{Ord}} V_i$$

within \mathbf{Class} . The reason is that for all $i \leq \aleph_\infty$ we can easily prove by transfinite induction that

$$\text{card } V_i \geq 2^i$$

Thus, V_{\aleph_∞} is not a class.

Remark 3.17. In spite of the three proofs mentioned above, we present a new proof, based on ideas of Peter Gumm and Tobias Schröder [GS] since it is the shortest and clearest one, and it gives a sort of concrete description: a terminal H -coalgebra is obtained from a terminal H_Σ -coalgebra via a suitable congruence. Recall that for every H -coalgebra $\alpha : A \rightarrow HA$ a *congruence* is a quotient $e : A \rightarrow A/\sim$ of A in \mathbf{Class} for which a (necessarily unique) structure map $\bar{\alpha} : A/\sim \rightarrow H(A/\sim)$ exists turning e into a homomorphism:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & HA \\ e \downarrow & & \downarrow He \\ A/\sim & \xrightarrow{\bar{\alpha}} & H(A/\sim) \end{array}$$

Recall further that a nice description of terminal H_Σ -coalgebras is known, which works for large signatures as well as for small ones: let

$$T_\Sigma$$

be the class of all (small) Σ -trees. (In comparison to I_Σ , we just drop the well-foundedness.) This is, like I_Σ , a Σ -algebra w.r.t. tree tupling — and since in both cases tree-tupling is actually an isomorphism we can invert it to the structure map $\tau_\Sigma : T_\Sigma \rightarrow H_\Sigma T_\Sigma$ of a coalgebra. And that coalgebra is terminal.

Proposition 3.18. *Every endofunctor H of \mathbf{Class} , represented as a quotient $\varepsilon : H_\Sigma \rightarrow H$ (as in Proposition 3.2) has a terminal coalgebra, viz, the quotient of the H -coalgebra*

$$T_\Sigma \xrightarrow{\tau_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon_{T_\Sigma}} HT_\Sigma$$

modulo the largest congruence.

Proof.

- (1) The largest congruence exists. In fact, the pushout of all congruences of T_Σ is easily seen (due to the universal property of pushouts) to be a congruence.
- (2) Given the largest congruence $e : T_\Sigma \rightarrow T/\sim$, the corresponding coalgebra $\bar{\tau}_\Sigma : T/\sim \rightarrow H(T/\sim)$ is terminal. In fact, given a coalgebra $\beta : B \rightarrow HB$ the uniqueness of a homomorphism from B to T/\sim follows from the observation that given two homomorphisms $f_1, f_2 : B \rightarrow T/\sim$, then a coequalizer $c : T/\sim \rightarrow T/\approx$ of f_1, f_2 in \mathbf{Class} yields a congruence $ce : T_\Sigma \rightarrow T/\approx$, thus, \approx and \sim coincide, which means $f_1 = f_2$. The existence of a homomorphism is proved by choosing a splitting of the epimorphism ε_B :

$$u : HB \rightarrow H_\Sigma B \quad \text{with } \varepsilon_B u = id$$

The unique homomorphism, f , of the H_Σ -coalgebra $B \xrightarrow{\beta} HB \xrightarrow{u} H_\Sigma B$ yields a homomorphism, ef , of H -coalgebras:

$$\begin{array}{ccccc}
 & & & & HB \\
 & & & \nearrow & \downarrow Hf \\
 B & \xrightarrow{\beta} & HB & \xrightarrow{u} & H_\Sigma B \\
 \downarrow f & & \downarrow H_\Sigma f & \nearrow \varepsilon_{T_\Sigma} & \downarrow He \\
 T_\Sigma & \xrightarrow{\tau_\varepsilon} & H_\Sigma T_\Sigma & & HT_\Sigma \\
 \downarrow e & & & & \downarrow He \\
 T_\Sigma / \sim & \xrightarrow{\tau_\Sigma} & H(T_\Sigma / \sim) & &
 \end{array}$$

□

Corollary 3.19. *Every endofunctor H of **Class** is a covariator, i.e., a cofree H -coalgebra on every class exists.*

In fact, a cofree H -coalgebra on A is just a terminal coalgebra of $H(-) \times A$.

Remark 3.20. The category $\mathbf{Coalg} H$ has all small limits and all class-indexed colimits for every endofunctor H of **Class**: the colimits are (obviously) created by the forgetful functor. The existence of limits follows, if H preserves monomorphisms, from the dualization of Theorem 16.5 of [Ke]. For general H use the result of Věra Trnková cited in the proof of Theorem 2.2 above.

4. ALL FUNCTORS GENERATE COMPLETELY ITERATIVE MONADS

In this section we assume that the reader is acquainted with the concept of an iterative theory (or iterative monad) of Calvin Elgot, and the coalgebraic treatment of completely iterative monads in [M] or [AAMV]. In [AAMV] we worked with endofunctors H such that a terminal coalgebra, TX , of the endofunctor $H(-) + X$ exists for every X . Such functors were called *iteratable*. In the category of classes this concept need not be used:

Corollary 4.1. *Every endofunctor of **Class** is iteratable.*

This follows from Proposition 3.18 applied to $H(-) + X$.

Recall from [M] or [AAMV] that the coalgebra structure of TX , $TX \xrightarrow{\cong} HTX + X$, turns TX into a coproduct of HTX and X , where the coproduct inclusions are denoted by

$$\tau_X : HTX \longrightarrow TX \text{ (} TX \text{ is an } H\text{-algebra)}$$

and

$$\eta_X : X \longrightarrow TX \text{ (} X \text{ is contained in } TX)$$

It turns out that this is part of a monad $\mathbb{T} = (T, \eta, \mu)$. This monad is *completely iterative*, i.e., for every “equation” morphism $e : X \longrightarrow T(X + Y)$ which is guarded, i.e., it factorizes through the coproduct injection

$$HT(X + Y) + Y \hookrightarrow HT(X + Y) + X + Y = T(X + Y)$$

there exists a unique *solution*. That is, a unique morphism e^\dagger for which the following square

$$\begin{array}{ccc}
 X & \xrightarrow{e^\dagger} & TY \\
 e \downarrow & & \uparrow \mu_Y \\
 T(X + Y) & \xrightarrow{T[e^\dagger, \eta_Y]} & TTY
 \end{array}$$

commutes. And in [AAMV] it has been proved that \mathbb{T} can be characterized as a free completely iterative monad on H .

Example 4.2. Let Σ be a (possibly large, infinitary) signature, i.e., a class of operation symbols together with a function $\text{ar}(\cdot)$ assigning a small cardinal to every symbol σ . Put $\Sigma_n = \{\sigma \mid \text{ar}(\sigma) = n\}$. The polynomial functor

$$H_\Sigma : X \mapsto \prod_{\sigma \in \Sigma} X^{\text{ar}(\sigma)}$$

generates the following completely iterative monad \mathbb{T}_Σ :

$T_\Sigma Y$ is the Σ -algebra of all Σ -trees on Y , i.e., small trees with leaves labelled in $\Sigma_0 + Y$ and nodes with $n > 0$ children labelled in Σ_n ,

and

η_Y is the singleton-tree embedding.

The fact that \mathbb{T}_Σ is completely iterative just restates the well-known property of tree algebras: all iterative systems of equations that are guarded (i.e., do not contain equations $x \approx x'$ where x and x' are variables) have unique solutions.

Corollary 4.3. *All free completely iterative monads on **Class** are quotient monads of the tree-monads \mathbb{T}_Σ (for all signatures Σ).*

In fact, every endofunctor H of **Class** is a quotient of H_Σ for a suitable signature Σ (denote by Σ_n the class Hn). It follows that a free completely iterative monad on H is a quotient of \mathbb{T}_Σ , see [A₂].

5. TERMINAL COALGEBRA OF THE POWER-SET FUNCTOR

We apply the above results to non-labelled transition systems, i.e., to coalgebras of the power-set functor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$. It has been noticed by several authors, e.g., [AM], [B], [JPTWW], [RT], [W] that \mathcal{P}^∞ has a very natural *weakly terminal* coalgebra B (i.e., such that every \mathcal{P} -coalgebra A has at least one homomorphism from A to B): the coalgebra of all small extensional (see 3.8) trees. Throughout this section trees are always taken up to (graph) isomorphism. Thus, shortly, a tree is extensional if and only if distinct siblings define distinct subtrees.

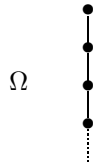
The weakly terminal coalgebra B has as elements all small extensional trees, and the coalgebra structure

$$\beta : B \rightarrow \mathcal{P}^\infty B$$

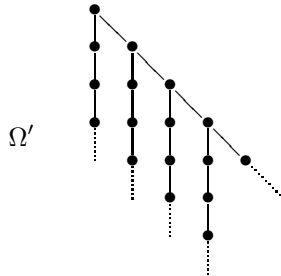
is the inverse of tree tupling, i.e., β assigns to every tree t the set of all children of t .

We know from Theorem 3.15 that a terminal coalgebra for \mathcal{P}^∞ exists. Since B is weakly terminal, it follows that a terminal coalgebra is a quotient of B modulo the *bisimilarity equivalence* \sim (i.e., the largest bisimulation on B). We are going to describe this equivalence \sim . We start by describing one interesting class.

Example 5.1. An extensional tree t is bisimilar to the following tree



if and only if all paths in t are infinite. Thus, for example, the following tree



is bisimilar to Ω . This illustrates that the bisimilarity equivalence is non-trivial. We prove $\Omega \sim \Omega'$ below.

Remark 5.2. For the finite-power-set functor \mathcal{P}_f a nice description of a terminal coalgebra has been presented by Michael Barr [B]: let B_f denote the coalgebra of all finitely branching extensional trees. This is a small subcoalgebra of our (large) coalgebra B . We call two trees b, b' in B_f *Barr-equivalent*, notation

$$b \sim_0 b'$$

provided that for every natural number n the tree $b|_n$ obtained by cutting b at level n has the same extensional quotient (sse 3.8) as the tree $b'|_n$. For example

$$\Omega \sim_0 \Omega'$$

Barr proved that the quotient coalgebra

$$B_f / \sim_0$$

is a terminal \mathcal{P}_f -coalgebra — that is, \sim_0 is the bisimilarity equivalence on B_f .

We define, for every small ordinal number i , the following equivalence relation \sim_i on B :

\sim_0 is the Barr-equivalence

and in case $i > 0$

$t \sim_i s$ iff for all $j < i$ the following hold:

(1) for each child t' of t there exists a child s' of s such that $t' \sim_j s'$

and

(2) vice versa.

Remark 5.3. We shall show below that the bisimilarity equivalence \sim is just the intersection of all \sim_i . Notice that this intersection is just the usual construction of a greatest fixed point. Indeed, consider the collection Rel of all binary relations on B . This collection, ordered by set-inclusion, is a class-complete lattice. Define $\Phi : \text{Rel} \rightarrow \text{Rel}$ as follows:

$$t \Phi(R) s \quad \text{iff} \quad \text{for every child } t' \text{ of } t \text{ there exists a child } s' \text{ of } s \text{ such that } t' R s', \text{ and vice versa.}$$

Observe that Φ is a monotone function. Moreover, a binary relation R is a fixed point of Φ if and only if R is a bisimulation on B . Notice that the definition of \sim_i is just an iteration of Φ on the largest equivalence relation \approx_0 (i.e., $B \times B$) shifted by ω steps: we have

$$\sim_0 = \Phi^{(\omega)}(\approx_0)$$

where for every relation R the iterations $\Phi^{(i)}(R)$, $i \in \text{Ord}$, are defined inductively as follows: $\Phi^{(0)}(R) = R$, the isolated step is $\Phi^{(i+1)}(R) = \Phi(\Phi^{(i)}(R))$, and for limit ordinals $\Phi^{(i)}(R) = \bigcap_{j < i} \Phi^{(j)}(R)$. Consequently, $\sim_i = \Phi^{(\omega+i)}(\approx_0)$.

That we are indeed constructing the largest fixed point for Φ follows from the following

Lemma 5.4. Φ preserves intersections of descending Ord-chains.

Proof. Let $(R_i)_{i \in \text{Ord}}$ be a descending chain in Rel and let

$$R = \bigcap_{i \in \text{Ord}} R_i$$

be its intersection. We show that $\Phi(R) = \bigcap_{i \in \text{Ord}} \Phi(R_i)$. In fact, the inclusion from left to right is obvious. To show the inclusion from right to left, suppose that $t \Phi(R_i) s$ holds for all $i \in \text{Ord}$. Let t' be any child of t . Then, for any ordinal number $i \in \text{Ord}$ there exists a child s'_i of s with $t' R_i s'_i$. Since s has only a small set of children the set $\{s'_i \mid i \in \text{Ord}\}$ is small, too. Therefore there is a cofinal subset C of Ord such that $\{s'_i \mid i \in C\}$ has only one element, s' say. It follows that $t' R_i s'$ for all $i \in \text{Ord}$. Hence, $t \Phi(R) s$, as desired. \square

Theorem 5.5. Two trees $t, s \in B$ are bisimilar iff $t \sim_i s$ holds for all small ordinals i .

Proof. It follows from Lemma 5.4 that the intersection of all $\sim_i = \Phi^{(i)}(\sim_0)$, $i \in \text{Ord}$ is a fixed point of Φ .

Next form the quotient coalgebra B/\sim . Since B is weakly terminal, so is B/\sim . In order to establish that B/\sim is a terminal \mathcal{P}^∞ -coalgebra we must show that for any \mathcal{P}^∞ -coalgebra (X, ξ) and any two coalgebra homomorphisms $h, k : (X, \xi) \rightarrow (B, \beta)$ we have $h(x) \sim k(x)$ for all $x \in X$. We show this by transfinite induction. We write

$$\beta(k(x)) = \{s_j^x \mid j \in J_x\} \quad \text{and} \quad \beta(h(x)) = \{t_i^x \mid i \in I_x\}$$

for the sets of children of $k(x)$ and $h(x)$, respectively. Since h and k are coalgebra homomorphisms we have

$$\{s_j^x \mid j \in J_x\} = \{kx_\ell \mid \ell \in L_x\} \quad \text{and} \quad \{t_i^x \mid i \in I_x\} = \{hx_\ell \mid \ell \in L_x\},$$

where $\xi(x) = \{x_\ell \mid \ell \in L_x\}$.

First step, $i = 0$: We will show that $k(x) \sim_0 h(x)$, i.e., $E(k(x)|_n) = E(h(x)|_n)$ for all $n < \omega$ by induction on n . The statement is obvious for $n = 0$. For the induction step observe that

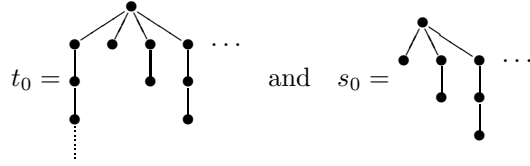
$$\begin{aligned} \{E(s_j^x|_n) \mid j \in J_x\} &= \{E(kx_\ell|_n) \mid \ell \in L_x\} \\ &= \{E(hx_\ell|_n) \mid \ell \in L_x\} \\ &= \{E(t_i^x|_n) \mid i \in I_x\} \end{aligned}$$

by the induction hypothesis. Hence, $E(k(x)|_{n+1})$ and $E(h(x)|_{n+1})$ have the same sets of children and therefore are equal.

Induction step: Suppose now that $i > 0$ is any ordinal number and that for all $x \in X$, $k(x) \sim_j h(x)$ holds for all $j < i$. Consider any child s' of $k(x)$, i.e., $s' = kx_\ell$ for some $x_\ell \in \xi(x)$. Then $t' = hx_\ell$ is a child of $h(x)$ such that $s' \sim_j t'$ for all $j < i$.

Hence, we obtain $k(x) \sim_i h(x)$ for all $i \in \text{Ord}$, which implies the desired result. \square

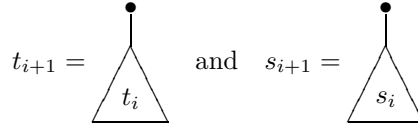
Remark 5.6. Barr showed that \sim_0 is the bisimilarity equivalence on the set of finitely branching trees. However, it is not a bisimulation on B . In order to see this notice that it suffices to find trees that are in \sim_0 but not in \sim_1 . Consider the following trees



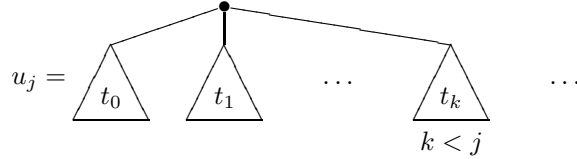
We clearly have $t \sim_0 s$. But $t_0 \not\sim_1 s_0$, since t_0 has a child which is an infinite path while s_0 does not.

Definition 5.7. We define trees t_i and s_i for all small ordinals i for which we show below that they are equivalent under \sim_i but not under \sim_{i+1} .

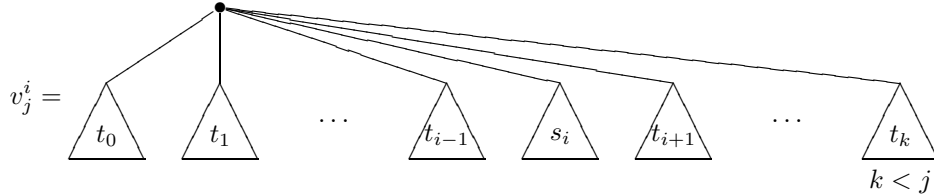
- (1) We start with the trees t_0 and s_0 from the previous remark.
- (2) Given t_i and s_i we define



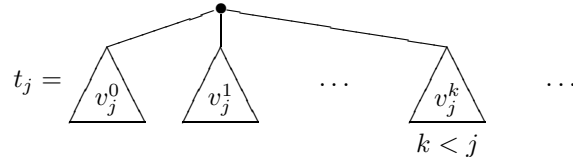
- (3) For every limit ordinal j we use the following auxilliary trees (where $i < j$ is arbitrary)



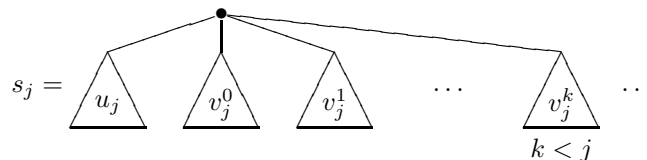
and



and we define



and



Theorem 5.8. *None of the equivalences \sim_i is a congruence.*

Proof. We prove $t_i \sim_i s_i$ but $t_i \not\sim_{i+1} s_i$ for all ordinals i . This proves the theorem.

- (1) Proof of $t_i \sim_i s_i$. We proceed by transfinite induction on i .

Initial case: $t_0 \sim_0 s_0$ — clear.

Isolated case: $t_i \sim_i s_i$ clearly implies $t_{i+1} \sim_{i+1} s_{i+1}$.

Limit case: Let j be a limit ordinal with $t_i \sim_i s_i$ for all $i < j$. Then, obviously, $u_j \sim_i v_j^i$ for all $i < j$, thus, $u_j \sim_j v_j^j$, which implies $t_j \sim_j s_j$.

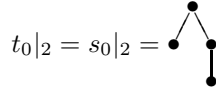
- (2) We need some auxilliary facts about cuttings $w|_n$ of trees w at level n :

- (a) For $n = 1$ all the trees t_i, s_i, u_j, v_j^i cut to

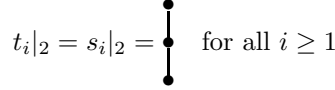


because they have all more than one vertex — this is obvious.

- (b) We have



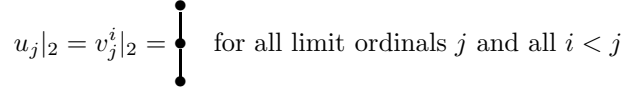
and



$t_i|_2 = s_i|_2 =$ for all $i \geq 1$

The first statement is obvious, and so is the second one for isolated ordinals i . For limit ordinals it follows from (a).

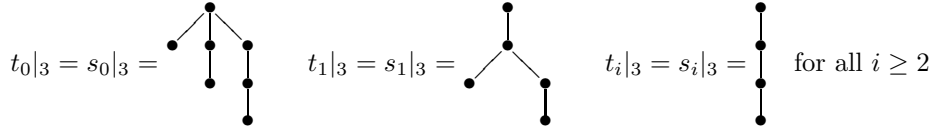
- (c) We have



$u_j|_2 = v_j^i|_2 =$ for all limit ordinals j and all $i < j$

This follows from (b).

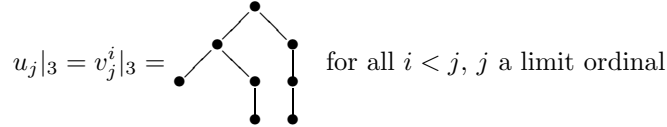
- (d) We have



$t_0|_3 = s_0|_3 =$ $t_1|_3 = s_1|_3 =$ $t_i|_3 = s_i|_3 =$ for all $i \geq 2$

The last statement follows from (c).

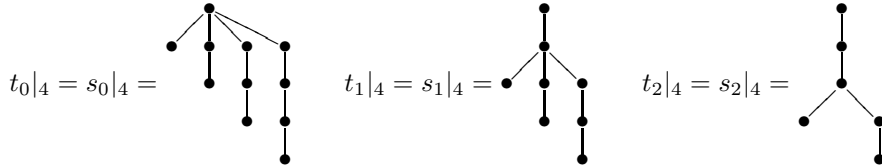
- (e) We have



$u_j|_3 = v_j^i|_3 =$ for all $i < j, j$ a limit ordinal

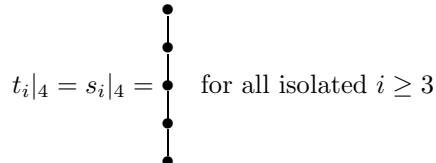
This follows from (b).

- (f) We have



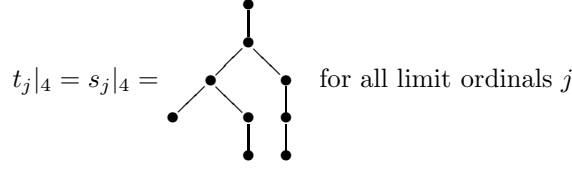
$t_0|_4 = s_0|_4 =$ $t_1|_4 = s_1|_4 =$ $t_2|_4 = s_2|_4 =$

and



$t_i|_4 = s_i|_4 =$ for all isolated $i \geq 3$

as well as



The last statement follows from (e), the last but one from (d).

(3) We prove

$$t_i \not\sim_{i+2} t_k \quad \text{and} \quad s_i \not\sim_{i+2} t_k \quad \text{for all ordinals } i < k$$

We proceed by transfinite induction on k :

1. Initial case: there is nothing to prove if $k = 0$.
2. Isolated case: $t_i \not\sim_{i+2} t_{k+1}$ is clear if $i = 0$ (in fact, $t_0 \not\sim_0 t_{k+1}$ because $t_0|_2 \neq t_{k+1}|_2$, see (2b) and if i is a limit ordinal ($t_i \not\sim_0 t_{k+1}$ because $t_i|_4 \neq t_{k+1}|_4$, see (2f)). If i is an isolated ordinal, then $t_{i-1} \not\sim_{i+1} t_k$ implies $t_i \not\sim_{i+2} t_{k+1}$. Analogously with $s_i \not\sim_{i+2} t_k$.
3. Limit case: let k be a limit ordinal. We proceed by transfinite induction on i .

3.1 Initial case: $t_0 \not\sim_2 t_k$ because $t_0|_2 \neq t_k|_2$, see (2b). Analogously $s_0 \not\sim_2 t_k$.

3.2 Isolated case: $t_{i+1} \not\sim_{i+3} t_k$ because $t_{i+1}|_4 \neq t_k|_4$, see (2f). Analogously $s_{i+1} \not\sim_{i+3} t_k$.

3.3 Limit case: let $j < k$ be a limit ordinal. Assuming $t_j \not\sim_{j+2} t_k$, we derive a contradiction.

The child u_k of t_k must be \sim_{j+1} -equivalent to a child of t_j , i.e.,

$$\text{either } u_j \sim_{j+1} u_k, \text{ or } v_j^i \sim_{j+1} u_k \text{ for some } i < j.$$

The first possibility implies that the child t_j of u_k is \sim_j -equivalent to a child t_l of u_j , $l < j$. Thus, we have

$$t_l \sim_j t_j \text{ for } l < j < k.$$

This contradicts to the fact that, by induction, $t_l \not\sim_{l+2} t_j$ (and $l + 2 < j$). Analogously with the second possibility, $v_j^i \sim_{j+1} u_k$, where the only case that we have to consider extra is the child s_i of v_j^i — however,

$$s_i \sim_j t_j$$

is also a contradiction since, by induction, $s_i \not\sim_{i+2} t_j$ (and $l + 2 < j$).

Finally, assuming $s_j \sim_{j+2} t_k$, we derive a contradiction analogously, the only new case to consider here is that the child t_j of t_k is \sim_j -equivalent to the child u_j of s_j :

$$u_j \sim_j t_j$$

This, however, is a contradiction again: we have $u_j \not\sim_0 t_j$ because $u_j|_3 \neq t_j|_3$, see (2d,e).

(4) Proof of $t_i \not\sim_{i+1} s_i$. We proceed by transfinite induction on i .

Initial case: $t_0 \not\sim_1 s_0$ by our choice of trees t_0 and s_0 .

Isolated case: From $t_i \not\sim_{i+1} s_i$ it follows immediately that $t_{i+1} \not\sim_{i+2} s_{i+1}$.

Limit case: Let j be a limit ordinal with $t_j \sim_{j+1} s_j$. We derive a contradiction. The child u_j of s_j is \sim_0 -equivalent to a child of t_j . That is,

$$u_j \sim_j v_j^k \text{ for some } k < j.$$

This implies that the child t_k of u_j is \sim_{k+2} -equivalent to some child of v_j^k , i.e.,

$$\text{either } t_k \sim_{k+2} s_k \text{ or } t_k \sim_{k+2} t_l \text{ for some } l \neq k, l < j.$$

The first case does not happen: by induction hypothesis, $t_k \not\sim_{k+1} s_k$. The second case contradicts to (3): if $k < l$, and for $l < k$ we know from (3) that $t_l \not\sim_{l+2} t_k$, thus, again $t_l \not\sim_{k+2} t_k$. \square

Remark 5.9. We have described a terminal coalgebra of \mathcal{P}^∞ as the coalgebra of all extensional trees modulo the congruence $\bigcap_{i \in \text{Ord}} \sim_i$. Since none of the equivalences is a congruence, we see no hope in obtaining a nicer description of a terminal \mathcal{P}^∞ -coalgebra in well-founded set theory.

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