Connectivity of discrete planes

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Abstract

Studying connectivity of discrete objects is a major issue in discrete geometry and topology. In the present work we deal with connectivity of discrete planes in the framework of Reveillès analytical definition [11]. Accordingly, a discrete plane is a set $P(a, b, c, \mu, \omega)$ of integer points (x, y, z) satisfying the Diophantine inequalities $0 \leq ax + by + cz + \mu < \omega$. The parameter $\mu \in \mathbf{Z}$ estimates the plane intercept while $\omega \in \mathbf{N}$ is the plane thickness. Given three integers (plane coefficients) a, b, and c with $0 \leq a \leq b \leq c$, one can seek the maximal ω for which the discrete plane $P(a, b, c, \mu, \omega)$ is disconnected. We call this remarkable topological invariable the *connectivity number* of $P(a, b, c, \mu, \omega)$ and denote it $\Omega(a, b, c)$. Despite several attempts over the last ten years to determine the connectivity number, this is still an open question. In the present paper we propose a solution to the problem. For this, we first investigate some combinatorial properties of discrete planes. These structural results facilitate the deeper understanding of the discrete plane structure. On this basis, we obtain a series of results, in particular, we provide an explicit solution to the problem under certain conditions. We also obtain exact upper and lower bounds on $\Omega(a, b, c)$ and design an $O(a \log b)$ algorithm for its computation.

Keywords: *discrete plane, discrete line, connectivity of discrete object, connectivity number*

1 Introduction

Deciding whether a given discrete object is connected or tunnel-free is an important issue in discrete geometry and topology. Usually such sort of problems are resolved algorithmically. A rare case admitting a simple answer is when the object is a discrete straight line. For discrete lines the notions of tunnel-freedom and connectivity are equivalent: a discrete straight line is tunnel-free if and only if it is connected. Such a simple dependence, however, does not hold for discrete planes: a discrete plane may have tunnels and still be connected.

Tunnel-freedom matters, both of discrete lines and planes, have always been intuitively clear, and indeed very simple solutions to the tunnel-freedom testing problem are available. However, this is not the case with discrete plane connectivity. Thus, for instance, it has been unclear under what kind of general conditions and when exactly a discrete plane fails to be connected, as it is getting "thinner" when certain voxels are removed from it. Moreover, it has not been very clear in what terms the question had to be asked and the answer sought. Part of the difficulty laid in the fact that all existing definitions of a discrete plane were "algorithmic." While being quite satisfactory regarding various practical purposes, it is not always easy to use them for obtaining structural results.

A promising approach to defining discrete objects is the one based on their *analytical description*. In [11] Reveillès proposed *analytical* definitions of discrete lines. A *2D arithmetic line* $L(a, b, \mu, \omega)$ is

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defined as a set of integer points (x, y) satisfying a double linear Diophantine inequality of the form $0 \le ax + by + \mu < \omega$. Here $\mu \in \mathbf{Z}$ is an *internal translation constant* which estimates the line intercept, while $\omega \in \mathbf{N}$ is the *arithmetic thickness* of the line. Similarly, an *arithmetic plane* is a set $P(a, b, c, \mu, \omega)$ of integer points (x, y, z) satisfying $0 \le ax + by + cz + \mu < \omega$, where the parameters $\mu \in Z$ and $\omega \in \mathbf{N}$ have similar interpretation. $L(a, b, \mu, \omega)$ and $P(a, b, c, \mu, \omega)$ can be regarded as discretizations of a line $ax + by = \mu$ and a plane $ax + by + cz = \mu$, respectively. It can be shown that if $\omega = \max(|a|, |b|)$ (resp. $\omega = \max(|a|, |b|, |c|)$, then the above definitions are equivalent to the well-known classical definitions of discrete lines and planes. (See [12] for getting acquainted with different approaches to define digital straightness, and [3] for a study on digital flatness.)

Certain advantages of the analytical definitions are discussed in the literature (see, e.g., [2, 6]). Their main worth seems to lie in the fact that one can study an object in terms of a few parameters that define it. This may significantly facilitate the geometric and analytic reasoning and help describe theoretical results in a more rigorous and elegant form. For example, one can easily show that a discrete line $L(a, b, \mu, \omega)$ has tunnels (and, thus, is disconnected) if and only if $\omega < \max(|a|, |b|)$. Similarly, a discrete plane $P(a, b, c, \mu, \omega)$ has tunnels, if and only if $\omega < \max(|a|, |b|, |c|)$ [1]. Analytic definitions may also help arise new theoretical questions, whose rigorous formulation would be difficult by other means. For instance, given three integers (plane coefficients) a, b, and c, one can look for the maximal value ω for which the discrete plane $P(a, b, c, \mu, \omega)$ is disconnected. Thus the notion of discrete plane connectivity can be properly formalized and studied. This problem is important since, on the one hand, discrete plane is a very fundamental primitive in volume modeling (in particular, in medical imaging) and properties of digital flatness are of a wide interest from theoretical perspective. On the other hand, connectivity is a principal topological characteristic, crucial for the deeper understanding the properties of a given class of objects and, possibly, for designing new more powerful visualization techniques.

Discrete plane connectivity, however, cannot be characterized by a condition as simple as the one above characterizing tunnel-freedom. To our knowledge of the available literature and according to our personal communications, this problem is still open, although within the last ten years or more, several researchers (including Reveillès, among others) have attempted to resolve it. So it becomes a challenge to achieve certain progress towards its solution.

In the present paper we describe our effort for resolving discrete plane connectivity problem in terms of the above analytical definition. In Section 2, we recall some notions and facts to be used in the sequel. In Section 3, we define connectivity number. In the subsequent sections we present our solution to the problem. Specifically, in Section 4, we investigate some useful combinatorial properties of discrete planes which we use further in the paper. These properties may be of interest in their own, since they help understand more deeply the structure of the discrete plane. In Section 5, we handle some important special cases. In Section 6, we provide an explicit solution to the problem under certain conditions. In Section 7, we obtain reachable upper and lower bounds for the connectivity number. In Section 8, we propose an algorithm for computing the connectivity number. The algorithm runs in time $O(a \log b)$ for a discrete plane with coefficients a, b, c, $a \le b \le c$. In the last Section 9, we conclude with some final remarks and open questions.

2 Preliminaries

In this section we introduce some notions and denotations and recall some well-known facts to be used in the sequel.

2.1 Basic notions of digital topology

Discrete coordinate plane consists of unit squares (pixels), centered on the integer points of the two-dimensional Cartesian coordinate system in the plane. *Discrete coordinate space* consists of unit cubes (voxels), centered on the integer points of the three-dimensional Cartesian coordinate system in the space. The edges of a pixel/voxel are parallel to the coordinate axes. The pixels/voxels are identified with the coordinates of their centers. Sometimes they are called *discrete points*, or simply *points*, for short. A set of discrete points is usually referred to as a *discrete object*.

A j-dimensional facet of a pixel/voxel will be called j*-facet*, for some j, $0 \leq j \leq n-1$ ($n = 2$ or 3). Thus the 0-facets of a voxel v are its vertices, the 1-facets are its edges, and the 2-facets of a 3D polytope are its 2D faces.

Two pixels/voxels are called j*-adjacent* if they share a j-facet. In this paper we will call two pixels *adjacent* if they are at least 0-adjacent. A k*-path* in a discrete object A is a sequence of pixels/voxels from A such that every two consecutive pixels/voxels are k-adjacent. Two pixels/voxels are k*-connected* if there is a k-path between them. A discrete object A is k*-connected* if there is a k-path connecting any two pixels/voxels of A. A discrete object is said to be *connected* if it is at least 0-connected. Otherwise it is *disconnected*.[1]

Let D be a subset of a discrete object A. If $A - D$ is not k-connected then the set D is said to be k*-separating* in A. (In particular, the empty set k-separates any set A which is not k-connected.) Let a set of pixels/voxels A be k-separating in a discrete object B but not j-separating in B. Then A is said to have j*-tunnels* for any $j < k$. A discrete object without any k-tunnels is called k*-tunnel-free*[2]. An object that has no tunnels for any k, $0 \leq k \leq 2$, is called *tunnel-free*, for short.

2.2 2D arrays and tilings

Let X be an *image* defined on Z^2, i.e., X is a mapping from Z^2 to a certain alphabet Σ. It can be considered as the discrete coordinate plane whose pixels are labeled by symbols from Σ. We will alternatively call X an *array on* Z^2 *over the alphabet* Σ. A *point* of X is a pair of integers (i, j) for a row i and a column j. An *element* of X at the point (i, j) is $X[i, j] \in \Sigma$. Two elements of an array are called *adjacent* if the corresponding points (pixels) are adjacent.

Let $s \subseteq Z^2$. Given an array X on Z^2, by $X[s]$ we denote the restriction of X to s. $X[s]$ is *connected* if s is connected. We will call $X[s]$ *factor of* X *on* s. In what follows, we will consider factors which are discretizations of lines, line segments, rectangles, or parallelograms. A rectangular factor of size $m \times n$ will be called an $m \times n$*-array*, or a *block*.

An $m \times n$*-array* A is *primitive* if setting $A = \begin{array}{|ccc|} \hline \mathrm{W} & \dots & \mathrm{W} \\ \dots & \dots & \dots \\ \mathrm{W} & \dots & \mathrm{W} \\ \hline \end{array}$, where A has k rows and l columns, implies $k = 1$, $l = 1$, i.e., A collapses to a single block W of size $m \times n$.

An array X on Z^2 is *tiled* by a *tile* W if $X = \begin{array}{|cccc|} \hline \dots & \dots & \dots & \dots \\ \dots & \mathrm{W} & \mathrm{W} & \dots \\ \dots & \mathrm{W} & \mathrm{W} & \dots \\ \dots & \dots & \dots & \dots \\ \hline \end{array}$ for some block W.

[1] Classically, 0-adjacent/connected (resp. 1-adjacent/connected) pixels are called 8-adjacent/connected (resp. 4-adjacent/connected). In dimension 3, 0-adjacent/connected (resp. 1 or 2-adjacent/connected) voxels are called 26-adjacent/connected (resp. 18 or 6-adjacent/connected).

[2] Classically, in dimension two, a 0-tunnel (resp. 1- tunnel) is called 8-tunnel (resp. 4-tunnel). In dimension three, a 0-tunnel (resp. 1- or 2-tunnel) is called 26-tunnel (resp. 18- or 6-tunnel).

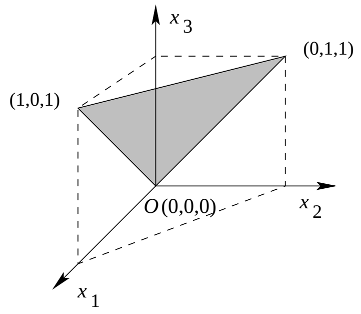


Figure 1: A plane forming an angle arctan $\sqrt{2}$ with the plane Ox_1x_2.

2.3 Discrete lines and planes

As mentioned in the Introduction, the classical definitions of a discrete line correspond to an arithmetic line $L(a, b, \mu, \max(|a|, |b|))$. Such a line is 0-connected and is the thinnest possible 1-tunnel-free arithmetic line. We have the following result.

Theorem 2.1 *(Lunnon and Pleasants 1991 [10]) All discretizations of straight lines with the same rational slope $\alpha \neq 0, \infty$ are equivalent up to a translation with an appropriate translation vector.*

In analytical terms, we have the following formulation.

Theorem 2.2 *All discrete lines $L(a, b, \mu, \omega)$ for $\mu = \pm 1, \pm 2, \ldots$ are equivalent up to a translation.*

An arithmetic line $L(a, b, \mu, |a| + |b|)$ is called *standard*. It is always 1-connected and 1-tunnel-free [11]. An arithmetic naive plane $P(a, b, c, \mu, \max(|a|, |b|, |c|))$ is always 1-connected, and is the thinnest possible 2-tunnel-free arithmetic plane (see, e.g., [4]). We have the following analog of Theorem 2.2.

Theorem 2.3 *All discrete planes $P(a, b, c, \mu, \omega)$ for $\mu = 0, \pm 1, \pm 2, \ldots$ are equivalent up to a translation with an appropriate translation vector.*

If not specified otherwise, we will assume throughout that the coefficients of a generic discrete plane $P(a, b, c)$ satisfy the conditions

$$0 < a < b < c \text{ and } gcd(a, b, c) = 1. \quad (1)$$

We will also suppose that the corresponding Euclidean plane P makes with the coordinate plane Oxy an angle θ with

$$0 \leq \theta \leq \arctan \sqrt{2}. \quad (2)$$

(See Figure 1.) Because of the well-known symmetry of the discrete space, the above conditions do not appear as restriction of the generality.

2.4 Jumps

In a discrete plane, configuration of two voxels such as the one in Figure 2, is called a *jump*. Jumps and related matters have been studied in [5]. Note that a discrete plane may have jumps even if it satisfies Condition (2). The occurrences of the jumps in the space is latticewise. We have the following fact [5].

Theorem 2.4 *A discrete plane $P(a, b, c)$ with $c = \max(a, b, c)$ contains jumps if and only if $c < a + b$.*

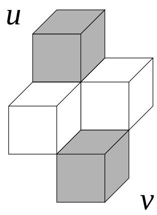


Figure 2: Possible configuration of voxels in a discrete plane satisfying Condition (2). The voxels u and v form a jump.

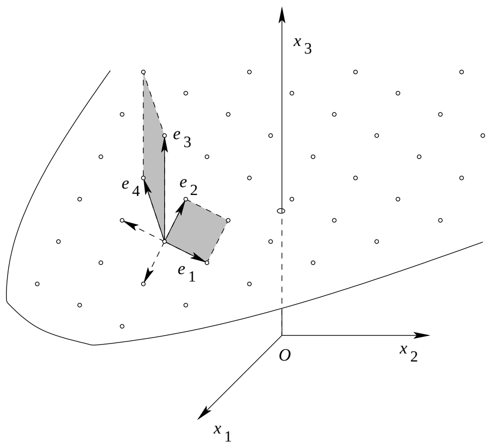


Figure 3: The 2D integer lattice Λ in the plane $\mathcal{P}$ and some of its bases.

2.5 Integer lattices and arithmetic planes

Consider the plane

$$\mathcal{P} : ax + by + cz = d,$$

where a, b, c, d are rational numbers. Without loss of generality suppose that a, b, c, d are integers. It is well-known that if $gcd(a, b, c)$ divides d, then $\mathcal{P}$ contains infinitely many integer points which form a 2-dimensional lattice $\Lambda(a, b, c)$ that is a sublattice of $\mathbf{Z}^3$ (see Figure 3). In other words, $\Lambda(a, b, c)$ has basis of two linearly independent integer vectors. Λ has different bases which feature different parallelogram partitions of $\mathcal{P}$ (see Figure 3). It is a well-known fact from lattice theory that the lattice cells have the same area for all possible bases. More precisely, assume without loss of generality that $gcd(a, b, c) = 1$. Then the area of a parallelogram corresponding to a basis cell is equal to $\max(a, b, c) = c$. Now consider the discrete planes $\mathcal{P}_i : ax + by + cz = d + i,\ 1 \leq i < \omega$. They are all parallel to $\mathcal{P}$. Therefore a generic plane $\mathcal{P}_i$ contains a lattice $\Lambda_i(a, b, c)$ which is equivalent to $\Lambda(a, b, c)$. An arithmetic plane $P(a, b, c, \omega)$ consists of the union of all voxels centered at the integer points of the lattices $\Lambda(a, b, c), \Lambda_1(a, b, c), \ldots, \Lambda_{\omega-1}(a, b, c)$. Note that the number of the involved lattices equals exactly the plane thickness ω.

3 Defining connectivity number

Since we are interested in topological properties of discrete planes and because of Theorem 2.3, we may consider without loss of generality discretizations of planes through the origin, i.e., of the form $\mathcal{P} : ax + by + cz = 0$. Then the corresponding discrete plane with a thickness ω will be denoted $P(a, b, c, \omega)$. If $P(a, b, c, \omega)$ is a naive plane, i.e., if $\omega = \max(|a|, |b|, |c|)$, then we will denote it $P(a, b, c)$, for short.

Now we give the following basic definition.

Definition 3.1 *Consider the function $\Omega : Z^3 \mapsto Z_+$ defined as follows:*

$$\Omega(a, b, c) = \max\{ \ \omega \ : \ \text{the discrete plane } P(a, b, c, \omega) \text{ is disconnected } \}. \tag{3}$$

Thus $\omega = \Omega(a, b, c) + 1$ is the least integer for which the discrete plane $P(a, b, c, \omega)$ is connected. For a particular choice of $a, b,$ and c, we call $\Omega(a, b, c)$ the **connectivity number** *relative to the class of discrete planes $\mathcal{C}(a, b, c) = \{ \ P(a, b, c, \omega) \ : \ \omega = 0, 1, 2, \dots \ \}$.*

Remark 3.1 *Since a (naive) discrete plane $P(a, b, c)$ is always connected, we have $\Omega(a, b, c) \leq c$. The thickness c of $P(a, b, c)$ determines the range for the connectivity number relative to the class $\mathcal{C}(a, b, c)$, and the number $\Omega(a, b, c)$ indicates when exactly a discrete plane $P(a, b, c)$ gets disconnected when it is "losing thickness." Therefore, sometimes $\Omega(a, b, c)$ will be refered to as* connectivity number of *$P(a, b, c)$, for short.*

Note also that the connectivity number is defined for arbitrary integer $a, b,$ and c, not necessarily satisfying conditions (1). Thus, for instance, we have $\Omega(a, b, c) = \Omega(b, a, c)$.

In order to get prepared for work on connectivity number computation, we need to study some structural properties of discrete planes.

4 Structural properties of discrete planes

In this section we investigate some structural properties of discrete planes. Sections 4.1 and 4.2 comprise some basic facts. Section 4.3 introduces an important lemma.

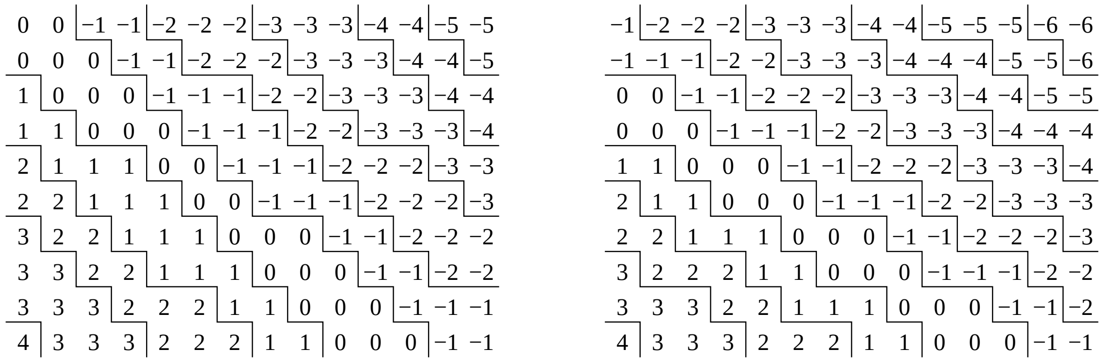
4.1 Level line code

Although the naive plane $P(a, b, c)$ is a 3D object, it admits a 2D representation by its *level lines.* They are determined by projecting voxels' z-coordinates on the plane Oxy. (See Figure 4a.) Because of Condition (2), each pixel of Oxy is a projection of exactly one voxel from $P(a, b, c)$. All level lines form the *level line code* of $P(a, b, c)$.

We have the following proposition.

Proposition 4.1 *Let a naive plane $P(a, b, c) : 0 \leq ax + by + cz < c$ be given.*

1. (a) *For a fixed value $z = z_0 \in Z$, the projection $P(a, b, c)_{z=z_0} : 0 \leq ax+by+cz_0 < c$ of $P(a, b, c)$ on Oxy is a discrete line $L_{z_0} = L(a, b, -cz_0, c)$. If $c = a + b$, then L is standard, and if $c > a + b$, then L is thicker than standard. If $c < a + b$, then L is thicker than naive and thinner than standard.*



a) b)

Figure 4: a) Level line code for $P(6,7,16)$. All level lines have identical shape since $gcd(6,7) = 1$. b) Level line code for $P(6,9,16)$. Different level lines may be different by shape since $gcd(6,9) = 3 \neq 1$.

(b) *As z runs over the integers, the lines L_0, $L_{\pm 1}, L_{\pm 2}, \ldots$ form a partition Π of the discrete plane Oxy. This partition defines an equivalence relation. Any equivalence class of this relation corresponds to a discrete line obtained for a certain particular value of z.*

(c) *If $gcd(a,b) = 1$, then all lines L_0, $L_{\pm 1}, L_{\pm 2}, \ldots$ are equivalent up to translation. Otherwise, let $gcd(a,b) = d \neq 1$. Then the partition Π features two different patterns (discrete lines). Any set of d consecutive discrete lines appear periodically in the partition. See for illustration Figure 4b. See also Figure 11 of Section 7.*

2. *For a fixed value $x = x_0 \in Z$, the projection $P(a,b,c)_{x=x_0} : 0 \leq ax_0 + by + cz < c$ of $P(a,b,c)$ on Oyz is a naive line $M_{x_0} = M(b, c, -ax_0, c)$. As x runs over the integers, the lines M_0, $M_{\pm 1}, M_{\pm 2}, \ldots$ form a partition of the discrete plane, and all lines of the partition are equivalent up to translation.*

3. *For a fixed value $y = y_0 \in Z$, the projection $P(a,b,c)_{y=y_0} : 0 \leq ax + by_0 + cz < c$ of $P(a,b,c)$ on Oxz is a naive line $N_{y_0} = M(a, c, -by_0, c)$. As y runs over the integers, the lines N_0, $N_{\pm 1}, N_{\pm 2}, \ldots$ form a partition of the discrete plane, and all lines of the partition are equivalent up to translation.*

We remark that the 2nd and 3rd statements of the above proposition are available in [7, 8]. A version of the first one is given as well, stating that for all values of z_0 the corresponding discrete lines are thick and equivalent to each other. We notice that such a claim is correct only if $gcd(a,b) = gcd(b,c) = gcd(a,c) = 1$, which is a restriction of the generality. We notice also that the obtained discrete lines may not be "thick" in the usual sense. If $c < a + b$, then a line may be thicker than a naive but thinner than standard, i.e., may have 0-tunnels.

4.2 Array of remainders

A discrete plane can also be represented by *array of remainders*, which is obtained as follows. Let $(x_0, y_0, z_0) \in P(a,b,c)$. We assign to the corresponding point $(x_0, y_0) \in Oxy$ the value $v(x_0, y_0) =$

9	15	5	11	1	7	13	3	9	15	5	11	1	7
2	8	14	4	10	0	6	12	2	8	14	4	10	0
11	1	7	13	3	9	15	5	11	1	7	13	3	9
4	10	0	6	12	2	8	14	4	10	0	6	12	2
13	3	9	15	5	11	1	7	13	3	9	15	5	11
6	12	2	8	14	4	10	0	6	12	2	8	14	4
15	5	11	1	7	13	3	9	15	5	11	1	7	13
8	14	4	10	0	6	12	2	8	14	4	10	0	6
1	7	13	3	9	15	5	11	1	7	13	3	9	15
10	0	6	12	2	8	14	4	10	0	6	12	2	8

a)

11	1	7	13	3	9	15	5	11	1	7	13	3	9
2	8	14	4	10	0	6	12	2	8	14	4	10	0
9	15	5	11	1	7	13	3	9	15	5	11	1	7
0	6	12	2	8	14	4	10	0	6	12	2	8	14
7	13	3	9	15	5	11	1	7	13	3	9	15	5
14	4	10	0	6	12	2	8	14	4	10	0	6	12
5	11	1	7	13	3	9	15	5	11	1	7	13	3
12	2	8	14	4	10	0	6	12	2	8	14	4	10
3	9	15	5	11	1	7	13	3	9	15	5	11	1
10	0	6	12	2	8	14	4	10	0	6	12	2	8

b)

Figure 5: a) Arrays of remainders $A(6,7,16)$ for $P(6,7,16)$. Since $gcd(6,7) = 1$, all equivalence classes contain numbers in the whole range from 0 to 15. b) Arrays of remainders $A(6,9,16)$ for $P(6,9,16)$. Since $gcd(6,9) = 3$, all numbers in a particular equivalence class of $A(6,9,16)$ are the same modulo 3 and fall within the range $[0,15]$. Classes with remainders 0, 1, and 2 appear consecutively. Lines corresponding to remainders 1 and 2 modulo 3 are equivalent to each other, but different from the line corresponding to remainder 0 modulo 3. These figures illustrate also the fundamental Lemma 4.4 (the Symmetry Lemma) from Section 4.3. The arrays $A(6,7,16)$ and $A(6,9,16)$ are symmetric to each other with respect to the second coefficient. They contain identical rows whose rowwise equivalence classes are the same. Vertically, the two arrays are symmetric with respect to any horizontal row.

$r(x_0,y_0,z_0) = ax_0 + by_0 + cz_0$. Thus we obtain an array of remainders modulo c, whose values are determined by the linear function $ax + by + cz$.

Any $r(x_0,y_0,z_0)$ admits the following interpretation. By definition, the points of a discrete plane $P(a,b,c)$ belong to c different Euclidean planes $P_0 : ax + by + cz = 0$, $P_1 : ax + by + cz = 1$, ..., $P_c : ax + by + cz = c$. Then $r(x_0,y_0,z_0)$ is the index i of the plane $P_i : ax + by + cz = i$ to which the voxel (x_0,y_0,z_0) belongs. The obtained array on Z^2 will be denoted $A(a,b,c)$ and called *array of remainders for* $P(a,b,c)$. Arrays of remainders have been considered by Debled [7].

Regarding the level line code consider above, the remainders corresponding to any level line are numbers in the range $[0, c-1]$. See Figure 5a. The next lemma follows directly from the definition of array of remainders. It will be used repeatedly in the sequel.

Lemma 4.1 *Let $P(a,b,c) : 0 \le ax + by + cz < c$ be a naive plane. Consider the array of remainders $A(a,b,c)$ for $P(a,b,c)$. Let (x,y) be a point of $A(a,b,c)$ with value s, $0 \le s \le c-1$.*

1. *Let the points (x,y) and (x',y') belong to the same equivalence class (discrete line) of the partition Π, where (x',y') labels one of the following points: $(x,y+1),(x,y-1),(x-1,y),(x+1,y),(x+1,y+1),(x+1,y-1),(x-1,y+1)$, or $(x-1,y-1)$. Then the value of (x',y') is respectively, $s+b, s-b, s-a, s+a, s+b+a, s-b+a, s+b-a$, or $s-b-a$.*

2. *Let now the points (x,y) and (x'',y'') belong to different equivalence classes (discrete lines) of the partition Π, where (x'',y'') labels one of the points: $(x,y+1),(x,y-1),(x-1,y),(x+1,y),(x+1,y+1),(x+1,y-1),(x-1,y+1)$, or $(x-1,y-1)$. Then the value of (x'',y'') is respectively, $s+b-c, s-b+c, s-a+c, s+a-c, s+b+a-c, s-b+a+c, s+b-a-c$, or $s-b-a+c$.*

Now we can state the following proposition.

Proposition 4.2 *Let $P(a,b,c) : 0 \leq ax+by+cz < c$ be a naive plane. Consider the array of remainders $A(a,b,c)$.*

1. *Let $gcd(a,b) = 1$. According to Proposition 4.1, all discrete lines of the partition Π are equivalent. Each of them involves the numbers $0,1,2,\ldots,c-1$. See Figure 5a.*

2. *Let $gcd(a,b) = d \neq 1$. According to Proposition 4.1, the partition Π features two different discrete line patterns, as any d consecutive discrete lines $D_0, D_1, \ldots, D_{d-1}$ appear periodically in the partition.*

   Moreover, for some permutation $(i_0, i_1, \ldots, i_{d-1})$ of the indexes $0,1,\ldots,d-1$, for any $k : 0 \leq k \leq d-1$, the line $D_{i_k}, 0 \leq k \leq d-1$ involves only integers in the range $[0, c-1]$ equal to k modulo d. See Figure 5b.

Sketch of Proof Only the last statement of the theorem needs proof. Consider first the discrete line D_0 containing an element 0 (i.e., a point $p_0 \in A(a,b,c)$ with a value $v(p_0) = 0$). Since $gcd(a,b) = d$, Part 1 of Lemma 4.1 implies that all other numbers appearing in D_0 will be multiples of $d = gcd(a,b)$. Now let us move rightward starting from the point p_0, until we reach the rightmost point p'_0 of D_0 in the same row (i.e., just the last one before the vertical wall of the border between D_0 and its neighbor to the right). As mentioned, $v(p'_0)$ is multiple of d, i.e., $v(p'_0) = k_0 d$ for some nonnegative integer k, as are all other elements between p_0 and p'_0.

The point p_1 to the right of p'_0 belongs to the neighboring class D_1. By Part 2 of Lemma 4.1, we have $v(p_1) = v(p'_0) + r = k_0 d + r$, where $r = a - c$. Clearly, $k_0 d + r$ is different than 0 modulo d.

Let us keep moving to the right until reaching the rightmost point $p'_1 \in D_1$. By Part 1 of 4.1, we have that $v(p'_1) = k_1 d + r$ for certain nonnegative integer k_1. (Clearly, all points p between p_1 and p'_1 satisfy $v(p) \equiv v(p_1)(mod\ d)$.) Then by Part 2 of Lemma 4.1, we get $v(p_2) = k_1 d + 2r$.

Continuing this process, we obtain that the elements of the consecutive lines $D_1, D_2, \ldots, D_{d-1}$ contain elements of the form $k_0 d, k_1 d + r, k_2 + 2r, \ldots, k_{d-1} d + (d-1)r$, respectively. What remains to show is that all such kind of values must be different modulo d.

Assume the opposite, i.e., that for some $i, j : 0 \leq i < j \leq d-1$, we have $k_i d + ir \equiv k_j d + jr(mod\ d)$. It is easy to see that this last equivalenve holds if and only if $ir \equiv jr(mod\ d)$ iff the number $ir - jr = (i-j)r$ is divisible by d. Since $gcd(a,b) = d$ and $gcd(a,b,c) = 1$, we have that $r = a - c$ is not divisible by d. Then $(i-j)r$ is divisible by d iff $i - j$ is divisible by d. Since d cannot be a divisor of $i - j$, we have reached a contradiction. □

Remark 4.1 *The equivalent discrete lines (i.e., those containing the same values) form an equivalence class. Thus we have $gcd(a,b)$ equivalence classes overall. Note that the equivalence relation defined on $A(a,b,c)$ is different from the one defined through level line code.*

In order to state the next proposition, we need a simple lemma.

Lemma 4.2 *If $gcd(a,b,c) = 1$, then $gcd(a,c).gcd(b,c)$ is a divisor of c.*

Proof Denote $d_1 = gcd(a,c)$, $d_2 = gcd(b,c)$. The condition $gcd(a,b,c) = 1$ implies $gcd(d_1,d_2) = 1$ (For if, let $gcd(d_1,d_2) = d \neq 1$. Then d is a divisor of a, b, and c - a contradiction.) We have that d_1 and d_2 divide c. Then $d_1 d_2$ divides c as well. □

1	27	39	1	13	25	37	49	11	23	35	47	9	21	33	45	7	19	31	43	5	17	29	41	3	15
0	**12**	**24**	**36**	**48**	**10**	**22**	**34**	**46**	**8**	**20**	**32**	**44**	**6**	**18**	**30**	**42**	**4**	**16**	**28**	**40**	**2**	**14**	**26**	**38**	**0**
1	47	9	21	33	45	7	19	31	43	5	17	29	41	3	15	27	39	1	13	25	37	49	11	23	35
0	**32**	**44**	**6**	**18**	**30**	**42**	**4**	**16**	**28**	**40**	**2**	**14**	**26**	**38**	**0**	**12**	**24**	**36**	**48**	**10**	**22**	**34**	**46**	**8**	**20**
1	17	29	41	3	15	27	39	1	13	25	37	49	11	23	35	47	9	21	33	45	7	19	31	43	5
0	**2**	**14**	**26**	**38**	**0**	**12**	**24**	**36**	**48**	**10**	**22**	**34**	**46**	**8**	**20**	**32**	**44**	**6**	**18**	**30**	**42**	**4**	**16**	**28**	**40**
1	37	49	11	23	35	47	9	21	33	45	7	19	31	43	5	17	29	41	3	15	27	39	1	13	25
0	**22**	**34**	**46**	**8**	**20**	**32**	**44**	**6**	**18**	**30**	**42**	**4**	**16**	**28**	**40**	**2**	**14**	**26**	**38**	**0**	**12**	**24**	**36**	**48**	**10**
1	7	19	31	43	5	17	29	41	3	15	27	39	1	13	25	37	49	11	23	35	47	9	21	33	45
0	**42**	**4**	**16**	**28**	**40**	**2**	**14**	**26**	**38**	**0**	**12**	**24**	**36**	**48**	**10**	**22**	**34**	**46**	**8**	**20**	**32**	**44**	**6**	**18**	**30**
	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0

Figure 6: Illustration to Proposition 4.3. Structure of a primitive tile of the array $A(12, 15, 50)$. All numbers are in the range $[0, 49]$. The rows are alternately composed by odd and even numbers (marked by 1 and 0, respectively). The columns are alternately composed by remainders modulo 5 (marked by 0,1,2,3, and 4, respectively).

Proposition 4.3 *Let $P(a, b, c) : 0 \leq ax + by + cz < c$ be a naive plane. Consider the array of remainders $A(a, b, c)$ for $P(a, b, c)$.*

1. *Let $gcd(a, c) = gcd(b, c) = 1$. Then $A(a, b, c)$ is tiled by a primitive tile T of size $c \times c$, as every row or column of T is a permutation of the numbers $0, 1, \ldots, c - 1$.*

2. *Let $gcd(a, b) \neq 1$, $gcd(a, c) \neq 1$. Then $A(a, b, c)$ is tiled by a primitive tile T of size $\frac{c}{gcd(a,c)} \times \frac{c}{gcd(b,c)}$. Moreover, the tile T has the following structure.*

   (a) *Let $k = \frac{c}{gcd(a,c).gcd(b,c)}$. According to Lemma 4.2, k is integer. Then T can be vertically partitioned into k subarrays $T_1, T_2, \ldots, T_k$, each containing $gcd(b, c)$ columns. Every block $T_j, 1 \leq j \leq k$, contains $gcd(b, c)$ columns, while every column contains $\frac{c}{gcd(b,c)}$ numbers, i.e., c elements overall. For a particular column of T_j, all its entries have the same remainder modulo $gcd(b, c)$, and all columns of T_j cover all possible remainders.*

   (b) *Analogously, T can be horizontally partitioned into k subarrays $T'_1, T'_2, \ldots, T'_k$, each containing $gcd(a, c)$ columns. Every block $T'_j, 1 \leq j \leq k$, contains $gcd(a, c)$ columns, while every column contains $\frac{c}{gcd(a,c)}$ numbers, i.e., c elements overall. For a particular column of T'_j, all its entries have the same remainder modulo $gcd(a, c)$, and all columns of T'_j cover all possible remainders.*

   (c) *If T is partitioned both horizontally and vertically, then T gets partitioned into k^2 blocks, each of size $gcd(a, c).gcd(b, c)$. Each such a block appears k times in T, exactly one time in every row and every column of the partition.*

See for illustration Figure 6.

Proof Follows from Lemma 4.1 and number-theoretical arguments similar to those used in the proof of Proposition 4.2. $\square$

We will study a discrete plane connectivity by examining the connectivity of its array of remainders. Note however that two points from the level line code may be connected while the corresponding voxels are disconnected. Specifically, if a discrete plane contains jumps, then connectivity of a set of

−1	−2	−2	−3	−3	−3	−4	−4	−5	−5	−6	−6	−7	−7
−1	−1	−1	−2	−2	−3	−3	−4	−4	−5	−5	−6	−6	−6
0	0	−1	−1	−2	−2	−3	−3	−4	−4	−4	−5	−5	−6
1	0	0	−1	−1	−2	−2	−2	−3	−3	−4	−4	−5	−5
1	1	0	0	0	−1	−1	−2	−2	−3	−3	−4	−4	−5
2	2	1	1	0	0	−1	−1	−2	−2	−3	−3	−3	−4
3	2	2	1	1	0	0	−1	−1	−1	−2	−2	−3	−3
3	3	2	2	1	1	1	0	0	−1	−1	−2	−2	−3
4	3	3	3	2	2	1	1	0	0	−1	−1	−2	−2
5	4	4	3	3	2	2	1	1	0	0	0	−1	−1

Figure 7: Connected subset (in gray) of the level code of a discrete plane $P(5,7,11)$ corresponds to a disconnected set of voxels of $P(5,7,11)$, since $a+b = 5+7 = 12 > c = 11$.

voxels' projections over Oxy does not imply connectivity of the set of voxels itself (see Figure 7). In the next section we will show how one can legally substitute the original 3D problem about discrete plane connectivity with a 2D one about connectivity of the corresponding array of remainders.

4.3 Symmetry Lemma

We start with a simple fact.

Lemma 4.3 *Consider the discrete planes* $P(a,b,c) : 0 \leq ax + by + cz < c$ *and* $P(a, c-b, c) : 0 \leq ax + (c-b)y + cz < c$ *and their arrays of remainders* $A(a,b,c)$ *and* $A(a,c-b,c)$*, respectively. Let* (x,y) *be a point of* $A(a,b,c)$ *with a value* $v(x,y) = s$*, and* (x', y') *a point of* $A(a, c-b, c)$ *with the same value* $v(x', y') = s$*. Then* (x,y) *and* $(x, y \pm 1)$ *belong to the same equivalence class of* $A(a,b,c)$ *if and only if* (x', y') *and* $(x', y' \mp 1)$ *belong to different equivalence classes of* $A(a, c-b, c)$*.*

Proof Let (x,y) and $(x, y+1)$ belong to the same equivalence class of $A(a,b,c)$. By Lemma 4.1, $v(x, y+1) = s + b < c$, which is equivalent to $v(x', y'-1) = s - (c-b) < 0$, i.e., (x', y') and $(x', y' \mp 1)$ belong to different equivalence classes of $A(a, c-b, c)$.

Analogously, if (x,y) and $(x, y-1)$ belong to the same equivalence class of $A(a,b,c)$, then $v(x, y-1) = s - b > 0$, which is equivalent to $v(x', y'+1) = s + (c-b) > c$, i.e., (x', y') and $(x', y' \pm 1)$ belong to different equivalence classes of $A(a, c-b, c)$. $\square$

Remark 4.2 *A "horizontal" version of the above statement holds for the discrete planes* $P(a,b,c)$ *and* $P(c-a, b, c)$*. Both imply an analogous statement (in "horizontal"* and *"vertical" dimensions) for the planes* $P(a,b,c)$ *and* $P(c-a, c-b, c)$*.*

We now prove the following fundamental lemma.

Lemma 4.4 *(Symmetry Lemma)* $A(a,b,c) = A(c-a,b,c) = A(a,c-b,c) = A(c-a,c-b,c)$*.*

Proof

1. First we will show that $A(a, b, c) = A(a, c - b, c)$.

   Since $P(a, b, c)$ and $P(a, c - b, c)$ have the same first and third coefficients, it follows that the corresponding arrays $A(a, b, c)$ and $A(a, c - b, c)$ are composed by the same set of rows. We will show that both arrays are, in fact, identical.

   Let (x, y) be an arbitrary point of $A(a, b, c)$ with value $v(x, y) = s$. Consider the point $(x, y+1)$. Assume that (x, y) and $(x, y+1)$ belong to the same equivalence class in $A(a, b, c)$ (i.e., $s+b < c$). Then by Lemma 4.1, $v(x, y+1) = s+b$. Let (x', y') be a point of $A(a, c-b, c)$ with the same value $v(x', y') = s$ as (x, y). Then by Lemma 4.3, (x', y') and $(x', y' - 1)$ belong to different equivalent classes of $A(a, c - b, c)$. Then, again by Lemma 4.1, $v(x', y' - 1) = s - (c - b) + c = s + b$, i.e., the same as the value $v(x, y + 1)$.

   Similarly, consider the point $(x, y - 1)$, and assume that (x, y) and $(x, y - 1)$ belong to the same equivalence class in $A(a, b, c)$ (i.e., $s - b > 0$). We have $v(x, y - 1) = s - b$. Then (x', y') and $(x', y'+1)$ belong to different equicalent classes of $A(a, c-b, c)$, and $v(x', y'+1) = s+(c-b)-c = s - b$, which is the same as the value $v(x, y - 1)$.

   Now let (x, y) and $(x, y + 1)$ belong to different equivalence classes of $A(a, b, c)$ (i.e., $s + b > c$). Then by Lemma 4.1, $v(x, y+1) = s+b-c$. If (x', y') is a point of $A(a, c-b, c)$ with $v(x', y') = s$, then Lemma 4.3 implies that (x', y') and $(x', y' - 1)$ belong to the same equicalent class of $A(a, c - b, c)$. Then by Lemma 4.1, $v(x', y' + 1) = s - (c - b) + c = s + b - c$, which is the same as the value $v(x, y + 1)$.

   Similarly, consider the point $(x, y - 1)$, and assume that (x, y) and $(x, y - 1)$ belong to different equivalence classes of $A(a, b, c)$ (i.e., $s - b < 0$). We have $v(x, y - 1) = s - b + c$. Then (x', y') and $(x', y' + 1)$ belong to the same equivalence class of $A(a, c - b, c)$, and $v(x', y' + 1) = s + (c - b) = s - b + c$, that is, the same as the value $v(x, y - 1)$.

   Thus we can conclude that the array $A(a, c - b, c)$ can be obtained from the array $A(a, b, c)$ by symmetry w.r.t. an arbitrary row of $A(a, b, c)$. Hence, the arrays $A(a, b, c)$ and $A(a, c - b, c)$ are equivalent. See Figure 5.

2. In an analogous way it follows that $A(c - a, b, c) = A(a, b, c)$.

3. From Part 1 we have that $A(a, c - b, c) = A(a, b, c)$. Applying Part 2 to $A(a, c - b, c)$, we get $A(c - a, c - b, c) = A(a, c - b, c)$. Hence, $A(a, b, c) = A(c - a, c - b, c)$. □

Remark 4.3 *We remark that the Euclidean planes $ax + by + cz = 0$, $(c - a)x + by + cz = 0$, $ax + (c - b)y + cz = 0$, and $(c - a)x + (c - b)y + cz = 0$ are, in general, quite different. Nevertheless, in view of Lemma 4.4, we will call the corresponding discrete planes $P(a, b, c)$, $P(c - a, b, c)$, $P(a, c - b, c)$, and $P(c - a, c - b, c)$* symmetric *to each other.*

If one or both coefficients a and b are larger than $c/2$, then one can consider an appropriate symmetric plane $P(a, c - b, c)$ or $P(c - a, c - b, c)$ for which the first two coefficients do not exceed $c/2$.

As already mentioned, connectivity of a level line code does not directly imply connectivity of the corresponding discrete plane $P(a, b, c)$, since the latter may contain jumps. By Lemma 2.4, this is possible if and only if $a + b > c$. If this last inequality holds, one can consider the discrete plane $P(c - a, c - b, c)$. For $a + b > c$, we have $(c - a) + (c - b) < c$, i.e., $P(c - a, c - b, c)$ does not have jumps.

By the Symmetry Lemma, $A(a,b,c) = A(c-a, c-b, c)$. Thus possible voxel disconnectedness of a subset of the level line code does not imply disconnectedness of the corresponding array of remainders. Hence, without loss of generality we may assume that the considered discrete plane is jump-free and thus legitimately use its array of remainders in the connectivity test. With this in mind, we can state the following corollary.

Corollary 4.1 *(Symmetry Lemma II)*
$\Omega(a,b,c) = \Omega(c-a,b,c) = \Omega(a,c-b,c) = \Omega(c-a,c-b,c)$.

5 Special cases. Connectivity number of Graceful planes

Explicit formulas for $\Omega(a,b,c)$ in some important special cases are summarized in the following proposition.

Proposition 5.1

1. $\Omega(a,a,a) = 0$;
2. $\Omega(0,b,c) = c-1$;
3. $\Omega(a,b,b) = b-1$;
4. $\Omega(a,a,c) = c-a-1$;
5. $\Omega(a,b,2b) = b-1$.

Proof

1. Follows from the equality $\Omega(a,a,a) = \Omega(1,1,1) = 0$;
2. Since one of the coefficients is zero, the problem is essentially two- dimensional, and the equality follows from the well-known result about discrete line connectivity (see the Introduction).
3. Symmetry Lemma II implies $\Omega(a,b,b) = \Omega(a,0,b)$. Then the claim follows from statement 2.
4. Since $a = b$, the array of remainders $A(a,a,c)$ is built by adjacent "diagonals," each of them composed by replicas of the same number. (See Figure 8a). Thus the points of some diagonals will be labeled by 0, others by 1, and so on, up to diagonals labeled by $c-1$. Clearly, as ω gets smaller than c, first the points from the diagonals containing $(c-1)$'s will be dropped, then those from diagonals containing $(c-2)$'s, etc. One can easily realize that $A(a,b,c)$ will become disconnected as soon as two adjacent diagonals vanish. Note that such two diagonals form a "diagonal" standard line.

   Consider a horizontal sequence S of c elements, starting from a point with value 0 and ending to the right with the element before the next 0 in the same horizontal row. Clearly, a diagonal standard line which vanishes first is determined by two consecutive elements of S, that vanish *as a couple* before any other couple of elements.

   Denote the value of a point $x \in S$ by $v(x)$. We will show that the first couple of points $\{x-1, x\}$ that vanishes is the one for which $v(x-1) = c-a-1$ and $v(x) = c-1$. If $a = b = 1$, the statement is obvious since $v(x-1) = c-2$ and $v(x) = c-1$. (see Figure 8b). Thus we will assume that $a = b \geq 2$.

0	5	10	3	8	1	6	11	4	9	2	7	0	5
7	0	5	10	3	8	1	6	11	4	9	2	7	0
2	7	0	5	10	3	8	1	6	11	4	9	2	7
9	2	7	0	5	10	3	8	1	6	11	4	9	2
4	9	2	7	0	5	10	3	8	1	6	11	4	9
11	4	9	2	7	0	5	10	3	8	1	6	11	4
6	11	4	9	2	7	0	5	10	3	8	1	6	11
1	6	11	4	9	2	7	0	5	10	3	8	1	6
8	1	6	11	4	9	2	7	0	5	10	3	8	1
3	8	1	6	11	4	9	2	7	0	5	10	3	8

a)

0	1	2	3	4	5	0	1	2	3	4	5	0	1
5	0	1	2	3	4	5	0	1	2	3	4	5	0
4	5	0	1	2	3	4	5	0	1	2	3	4	5
3	4	5	0	1	2	3	4	5	0	1	2	3	4
2	3	4	5	0	1	2	3	4	5	0	1	2	3
1	2	3	4	5	0	1	2	3	4	5	0	1	2
0	1	2	3	4	5	0	1	2	3	4	5	0	1
5	0	1	2	3	4	5	0	1	2	3	4	5	0
4	5	0	1	2	3	4	5	0	1	2	3	4	5
3	4	5	0	1	2	3	4	5	0	1	2	3	4

b)

Figure 8: Illustration to the proof of Proposition 5.1d. The gray traces exhibit where a disconnection first occurs. a) Array $A(5, 5, 12)$. b) Array $A(1, 1, 6)$.

The value $c-1$ is the maximal possible. Since $a \geq 2$, the points of $A(a, b, c)$ corresponding to $x-1$ and x belong to the same equivalence class. Then $v(x-1) = c-a-1$ follows from the first part of Lemma 4.1.

We will show now that for any other couple of points $\{y-1, y\}$ we have $\min(v(y-1), v(y)) < c-a-1$. If the points in $A(a, b, c)$ corresponding to $y-1$ and y belong to the same equivalence class, then according to Part 1 of Lemma 4.1, $v(y-1) = v(y) - a < c-2-a < v(x-1)$.

Let now the points corresponding to $y-1$ and y belong to different equivalence classes. Then $v(y-1) = v(y) + a - c < (c-1) + a - c = a - 1$. Since $a + b = 2a < c$, we have $a < c/2$. Under this condition, $v(y-1) = a - 1 < v(x-1) = c - a - 1$, which completes the proof.

5. This case can be handled similarly to the previous one and is left for an exercise. $\square$

Statement 4 of Proposition 5.1 implies an explicit solution for an important class of discrete planes, called graceful. Specifically, a *graceful plane* is a discrete plane $P(a, b, \omega)$ with $c = a + b$. The graceful planes have been introduced in [5] and used for designing thin tunnel-free discretizations of polyhedral surfaces. We have the following immediate fact.

Corollary 5.1 *A graceful plane $P(a, b, a+b)$ has a connectivity number $\Omega(a, b, a+b) = b - 1$.*

Proof By Symmetry Lemma II, $\Omega(a, b, a+b) = \Omega(b, b, a+b) = \Omega(a, a, a+b)$. Then the claim follows from statement 4 of Proposition 5.1. $\square$

6 On explicit solution

6.1 Subsidiary constructions

Let A be a 2D array (finite or infinite) and $p = (x_0, y_0)$, $q = (x_m, y_m)$ two points of A. Let, for definiteness, $x_0 \leq x_m$ and $y_0 \leq y_m$. We call the sequence of points $P = \langle (x_0, y_0) = p, (x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m) = q \rangle$ a *stairwise path* between p and q if the coordinates of two consecutive points (x_i, y_i) and (x_{i+1}, y_{i+1}),

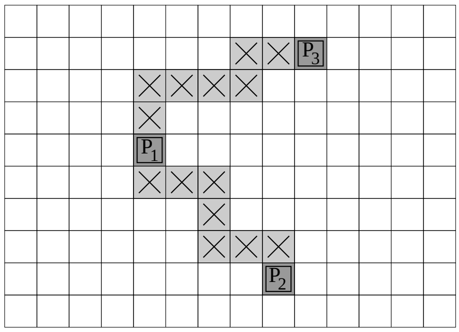


Figure 9: Two stairwise paths marked by shadowed $\times$ sign: one between the points P_1 and P_2, and another between the points P_1 and P_3.

$0 \leq i \leq m-1$, satisfy either $x_{i+1} = x_i, y_{i+1} = y_i + 1$, or $x_{i+1} = x_i + 1, y_{i+1} = y_i$ (see Figure 9). The number m is the *length* of the path.

For all other possible mutual locations of p and q, a stairwise path is defined similarly. For instance, if $x_0 \leq x_m$ and $y_0 \geq y_m$, the condition is either $x_{i+1} = x_i, y_{i+1} = y_i - 1$, or $x_{i+1} = x_i + 1, y_{i+1} = y_i$ (see Figure 9).

Consider now the array of remainders $A(a,b,c)$ together with its equivalence classes described in Proposition 4.2. The points of $A(a,b,c)$ which contain the value $\Omega(a,b,c)$ will be called the *plugs* of $A(a,b,c)$. The points containing the maximal possible value $c-1$ will be called the *maximal points* of $A(a,b,c)$.

Assume for a moment that c is "enough large" compared to a and b. More precisely, we will suppose that $c \geq a+2b$. Then the discrete lines corresponding to the equivalence classes are thicker than standard. In particular, if $c = a+2b = (a+b)+b$, then a particular equivalence class C will be a disjoint union of one standard and one naive line. Note that in this case there are two different possible partitions of this kind: one can consider the standard line to be above the naive, and vice versa. In the first case we will call the standard line *upper standard line* for the class C, while in the second case we will call it *lower standard line* for C.

Any discrete line $S = S(a,b,\omega = a+b,\mu)$ with $0 < a < b$ is composed by *strips*, where a strip is any horizontal sequence of points. Let a particular strip be composed by the points $P_1, P_2, \ldots, P_k$, read from left to the right. Then P_1 is a *leftmost point* and P_k is a *rightmost point* of S.

Similarly, if $c > a+2b$, then C can be partitioned in two different fashions into disjoint union of one standard line and another line which is thicker than naive. Consider then a class C which contains maximal points of $A(a,b,c)$, where $c \geq a+2b$. We have $C = S \cup L$, where S is the standard line containing maximal points of $A(a,b,c)$, and L is a discrete line that is naive or thicker than naive. A point $P \in S$ with a minimal value will be called a *core of the class C* (see Figure 10).

Keeping in mind the properties of $A(a,b,c)$ from Section 4.2, we can state the following lemma.

Lemma 6.1 *Let P_1 and P_2 be two consecutive maximal points belonging to an equivalence class C. Let $S \subseteq C$ be the standard line containing P_1 and P_2, and $\bar{S}(P_1,P_2) \subset S$ the stairwise path between P_1 and P_2. Then all points of S have different values.*

We are now able to obtain a number of results presented in the next section.

25	5	12	19	**26**	6	13	20	0	7	14	21	1	8
15	22	2	9	16	23	3	10	17	24	4	11	18	25
5	12	19	**26**	6	13	20	0	7	14	21	1	8	15
22	2	9	16	23	3	10	17	24	4	11	18	25	5
12	19	**26**	6	13	20	0	7	14	21	1	8	15	22
2	9	16	23	3	10	17	24	4	11	18	25	5	12
19	**26**	6	13	20	0	7	14	21	1	8	15	22	2
9	16	23	3	10	17	24	4	11	18	25	5	12	19
26	6	13	20	0	7	14	21	1	8	15	22	2	9
16	23	3	10	17	24	4	11	18	25	5	12	19	**26**
6	13	20	0	7	14	21	1	8	15	22	2	9	16
23	3	10	17	24	4	11	18	25	5	12	19	**26**	6
13	20	0	7	14	21	1	8	15	22	2	9	16	23
3	10	17	24	4	11	18	25	5	12	19	**26**	6	13

Figure 10: A stairwise path between two maximal points of value 26 in array $A(7, 10, 27)$. The path (in dark gray) is a part of an upper standard line (in gray) through the two maximal points. The core of the class has value 10. It coincides with a plug of $A(7, 10, 27)$. A core is marked by $\bigcirc$ and a plug by $\diamond$.

6.2 Main results

In this section we provide an explicit solution for a broad classes of instances. We have the following theorem.

Theorem 6.1 *Let $c \geq a + 2b$. Then*

$$\Omega(a, b, c) = c - a - b + gcd(a, b) - 1. \tag{4}$$

Proof Let the points $P_1, P_2 \in C$, the standard line S, and the stairwise path $\bar{S}(P_1, P_2)$ be as in Lemma 6.1. This last lemma implies that $\bar{S}$ contains a unique core of C.

Clearly, when ω decreases starting from $c - 1$ and going downwards, first the points from the standard line S will vanish from $A(a, b, c)$. Consider first what happens when $c = a + 2b$. As already discussed in Section 6.1, the complement of S to C is a naive line L which is "below" S. Moreover, the mutual location of S and L within the class C implies the following property: The 4-neighbors of any pixel from S are points which belong either to S or to L. See Figure 10. Therefore, if the points from S are removed from C, all points from the naive line L will get disconnected from the points of the next equivalence class "above" C. Obviously, this will also hold when $c > a + 2b$.

All equivalence classes are discrete lines and therefore are periodic. The period length of a class is equal to $a + b$ which is the length of the path between two consecutive maximal points of C. Therefore, the disconnectedness considered above propagates along all the class C. On the other hand, as we have seen in Section 4.3, the array of remainders $A(a, b, c)$ is periodic. More precisely, the class C appears periodically, in a way that if we start counting from it, every $gcd(a, b)$th class is equivalent to C. Thus we obtain that if $c \geq a + 2b$, the array $A(a, b, c)$ gets disconnected, if the points of the standard line S are removed from it.

What remains to show is that $\Omega(a, b, c) = c - a - b + gcd(a, b) - 1$. Clearly, the value of $\Omega(a, b, c)$ is equal to the value of a core of a class C that contains maximal values. In other words, we have that the set of plugs of $A(a,b,c)$ and the set of the cores of all classes containing maximal elements, coincide. If $gcd(a,b)=1$, then $\Omega(a,b,c)=c-a-b=c-a-b+gcd(a,b)-1$, since $A(a,b,c)$ becomes disconnected when points with values $c-1,c-2,\ldots,c-a-b$ are removed from it.

Now let $gcd(a,b)=d\neq 1$. Consider again the points in a stairwise path $\bar{S}(P_1,P_2)$ between two consecutive maximal points in a class C. We have that $\bar{S}$ contains $\frac{a+b}{gcd(a,b)}$ points whith values $c-1, c-1-\gcd(a,b), c-1-2\gcd(a,b),\ldots,f$, where the last value f is equal to

$$c-1-\left(\frac{a+b}{gcd(a,b)}-1\right)gcd(a,b)=c-a-b+gcd(a,b)-1.$$

See Figure 11. Thus we obtained that if $c\geq a+2b$, then $\Omega(a,b,c)=c-a-b+gcd(a,b)-1$. □

Combining the above theorem and the Symmetry Lemma II, one can extend the class of problems for which an explicit solution is available. The following proposition provides such a solution when the condition of Theorem 6.1 does not hold. As before, w.l.o.g. we assume that $c>a+b$.

Proposition 6.1 *Let $c<2b-a$. Then $\Omega(a,b,c)=b-a+gcd(a,c-b)-1$.*

Proof Consider an instance with coefficients $a,c-b,c$. Since $a+b<c$, we have $a<c-b<c$. By Symmetry Lemma II, we have $\Omega(a,c-b,c)=\Omega(a,b,c)$. We also have $c<2b-a \iff a+2(c-b)<c$. Hence the instance $(a,c-b,c)$ satisfies the condition of Theorem 6.1. Then we obtain $\Omega(a,b,c)=\Omega(a,c-b,c)=c-a-(c-b)+gcd(a,c-b)-1=b-a+gcd(a,c-b)-1$. □

Note that because of the symmetry properties of the arrays of remainders, an explicit solution can be found also in some cases that are not covered by Theorem 6.1 and Proposition 6.1 (e.g., when $c<a+b$). To illustrate, below we offer a direct proof ot one more fact similar to Proposition 6.1.

Proposition 6.2 *Let $c<a+\frac{b}{2}$. Then $\Omega(a,b,c)=b+a-c+gcd(c-b,c-a)-1$.*

Proof The proof is similar to the one of Proposition 6.1. Consider the instance with coefficients $c-b,c-a,c$, where $c-b<c-a<c$. By Symmetry Lemma II, $\Omega(c-b,c-a,c)=\Omega(a,b,c)$. We also have $c<a+\frac{b}{2} \iff (c-b)+2(c-a)<c$. Hence, the instance $(c-b,c-a,c)$ satisfies the condition of Theorem 6.1. Then we obtain $\Omega(a,b,c)=\Omega(c-b,c-a,c)=c-(c-b)-(c-a)+gcd(c-b,c-a)-1=b+a-c+gcd(c-b,c-a)-1$. □

7 General upper and lower bounds

In this section we obtain reachable upper and lower bounds for the connectivity number. We will suppose that the plane coefficients a,b,c satisfy the conditions $a\neq b$ and $c<a+2b$. This is not a loss of generality since we already provided an explicit solution for the opposite cases.

Let us first note that Theorem 6.1 and its proof imply the following corollary.

Corollary 7.1 *$\Omega(a,b,c)\geq c-a-b+gcd(a,b)-1$.*

Furthermore, we have the following theorem.

Theorem 7.1 *Let w.l.o.g. $a+b<c<a+2b$. Then $a-1\leq\Omega(a,b,c)\leq b-1$.*

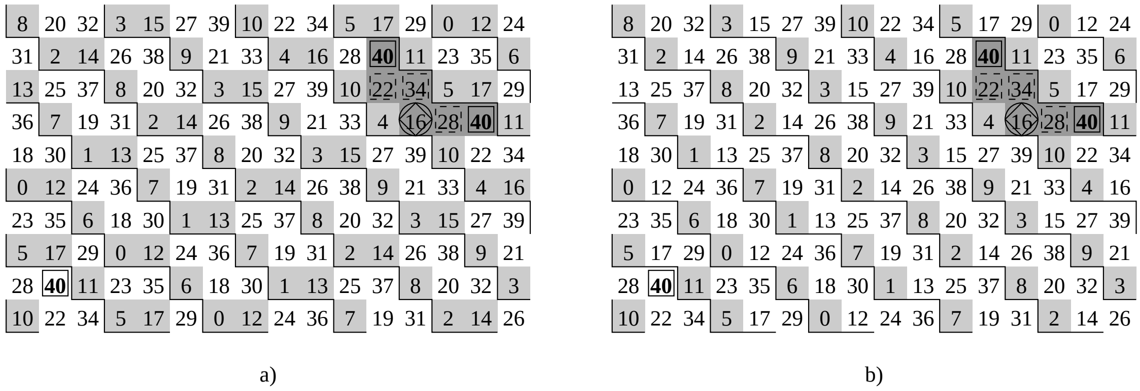


Figure 11: Illustration to the proof of Theorem 7.1. a) Array $A(12, 18, 41)$. The lower naive lines are in gray. They contain points with values in the range $[0, b-1] = [0, 17]$ and constitute a connected set. b) The leftmost points of the same array are in gray. They form a disconnected set. Their values are in the range $[0, a-1] = [0, 11]$. A dark gray stairwise path between two maximal points is shown in both figures. A core is marked by $\bigcirc$ and a plug by $\diamond$.

Proof

1. *Upper bound.* Consider a generic class C of $A(a, b, c)$. Partition as before C into a naive line L and a discrete line T which is "above" L. The values of the points of L are numbers in the range $[0, b-1]$. Note that if $gcd(a, b) = 1$, for every class C the values of the points of L are $0, 1, 2, \ldots, b-1$. If $gcd(a, b) = d \neq 1$, then according to Proposition 4.2, the values of the points of L are the corresponding remainders modulo b. See Figure 11.

   Consider now two arbitrary neighboring classes C_1 and C_2. Let L_1 and L_2 be the naive lines associated with them. Let $T_1 = C_1 - N_1$ and $T_2 = C_2 - N_2$. Since $c < a + 2b$, T_1 and T_2 are discrete lines which are thinner than standard. Then at certain locations points from L_1 and L_2 will share sides or vertices. Then the set of points of the lines L, over all equivalence classes of $A(a, b, c)$, will be connected. Hence, $\Omega(a, b, c) \leq b - 1$. The bound is reachable, which follows from Proposition 5.1, Part 5.

2. *Lower bound.* The lower bound will follow from certain considerations and results of the next section. $\square$

8 An $O(a \log b)$ algorithm for $\Omega(a, b, c)$ computation

In this section we propose an algorithm which computes $\Omega(a, b, c)$ with $O(a \log b)$ arithmetic operations. We have seen that if $c \geq 2b + a$ or $c \leq 2b - a$, then the plugs of $A(a, b, c)$ are the cores of those classes of $A(a, b, c)$ which contain maximal points. In view of Remark 4.3, further we may assume that $2b < c < 2b + a$.

We have already noticed that if c is large enough compared to a and b, then as ω decreases, $P(a, b, c)$ gets disconnected along an upper standard line within the same equivalence class. This is so because for a large c, the classes are wide enough to ensure such sort of disconnectedness. For smaller

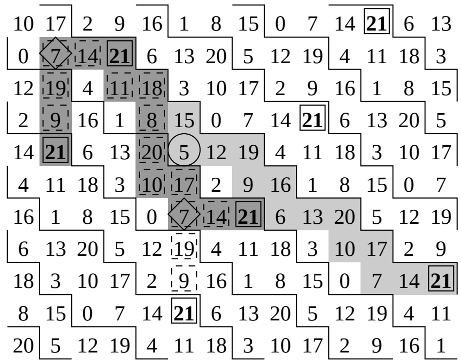


Figure 12: Illustration to Connectivity-search algorithm on array $A(7, 10, 22)$. Both the Down-right and Up-right searches find a solution $\Omega(7, 10, 22) = 7$. The corresponding stairwise paths are in dark gray. The core has value 5. A core is marked by $\bigcirc$ and a plug by $\diamond$.

c, however, the classes are narrower, therefore their upper standard lines can interfere. Specifically, let C_1 and C_2 be two neighboring classes and S_1 and S_2 their upper standard lines. Let P' and P'' be two consecutive maximal points of C_1 and $\bar{S}_1(P', P'') \subset S_1$ the stairwise path between P' and P''. Now imagine that we want to move from P' to another maximal point (possibly, P''). Remember that one can move from one point to another if they share a side. Moreover, we want that the minimal value we meet be maximal. It is easy to realize that this value will be the connectivity number.

If $c < 2b + a$, then some points from S_1 share sides with points from S_2. In such a case it may be possible at a certain point to pass from a point of $\bar{S}_1(P', P'')$ to a point from S_2, and then keep moving within the class C_2. This way one can evade points of $\bar{S}_1(P', P'')$ with smaller values. Such kind of direct passage from one class to another will be called a *shortcut* (see Figure 12). After one or several shortcuts are made, one can reach a maximal point belonging to an equivalence class different than C_1 (and, possibly, than C_2). Thus disconnectedness of $A(a, b, c)$ can emerge along a stairwise path composed by stairwise pieces from different consecutive equivalence classes.

Let $P = (x_0, y_0)$ be a maximal point of $A(a, b, c)$. We will search a stairwise path T from P to another maximal point of $A(a, b, c)$. The path will have the property that the minimal value found in it will be maximal, over all possible paths connecting P with other maximal points. Since the maximal points of $A(a, b, c)$ form a 2D lattice in the plane, it is clear that the search may be restricted to the discrete half-plane consisting of the integer points (x, y) with $x \geq x_0$. Thus the search of a stairwise path can be directed to "down-right" or to "up-right."

Following the above intuitive explanations, below we outline an algorithm computing $\Omega(a, b, c)$.

Connectivity-search algorithm

The algorithm performs consecutively down-right and up-right search. The output of each search is an integer number, as the maximal one is $\Omega(a, b, c)$. For the sake of clarity, we describe the algorithm in terms of constructing a stairwise path $T = \langle P_1, P_2, \dots, P_k \rangle$.

Down-right search

Step (1)

The search starts from a maximal point $P_1 = (x_1, y_1)$ belonging to a certain equivalence class C_1. We have $v(P_1) = c - 1$. P_1 belongs to the stairwise path T.

Then move to the point $P'_1 = (x_1, y_1 - 1)$ just below P_1. Next move to the rightmost point of C in the same row and denote it P_2. P_2 belongs to the path, together with all points of the strip with end-points P_1 and P_2.

Step (i)

Let $P_i = (x_i, y_i)$ be a rightmost point from a certain equivalence class C'. Let P_i be the last point included in the path T. (Note that each of the possibilities $C' = C$ and $C' \neq C$ may take place.)

Move to the point $P'_i = (x_i, y_i - 1)$ just below P_i. P'_i belongs to T. Next find the value $v(Q)$ of the rightmost point $Q \in C'$ in the same row.

Check whether the point $P''_i = (x_i, y_i - 2)$ just below P', belongs to C'. If $P''_i \in C'$, then set $P_{k+1} = Q$. In this case, all points of the strip stretching between P'_i and P_{k+1} belong to T. Else, compute the value $v(P''_k)$ and set

$$P_{k+1} = \begin{cases} Q, & \text{when } v(Q) > v(P''_k) \\ P''_k, & \text{otherwise} \end{cases} .$$

Note that if $P_{k+1} = P''_k$, we have a shortcut.

Repeate Step (i) until a maximal point P_{k+1} with value $v(P_{k+1}) = c-1$ is reached. Then set $\Omega' = \min\{v(P_i) : P_i \in T\}$. See Figures 12 and 13a.

Up-right search

This search is similar to the bottom-right search, therefore we only briefly sketch it.

The search starts from a maximal point $P_1 = (x_1, y_1) \in C_1$. Move upward to the point $P'_1 = (x_1, y_1 + 1)$. It clearly belongs to another equivalence class $C_2 \neq C_1$. Then find the rightmost point Q in the row of P'_1. Next go to the uppermost point $P''_1 = (x_1, y_1 + 2) \in C_2$. Set

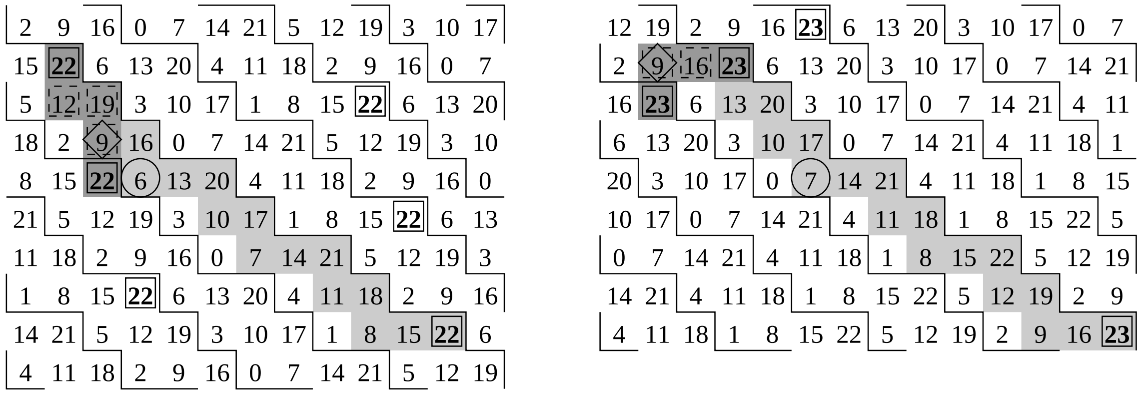
$$P_2 = \begin{cases} Q, & \text{when } v(Q) > v(P''_1) \\ P''_1, & \text{otherwise} \end{cases} .$$

A generic ith step is analogous to the first one. Take consecutive steps until another maximal point is reached. Set Ω'' to be equal to the minimal element in the constructed path between the two maximal points. See Figures 12 and 13b.

After running both searches, we determine the connectivity number as $\Omega(a, b, c) = \max(\Omega', \Omega'')$.

Note that the values found during a Down-right search and an Up-right search can be equal or different (see Figures 12 and 13).

Remark 8.1 *Since $c > 2b$ and the points P_1 in Step (1) and P_i in Step (i) are rightmost points, it follows that the points P'_1 and P'_i always belong to the classes C_1 and C', respectively.*



a) b)

Figure 13: Illustration to Connectivity-search algorithm. a) Array $A(7,10,23)$. The solution is found by the Down-right search. The obtained stairwise path is in dark gray. It contains a plug with a value $\Omega(7,10,23)=9$. The core has value 6. Up-right search does not find a solution. b) Array $A(7,10,24)$. The solution is found by the Up-right search. The obtained stairwise path is in dark gray. It contains a plug with a value $\Omega(7,10,23)=9$. The core has value 7. Down-right search does not find a solution. A core is marked by $\bigcirc$ and a plug by $\diamond$.

In Step (1) the search starts from a maximal point with value $c-1$. Moreover, since $a<b$, we have $2b<c<2b+a<3b-1$. Then there are exactly two points under P_1, which belong to the same class C_1. Therefore, in the first step shortcut is impossible.

Remark 8.2 *Let P be a rightmost point of a class C. It is easy to see that the minimal possible value for P is $c-a+1$. Also, if S is a standard line between two maximal points of C, then the minimal possible value of a point of S is $c-a-b+1$. It is also clear that the maximal possible value of a leftmost point of C is $a-1$.*

Remark 8.3 *The Down-right and Up-right searches can be easily rephrased in terms of reversed Up-left and Down-left searches, respectively. Thus if we apply an Up-left/Down-left search, we will obtain a stairwise path which is equivalent to one obtained through a Down-right/Up-right search.*

The following proposition implies the lower bound of Theorem 7.1.

Proposition 8.1 *Leftmost points of a class C cannot belong to the path T constructed by the Connectivity-search algorithm.*

Proof Consider a point $Q=(x,y)\in T$. By contradiction, assume that Q is a leftmost point in C. This is possible only if the point $P=(x,y+1)$ above Q is a rightmost point. For if, assume by contradiction that P is not rightmost. Then, since $b>a$, the point to the right of P would have a larger value than Q. Then Q would not be selected by the algorithm, which is a contradiction.

We have $v(P)=v(Q)+b$. The point $P'=(x-1,y+1)$ to the left from P has value $v(P')=v(Q)+b-a$. It is easy to see that $P'\in T$. For if, the point $P''=(x,y+2)\in T$. By Lemma 4.1, we have $v(P'') = v(P) + b - c$, while $v(P') = v(P) - a + c > v(Q)$. Hence, if one applies the reverse Up-left search, the point $v(P')$ will be included in T instead of P'' - a contradiction.

Since Q is a leftmost point, the point $Q' = (x-1, y)$ to the left from it is a rightmost point belonging to a class C' which is neighboring to C.

By Lemma 4.1, $v(Q') = (v(Q) + b - a) - a + c = v(Q) + b + c - 2a > v(Q)$. Then the algorithm would make a shortcut from P' to Q' and Q would not be included in the path T - a contradiction.

One can follow the above reasoning with the help of one of Figures 11-13. □

Now we can state the following theorem.

Theorem 8.1 *The Connectivity-search algorithm computes $\Omega(a, b, c)$ with $O(a \log b)$ arithmetic operations. Within a model with a unit cost floor operation, the algorithms complexity is $O(a)$.*

Proof

1. *Correctness of the algorithm.* First we show that the algorithm always constructs a stairwise path between two maximal points of $A(a, b, c)$. This follows from the fact that the value of every subsequent element that enters the path T is different from the values of the elements already in the path. To see this, assume the opposite, i.e., that the algorithm starts from a maximal point P and while traversing the path T, it includes in T two points A and B that are not maximal but have the same value $v(A) = v(B) = h < c - 1$. Denote by $T(A, B)$ the stairwise path between A and B built by the algorithm. Since a discrete plane with rational coefficients is periodic, the conditions for the algorithm at the points A and B are identical. Consequently, after reaching the point B, it will produce an infinite sequence of adjacent replicas of $T(A, B)$ crossing the whole discrete plane in direction down-right. Keeping Remark 8.3 in mind, now let us run an Up-left search starting from A. We have to get an infinite sequence of adjacent replicas of $T(A, B)$ crossing the whole discrete plane in direction up-left. Thus the algorithm will run forever and produce infinite stairwise path which does not contain a point with a value $c - 1$. This contradicts the fact that if we run from A an Up-left search (that is, the reverse of the Down-right search), we have to reach the starting point P.

   Furthermore, by construction, the minimal value contained in the path T is as large as possible, over all possible stairwise paths connecting two maximal elements of $A(a, b, c)$. This follows from the fact that after a shortcut, say, within a Down-right search, the algorithm proceeds from the point $P_{i+1} = P_i''$ whose value is strictly larger than the corresponding rightmost point Q in the upper row. Thus, in turn, the next point P_{i+2} which is below P_{i+1} will have a larger value than the value of the point below Q. This ensures that the obtained path has the desired property.

2. *Complexity bound.* We consider the Down-right search, the Up-right search complexity analysis being similar.

   From the algorithm description it is clear that a shortcut can only decrease the number of iterations. In case of a shortcut, the comparison $v(Q) > v(P_k'')$ takes only a constant time and does not influence the algorithms complexity. Therefore we can consider the case when no shortcuts are performed, which holds if $c \geq a + 2b$. In this case the algorithm traverses a standard line segment between two consecutive maximal points of a certain class.

   Consider then the complexity of a generic iteration - Step (i), which starts at a rightmost point P_i. Lemma 4.1 implies that finding the value $v(P')$ of the lower point P' takes $O(1)$ operations. The value of the rightmost point Q in the same row is $v(Q) = v(P') + \lfloor \frac{(c-1)-v(P')}{a} \rfloor a$. From

Remark 8.3 we have that $v(P') \geq c-a-b+1$. Then $\frac{(c-1)-v(P')}{a} \leq \frac{(c-1)-(c-a-b+1)}{a} = O(\frac{b}{a})$. Since computing the greatest integer in a rational number x requires $\Theta(\log x)$ arithmetic operations [13], we get that the value $v(Q)$ can be computed in $O(\log b)$ time. Now we observe that the path T contains exactly $O(a)$ rightmost points, which follows from [11]. Then the overall time complexity of the Connectivity-search algorithm becomes $O(a \log b)$, as this bound is reached for the class of inputs with $c \geq a + 2b$.

Clearly, within a model with a unit cost floor operation, the algorithms complexity is $O(a)$. □

Remark 8.4 *If the minimal coefficient a is bounded by a constant, then Connectivity- search algorithm determines the discrete plane connectivity number in* $O(\log b)$ *arithmetic operations.*

Note that the computation of the explicit solution given by formula (4) requires $\Theta(\log b)$ *operations, since this is the complexity of computing the greatest common divisor of two integers a and b [9].*

9 Concluding remarks

Discrete plane is a very basic primitive in discrete modeling. Naturally, related investigations have been carried out by several authors and as a result some useful properties have been obtained. One should admit, however, that structural results which are really deep and valuable from mathematical point of view are almost missing. Moreover, various fundamental concepts and theorems about discrete lines do not have counterparts about planes. (See [12] for a survey on such kind of concepts and results.) Filling up such sort of gaps is seen as an important further task.

In this paper we studied some combinatorial and topological properties of discrete planes, in particular we proposed a solution to the discrete plane connectivity problem. For inputs (a, b, c) satisfying $c \in [b, 2b-a] \cup [2b+a, +\infty)$ (Case 1) the solution is given in explicit form and is computable in optimal $O(\log b)$ time, while for inputs with $c \in (2b-a, 2b+a)$ (Case 2) the solution is found algorithmically in $O(a \log b)$ time. Thus, for any fixed pair of coefficients a, b and a variable c, an explicit solution exists for any value of c, except for a finite number of problem samples which fall within Case 2. It is still unclear to us whether in that case one can do better. Thus two natural questions arise.

Question 1: What is the optimal time to compute $\Omega(a, b, c)$ in Case 2? In particular, is it possible to compute $\Omega(a, b, c)$ in $O(\log b)$ time?

Question 2: Is it possible in Case 2 to explicitly express $\Omega(a, b, c)$ by a formula involving the given coefficients and elementary analytical or number-theoretical functions of them?

It seems to us that answering the above questions will require immaculate understanding of the deepest topological and combinatorial properties of discrete plane.

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