# A note on the Burrows-Wheeler transformation 

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#### Abstract

We relate the Burrows-Wheeler transformation with a result in combinatorics on words known as the Gessel-Reutenauer transformation.


## 1 Introduction

The Burrows-Wheeler transformation is a popular method used for text compression 2. The rough idea is to encode a text in two passes. In the first pass, the text $w$ is replaced by a text $T(w)$ of the same length obtained as follows: list the cyclic shitfs of $w$ in alphabetic order as the rows $w_{1}, w_{2}, \ldots, w_{n}$ of an array. Then $T(w)$ is the last column of the array. In a second pass, a simple encoding allows to compress $T(w)$, using a simple method like run-length or move-to-front encoding. Indeed, adjacent rows will often begin by a long common prefix and $T(w)$ will therefore have long runs of identical symbols. For example, in a text in english, most rows beginning with 'nd' will end with 'a'. We refer to [11 for a complete presentation of the algorithm and an analysis of its performances. It was remarked recently by S. Mantaci, A. Restivo and M. Sciortino [10] that this transformation was related with notions in combinatorics on words such as Sturmian words. Similar considerations were developped in [1] in a different context. The results presented here are also close to the ones of 4].

In this note, we study the transformation from the combinatorial point of view. We show that the Burrows-Wheeler transformation is a particular case of a bijection due to I.M. Gessel and C. Reutenauer which allows the enumeration of permutations by descents and cyclic type (see 9 ).

The paper is organized as follows. In the first section, we describe the Burrows-Wheeler transformation. The next section describes the inverse of the transformation with some emphasis on the computational aspects. The last section is devoted to the link with the Gessel-Reutenauer correspondance.

## 2 The Burrows-Wheeler transformation

The principle of the method is very simple. We consider an ordered alphabet $A$. Let $w=a_{1} a_{2} \cdots a_{n}$ be a word of length $n$ on the alphabet $A$. The Parikh vector
of a word $w$ on the alphabet $A$ is the integer vector $v=\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ where $n_{i}$ is the number of occurrences of the $i$-th letter of $A$ in $w$. We suppose $w$ to be primitive, i.e. that $w$ is not a power of another word. Let $w_{1}, w_{2}, \ldots, w_{n}$ be the sequence of conjugates of $w$ in increasing alphabetic order. Let $b_{i}$ denote the last letter of $w_{i}$, for $i=1, \ldots, n$. Then the Burrows-Wheeler transform of $w$ is the word $T(w)=b_{1} b_{2} \cdots b_{n}$.

Example 1 Let $w=a b r a c a d a b r a$. The list of conjugates of $w$ sorted in alphabetical order is represented below.

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ |
| 2 | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ |
| 3 | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ |
| 4 | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ |
| 5 | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ |
| 6 | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ |
| 7 | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ |
| 8 | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ |
| 9 | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ |
| 10 | $r$ | $a$ | $a$ | $b$ | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ |
| 11 | $r$ | $a$ | $c$ | $a$ | $d$ | $a$ | $b$ | $r$ | $a$ | $a$ | $b$ |

The word $T(w)$ is the last column of the array. Thus $T(w)=r d a r c a a a a b b$.
It is clear that $T(w)$ depends only on the conjugacy class of $w$. Therefore, in order to study the correspondance $w \mapsto T(w)$, we may suppose that $w$ is a Lyndon word, i.e. that $w=w_{1}$. Let $c_{i}$ denote the first letter of $w_{i}$. Thus the word $z=c_{1} c_{2} \cdots c_{n}$ is the nondecreasing rearrangement of $w$ (and of $T(w)$ ).

Let $\sigma$ be the permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i)=j$ iff $w_{j}=$ $a_{i} a_{i+1} \cdots a_{i-1}$. In other terms, $\sigma(i)$ is the rank in the alphabetic order of the $i$-th circular shift of the word $w$.

Example 1 (continued)
We have

$$
\sigma=\left(\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 3 & 7 & 11 & 4 & 8 & 5 & 9 & 2 & 6 & 10
\end{array}\right)
$$

By definition, we have for each index $i$ with $1 \leq i \leq n$

$$
\begin{equation*}
a_{i}=c_{\sigma(i)} \tag{1}
\end{equation*}
$$

We also have the following formula expressing $T(w)$ using $\sigma$

$$
\begin{equation*}
b_{i}=a_{\sigma^{-1}(i)-1} \tag{2}
\end{equation*}
$$

Indeed, $b_{\sigma(j)}$ is the last letter of $w_{\sigma(j)}=a_{j} a_{j+1} \cdots a_{j-1}$, whence $b_{\sigma(j)}=a_{j-1}$ which is equivalent to the above formula.

Let $\pi=P(w)$ be the permutation defined by $\pi(i)=\sigma\left(\sigma^{-1}(i)+1\right)$ where the addition is to be taken $\bmod n$. Actually, $\pi$ is just the permutation obtained by writing $\sigma$ as a word and interpreting it as an $n$-cycle. Thus, we have also $\sigma(i)=\pi^{i-1}(1)$ and

$$
\begin{equation*}
a_{i}=c_{\pi^{i-1}(1)} \tag{3}
\end{equation*}
$$

Example 1 (continued)
We have, written as a cycle

$$
\pi=\left(\begin{array}{ccccccccccc}
1 & 3 & 7 & 11 & 4 & 8 & 5 & 9 & 2 & 6 & 10
\end{array}\right)
$$

and as an array $\pi=\left(\begin{array}{ccccccccccc}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 6 & 7 & 8 & 9 & 10 & 11 & 5 & 2 & 1 & 4\end{array}\right)$
Substituting in Formula (2) the value of $a_{i}$ given by Formula (1), we obtain $b_{i}=c_{\sigma\left(\sigma^{-1}(i)-1\right)}$ which is equivalent to

$$
\begin{equation*}
c_{i}=b_{\pi(i)} \tag{4}
\end{equation*}
$$

Thus the permutation $\pi$ transforms the last column of the array of conjugates of $w$ into the first one. Actually, it can be noted that $\pi$ transforms any column of this array into the following one.

The computation of $T(w)$ from $w$ can be done in linear time. Indeed, provided $w$ is chosen as a Lyndon word, the order between the conjugates is the same as the order between the corresponding suffixes. The computation of the permutation $\sigma$ results from the suffix array of $w$ which can be computed in linear time [3] on a fixed alphabet. The corresponding result on the alphabet of integers is a more recent result. It has been proved independently by three groups of researchers, 7], 8] and [6].

## 3 Inverse transformation

We now show how $w$ can be recovered from $T(w)$. For this, we introduce the following notation. The rank of $i$ in the word $y=b_{1} b_{2} \cdots b_{n}$, denoted $\operatorname{rank}(i, y)$ is the number of occurrences of the letter $b_{i}$ in $b_{1} b_{2} \cdots b_{i}$.

We observe that for each index $i$, and for the aforementioned words $y=$ $b_{1} b_{2} \cdots b_{n}$ and $z=c_{1} c_{2} \cdots c_{n}$

$$
\begin{equation*}
\operatorname{rank}(i, z)=\operatorname{rank}(\pi(i), y) \tag{5}
\end{equation*}
$$

Indeed, we first note that for two words $u, v$ of the same length and any letter $a$, one has $a u<a v \Leftrightarrow u a<v a \quad(\Leftrightarrow u<v)$. Thus for all indices $i, j$

$$
\begin{equation*}
i<j \text { and } c_{i}=c_{j} \Rightarrow \pi(i)<\pi(j) \tag{6}
\end{equation*}
$$

Hence, the number of occurrences of $c_{i}$ in $c_{1} c_{2} \cdots c_{i}$ is equal to the number of occurrences of $b_{\pi(i)}=c_{i}$ in $b_{1} b_{2} \cdots b_{\pi(i)}$.

To obtain $w$ from $T(w)=b_{1} b_{2} \cdots b_{n}$, we first compute $z=c_{1} c_{2} \cdots c_{n}$ by rearranging the letters $b_{i}$ in nondecreasing order. Property (5) shows that $\pi(i)$ is the index $j$ such that $c_{i}=b_{j}$ and $\operatorname{rank}(j, y)=\operatorname{rank}(i, z)$. This defines the permutation $\pi$, from which $\sigma$ can be reconstructed. An algorithm computing $\pi$ from $y=T(w)$ is represented below.

```
Permutation \(\left(b_{1} b_{2} \cdots b_{n}\right)\)
    \(c \leftarrow \operatorname{SORT}\left(b_{1} b_{2} \cdots b_{n}\right)\)
    for \(i \leftarrow 1\) to \(n\) do
    if \(i=1\) or \(c_{i-1} \neq c_{i}\) then
        \(j \leftarrow 0\)
    do \(\quad j \leftarrow j+1\)
    while \(b_{j} \neq c_{i}\)
    \(\pi(i) \leftarrow j\)
return \(\pi\)
```

This algorithm can be optimized to a linear-time algorithm by storing the first position of each symbol in the word $z$.

Finally $w$ can be recovered from $z=c_{1} c_{2} \cdots c_{n}$ and $\pi$ by Formula (3). The algorithm allowing to recover $w$ is represented below.


The computation of $w$ is not possible without the Parikh vector or equivalently the word $z$. One can however always compute the word $w$ on the smallest possible alphabet associated with permutation $\pi$ (this is the computation described in (1).

## 4 Descents of permutations

A descent of a permutation $\pi$ is an index $i$ such that $\pi(i)>\pi(i+1)$. We denote by $\operatorname{des}(\pi)$ the set of descents of the permutation $\pi$. It is clear by Property (6) that if $i$ is a descent of $P(w)$, then $c_{i} \neq c_{i+1}$. Thus, the number of descents of $\pi$ is at most equal to $k-1$ where $k$ is the number of symbols appearing in the word $w$.

Example 1 (continued) The descents of $\pi$ appear in boldface.

$$
\pi=\left(\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \mathbf{7} & \mathbf{8} & \mathbf{9} & 10 & 11 \\
3 & 6 & 7 & 8 & 9 & 10 & 11 & 5 & 2 & 1 & 4
\end{array}\right)
$$

Thus $\operatorname{des}(\pi)=\{7,8,9\}$.

Let us fix an ordered alphabet $A$ with $k$ elements for the rest of the paper. Let $w$ be a word and $v=\left(n_{1}, n_{2}, \ldots, n_{k}\right)$ be the Parikh vector of $w$. We say that $v$ is positive if $n_{i}>0$ for $i=1,2, \ldots, k$. We denote by $\rho(v)$ the set of integers $\rho(v)=\left\{n_{1}, n_{1}+n_{2}, \ldots, n_{1}+\cdots+n_{k-1}\right\}$. When $v$ is positive, $\rho(v)$ has $k-1$ elements. Let $\pi=P(w)$ and let $v$ be the Parikh vector of $w$. It is clear by Formula 6 that we have the inclusion $\operatorname{des}(\pi) \subset \rho(v)$.

Example 1 (continued) The Parikh vector of the word $w=$ abracadabra is $v=(5,2,1,1,2)$ and $\rho(v)=\{5,7,8,9\}$.

The following statement results from the preceding considerations.
Theorem 1 For any positive vector $v=\left(n_{1}, n_{2}, \cdots, n_{k}\right)$ with $n=n_{1}+\cdots+n_{k}$, the map $w \mapsto \pi=P(w)$ is one to one from the set of conjugacy classes of primitive words of length $n$ on $A$ with Parikh vector $v$ onto the set of cyclic permutations on $\{1,2, \ldots, n\}$ such that $\rho(v)$ contains $\operatorname{des}(\pi)$.

This result is actually a particular case of a result stated in [9] and essentially due to I. Gessel and C. Reutenauer [5]. The complete result (9], Theorem 11.6.1 p. 378) establishes a bijection between words of type $\lambda$ and pairs $(\pi, E)$ where $\pi$ is a permutation of type $\lambda$ and $E$ is a subset of $\{1,2, \ldots, n-1\}$ with at most $k-1$ elements containing $\operatorname{des}(\pi)$. The type of a word $w$ of length $n$ is the partition of $n$ realized by the length of the factors of its nonincreasing factorization in Lyndon words. The type of a permutation is the partition resulting of the length of its cycles. Thus, Theorem corresponds to the case where $w$ is a Lyndon word (i.e. $\lambda$ has only one part) and $\pi$ is circular.

We illustrate the general case of an arbitrary word with an example for the sake of clarity. For example, the word $w=a b a a b$ has the nonincreasing factorization in Lyndon words $w=(a b)(a a b)$. Thus $w$ has type (3,2). The corresponding permutation of type $(3,2)$ is $\pi=(35)(124)$. Actually, the permutation $\pi$ is obtained as follows. Its cycles correspond to the Lyndon factors of $w$. The letters are replaced by the rank in the lexicographic order of the cyclic iterates of the conjugates. In our example, we obtain

| 1 |  | $a$ | $a$ | $b$ | $a$ | $a$ | $b$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\cdots$ |  |  |  |  |  |  |  |
| 2 | $a$ | $b$ | $a$ | $a$ | $b$ | $a$ | $\cdots$ |
| 3 |  | $a$ | $b$ | $a$ | $b$ | $a$ | $b$ |
| $\cdots$ |  |  |  |  |  |  |  |
| 4 | $b$ | $a$ | $a$ | $b$ | $a$ | $a$ | $\cdots$ |
| 5 | $b$ | $a$ | $b$ | $a$ | $b$ | $a$ | $\cdots$ |

We have $\operatorname{des}(\pi)=\{3\}$ which is actually included in $\rho(v)=\{3,5\}$.
We may observe that when the alphabet is binary, i.e. when $k=2$, Theorem 1 takes a simpler form: the map $w \mapsto P(w)$ is one-to-one from the set of primitive binary words of length $n$ onto the set of circular permutations on $\{1,2, \ldots, n\}$ having one descent.

In the general case of an arbitrary alphabet, another possible formulation is the following. Let us say that a word $b_{1} b_{2} \cdots b_{n}$ is co-Lyndon if the permutation
$\pi$ built by Algorithm Permutation is an $n$-cycle. It is clear that the map $w \mapsto T(w)$ is one-to-one from the set of Lyndon words of length $n$ on $A$ onto the set of co-Lyndon words of length $n$ on $A$.

The properties of co-Lyndon words have never been studied and this might be an interesting direction of research.

Example 2 The following array shows the correspondance between Lyndon and co-Lyndon words of length 5 on $\{a, b\}$. The permutation $\pi$ is shown on the right.

| Lyndon | co-Lyndon |  |
| :---: | :---: | :---: |
| aaaab | baaaa | $(12345)$ |
| $a a a b b$ | $b a a b a$ | $(12354)$ |
| aabab | $b b a a a$ | $(13524)$ |
| aabbb | babba | $(12543)$ |
| ababb | $b b b a a$ | $(14253)$ |
| $a b b b b$ | $b b b b a$ | $(15432)$ |

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