Determination of Q-convex sets by X-rays

Alain Daurat

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Abstract

In this paper, the problem of the determination of lattice sets from X-rays is studied. We define the class of Q-convex sets along a set $\mathcal{D}$ of directions which generalizes classical lattice convexity and we prove that for any $\mathcal{D}$, the X-rays along $\mathcal{D}$ determine all the convex sets if and only if it determines all the Q-convex sets along $\mathcal{D}$. As a consequence, any algorithm which reconstructs Q-convex sets from X-rays can be used to reconstruct convex lattice sets from X-rays along directions which provide uniqueness. This gives a constructive answer to the discrete version of Hammer's X-ray problem.

The aim of tomography is to reconstruct a 3D object from 2D X-ray pictures, the grey-level of each point of the X-ray picture corresponding to the integral of the density of the 3D object on a straight line. In many cases the reconstruction can be done slice by slice, so it is sufficient to consider the analogue problem in the plane: how can a 2D object be reconstructed from its 1D X-rays ?

The reconstruction from a lot of X-ray pictures has been much studied since the beginning of 20th century and is applied intensively since the 1970s in computerized tomography (see for example [16]).

Sometimes, we only have a few X-ray pictures (for example we only have two pictures in Biplane Angiography [20]). In this case we must impose to the object to be reconstructed some properties. The strongest properties we can impose to the set are homogeneity and convexity. The homogeneity permits to simply model the object by a subset of the plane. In 1961 Hammer posed the following problem: how many X-ray pictures are needed to reconstruct a convex set ([15]) ? This problem was solved in 1980 by Gardner and McMullen:

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[1] A large part of this work has been made when the author prepared his PhD-Thesis at the LLAIC1, IUT de Clermont-Ferrand, France. ([8])

a planar convex set is completely determined by four X-rays in suitable directions ([12]). But the proof of this determination is not constructive. Some algorithms which reconstruct the convex sets are described in [19], [10, theorem 1.2.28] but they are not completely satisfactory (see [19, Example 1,2], [10, note 1.2]).

In fact, computability of continuous objects is much more difficult than the one of discrete objects, so it is natural to study discretized versions of tomographic problems: the reconstructed image is a subset of the discrete plane $\mathbb{Z}^2$ (called lattice set), the X-rays are the numbers of points along the lines in a given direction. Moreover this formulation is near to some problems in electron microscopy, where the points correspond exactly to the atoms (see [21,18]). Since the beginning of the 1990s many problems in *"discrete tomography"* have been studied (for an overview see [17]): especially, Gardner-McMullen's result has been extended to the discrete case by Gardner-Gritzmann ([11]): lattice sets which are the intersection between a convex polygon and the discrete plane (called *convex lattice sets*) are completely determined by four X-rays in suitable directions, or seven X-rays in any directions. But no polynomial-time algorithm which reconstructs the convex lattice set in this case has been found ([14]). Nevertheless in [1,2,5], polynomial-time algorithms permit to reconstruct lattice sets which satisfy properties which are linked but not equivalent to lattice convexity. In particular in [5], the introduced class, Q-convexity is a property which depends on the set of the directions of the X-rays.

In this paper we extend Gardner-Gritzmann's uniqueness result to the class of Q-convex sets: precisely in a first section of this paper we precise the original uniqueness result by giving a characterization of all the sets of directions which provide uniqueness for the convex lattice sets. In a second part we extend the results of Gardner and Gritzmann to Q-convex sets, and finally we show how this result permits to the polynomial-time algorithm of [5] to reconstruct lattice convex sets from X-rays.

1 Preliminaries

Notations

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{F}_m, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ will denote respectively the sets of non-negative integers, integers, integers modulo m, rational numbers, real numbers, complex numbers. Thus $\mathbb{R}^2, \mathbb{Z}^2$ denote respectively the euclidean plane and the discrete plane. If E is a finite set, we denote by $|E|$ the cardinality of E. If x is a real number, $\lfloor x \rfloor$ designs the greatest integer smaller or equal to x.

Lattice direction

A direction is an equivalence class for the relation of parallelism on the straight lines of the plane. It can be given by an equation $\lambda x + \mu y = \text{const.}$ or by a directing vector $(-\mu, \lambda)$ or by the slope $-\frac{\lambda}{\mu} \in \mathbb{R} \cup \{\infty\}$. If λ and μ are integers then, the direction is called a *lattice direction*, and we can suppose that λ and μ are coprime. In this paper we will identify a direction p with its equation $p(x, y) = \lambda x + \mu y$ or with its slope $p = -\frac{\lambda}{\mu}$.

If p and q are two lattice directions, we denote by $\langle i, j \rangle_{p,q}$ (or $\langle i, j \rangle$ if there is no ambiguity) the point M which satisfies $p(M) = i$ and $q(M) = j$. It must be noticed that the point $\langle i, j \rangle_{p,q}$ can be outside $\mathbb{Z}^2$, even if i and j are integers.

X-rays

We recall that a lattice set is a non-empty finite subset of $\mathbb{Z}^2$. The *X-ray* of a lattice set F in a lattice direction p is the function $X_pF(i) : \mathbb{Z} \to \mathbb{N}$ defined by: $X_pF(i) = |\{N \in F \,:\, p(N) = i\}|$.

Convexity

A lattice set F is *line-convex* along a direction p if the intersection of any line of direction p and F is the set of the points with integer coordinates of a straight line segment. A lattice set is convex if it is the intersection between $\mathbb{Z}^2$ and its convex hull.

Determination of a class of lattice sets by X-rays in the directions of $\mathcal{D}$

We suppose that $\mathcal{E}$ is a class of subsets of $\mathbb{Z}^2$. The set $\mathcal{D}$ of directions *determines* the class $\mathcal{E}$ if for any sets E_1 and E_2 of $\mathcal{E}$ we have:

$$(\forall p \in \mathcal{D}\ X_pE_1 = X_pE_2) \Longrightarrow E_1 = E_2.$$

Cross ratio

A linear transformation of $\mathbb{R}^2$ is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which can be written $f(x, y) = (ax+by, cx+dy)$ with $a, b, c, d \in \mathbb{R}$. If a, b, c, d are rational, then f is said to be rational. A projective transformation is the trace on the directions of a linear transformation of $\mathbb{R}^2$.

Given three distinct directions p_1, p_2, p_3, there is always a projective transformation ϕ such that the images by ϕ of these three directions have the slopes

$\infty, 0, 1$. If p_4 is a fourth direction, then $\phi(p_4)$ is called the cross ratio of the directions p_1, p_2, p_3, p_4 and is denoted $\left[\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right]$.

If we represent the directions by their slopes we have:

$$\left[\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right] = \frac{p_3 - p_1}{p_3 - p_2} : \frac{p_4 - p_1}{p_4 - p_2} = \frac{(p_3 - p_1)(p_4 - p_2)}{(p_3 - p_2)(p_4 - p_1)}$$

with evident conventions when there is ∞.

The quadruplet (p_1, p_2, p_3, p_4) of four distinct directions is said to be in order if the sequence of line-angles $(p_1, p_2), (p_1, p_3), (p_1, p_4) \in]0, \pi[$ is increasing or decreasing. This property is equivalent with $\left[\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right] > 1$ and is preserved after a bijective projective transformation.

For more details about cross-ratios see [4, Paragraph 6.1].

Polygons

A convex polygon is the convex hull of a finite set of points of $\mathbb{R}^2$. If $\mathcal{D}$ is a set of directions, a $\mathcal{D}$-polygon P is a convex polygon such that any line of direction in $\mathcal{D}$ contains zero or two vertices of P. An affinely regular polygon is the non-singular affine image of a regular polygon.

2 Determination of convex lattice sets

In this section we shall summarize the results of [11] by the following theorem:

Theorem 1. *Let $\mathcal{D}$ be a finite set of lattice directions. The seven following statements are equivalent:*

- *(1) The set $\mathcal{D}$ does not determine the class of the lattice convex sets.*
- *(2) There exists a $\mathcal{D}$-polygon whose vertices are in $\mathbb{Z}^2$.*
- *(3) There exists a $\mathcal{D}$-polygon whose vertices are in $\mathbb{Q}^2$.*
- *(4) There exists a $\mathcal{D}$-polygon.*
- *(5) There exists an affinely regular $\mathcal{D}$-polygon.*
- *(6) The cross-ratio of any four directions in $\mathcal{D}$, arranged in order, is in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$.*
- *(7) There exists a bijective* rational *linear transformation ϕ of $\mathbb{R}^2$ such that the images of the directions of $\mathcal{D}$ by ϕ have slopes in $\{\infty, 0, 1, \frac{3}{2}, 2, 3\}$.*

Proof. • (1) $\Leftrightarrow$ (2) is Theorem 5.5 of [11].
• (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) clear.

- (4) $\Rightarrow$ (5) is proved in [12, Lemmas 5,6] or [10, p34-36]. The implication (9) $\Rightarrow$ (5) of Theorem 12 will be a generalization of this implication.
- (5) $\Rightarrow$ (6) is Theorem 4.5 of [11].
- (6) $\Rightarrow$ (7): let p_1, p_2, p_3 be three consecutive directions of $\mathcal{D}$, which means that for any other $d \in \mathcal{D}$ the sequence (p_1, p_2, p_3, d) is in order (*i.e.* $\begin{bmatrix} p_1 & p_2 \\ p_3 & d \end{bmatrix} > 1$).

  There exists a bijective linear transformation ϕ_1 such that $\phi_1(p_1) = \infty, \phi_1(p_2) = 0, \phi_1(p_3) = 1$. The directions p_1, p_2, p_3 are lattice directions, so ϕ_1 is a rational linear transformation. From the statement (6) it follows that: $\{\infty, 0, 1\} \subseteq \phi_1(\mathcal{D}) \subseteq \{\infty, 0, 1, \frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$.

  We have:

  · $\begin{bmatrix} \infty & 0 \\ \frac{4}{3} & \frac{3}{2} \end{bmatrix} = \frac{9}{8}$ so $\{\frac{4}{3}, \frac{3}{2}\} \nsubseteq \phi_1(\mathcal{D})$.

  · $\begin{bmatrix} \infty & 0 \\ \frac{4}{3} & 3 \end{bmatrix} = \frac{9}{4}$ so $\{\frac{4}{3}, 3\} \nsubseteq \phi_1(\mathcal{D})$.

  · $\begin{bmatrix} \infty & 0 \\ \frac{3}{2} & 4 \end{bmatrix} = \frac{8}{3}$ so $\{\frac{3}{2}, 4\} \nsubseteq \phi_1(\mathcal{D})$.

  · $\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} = \frac{9}{8}$ so $\{3, 4\} \nsubseteq \phi_1(\mathcal{D})$.

  · $\begin{bmatrix} \infty & 1 \\ \frac{4}{3} & 4 \end{bmatrix} = 9$ so $\{\frac{4}{3}, 4\} \nsubseteq \phi_1(\mathcal{D})$.

  So $\phi_1(\mathcal{D})$ is included in one of the following sets:

  $$F_1 = \{\infty, 0, 1, \frac{3}{2}, 2, 3\}, \quad F_2 = \{\infty, 0, 1, \frac{4}{3}, 2\}, \quad F_3 = \{\infty, 0, 1, 2, 4\}$$

  · If $\phi_1(\mathcal{D}) \subseteq F_1$, we take $\phi = \phi_1$.

  · If $\phi_1(\mathcal{D}) \subseteq F_2$, as $\phi_2(F_2) = \{\infty, 0, \frac{3}{2}, 2, 3\} \subseteq F_1$ with $\phi_2(x, y) = (x, \frac{3y}{2})$, we take $\phi = \phi_2 \circ \phi_1$.

  · If $\phi_1(\mathcal{D}) \subseteq F_3$, as $\phi_3(F_3) = \{\frac{3}{2}, 0, \infty, 3, 2\} \subseteq F_1$ with $\phi_3(x, y) = (-2x + 2y, 3y)$, we take $\phi = \phi_3 \circ \phi_1$.
- (7) $\Rightarrow$ (3) The image by ϕ^{-1} of the polygon of Figure 1 is a $\mathcal{D}$-polygon.

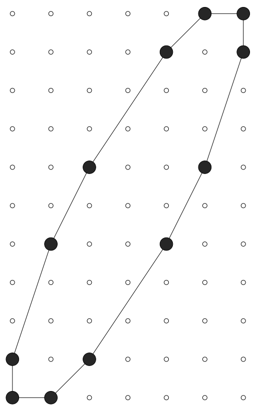


Fig. 1. A $\mathcal{D}$-polygon whose vertices are in $\mathbb{Z}^2$ with $\mathcal{D} = \{\infty, 0, 1, \frac{3}{2}, 2, 3\}$.

$\square$

Remark 2. If $\mathcal{D} \geq 7$ then statement (7) is impossible, so $\mathcal{D}$ determines the lattice convex sets: we find again Theorem 5.7 (iii) of [11].

Remark 3. The implication (6) $\Rightarrow$ (2), restricted to the sets of four directions, has been enunciated in Proposition 4.8 of [3].

Corollary 4. *Let $\mathcal{D}$ be any set of lattice directions. Then $\mathcal{D}$ determines the lattice convex sets if and only if there is a subset of $\mathcal{D}$ of four directions determining the lattice convex sets.*

3 Determination of Q-convex sets by X-rays

3.1 Definitions

Q-convexity along two directions

For this definition we must fix two lattice directions p and q.

For any $M \in \mathbb{Z}^2$, we can define the four quadrants around M along the directions p and q by:

$$\begin{aligned} R_0^{pq}(M) &= \{N \in \mathbb{Z}^2 \mid p(N) \leq p(M) \text{ and } q(N) \leq q(M)\} \\ R_1^{pq}(M) &= \{N \in \mathbb{Z}^2 \mid p(N) \geq p(M) \text{ and } q(N) \leq q(M)\} \\ R_2^{pq}(M) &= \{N \in \mathbb{Z}^2 \mid p(N) \geq p(M) \text{ and } q(N) \geq q(M)\} \\ R_3^{pq}(M) &= \{N \in \mathbb{Z}^2 \mid p(N) \leq p(M) \text{ and } q(N) \geq q(M)\} \end{aligned}$$

(see Figure 2).

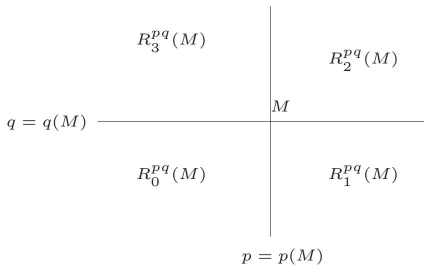
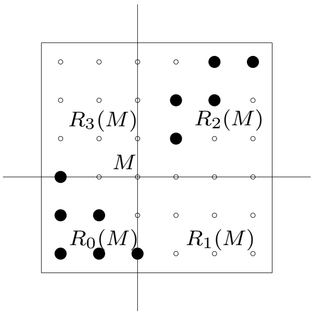


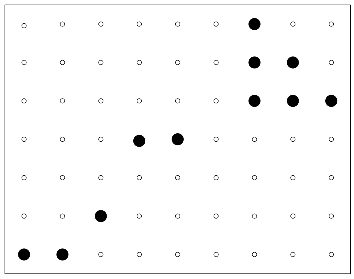
Fig. 2. The four quadrants along $p = x$ and $q = y$.

Remark 5. For any point M, the function $(i, p, q) \mapsto R_i^{pq}(M)$ applied to the triplets $(0, p, q)$, $(1, -p, q)$, $(2, -p, -q)$, $(3, p, -q)$, $(0, q, p)$, $(1, -q, p)$, $(2, -q, -p)$, $(3, q, -p)$ gives the same quadrant. In the following we will identify these triplets.

Definition 6. A lattice set E is *Q-convex* (quadrant-convex) along $\mathcal{D} = \{p, q\}$ if $R_k^{pq}(M) \cap E \neq \varnothing$ for all $k \in \{0, 1, 2, 3\}$ implies $M \in E$.



(a)



(b)

Fig. 3. a) A lattice set which is line-convex along x and y, but not Q-convex along $\{x, y\}$. b) A lattice set which is Q-convex along $\{x, y\}$.

Every Q-convex set along $\mathcal{D}$ is line-convex along the directions of $\mathcal{D}$, but line-convexity is not a sufficient property to be Q-convex (see Figure 3). Every lattice convex set is Q-convex along any pair of directions, so Q-convexity is an intermediate property between line-convexity and usual convexity.

The intersection of two Q-convex sets along $\mathcal{D}$ is also Q-convex along $\mathcal{D}$ so we can define the Q-convex hull along $\{p, q\}$ of a set E, denoted $QCONV_{pq}(E)$.

Extension to 3 directions and more

If $\mathcal{D}$ is any finite set of directions, then Q-convexity along $\mathcal{D}$ can be defined as follows:

Definition 7. A lattice set E is *Q*-convex along $\mathcal{D}$ if it is *Q*-convex along any pair of directions included in $\mathcal{D}$. (see Figure 4)

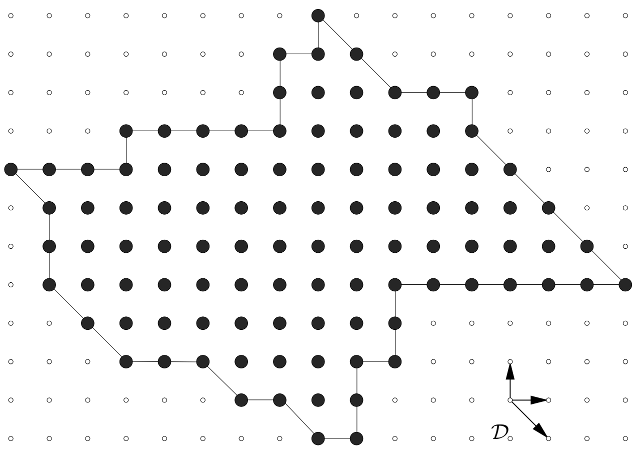


Fig. 4. A Q-convex set along $\mathcal{D} = \{x, y, x + y\}$

The ASP are a generalization of the quadrants to the sets of more than two directions:

Definition 8. An ASP (*almost-semi-plane*) along $\mathcal{D}$ is a quadrant $\Pi = R_i^{pq}(M)$ with $p, q \in \mathcal{D}$ such that for any direction $r \in \mathcal{D}$ there exists a semi-line of direction r with starting point M which is contained in Π (see Figure 5).

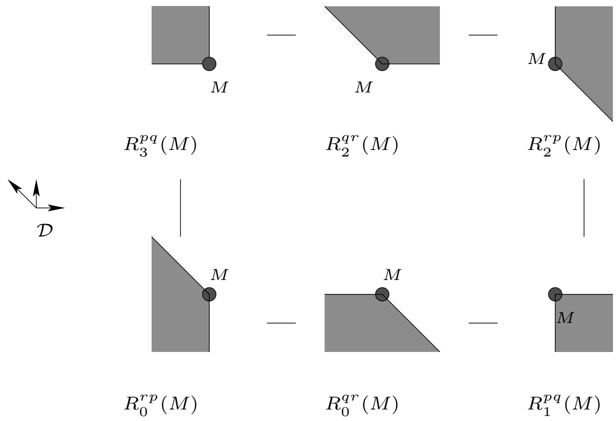


Fig. 5. The six ASP along $\mathcal{D} = \{p, q, r\}$, $p = x$, $q = y$, $r = x + y$. (The ASP are the grey regions.)

An ASP is a maximum element of the set $\{R_i^{pq}(M) \ : \ p, q \in \mathcal{D}\}$ ordered by inclusion.

The property for $R_i^{pq}(M)$ to be an ASP does not depend on M. We denote by $\mathcal{A}_\mathcal{D}$ the set of (i, p, q) such that R_i^{pq} is an ASP. Two ASP are said to be consecutive if their union is a semi-plane. The associated graph is cyclic, more precisely: suppose we have ordered the directions of $\mathcal{D}$ by decreasing angle with the y-axis: $\mathcal{D} = \{p_0, p_2, \ldots, p_{n-1}\}$ where $p_0 = a_0x + b_0y$ with $a_0 > 0$ or $(a_0, b_0) = (0, -1)$, $p_i = a_ix + b_iy$ with $b_i > 0$ for $i > 0$ and $\infty \geq \frac{b_i}{a_i} > \frac{b_{i+1}}{a_{i+1}} > -\infty$. The set $\mathcal{A}_\mathcal{D}$ and its associated cyclic graph are the following:

$$\begin{array}{ccccccc} (3, p_0, p_1) & - & (2, p_1, p_2) & - \cdots - & (2, p_{n-2}, p_{n-1}) & - & (2, p_{n-1}, p_0) \\ | & & & & & & | \\ (0, p_{n-1}, p_0) & - & (0, p_{n-2}, p_{n-1}) & - \cdots - & (0, p_1, p_2) & - & (1, p_0, p_1) \end{array}$$

(see Figure 5).

Remark 9. If E is a Q-convex set along $\mathcal{D}$ and $M \notin E$, then, in many cases, there is an ASP along M which does not contain any point of E, but it is not true in the general case (see Figure 6). The sets which satisfy this property are said to be strongly Q-convex and are used in [6].

$\mathcal{D}$-sequences

The $\mathcal{D}$-sequences generalize the $\mathcal{D}$-polygons.

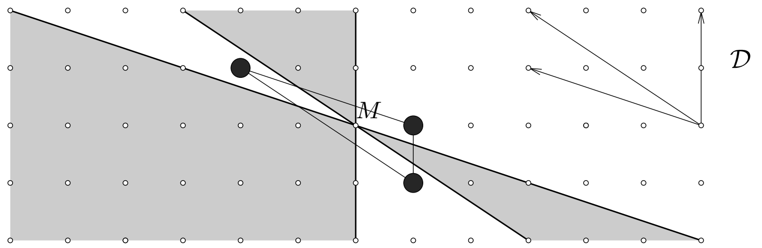


Fig. 6. A Q-convex set E along $\mathcal{D} = \{x + 3y, x, 2x + 3y\}$, but around the point M there is no ASP which is disjoint with E.

Definition 10. A $\mathcal{D}$-sequence is a sequence $(A_k)_{k\in\mathbb{F}_m}$ of m points of $\mathbb{R}^2$ such that m is even and for any p in $\mathcal{D}$ there is an $s \in \mathbb{F}_m$ such that

$$\begin{array}{ccccccc} p(A_{s-1}) & < & p(A_{s-2}) & < \cdots < & p(A_{s-\frac{m}{2}}) \\ \shortparallel & & \shortparallel & & \shortparallel \\ p(A_s) & < & p(A_{s+1}) & < \cdots < & p(A_{s+\frac{m}{2}-1}) \end{array}$$

We can see that the sequences of the vertices of a $\mathcal{D}$-polygon are $\mathcal{D}$-sequences, but $\mathcal{D}$-sequences are not always vertices of $\mathcal{D}$-polygons. (see Figure 7)

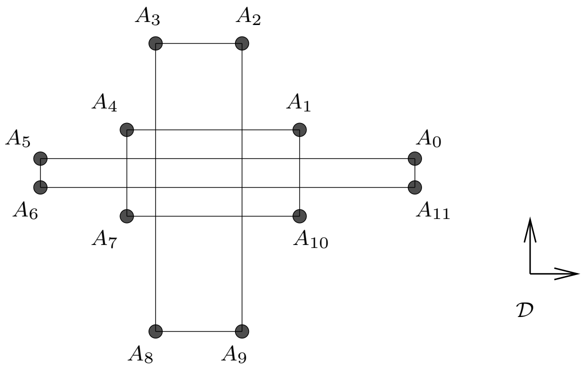


Fig. 7. A $\mathcal{D}$-sequence with $\mathcal{D} = \{x, y\}$. With the notations of the definition, for $p = x$ we take $s = 6$, and for $p = y$, $s = 9$.

3.2 Results

Now we can give the main result of this paper:

Theorem 11. *Let $\mathcal{D}$ be a finite set of lattice directions. The set $\mathcal{D}$ determines the class of the Q-convex sets along $\mathcal{D}$ if and only if it determines the class of the lattice convex sets.*

This theorem is a corollary of an extension of Theorem 1:

Theorem 12. *Let $\mathcal{D}$ be a set of lattice directions such that $|\mathcal{D}| \geq 2$. Then the seven statements of Theorem 1 are equivalent to the two following ones:*

(8) The set $\mathcal{D}$ does not determine the class of the Q-convex sets along $\mathcal{D}$.
(9) There exists a $\mathcal{D}$-sequence.

The equivalence (1) $\Leftrightarrow$ (8) is exactly Theorem 11.

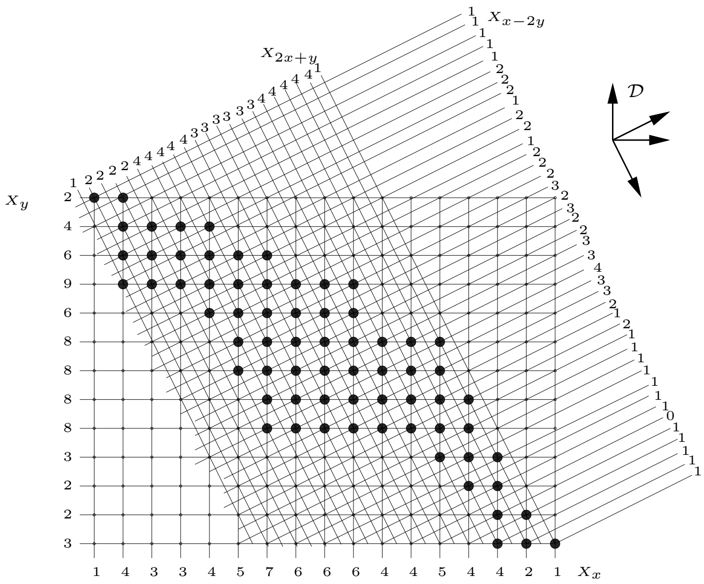


Fig. 8. The set of directions $\mathcal{D} = \{x, y, 2x+y, x-2y\}$ does not satisfy the equivalent statements of Theorem 12 because the cross-ratio of these directions arranged in order is $\frac{5}{4} \notin \{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$. Therefore the set of the figure is the only Q-convex set along $\mathcal{D}$ which has the given X-rays

We shall prove (1) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (5). The implication (1) $\Rightarrow$ (8) is clear since every lattice convex set is Q-convex along $\mathcal{D}$.

Subsections 3.3 and 3.4 are devoted to the proofs of implications (8) $\Rightarrow$ (9) and (9) $\Rightarrow$ (5) respectively.

3.3 Construction of a $\mathcal{D}$-sequence

In this subsection we prove the implication (8) $\Rightarrow$ (9). So we suppose that we have a set $\mathcal{D}$ of lattice directions which satisfies the statement (8) of Theorem 12. By hypothesis there exist two sets F^+ and F^- which are Q-convex along $\mathcal{D}$ and which have the same X-rays in $\mathcal{D}$. We define:

$$E^+ = F^+ \setminus F^-, \qquad E^- = F^- \setminus F^+.$$

We have to construct a $\mathcal{D}$-sequence from the two sets E^+ and E^-. The points of the $\mathcal{D}$-sequence will be gravity centers of equivalence classes for a well-chosen equivalence relation on $E^+ \cup E^-$ (see Figure 9).

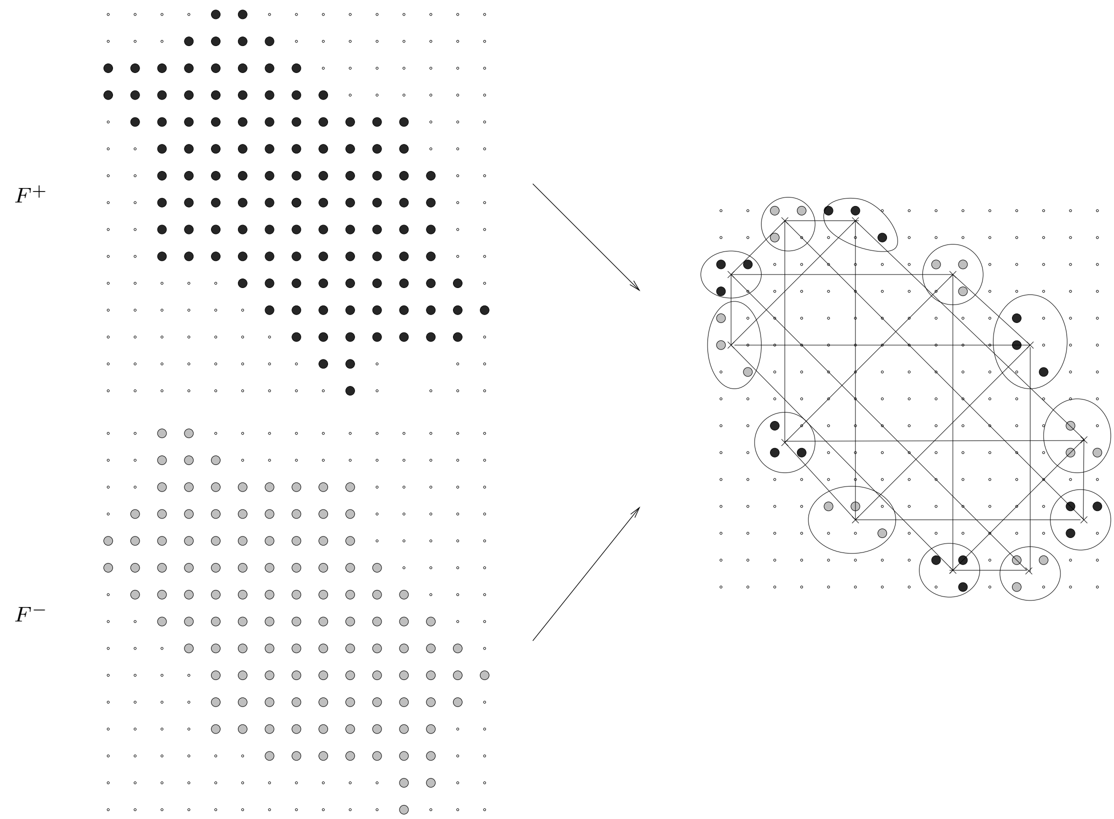


Fig. 9. The construction of a $\mathcal{D}$-sequence from two Q-convex sets which have the same X-rays. ($\mathcal{D} = \{x, y, x+y, x-y\}$)

The first lemma will be used in whole Subsection 3.3.

Lemma 13. *There does* not *exist any point* $M \in E^-$ *such that* $R_i^{pq}(M) \cap E^- = \{M\}$ *and* $R_{(i+2) \mod 4}^{pq}(M) \cap E^+ = \varnothing$. *Symmetrically there does* not *exist any point* $M \in E^+$ *such that* $R_i^{pq}(M) \cap E^+ = \{M\}$ *and* $R_{(i+2) \mod 4}^{pq}(M) \cap E^- = \varnothing$.

Proof. We suppose that a point $M \in E^-$ and integer i satisfy $R_i^{pq}(M) \cap E^- = \{M\}$ and $R_{(i+2) \mod 4}^{pq}(M) \cap E^+ = \varnothing$. By Remark 5 we can suppose, after replacing p by $-p$ and/or q by $-q$ if necessary, that $i = 0$. Let

$$\begin{aligned}
n_0^+ &= |\{N \in E^+ \ : p(N) < p(M) \text{ and } q(N) < q(M)\}| \\
n_2^- &= |\{N \in E^- \ : p(N) > p(M) \text{ and } q(N) > q(M)\}| \\
n_1^+ &= |\{N \in E^+ \ : p(N) > p(M) \text{ and } q(N) < q(M)\}| \\
n_1^- &= |\{N \in E^- \ : p(N) > p(M) \text{ and } q(N) < q(M)\}| \\
u &= X_pE^+(p(M)) = X_pE^-(p(M)) \\
v &= X_qE^+(q(M)) = X_qE^-(q(M)).
\end{aligned}$$

We have $X_q E^+ = X_q E^-$ so:

$$n_0^+ + u + n_1^+ = \sum_{k<q(M)} X_q E^+(k) = \sum_{k<q(M)} X_q E^-(k) = n_1^- \tag{1}$$

and similarly by $X_p E^+ = X_p E^-$

$$n_1^- + (v-1) + n_2^- = \sum_{k>p(M)} X_p E^-(k) = \sum_{k>p(M)} X_p E^+(k) = n_1^+ \tag{2}$$

By summing (1) and (2) we obtain

$$n_0^+ + n_2^- + u + v - 1 = 0. \tag{3}$$

The point M is in E^-, so $u \geq 1$ and $v \geq 1$, which contradicts (3). $\square$

Let $p \in \mathcal{D}$. We have $X_p E^- = X_p E^+$. So for any point $M \in E^+$, there exists a point $N \in E^-$ such that $p(M) = p(N)$. We denote one of these points by M_p. Similarly for any point $M \in E^-$ there is a point $M_p \in E^+$ such that $p(M_p) = p(M)$.

Lemma 14. *For any point $M \in E^+$, there exists one and only one $(i,p,q) \in \mathcal{A}_\mathcal{D}$ such that $R_i^{pq}(M) \cap E^- = \varnothing$. Symmetrically, for any $M \in E^-$, there exists one and only one $(i,p,q) \in \mathcal{A}_\mathcal{D}$ such that $R_i^{pq}(M) \cap E^+ = \varnothing$.*

Proof. Suppose that M is a point of E^+. For any pair of directions p and q, if for all i, $R_i^{pq}(M)$ contains one point of E^-, then by Q-convexity of F^- we have $M \in F^-$ which contradicts $M \in E^+$. So for any $p, q \in \mathcal{D}$, there exists i such that $R_i^{pq}(M) \cap E^- = \varnothing$. Consider (i,p,q) which maximizes $R_i^{pq}(M)$ (for the inclusion) among all the quadrants $R_i^{pq}(M)$ which satisfy $R_i^{pq}(M) \cap E^- = \varnothing$. By Remark 5 we can suppose that $i = 0$.

We make the hypothesis $(0,p,q) \notin \mathcal{A}_\mathcal{D}$. Thus there is a direction r such that the line $r = r(M)$ has only the point M in the quadrant $R_i^{pq}(M)$, and so $r = \alpha p + \beta q$ with $\alpha\beta > 0$. By replacing r by $-r$ if necessary, we can suppose $\alpha > 0, \beta > 0$.

Let us suppose that $p(M_r) < p(M)$. We shall show that $R_0^{qr}(M) = R_0^{pq}(M) \cup R_1^{pr}(M)$ contains no point of E^-, which will be in contradiction with the maximality of $R_0^{pq}(M)$. Indeed, no point of E^- can be in $R_1^{pr}(M)$ because otherwise for such a point N we would have $M \in QCONV_{rp}(M_r, M_p, N)$ (see Figure 10). So the hypothesis $(0,p,q) \notin \mathcal{A}_\mathcal{D}$ is inconsistent with $p(M_r) < p(M)$.

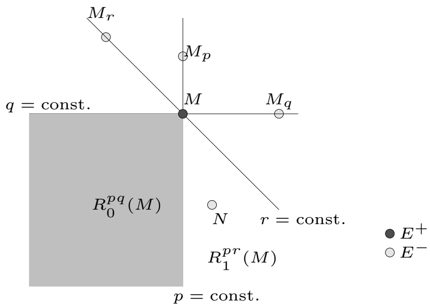


Fig. 10. Why is there an ASP around M which contains no point of E^- ?

If $p(M_r) > p(M)$ then $r(M_r) = r(M)$ and $r = \alpha p + \beta q$ with $\alpha, \beta > 0$ imply that $q(M_r) < q(M)$, so this case is reduced to the previous one by reversing p and q. Therefore we have proved the existence of $(i, p, q) \in \mathcal{A}_{\mathcal{D}}$ such that $R_i^{pq}(M) \cap E^- = \varnothing$.

Now we have to prove the uniqueness of such triplet $(i, p, q) \in \mathcal{A}_{\mathcal{D}}$. Suppose that there is another $(i', p', q') \in \mathcal{A}_{\mathcal{D}}$ which also satisfies $R_{i'}^{p'q'}(M) \cap E^- = \varnothing$. Then the region $R_i^{pq}(M) \cup R_{i'}^{p'q'}(M)$ contains one line $r = r(M)$ with $r \in \mathcal{D}$. But this line contains the point $M \in E^+$ and no point of E^- which is impossible because $X_r E^+ = X_r E^-$.

The case $M \in E^-$ can be proved in symmetric way. $\square$

By this lemma, we can partition E^+ and E^-:

$$E^+ = \bigcup_{(i,p,q) \in \mathcal{A}_{\mathcal{D}}} E^+_{i,p,q}\ , \qquad E^- = \bigcup_{(i,p,q) \in \mathcal{A}_{\mathcal{D}}} E^-_{i,p,q}$$

where

$$E^+_{i,p,q} = \{M \in E^+ \ : \ R_i^{pq}(M) \cap E^- = \varnothing\}, \qquad E^-_{i,p,q} = \{M \in E^- \ : \ R_i^{pq}(M) \cap E^+ = \varnothing\}.$$

Now we define a relation on the points of each $E^+_{i,p,q}$:

Definition 15. Two points $A, B \in E^+_{i,p,q}$ are equivalent ($A \sim B$) if there exists $N \in \mathbb{Q}^2$ such that $A, B \in R_i^{pq}(N)$ and $R_i^{pq}(N) \cap E^- = \varnothing$. A similar equivalence (also denoted $\sim$) is defined on $E^-_{i,p,q}$.

In fact, to test if two points A and B are equivalent, it is enough to check for only one point N if $R_i^{pq}(N) \cap E^- = \varnothing$. For example, if $i = 0$ then we take $N = \langle \max(p(A), p(B)), \max(q(A), q(B)) \rangle$.

Lemma 16. *The relation $\sim$ is an equivalence relation on $E^+ \cup E^-$. Moreover for any $\sim$-equivalence class C, there exists $(i, p, q) \in \mathcal{A}_{\mathcal{D}}, N \in \mathbb{Q}^2$ such that*

$C = (E^+ \cup E^-) \cap R_i^{pq}(N)$.

Proof. We only have to prove the transitivity of the relation $\sim$. Let $A, B, C \in E^+_{0,p,q}$ be three points such that $A \sim B$ and $B \sim C$.

Let $N_1 = \langle \max(p(A), p(B)), \max(q(A), q(B))\rangle_{p,q}$, $N_2 = \langle \max(p(B), p(C)), \max(q(B), q(C))\rangle_{p,q}$ and $N = \langle \max(p(N_1), p(N_2)), \max(q(N_1), q(N_2))\rangle_{p,q}$. We have $R_0^{pq}(N_1) \cap E^- = R_0^{pq}(N_2) \cap E^- = \varnothing$ and we must prove that $R_0^{pq}(N) \cap E^- = \varnothing$.

If $R_0^{pq}(N_1) \subseteq R_0^{pq}(N_2)$ or $R_0^{pq}(N_1) \supseteq R_0^{pq}(N_2)$ we have $N \in \{N_1, N_2\}$ and so $R_0^{pq}(N) \cap E^- = \varnothing$.

So we can suppose $p(N_1) < p(N_2)$ and $q(N_1) > q(N_2)$ (after exchanging A and C if necessary).

Thus $\max(p(A), p(B)) < \max(p(B), p(C))$ so $p(B) < \max(p(B), p(C))$ *i.e.* $p(B) < p(C)$, and then:

$$p(A) \leq p(B) = p(N_1) < p(C) = p(N_2)$$
$$\text{or}$$
$$p(B) \leq p(A) = p(N_1) < p(C) = p(N_2).$$

Similarly:

$$q(A) = q(N_1) > q(B) = q(N_2) \geq q(C)$$
$$\text{or}$$
$$q(A) = q(N_1) > q(C) = q(N_2) \geq q(B).$$

For any of the four combinations of these cases, we have $A \in R_3^{pq}(N_1) \cap R_0^{pq}(N_1)$, $B \in R_0^{pq}(\langle p(N_1), q(N_2)\rangle)$, $C \in R_1^{pq}(N_2) \cap R_0^{pq}(N_2)$.

Now we suppose that $E^- \cap R_0^{pq}(N)$ is non-empty. Let M be a point of $E^- \cap R_0^{pq}(N)$ which minimizes $p(M) + q(M)$.

We have $p(N_1) < p(M) \leq p(N_2)$ and $q(N_1) > q(M) \geq q(N_2)$, so $B \in R_0^{pq}(\langle p(N_1), q(N_2)\rangle) \subseteq R_0^{pq}(M)$ and $C \in R_1^{pq}(N_2) \subseteq R_1^{pq}(M)$. So if $q(M_p) \geq q(M)$ then $M \in QCONV_{pq}(B, C, M_p)$, and $M \in F^+$ which is impossible, and so $q(M_p) < q(M)$. Similarly we have $p(M_q) < p(M)$. Thus $R_2^{pq}(M) \cap E^+ = \varnothing$. By minimality of $p(M) + q(M)$ we also have $R_0^{pq}(M) \cap E^- = \{M\}$. Finally $R_2^{pq}(M) \cap E^+ = \varnothing$ and $R_0^{pq}(M) \cap E^- = \{M\}$ which is impossible by Lemma 13 so $E^- \cap R_0^{pq}(N) = \varnothing$ and $A \sim C$. (see Figure 11)

Now we consider an equivalence class $D \subseteq E^+_{0,p,q}$. Let A be a point of D which maximizes p, and B be a point of D which maximizes q. We have $A \sim B$ so $R_0^{pq}(N) \cap E^- = \varnothing$ where $N = \langle p(A), q(B)\rangle$. By definition of A and B we have

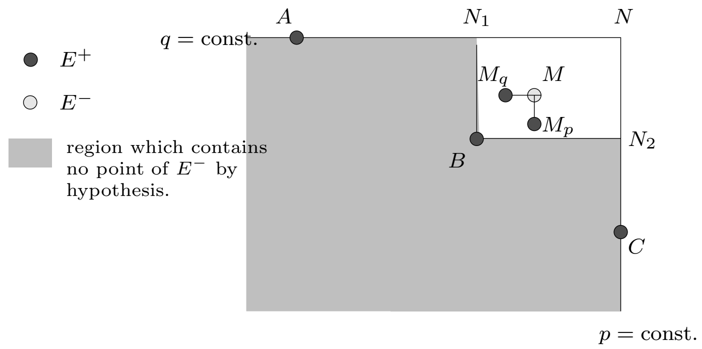


Fig. 11. Why do $A \sim B$ and $B \sim C$ imply $A \sim C$?

$D \subseteq R_0^{pq}(N)$, and by definition of $\sim$, if $M \in E^+ \cap R_0^{pq}(N)$ then $M \sim A$, so $D = R_0^{pq}(N) \cap (E^+ \cup E^-)$. $\square$

The following lemma shows that it is not necessary to suppose that $R_i^{pq}(N)$ is an ASP, in the definition of the relation $\sim$.

Lemma 17. *Let A, B be two points of E^+. If there exists a quadrant $R_i^{pq}(N)$ with $N \in \mathbb{Q}^2$ such that $R_i^{pq}(N) \cap E^- = \varnothing$ then $A \sim B$.*

Proof. Let $R_i^{pq}(N)$ be a quadrant which satisfies the conditions of Lemma 17 and which is maximum for the following order:

$$R_i^{rs}(N) \prec R_j^{r's'}(N') \text{ iff } R_i^{rs}(O) \subseteq R_j^{r's'}(O) \tag{4}$$

where O is the point $(0, 0)$ (this order neither depends on N nor on N').

We can suppose that $i = 0$ and $N = \langle \max(p(A), p(B)), \max(q(A), q(B)) \rangle$.

Suppose that $R_0^{pq}(N)$ is not an ASP, so there exists a direction r such that $r = \alpha p + \beta q$ with $\alpha\beta > 0$. We can suppose (like in Lemma 14) that $\alpha > 0, \beta > 0$.

- If $A = N$ then we have $B \in R_0^{pq}(A) \subseteq R_0^{pr}(A) \cap R_0^{qr}(A)$ so $R_0^{pr}(B) \subseteq R_0^{pr}(A)$ and $R_0^{qr}(B) \subseteq R_0^{qr}(A)$. If $p(A_r) < p(A)$ then $R_0^{qr}(A) \cap E^- = \varnothing$, and if $p(A_r) > p(A)$ then $R_0^{pq}(A) \cap E^- = \varnothing$. So $R_0^{pr}(A)$ or $R_0^{qr}(A)$ verifies the conditions of Lemma 17 and we have $R_0^{pr}(A) \not\prec R_0^{pq}(A)$ and $R_0^{qr}(A) \not\prec R_0^{pq}(A)$, so there is always a contradiction with the maximality of $R_0^{pq}(A)$.
- If $B = N$ then we can make the same proof as previously by exchanging A and B.
- The remaining cases are $N = \langle p(A), q(B) \rangle$ and $N = \langle p(B), q(A) \rangle$. Suppose the first case:

  Then we have $(p(A_r) > p(A)$ and $p(B_r) > p(B))$ or $(p(A_r) < p(A)$ and $p(B_r) < p(B))$ because otherwise:

· If $r(B) \leq r(A)$ then $A \in QCONV_{pr}(A_r, A_p, B_r)$ (see Figure 12) which is *impossible.*

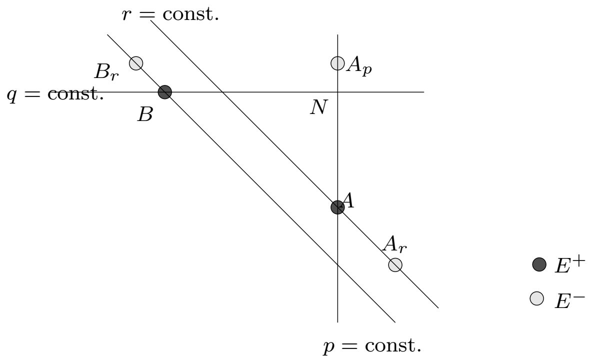


Fig. 12. Case $r(B) \leq r(A)$

· If $r(B) \geq r(A)$ then $B \in QCONV_{qr}(B_r, B_q, A_r)$ which is *impossible.*
We can deduce that $(R_0^{pr}(A) \cup R_0^{pr}(B)) \cap E^- = \varnothing$ or $(R_0^{qr}(A) \cup R_0^{qr}(B)) \cap E^- = \varnothing$. Suppose we are in the first case. Let $N' = \langle \max(p(A), p(B)), \max(r(A), r(B)) \rangle_{p,r}$, we have $R_0^{pr}(N') \subseteq R_0^{pr}(A) \cup R_0^{pq}(N) \cup R_0^{pr}(B)$ and so $R_0^{pr}(N') \cap E^- = \varnothing$ in contradiction with the maximality of $R_0^{pq}(N)$.

□

Lemma 18. *Let A, B be two points of $E^+_{i,p,q}$ such that $A \sim B$. Then for any direction $r \in \mathcal{D}$ and any point $M \in E^+$ such that $r(A) \leq r(M) \leq r(B)$, we have $M \in R_i^{pq}(M_r)$ and $M \sim A \sim B$.*

Proof. We suppose $i = 0$. Let $N = \langle \max(p(A), p(B)), \max(q(A), q(B)) \rangle_{p,q}$. If $M \in R_0^{pq}(N)$ then the conclusion of the lemma is clear. So we can suppose that $p(M) > \max(p(A), p(B))$ or $q(M) > \max(q(A), q(B))$.

Suppose that we are in the case $p(M) > \max(p(A), p(B))$. If $p(M_r) \leq p(M)$ then $M_r \in QCONV_{pr}(M, A, B)$. So $p(M_r) > p(M)$, and since $r = \alpha p + \beta q$ with $\alpha\beta \leq 0$ we also have $q(M_r) > q(M)$ and so $M \in R_0^{pq}(M_r)$. Moreover we have $q(M) \geq \min(q(A), q(B))$, because otherwise we cannot have $r(A) \leq r(M) \leq r(B)$, so A or B is in $R_0^{pq}(M)$ and so $M \sim A \sim B$.

Similarly if $q(M) > \max(q(A), q(B))$ then $q(M_r) > q(M)$, $p(M_r) > p(M)$ and $M \sim A \sim B$. □

Lemma 19. *Let $r \in \mathcal{D}$. If $A \sim B$ then $A_r \sim B_r$.*

Proof. Let A, B, r satisfy the conditions of the lemma. We denote $A' = A_r$, $B' = B_r$. We can suppose that $A, B \in E^+_{0,p,q}$ with $(0, p, q) \in \mathcal{A}_{\mathcal{D}}$. Because R_0^{pq} is an ASP, we have $r = \alpha p + \beta q$ with $\alpha\beta \leq 0$. We can also suppose that $\alpha \leq 0$ and $\beta \geq 0$. Let $N = \langle \max(p(A), p(B)), \max(q(A), q(B)) \rangle$. As $A \sim B$ we have

$R_0^{pq}(N) \cap E^- = \varnothing$. By exchanging A and B if necessary, we can suppose that $r(A') = r(A) \leq r(N) \leq r(B) = r(B')$.

There exist i' and j' such that $R_{i'}^{pq}(A') \cap E^+ = \varnothing$ and $R_{j'}^{pq}(B') \cap E^+ = \varnothing$, thus i', j' cannot be equal to zero because $A \in R_0^{pq}(A')$ and $B \in R_0^{pq}(B')$.

There are 9 other remaining cases:

- $i' = 1$, $j' = 1$. So we have $R_1^{pq}(A') \cap E^+ = \varnothing$. For any M in $R_1^{pr}(A')$, we have $A' \in QCONV_{pr}(A, A'_p, M)$ and so $M \notin E^+$ and $R_1^{pr}(A') \cap E^+ = \varnothing$. In the same way $R_1^{pr}(B') \cap E^+ = \varnothing$.
  - If $p(B') \leq p(A')$ then $R_1^{pr}(A') \subseteq R_1^{pr}(B')$ and so by Lemma 17 we have $A' \sim B'$.
  - Suppose now $p(B') \geq p(A')$. Let $N' = \langle p(A'), r(B') \rangle_{p,r}$. We suppose that $R_1^{pr}(N') \cap E^+ \neq \varnothing$. Let M be a point of $R_1^{pr}(N') \cap E^+$ which maximizes $p(M) - r(M)$. By Lemma 18 we have $p(M_r) \geq p(M)$. Moreover $r(M_p) \leq r(M)$ because otherwise $M \in QCONV_{pr}(A, M_p, M_r)$. So $R_3^{pr}(M) \cap E^- \neq \varnothing$, but by the maximality of M we have $R_1^{pr}(M) \cap E^+ = \{M\}$ which is impossible by Lemma 13, so $R_1^{pr}(N') \cap E^+ = \varnothing$, and then by Lemma 17, $A' \sim B'$ (see figure13).

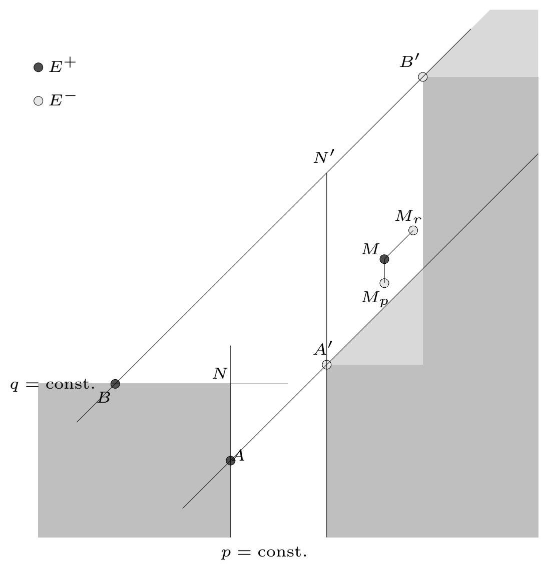


Fig. 13. Case $i' = 1$, $j' = 1$ and $p(B') \geq p(A')$.

- $i' = 1$, $j' = 2$. In this case we still have $R_1^{pr}(A') \cap E^+ = \varnothing$.
  - If $p(B') \leq p(A')$ then by Lemma 18, $p(A'_{pr}) \geq p(A'_p)$ and so $A'_p \in QCONV_{pr}(A', B', A'_{pr})$ which is *impossible*.
  - If $p(B') \geq p(A')$, in the same way, we have $B'_p \in QCONV_{pr}(A', B', B'_{pr})$ which is *impossible*.
- $i' = 1$, $j' = 3$. We have $R_1^{pr}(A') \cap E^+ = \varnothing$ and $R_2^{qr}(B') \cap E^+ = \varnothing$. By Lemma 18, we have $p(B'_{pr}) \geq p(B'_p)$ and $p(A'_{pr}) \geq p(A'_p)$.
  - If $p(A') \leq p(B')$ then $B'_p \in QCONV_{pr}(B'_{pr}, A', B')$ which is impossible.

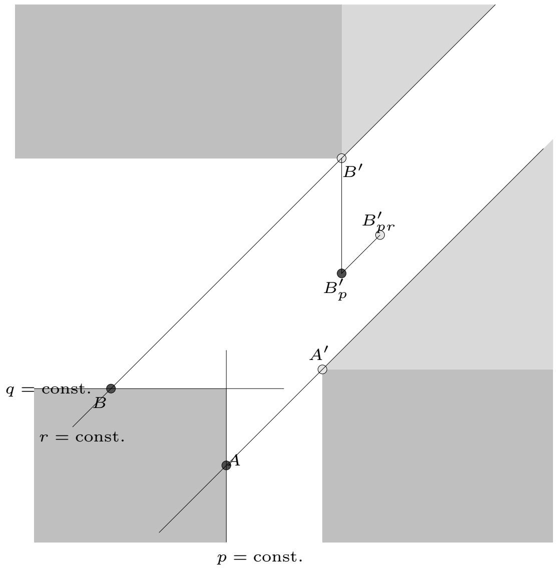


Fig. 14. Case $i' = 1$, $j' = 3$ and $p(A') \leq p(B')$

- · If $p(A') \geq p(B')$ then $A'_p \in QCONV_{pr}(A'_{pr}, A', B')$ which is impossible.
- $i' = 2$, $j' = 1$. We have $R_1^{pr}(B') \cap E^+ = \varnothing$.
  - · If $p(B') \geq p(A')$ then $B' \in R_2^{pq}(A')$, so $A' \sim B'$.
  - · If $p(B') \leq p(A')$ then $A' \in R_1^{pr}(B')$, so by lemma 17, $A' \sim B'$.
- $i' = 2$, $j' = 2$.
  - · If $p(A') \leq p(B')$, because $r(A') \leq r(B')$ we have $q(A') \leq q(B')$. And so $B' \in R_2^{pq}(A')$ therefore $A' \sim B'$.
  - · If $p(A') \geq p(B')$ and $q(A') \geq q(B')$ then $A' \in R_2^{pq}(B')$, and $A' \sim B'$.
  - · We suppose the remaining case: $p(A') \geq p(B')$ and $q(A') \leq q(B')$. We use the same argument as for the case $i' = 1, j' = 1$. Precisely, let $N' = \langle p(B'), q(A')\rangle_{p,q}$ and we suppose that $R_2^{pq}(N') \cap E^+ \neq \varnothing$. Let M be a point of $R_2^{pq}(N') \cap E^+$ which maximizes $p(M) + q(M)$. By lemma 18 $p(M_r) \geq p(M)$. We have $p(M_q) \geq p(M)$ because otherwise $M \in QCONV_{qr}(M_r, M_q, A')$, and $q(M_p) \geq p(M)$ because otherwise $M \in QCONV_{pr}(M_r, M_p, B')$. So $R_0^{pq}(M) \cap E^- = \varnothing$, and by maximality of M we have $R_2^{pq}(M) \cap E^+ = \{M\}$, but by lemma 13, this situation is impossible, so $R_2^{pq}(N') \cap E^+ = \varnothing$ and $A' \sim B'$.
- $i' = 2$, $j' = 3$. It is similar to the case $i' = 1, j' = 2$
- $i' = 3$, $j' = 1$. We have $R_2^{qr}(A') \cap E^+ = \varnothing$ and $R_1^{pr}(B') \cap E^+ = \varnothing$.
  - · If $p(A') \geq p(B')$ then $A' \in R_1^{pr}(B')$, so $A' \sim B'$.
  - · If $p(A') \leq p(B')$ then $B' \in R_2^{qr}(A')$, so $A' \sim B'$.
- $i' = 3$, $j' = 2$. It is similar to the case $i' = 2, j' = 1$.
- $i' = 3$, $j' = 3$. It is similar to the case $i' = 1, j' = 1$.

$\square$

For $(i, p, q) \in \mathcal{A}_{\mathcal{D}}$, we define $\mathcal{C}^+_{i,p,q}$ (*resp* $\mathcal{C}^-_{i,p,q}$) as the set of equivalence classes for the relation $\sim$ on $E^+_{i,p,q}$ (*resp* $E^-_{i,p,q}$) and $\mathcal{C}_{i,p,q}$ as $\mathcal{C}^+_{i,p,q} \cup \mathcal{C}^-_{i,p,q}$. So the sets of all the equivalence classes on E^+, E^-, $E^+ \cup E^-$ are:

$$\mathcal{C}^+ = \bigcup_{(i,p,q)\in\mathcal{A}_\mathcal{D}} (\mathcal{C}^+_{i,p,q}), \qquad \mathcal{C}^- = \bigcup_{(i,p,q)\in\mathcal{A}_\mathcal{D}} (\mathcal{C}^-_{i,p,q}), \qquad \mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- = \bigcup_{(i,p,q)\in\mathcal{A}_\mathcal{D}} (\mathcal{C}_{i,p,q}).$$

The previous lemma shows that for every class $C \in \mathcal{C}^+$ and any $r \in \mathcal{D}$ there exists another class, denoted $(C)_r \in \mathcal{C}^-$ such that $X_r C = X_r((C)_r)$. The application $C \mapsto (C)_r$ is a bijection from $\mathcal{C}^+$ onto $\mathcal{C}^-$, its inverse (also denoted $C \mapsto (C)_r$) is defined in the same way. In particular $|\mathcal{C}|$ is even.

Now we give a graph-structure to the set $\mathcal{C}$. If $C_1, C_2 \in \mathcal{C}$ then we say that $C_1 <_p C_2$ if for any $M_1 \in C_1, M_2 \in C_2$ we have $p(M_1) < p(M_2)$. We also define the relation $>_p$ by $C_1 >_p C_2 \iff C_2 <_p C_1$.

Lemma 20. *For any $(i,p,q) \in \mathcal{A}_\mathcal{D}$ the class $\mathcal{C}_{i,p,q}$ is non-empty.*

Proof. We can suppose $i = 0$. The class of the point M of $E^+ \cup E^-$ which minimizes $p(M) + q(M)$ is in $\mathcal{C}_{0,p,q}$. $\square$

Lemma 21. *For any $(i,p,q) \in \mathcal{A}_\mathcal{D}$ and $r \in \mathcal{D}$ the relation $<_r$ is a total strict order on $\mathcal{C}_{i,p,q} = \mathcal{C}^+_{i,p,q} \cup \mathcal{C}^-_{i,p,q}$. Moreover the graph associated to $(\mathcal{C}_{i,p,q}, <_r)$ is a chain which does not depend on r.*

Proof. We suppose $i = 0$. Let $C_1, C_2 \in \mathcal{C}_{0,p,q}$ with $C_1 \neq C_2$. By Lemma 16 there exist N_1, N_2 such that $C_1 = (E^+ \cup E^-) \cap R_0^{pq}(N_1)$ and $C_2 = (E^+ \cup E^-) \cap R_0^{pq}(N_1)$. So if $p(N_1) < p(N_2)$ then $q(N_2) > q(N_1)$ and so $C_1 <_p C_2$ and $C_1 >_q C_2$. Otherwise $C_1 >_p C_2$ and $C_2 <_q C_1$. Any direction $r \in \mathcal{D} \setminus \{p,q\}$ can be written $\alpha p + \beta q$ with $\alpha\beta < 0$ so we have $C_1 <_r C_2$ or $C_1 >_r C_2$.

So for any $r \in \mathcal{D}$, we have $C_1 <_r C_2$ or $C_2 <_r C_1$, thus $<_r$ is a total order. Moreover we have for any r:

$$\forall C_1, C_2 \in \mathcal{C}_{0,p,q}\ C_1 <_p C_2 \iff C_2 <_q C_1 \iff C_1 <_r C_2$$
$$\text{or}$$
$$\forall C_1, C_2 \in \mathcal{C}_{0,p,q}\ C_1 <_p C_2 \iff C_2 <_q C_1 \iff C_2 <_r C_1$$

So the graph-relation $C_1 \neq C_2$ and $\nexists C\ (C_1 <_r C <_r C_2$ or $C_2 <_r < C <_r C_1)$ does not depend on r. $\square$

Lemma 22. *Let r be any direction of $\mathcal{D}$ and $C_1, C_2 \in \mathcal{C}$. Then $C_1 <_r C_2$ or $C_2 <_r C_1$ or $C_1 = C_2$ or $(C_1)_r = C_2$.*

Proof. Let $C_1, C_2 \in \mathcal{C}^+$ such that $C_1 \neq C_2$ and $M \in C_1, N \in C_2$. By Lemma 18 we have $r(N) \notin [\min_{P\in C_1} r(P), \max_{P\in C_1} r(P)]$ and $r(M) \notin [\min_{P\in C_2} r(P), \max_{P\in C_2} r(P)]$, so $C_1 <_r C_2$ or $C_2 <_r C_1$.

By considering $(C_1)_r$ or $(C_2)_r$ the cases $C_1 \in \mathcal{C}^-$ or $C_2 \in \mathcal{C}^-$ are reduced to the previous one. $\square$

Definition 23. Two distinct classes $C_1, C_2 \in \mathcal{C}$ are said to be consecutive (denoted $C_1 \mathrm{CONS} C_2$) if one of the following statements is true:

- $C_1, C_2 \in \mathcal{C}_{i,p,q}$ and $\nexists C \in \mathcal{C}_{i,p,q}$ $(C_1 <_p C <_p C_2$ or $C_2 <_p < C <_p C_1)$,
- $C_1 \in \mathcal{C}_{i,p,q}$, $C_2 \in \mathcal{C}_{j,p,r}$, $C_1 = \min_{<_p} \mathcal{C}_{i,p,q}$, $C_2 = \min_{<_p} \mathcal{C}_{j,p,r}$, $R_i^{pq}(O) \cup R_j^{pr}(O) = \{M \ : \ p(M) \leq 0\}$,
- $C_1 \in \mathcal{C}_{i,p,q}$, $C_2 \in \mathcal{C}_{j,p,r}$, $C_1 = \max_{<_p} \mathcal{C}_{i,p,q}$, $C_2 = \max_{<_p} \mathcal{C}_{j,p,r}$, $R_i^{pq}(O) \cup R_j^{pr}(O) = \{M \ : \ p(M) \geq 0\}$.

Lemma 24. *Let C_1 and C_2 be two consecutive classes by the following property:*

$C_1 \in \mathcal{C}_{i,p,q}, C_2 \in \mathcal{C}_{j,p,r}, C_1 = \min_{<_p} \mathcal{C}_{i,p,q}, C_2 = \min_{<_p} \mathcal{C}_{j,p,r}, R_i^{pq}(O) \cup R_j^{pr}(O) = \{M \ : \ p(M) \leq 0\}.$

Then $C_1 = (C_2)_p$ and for any other $C \in \mathcal{C}$ we have $C_1 <_p C$ and $C_2 <_p C$.

Proof. Let $M \in E^+ \cup E^-$ such that $p(M)$ is minimum. We have $M \in \mathcal{C}_{i,p,q}$ or $M \in \mathcal{C}_{j,p,r}$ and then $M \in C_1$ or $M \in C_2$. So by Lemma 22 we have $C_1 = (C_2)_p$ and for any other $C \in \mathcal{C}$ we have $C_1 <_p C$ and $C_2 <_p C$. $\square$

By Lemmas 21, 20 and the cyclicity of $\mathcal{A}_\mathcal{D}$, the graph $(\mathcal{C}, \mathrm{CONS})$ is cyclic. Let $m = |\mathcal{C}|$ and $(C_k)_{k \in \mathbb{F}_m}$ such that $C_k \mathrm{CONS} C_{k+1}$. We recall that m is even.

Lemma 25. *For any direction $p \in \mathcal{D}$ there exists $s \in \mathbb{F}_m$ such that*

$$\begin{array}{c} C_{s-1} <_p C_{s-2} <_p \ldots <_p \ C_{s-\frac{m}{2}} \\ C_s \ <_p C_{s+1} <_p \ldots <_p C_{s+\frac{m}{2}-1} \end{array}$$

and $C_s = (C_{s-1})_p$, $C_{s+1} = (C_{s-2})_p, \ldots C_{s+\frac{m}{2}-1} = (C_{s-\frac{m}{2}})_p$.

Proof. We can suppose that $\mathcal{A}_\mathcal{D}$ has the form:

$$\begin{array}{ccccccc} (3,p_0,p_1) & - & (2,p_1,p_2) & - \ldots - & (2,p_{n-2},p_{n-1}) & - & (2,p_{n-1},p_0) \\ | & & & & & & | \\ (0,p_{n-1},p_0) & - & (0,p_{n-2},p_{n-1}) & - \ldots - & (0,p_1,p_2) & - & (1,p_0,p_1) \end{array}$$

with $p_0 = p$. By Lemma 24 the two classes $D = \min_{<_{p_0}} \mathcal{C}_{3,p_0,p_1}$ and $D' = \min_{<_{p_0}} \mathcal{C}_{0,p_{n-1},p_0}$ are consecutive so there exists s such that $\{C_{s-1}, C_s\} = \{D, D'\}$. We suppose for example $D = C_{s-1}$ and $D' = C_s$.

For any $i \geq 1$ we have $p = \alpha_i p_i + \beta_i p_{i+1}$ with $\alpha_i < 0$ and $\beta_i > 0$. Let $C_{s_1} = \max_{<_{p_1}} \mathcal{C}_{3,p_0,p_1} = \max_{<_p} \mathcal{C}_{3,p_0,p_1}$ and $C_{s_1+1} = \max_{<_{p_1}} \mathcal{C}_{2,p_1,p_2} = \min_{<_p} \mathcal{C}_{2,p_1,p_2}$. By Lemma 24 $C_{s_1+1} = (C_{s_1})_{p_1}$. For any $M \in C_{s_1}$, we have $p(M_{p_1}) > p(M)$ so $C_{s_1+1} >_p C_{s_1}$.

More generally if $C_{s_i} = \max_{<_{p_{i+1}}} \mathcal{C}_{2,p_i,p_{i+1}}$ and $C_{s_i+1} = \max_{<_{p_{i+1}}} \mathcal{C}_{2,p_{i+1},p_{i+2}}$. Then $C_{s_i+1} = (C_{s_i})_{p_{i+1}}$, and if $M \in C_{s_i}$ then $p_i(M_{p_{i+1}}) < p_i(M)$ and so $p(M_{p_{i+1}}) > p(M)$ and $C_{s_i} <_p C_{s_i+1}$. We deduce that there exist s' such that $C_{s'} = \max_{<_{p_0}} \mathcal{C}_{2,p_{n-1},p_0}$ and:

$$C_s <_p C_{s+1} <_p \ldots <_p C_{s'}.$$

On the same way, we can prove that there exists s'' such that $C_{s''} = \max_{<_{p_0}} \mathcal{C}_{1,p_0,p_1}$ and:

$$C_{s-1} <_p C_{s-2} <_p \ldots <_p C_{s''}.$$

We have $s'' = (s'+1) \mod m$ and $s' = s + \frac{m}{2} - 1, s'' = s - \frac{m}{2}$. Then we can see by recurrence that $(C_{s+k})_p = C_{s-1-k}$ for any $0 \leq k < \frac{m}{2}$. So this lemma has been proved. $\square$

Now we define $A_k = \frac{1}{|C_k|} \sum_{M \in C_k} M$. We have $p(A_k) = \frac{1}{|C_k|} \sum_{M \in C_k} p(M)$. So if $C_k <_p C_l$ then $p(A_k) < p(A_l)$, and if $C_k = (C_l)_r$ then $p(A_k) = p(A_l)$. We deduce from Lemma 25 that the sequence $(A_k)_{k \in \mathbb{F}_m}$ is a $\mathcal{D}$-sequence. So the implication (8)⇒(9) of Theorem 12 has been proved.

3.4 Construction of an affinely regular polygon

Now we shall prove the implication (9) ⇒ (5) of Theorem 12. So in this section, $\mathcal{D}$ is a set of directions such that $|\mathcal{D}| \geq 2$, and $(A_k)_{k \in \mathbb{F}_m}$ is a $\mathcal{D}$-sequence. This sequence can also be considered as an element of $\mathbb{C}^m$ after the identifications $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{F}_m = \{0, 1, \ldots, m-1\}$.

Definition 26. We define the function $\phi : \mathbb{C}^m \to \mathbb{C}^m$ by

$$\phi((M_k)_{k \in \mathbb{F}_m}) = \left(\frac{\frac{M_{k-1}+M_k}{2} + \frac{M_k+M_{k+1}}{2}}{2} \right)_{k \in \mathbb{F}_m} = \left(\frac{1}{4} M_{k-1} + \frac{1}{2} M_k + \frac{1}{4} M_{k+1} \right)_{k \in \mathbb{F}_m}$$

Lemma 27. *If $(M_k)_{k \in \mathbb{F}_m}$ is a $\mathcal{D}$-sequence then $(N_k)_{k \in \mathbb{F}_m} = \phi((M_k)_{k \in \mathbb{F}_m})$ is also a $\mathcal{D}$-sequence. More precisely if p is* any *direction of $\mathcal{D}$ and s is the index such that*

$$\begin{array}{ccccccc} p(M_{s-1}) & < & p(M_{s-2}) & < \ldots < & p(M_{s-\frac{m}{2}}) \\ \shortparallel & & \shortparallel & & \shortparallel \\ p(M_s) & < & p(M_{s+1}) & < \ldots < & p(M_{s+\frac{m}{2}-1}) \end{array} \tag{5}$$

then,

$$\begin{array}{ccccccc} p(N_{s-1}) & < & p(N_{s-2}) & < \ldots < & p(N_{s-\frac{m}{2}}) \\ \shortparallel & & \shortparallel & & \shortparallel \\ p(N_s) & < & p(N_{s+1}) & < \ldots < & p(N_{s+\frac{m}{2}-1}). \end{array}$$

Proof. For $0 \leq k < \frac{m}{2}$, we obtain from (5):

$$\begin{aligned} p(N_{s-k-1}) &= \Big(\frac{1}{4}p(M_{s-k-2}) + \frac{1}{2}p(M_{s-k-1}) + \frac{1}{4}p(M_{s-k})\Big) \\ &= \Big(\frac{1}{4}p(M_{s+k-1}) + \frac{1}{2}p(M_{s+k}) + \frac{1}{4}p(M_{s+k+1})\Big) \\ &= p(N_{s+k}) \end{aligned}$$

And moreover for $0 \leq k < \frac{m}{2} - 1$, by (5) we have:

$$p(M_{s+k-1}) \leq p(M_{s+k}) < p(M_{s+k+1}) \leq p(M_{s+k+2})$$

so $p(N_{s+k}) < p(N_{s+k+1})$. $\square$

Let G be the gravity center of $(A_k)_{k\in\mathbb{F}_m}$ and $(G)_{k\in\mathbb{F}_m} = (G, G, \ldots, G)$.

Lemma 28. *The sequence $\frac{\phi^n((A_k)_{k\in\mathbb{F}_m})-(G)_{k\in\mathbb{F}_m}}{\cos^{2n}(\frac{\pi}{m})}$ converges to an affinely regular $\mathcal{D}$-polygon as n tends to infinity.*

This result is a generalization of a result of Darboux. ([7])

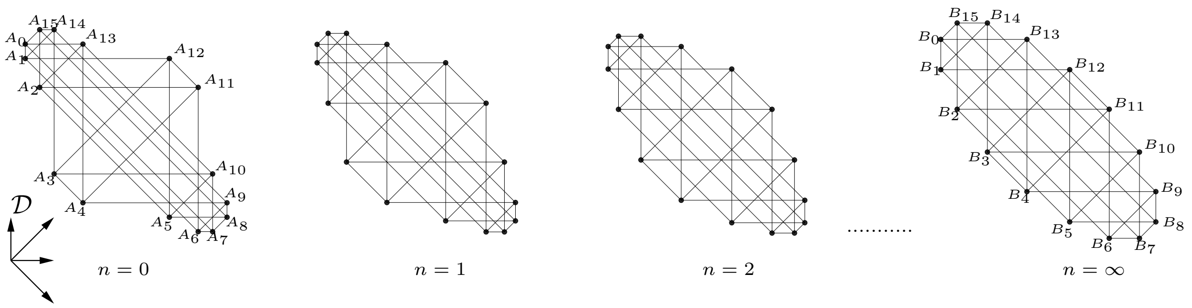


Fig. 15. The three first terms of a sequence $\frac{\phi^n((A_k)_{k\in\mathbb{F}_m})-(G)_{k\in\mathbb{F}_m}}{\cos^{2n}(\frac{\pi}{m})}$ and its limit. $(\mathcal{D} = \{x, y, x+y, x-y\})$

Proof. The function ϕ is linear. It is represented by the matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \dots & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{4} & 0 & 0 & \dots & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

This matrix is diagonalizable and has the eigenvectors $Y_j = (e^{i\frac{2\pi jk}{m}})_{k\in\mathbb{F}_m}$ associated to the eigenvalues $v_j = \cos^2(\frac{j\pi}{m})$.

Since the family $(Y_j)_{0\le j<m}$ is a basis of $\mathbb{C}^m$, there exist coefficients $\lambda_j \in \mathbb{C}$ such that $(A_k)_{k\in\mathbb{F}_m} = \sum_{j=0}^{m-1} \lambda_j Y_j$, or equivalently:

$$(A_k)_{k\in\mathbb{F}_m} = \lambda_0 Y_0 + \sum_{j=1}^{\frac{m}{2}-1} (\lambda_j Y_j + \lambda_{m-j}\overline{Y_j}) + \lambda_{\frac{m}{2}} Y_{\frac{m}{2}}$$

where $z \mapsto \overline{z}$ designs the conjugation on $\mathbb{C}$. Thus, for $n \ge 1$:

$$\phi^n((A_k)_{k\in\mathbb{F}_m}) = \lambda_0 Y_0 + \sum_{j=1}^{\frac{m}{2}-1} \cos^{2n}\left(\frac{j\pi}{m}\right)(\lambda_j Y_j + \lambda_{m-j}\overline{Y_j}).$$

(Notice that the eigenvalue associated to $Y_{\frac{m}{2}}$ is zero.)

Let $r = \min\{j \ge 1 : \lambda_j \ne 0 \text{ or } \lambda_{m-j} \ne 0\}$. If $r = \frac{m}{2}$ then all the points A_k are aligned, which is impossible because $(A_k)_{k\in\mathbb{F}_m}$ is a $\mathcal{D}$-polygon with $\mathcal{D} \ge 2$. So $1 \le r \le \frac{m}{2} - 1$.

We define the vector

$$(B_k)_{k\in\mathbb{F}_m} = \lim_{n\to\infty} \frac{\phi^n((A_k)_{k\in\mathbb{F}_m}) - \lambda_0 Y_0}{\cos^{2n}(\frac{r\pi}{m})}.$$

As $\cos(\frac{j\pi}{m}) > \cos(\frac{j'\pi}{m})$ for any $0 \le j < j' \le \frac{m}{2}$, we have $(B_k)_{k\in\mathbb{F}_m} = \lambda_r Y_r + \lambda_{m-r}\overline{Y_r}$, so $(B_k)_{k\in\mathbb{F}_m}$ is the image by the $\mathbb{R}$-linear transformation $\psi : z \mapsto \lambda_r z + \lambda_{m-r}\overline{z}$ of the polygon Y_r.

The linear transformation ψ is non-null so the sequence $(B_k)_{k\in\mathbb{F}_m}$ is not constant. Thus there exists a direction $p \in \mathcal{D}$ such that $(p(B_k))_{k\in\mathbb{F}_m}$ is not constant. By Lemma 27 we have:

$$\begin{array}{ccccc} p(B_{s-1}) & \le p(B_{s-2}) & \le \cdots \le & p(B_{s-\frac{m}{2}}) \\ \shortparallel & \shortparallel & & \shortparallel \\ p(B_s) & \le p(B_{s+1}) & \le \cdots \le & p(B_{s+\frac{m}{2}-1}). \end{array} \tag{6}$$

Since $(B_k)_{k\in\mathbb{F}_m} = \lambda_r Y_r + \lambda_{m-r}\overline{Y_r}$, there exist $b_1, b_2 \in \mathbb{R}$ such that

$$p(B_k) = b_1 \cos\left(\frac{2\pi r k}{m} + b_2\right).$$

The vector $(p(B_k))_{k\in\mathbb{F}_m}$ is not constant, and so by formula (6), we have:

$$p(B_{s-1}) = p(B_s) < p(B_{s+\frac{m}{2}-1}) = p(B_{s-\frac{m}{2}}).$$

But:

$$p(B_k) - p(B_{k-1}) = -2b_1 \sin\left(\frac{\pi r}{m}\right)\sin\left(\frac{\pi r(2k-1)}{m} + b_2\right)$$

therefore we can suppose $b_1 < 0$ and $b_2 = \frac{\pi r(1-2s)}{m}$.

The number $p(B_{s+k}) - p(B_{s+k-1}) = -2b_1 \sin\frac{\pi r}{m}\sin\frac{2\pi r k}{m}$ is non-negative for $k \in \{0, \ldots, \frac{m}{2}\}$. If $1 < r < \frac{m}{2}$ then $\left\lfloor \frac{m}{2r} + 1 \right\rfloor \in \{0, \ldots, \frac{m}{2}\}$ and $\sin\frac{2\pi r \left\lfloor \frac{m}{2r}+1 \right\rfloor}{m} < 0$. Thus the only possibility is $r = 1$.

The function gravity_center : $\mathbb{C}^m \to \mathbb{C},\ (z_k)_{k\in\mathbb{F}_m} \mapsto \frac{1}{m}\sum_{k\in\mathbb{F}_m} z_k$ is $\mathbb{C}$-linear so

$$\begin{aligned} G &= \text{gravity\_center}((A_k)_{k\in\mathbb{F}_m}) \\ &= \lambda_0 \text{gravity\_center}(Y_0) + \sum_{k\in\mathbb{F}_m\setminus\{0\}} \lambda_k \text{gravity\_center}(Y_k) \\ &= \lambda_0 \cdot 1 + \sum_{k\in\mathbb{F}_m\setminus\{0\}} \lambda_k \cdot 0 = \lambda_0. \end{aligned}$$

Then $(G)_{k\in\mathbb{F}_m} = \lambda_0 Y_0$ and so $(B_k)_{k\in\mathbb{F}_m}$ is the limit of the sequence $\left(\frac{\phi^n((A_k)_{k\in\mathbb{F}_m}) - (G)_{k\in\mathbb{F}_m}}{\cos^{2n}(\frac{\pi}{m})}\right)_n$ of Lemma 28.

So finally:

$$\lim_{n\to\infty} \frac{\phi^n((A_k)_{k\in\mathbb{F}_m}) - (G)_{k\in\mathbb{F}_m}}{\cos^{2n}(\frac{\pi}{m})} = (B_k)_{k\in\mathbb{F}_m} = \lambda_r Y_r + \lambda_{m-r}\overline{Y_r} = \lambda_1 Y_1 + \lambda_{m-r}\overline{Y_1}$$

Thus the vector $(B_k)_{k\in\mathbb{F}_m}$ is the image of a regular polygon by the $\mathbb{R}$-linear transformation $\psi : z \mapsto \lambda_1 z + \lambda_{m-1}\overline{z}$. Now we have to prove that this transformation is bijective. Suppose the converse.

Then the image, interpreted as points $\mathbb{R}^2$, of the $\mathbb{R}$-linear transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is not $\mathbb{R}^2$ and is a linear space so it is included in a line. So there exists a direction p such that $(p(B_k))_{k\in\mathbb{F}_m}$ is constant.

- We suppose $p \notin \mathcal{D}$. Let q, q' be two distinct directions of $\mathcal{D}$. The vector $(A_k)_{k\in\mathbb{F}_m}$ is a $\mathcal{D}$-polygon so there exist three *distinct* integers k_1, k_2, k_3 such that $q(A_{k_1}) = q(A_{k_2})$, $q'(A_{k_1}) = q'(A_{k_3})$. Moreover by Lemma 27 we can suppose $q(B_{k_1}) = q(B_{k_2})$ and $q'(B_{k_1}) = q'(B_{k_3})$. But B_{k_1}, B_{k_2}, B_{k_3} are aligned on a line whose direction is not q nor q' so $B_{k_2} = B_{k_1} = B_{k_3}$. We deduce that the k_1-th, k_2-th, k_3-th points of Y_1 have the same image by ψ, which is impossible because these points are not aligned and ψ is supposed to be non-null.

- We suppose $p \in \mathcal{D}$. Let r' be defined by:

$$r' = \min\{j : p(\lambda_j Y_j + \lambda_{m-j}\overline{Y_j}) \text{ is not a constant vector}\}.$$

We have

$$\frac{\phi^n((A_k)_{k\in\mathbb{F}_m}) - \left(\lambda_0 Y_0 + \sum_{j=1}^{r'-1}(\lambda_j Y_j + \lambda_{m-j}\overline{Y_j})\right)}{\cos^{2n}(\frac{r'\pi}{m})} \underset{n\to\infty}{\longrightarrow} \lambda_{r'}Y_{r'} + \lambda_{m-r'}\overline{Y}_{r'} = (C_k)_{k\in\mathbb{F}_m}$$

But $(A_k)_{k\in\mathbb{F}_m}$ is a $\mathcal{D}$-sequence and $p(\lambda_j Y_j + \lambda_{m-j}\overline{Y_j})$ is a constant vector for any $j < r'$, so again by Lemma 27, there exists an s such that:

$$\begin{array}{ccccccc} p(C_{s-1}) & \leq & p(C_{s-2}) & \leq & \ldots & \leq & p(C_{s-\frac{m}{2}}) \\ \shortparallel & & \shortparallel & & & & \shortparallel \\ p(C_s) & \leq & p(C_{s+1}) & \leq & \ldots & \leq & p(C_{s+\frac{m}{2}-1}). \end{array}$$

But, like for the vector $(B_k)_{k\in\mathbb{F}_m}$, it implies $r' = 1$ which contradicts that $(p(B_k))_{k\in\mathbb{F}_m}$ is constant.

We have proved that the transformation $\psi : z \mapsto \lambda_1 z + \lambda_{m-1}\overline{z}$ is bijective, so $(B_k)_{k\in\mathbb{F}_m}$ is an affinely regular polygon. It is also a $\mathcal{D}$-polygon by formula (6). □

So Theorem 12 has been proved.

3.5 Link with a conjecture

In this paragraph we describe a conjecture of [3]. Before, we recall some classical definitions:

A 4-path is a finite sequence $(M_0, M_1, \ldots, M_n)$ of points of $\mathbb{Z}^2$ such that $M_{i+1} - M_i$ is in the set $\{(\pm 1, 0), (0, \pm 1)\}$. A polyomino is a finite lattice set F which is 4-connected, which means that for any $A, B \in F$ there is a 4-path from A to B. A HV-convex set is a set which is line-convex along the horizontal and vertical directions.

Conjecture 29. *If $\mathcal{D}$ is a set of four directions containing the coordinate directions x and y, and such that the cross-ratio of these directions arranged in order is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$, then $\mathcal{D}$ determines the HV-convex polyominos.*

In fact Q-convexity is very linked to the HV-convex polyominos: Indeed, a direct consequence of proposition 2.3 of [5] is the following property:

Proposition 30. *Every HV-convex polyomino is Q-convex along $\mathcal{D} = \{x, y\}$.*

So it is natural to extend the previous conjecture by the following one:

Conjecture 31. *If $\mathcal{D}$ is a set of four directions such that the cross-ratio of these directions arranged in order is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$ and if $\mathcal{D}'$ is any pair of directions of $\mathcal{D}$ then $\mathcal{D}$ determines the class of the Q-convex sets along $\mathcal{D}'$.*

This conjecture has been checked in the case $\mathcal{D} = \{x, y, 2x + y, -x + 2y\}$ and $\mathcal{D}' = \{x, y\}$: counter-examples cannot be in the square $\{0, \ldots, 12\}^2$ ([9]).

Theorem 12 is weaker than this conjecture because in the theorem the convexity directions and the X-ray directions must be the same. Anyway it seems that the proof of this paper cannot be adapted easily to prove this conjecture (see Figure 16).

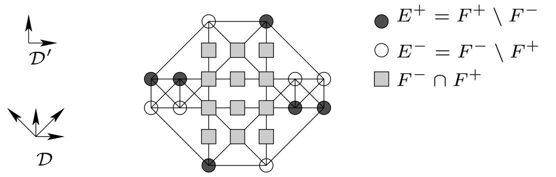


Fig. 16. The two sets F^+ and F^- are Q-convex along $\mathcal{D}' = \{x, y\}$, and have the same X-rays in $\mathcal{D} = \{x, y, x+y, x-y\}$. We cannot extract easily a $\mathcal{D}$-sequence from these sets.

4 Algorithmic consequences

In [5] the following problem

> **ReconstructionQconv($\mathcal{D}$)** where $\mathcal{D} = \{p_1, \ldots, p_d\}$
> **Instance:** d vectors $(f_{p_i}(min_{p_i}), f_{p_i}(min_{p_i} + 1), \ldots, f_{p_i}(max_{p_i}))_{1 \le i \le d}$.
> **Task:** Reconstruct a set which is Q-convex along $\mathcal{D}$ and satisfies $X_{p_i}F(j) = f_{p_i}(j)$ for all $i \in \{1, \ldots, d\}$ and $j \in \{min_{p_i}, \ldots, max_{p_i}\}$

is proved to be solved in $O(n^5)$ operations, where $n = \max(max_{p_i} - min_{p_i})$.

We also consider the more classical problem:

ReconstructionConv($\mathcal{D}$) where $\mathcal{D} = \{p_1, \ldots, p_d\}$
Instance: d vectors $(f_{p_i}(min_{p_i}), f_{p_i}(min_{p_i}+1), \ldots, f_{p_i}(max_{p_i}))_{1\le i\le d}$.
Task: Reconstruct a lattice convex set such that $X_{p_i}F(j) = f_{p_i}(j)$ for all $i \in \{1, \ldots, d\}$ and $j \in \{min_{p_i}, \ldots, max_{p_i}\}$.

Suppose that $\mathcal{D}$ determines the lattice convex sets, then by Theorem 12, $\mathcal{D}$ also determines the Q-convex sets along $\mathcal{D}$. So the solution of an instance of **ReconstructionConv($\mathcal{D}$)** is always the solution of the same instance for **ReconstructionQConv($\mathcal{D}$)**. Conversely if the solution of an instance of **ReconstructionQConv($\mathcal{D}$)** is a lattice convex set, then it is a solution of the same instance for **ReconstructionConv($\mathcal{D}$)**, otherwise it has no solutions. Since lattice convexity can be checked in a complexity less than $O(n^5)$ (see [13]), we have proved:

Theorem 32. *If $\mathcal{D}$ determines the lattice convex sets then* **ReconstructionConv($\mathcal{D}$)** *can be solved in $O(n^5)$ operations where $n = \max(max_{p_i} - min_{p_i})$.*

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