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Fault-free longest paths in star networks with conditional link faults

Ping-Ying Tsai [a], Jung-Sheng Fu [b], Gen-Huey Chen [c,*]

[a] Department of Computer Science and Information Engineering, Hwa Hsia Institute of Technology, Taipei, Taiwan, ROC
[b] Department of Electronics Engineering, National United University, Miaoli, Taiwan, ROC
[c] Department of Computer Science and Information Engineering, National Taiwan University, Taipei, 10764, Taiwan, ROC

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ABSTRACT

The star network, which belongs to the class of Cayley graphs, is one of the most versatile interconnection networks for parallel and distributed computing. In this paper, adopting the conditional fault model in which each node is assumed to be incident with two or more fault-free links, we show that an n-dimensional star network can tolerate up to $2n-7$ link faults, and be strongly (fault-free) Hamiltonian laceable, where $n \geq 4$. In other words, we can embed a fault-free linear array of length $n!-1$ ($n!-2$) in an n-dimensional star network with up to $2n-7$ link faults, if the two end nodes belong to different partite sets (the same partite set). The result is optimal with respect to the number of link faults tolerated. It is already known that under the random fault model, an n-dimensional star network can tolerate up to $n-3$ faulty links and be strongly Hamiltonian laceable, for $n \geq 3$.

1. Introduction

The star network [1], which belongs to the class of Cayley graphs [2], has been recognized as an attractive alternative to the hypercube. It possesses many favorable topological properties such as recursiveness, symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience (see [1]). They are all desirable when we are building an interconnection topology for a parallel and distributed system. Besides, the star network can embed rings [28], grids [18], trees [5], and hypercubes [27]. Efficient communication algorithms for shortest-path routing [28], multiple-path routing [9], broadcasting [26] and scattering [13] are also available.

A linear array, which is one of the most fundamental networks for parallel and distributed computation, is suitable for developing simple and efficient algorithms. Numerous algorithms that were designed on linear arrays for solving various algebraic problems and graph problems can be found in [21]. A linear array can be also used as a control/data flow structure for distributed computation in a network (refer to [3] for an example).

Since node faults and/or link faults may occur in networks, it is important to consider faulty networks. Previously, communication problems (e.g., routing [5,12], broadcasting [32], multicasting [25], and gossiping [10]), embedding problems [4,6,16,17,24,30], and fault diameters [8,19,29] were studied on various faulty networks. Among them, two fault models were adopted; one is the random fault model [5,8,10,12,16,17,24,25,32], and the other is the conditional fault model [4,6,19,29,30].

The *random fault model* assumed that the faults might occur anywhere without any restriction, whereas the *conditional fault model* assumed that the fault distribution must be subject to some constraint, e.g., that two or more fault-free links are incident to each node. As a consequence of the constraint, it is in general more difficult to solve problems under the conditional fault model than the random fault model.

* Corresponding author. Fax: +886 2 23628167.
E-mail address: ghchen@csie.ntu.edu.tw (G.-H. Chen).

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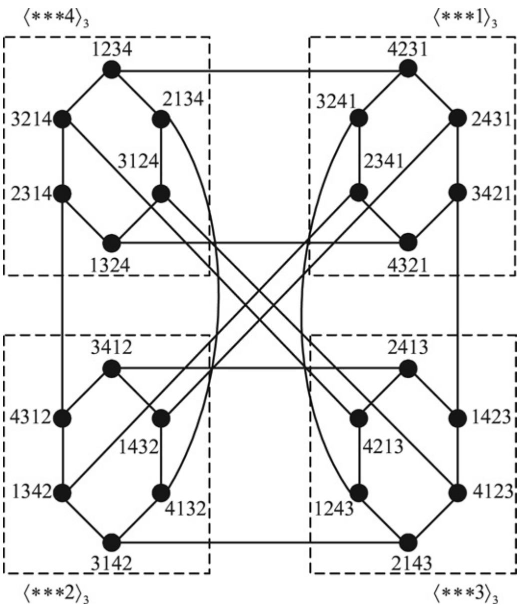

Fig. 1. S_4 and four embedded S_3's.

In this paper, under the conditional fault model and with the assumption of at least two fault-free links incident to each node, we show that an n-dimensional star network can tolerate up to $2n-7$ link faults, while retaining strongly (fault-free) Hamiltonian laceability, where $n \geq 4$. The result is optimal with respect to the number of link faults tolerated. For the same problem, at most $n-3$ link faults can be tolerated if the random fault model is adopted [24]. With our results, all parallel algorithms developed on a linear array of length $n!-1$ or $n!-2$ can be executed as well on an n-dimensional star network with up to $2n-7$ link faults.

Previous results under the conditional fault model are described as follows. With the same assumption as ours, an n-dimensional hypercube (n-cube for short) with $2n-5$ link faults is strongly (fault-free) Hamiltonian laceable [30], and an m-ary n-cube with $4n-5$ link faults has a fault-free Hamiltonian cycle [4]. On the other hand, with the assumption of each node having at least k fault-free neighbors, the minimum number of node faults whose removal may disconnect an n-cube is $(n-k)2^k$, where $1 \leq k \leq \lfloor n/2 \rfloor$ [20]. Such a minimum number was named the *restricted-node-connectivity* and denoted by *R_k-node-connectivity* [20]. The node-connectivity of an n-cube is known to be n. There is a lower bound of $m^d((n-d-1)(m-1)(s+1)+(m-s-1))$ on the R_k-node-connectivity of an m-ary n-cube [33], where $d = \lfloor k/(m-1) \rfloor$ and $s = k \bmod (m-1)$.

When $k = 1$, the R_1-node-connectivity of an n-cube (an m-ary n-cube) is $2n-2$ ($4n-2$ if $m \geq 4$, and $4n-3$ if $m = 3$) [11] ([7]), and the R_1-node-connectivities of cube-connected cycles, undirected binary de Bruijn networks and Kautz graphs are all greater by one at most than their node-connectivities [23]. Besides, the maximal diameters of an n-cube with $2n-3$ node faults and an n-dimensional star network with $2n-5$ node faults are $n+2$ [19] and $\lfloor 3(n-1)/2 \rfloor + 2$ [29], respectively. When they are fault-free, their diameters are n and $\lfloor 3(n-1)/2 \rfloor$, respectively.

In the next section, the structure of the star network is reviewed. Some necessary definitions, notations and previous results are also introduced. In Section 3, some new properties of the star network are derived in order to prove the main result. The proof of the main result is shown in Section 4. Finally, this paper concludes with some remarks in Section 5.

2. Preliminaries

It is convenient to represent a network as a graph G, where each vertex (edge) of G uniquely represents a node (link) of the network. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Throughout this paper, we use network and graph, node and vertex, link and edge, interchangeably. The following is a definition of star networks, in terms of graph theory.

Definition 1 (*[1]*). An n-dimensional star network, denoted by S_n, has the node set $V(S_n) = \{a_1a_2\cdots a_n | a_1a_2\cdots a_n$ is a permutation of $1, 2, \ldots, n\}$ and the link set $E(S_n) = \{(a_1a_2\cdots a_n, a_ia_2\cdots a_{i-1}a_1a_{i+1}\cdots a_n) | a_1a_2\cdots a_n \in V(S_n)$ and $2 \leq i \leq n\}$.

S_n has $n!$ nodes, each of degree $n-1$. S_1 is a node, S_2 is a link, and S_3 is a cycle of length six. S_4 is shown in Fig. 1. The link $(a_1a_2\cdots a_n, a_ia_2\cdots a_{i-1}a_1a_{i+1}\cdots a_n)$ is referred to as an *i-dimensional link*. We use $e^{(i)}(v)$ to denote the i-dimensional link that is incident to node v, and let $E^{(i)}(S_n) = \{e^{(i)}(v) | v \in V(S_n)\}$ be the set of all i-dimensional links in S_n. S_n is both node symmetric and link symmetric (see [2]).

It can be observed from Fig. 1 that S_4 contains four embedded S_3's, denoted by $\langle***1\rangle_3$, $\langle***2\rangle_3$, $\langle***3\rangle_3$ and $\langle***4\rangle_3$, respectively ($***1$, for example, represents any permutation of 1, 2, 3, 4 ending with 1). In general, S_n contains n embedded S_{n-1}'s.

For $1 \leq r \leq n-1$, an embedded S_r in S_n is denoted by $\langle s_1 s_2 \cdots s_n \rangle_r$, where $s_1 = *$ and there are exactly $r-1$ occurrences of $*$ in $s_2 s_3 \cdots s_n$. For example, $\langle *4*2\rangle_2$ denotes an embedded S_2 in S_4. When S_n is partitioned into $\langle *^{d-1}1*^{n-d}\rangle_{n-1}$, $\langle *^{d-1}2*^{n-d}\rangle_{n-1}, \ldots, \langle *^{d-1}n*^{n-d}\rangle_{n-1}$, S_n is said to be *partitioned along dimension d*, where $1 < d \leq n$ and $*^{d-1}$ ($*^{n-d}$) represents $d-1$ ($n-d$) consecutive $*$'s. These n embedded S_{n-1}'s are connected by d-dimensional links. When $d = n$, we use $\tilde{E}^{(n)}_{p,q}(S_n)$ to represent the set of those n-dimensional links in S_n that connect $\langle *^{n-1}p\rangle_{n-1}$ and $\langle *^{n-1}q\rangle_{n-1}$, where $p \neq q$. Clearly, we have $|\tilde{E}^{(n)}_{p,q}(S_n)| = (n-2)!$.

A path from vertex u ($= v_0$) to vertex v ($= v_k$) in a graph G, represented as $\langle v_0, v_1, \ldots, v_k\rangle$, is referred to as a *u–v path*. A path (cycle) in G is a *Hamiltonian path* (*Hamiltonian cycle*) if it contains every vertex of G exactly once. G is *bipartite* if there is a partition of $V(G)$ into $V_0(G)$ and $V_1(G)$ such that every $(u, v) \in E(G)$ has either $u \in V_0(G)$ and $v \in V_1(G)$ or $u \in V_1(G)$ and $v \in V_0(G)$. The two subsets $V_0(G)$ and $V_1(G)$ are referred to as *partite sets* of G. Star networks are known to be bipartite [1].

A bipartite graph G with $|V_0(G)| = |V_1(G)|$ is *Hamiltonian laceable* if it has a u–v Hamiltonian path for every $u \in V_0(G)$ and every $v \in V_1(G)$ [31], and *strongly Hamiltonian laceable* if it additionally has a longest u–v path of length $|V_0(G)|+|V_1(G)|-2$ for all $u, v \in V_0(G)$ or $u, v \in V_1(G)$ [15]. A Hamiltonian laceable graph G is *hyper Hamiltonian laceable* if for every $w \in V_0(G)(w \in V_1(G))$, G–w contains a u–v Hamiltonian path for all $u, v \in V_1(G)$ ($u, v \in V_0(G)$) [22]. Clearly, a hyper Hamiltonian laceable graph is strongly Hamiltonian laceable.

Given a vertex u in G, we define $N(u) = \{v|(u, v) \in E(G)\}$ to be the *neighborhood* of u, which is the set of vertices that are adjacent to u in G. The size of $N(u)$, i.e., $|N(u)|$, is the *degree* of u. The minimum vertex degree of G is denoted by $\delta(G) = \min\{|N(u)||u \in V(G)\}$. Let V' be a vertex subset of G. We define $N(V') = \bigcup_{u\in V'} N(u) - V'$ to be the *neighborhood* of V'. Besides, we use $G[V']$ to denote the subgraph of G induced by V'. Throughout this paper, we use F ($\subseteq E(S_n)$) to denote the set of link faults in S_n.

Lemma 1 (*[24]*). *$S_n - F$ is strongly Hamiltonian laceable if $|F| \leq n-3$, and hyper Hamiltonian laceable if $|F| \leq n-4$, where $n \geq 4$.*

Lemma 2 (*[14]*). *If $|F| \leq 2n-7$ and $\delta(S_n - F) \geq 2$, then there exists $1 < d \leq n$ such that $|E^{(d)}(S_n) \cap F| \geq 1$ and $\delta(\langle *^{d-1}q*^{n-d}\rangle_{n-1} - F) \geq 2$ for all $1 \leq q \leq n$, where $n \geq 4$.*

3. Properties and main result

In this section, we first introduce some properties of S_n. Then we present our main result.

Lemma 3. *Suppose $u = u_1u_2\cdots u_n \in V(\langle *^{n-1}q\rangle_{n-1})$, where $u_n = q$ and $n \geq 3$. For every $r \in \{1, 2, \ldots, n\} - \{u_1, q\}$, there exists $w \in V(\langle *^{n-1}r\rangle_{n-1})$ and $v \in N(u) \cap V(\langle *^{n-1}q\rangle_{n-1})$ such that $(w, v) = e^{(n)}(v)$.*

Proof. We assume $u_c = r$, where $1 < c < n$. Select $v = u_cu_2\cdots u_{c-1}u_1u_{c+1}\cdots u_n (\in N(u) \cap V(\langle *^{n-1}q\rangle_{n-1}))$ and $w = u_nu_2\cdots u_{c-1}u_1u_{c+1}\cdots u_c (\in V(\langle *^{n-1}r\rangle_{n-1}))$. Clearly, $(w, v) = e^{(n)}(v)$. □

Lemma 4. *Suppose that P is a path in $\langle *^{n-1}q\rangle_{n-1} - F$, where $n \geq 4$ and $1 \leq q \leq n$. If $|F| \leq 2n-7$ and P is Hamiltonian or of length $(n-1)!-2$, then P has a link (u, v) with $e^{(n)}(u), e^{(n)}(v) \notin F$.*

Proof. Suppose conversely that no such (u, v) can be found in P. Then, $|E^{(n)}(S_n) \cap F| \geq ((n-1)!-2)/2$, which is greater than $2n-7$ as $n \geq 4$, a contradiction. □

In subsequent discussion, we let $V_A = \bigcup_{r\in A} V(\langle *^{n-1}r\rangle_{n-1})$, where $A \subseteq \{1, 2, \ldots, n\}$.

Lemma 5. *Suppose that $A \subseteq \{1, 2, \ldots, n\}$ and $n \geq 5$. For any $s \in V(\langle *^{n-1}p\rangle_{n-1})$ and $t \in V(\langle *^{n-1}q\rangle_{n-1})$, there exists a longest s–t path of length $|A| \times (n-1)!-1$ or $|A| \times (n-1)!-2$ in $S_n[V_A] - F$, where $p, q \in A$ and $p \neq q$, provided the following two conditions hold:*

(1) *$|\tilde{E}^{(n)}_{i,j}(S_n) \cap F| < (n-2)!/2$ for all $i, j \in A$ and $i \neq j$;*

(2) *$\langle *^{n-1}r\rangle_{n-1} - F$ is strongly Hamiltonian laceable for every $r \in A$.*

Proof. Since the distance between node $ja_2*^{n-4}a_{n-1}i$ and node $ja_{n-1}*^{n-4}a_2i$ is three, they belong to different partite sets of $\langle *^{n-1}i\rangle_{n-1}$. For each link $(ja_2*^{n-4}a_{n-1}i, ia_2*^{n-4}a_{n-1}j)$ connecting $\langle *^{n-1}i\rangle_{n-1}$ with $\langle *^{n-1}j\rangle_{n-1}$, there exists another link $(ja_{n-1}*^{n-4}a_2i, ia_{n-1}*^{n-4}a_2j)$ connecting $\langle *^{n-1}i\rangle_{n-1}$ with $\langle *^{n-1}j\rangle_{n-1}$. It is implied that there are an equal number (i.e., $(n-2)!/2$) of nodes in $V_0(\langle *^{n-1}i\rangle_{n-1})$ and $V_1(\langle *^{n-1}i\rangle_{n-1})$, respectively, that are connected to $\langle *^{n-1}j\rangle_{n-1}$.

Suppose $A = \{a_1, a_2, \ldots, a_{|A|}\}$. A longest s–t path in $S_n[V_A] - F$ is shown in Fig. 2, where $a_1 = p$ and $a_{|A|} = q$ are assumed. Since $|\tilde{E}^{(n)}_{a_1,a_2}(S_n)| = (n-2)!$, two links in $\tilde{E}^{(n)}_{a_1,a_2}(S_n) - F$, one incident to $V_0(\langle *^{n-1}a_1\rangle_{n-1})$ and the other incident to $V_1(\langle *^{n-1}a_1\rangle_{n-1})$, can be found, as a consequence of (1). Hence, a link $(x_1, y_2) \in \tilde{E}^{(n)}_{a_1,a_2}(S_n) - F$ can be determined such that x_1 and s belong to different partite sets of $\langle *^{n-1}a_1\rangle_{n-1}$ and $y_2 \in V(\langle *^{n-1}a_2\rangle_{n-1})$. As a consequence of (2), there exists a Hamiltonian s–x_1 path in $\langle *^{n-1}a_1\rangle_{n-1} - F$.

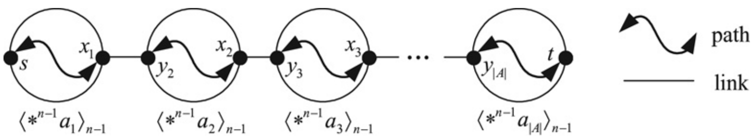

Fig. 2. A longest s–t path in $S_n[V_A] - F$.

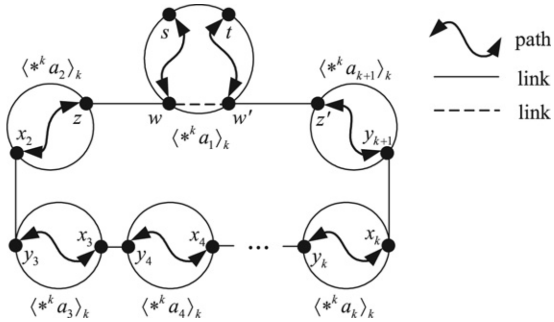

Fig. 3. A longest s–t path in S_n that contains (u, v), where $s, t \in V(\langle *^k q\rangle_k)$.

Similarly, by the aid of (1) and (2), links $(x_k, y_{k+1}) \in \tilde{E}^{(n)}_{a_k,a_{k+1}}(S_n) - F$ and Hamiltonian y_k–x_k paths in $\langle *^{n-1}a_k\rangle_{n-1} - F$ can be obtained, where $2 \le k \le |A| - 1$. All these Hamiltonian paths together with a longest $y_{|A|}$–t path in $\langle *^{n-1}a_{|A|}\rangle_{n-1} - F$ constitute a longest s–t path in $S_n[V_A] - F$. If the longest $y_{|A|}$–t path has length $(n-1)! - 1$, the length of the longest s–t path is computed as $(|A| - 1) \times ((n-1)! - 1) + (|A| - 1) + ((n-1)! - 1) = |A| \times (n-1)! - 1$. Similarly, if the longest $y_{|A|}$–t path has length $(n-1)! - 2$, the longest s–t path has length $|A| \times (n-1)! - 2$. □

Lemma 6. *Suppose that s, t are two distinct nodes of S_n and $(u, v) \in E(S_n)$, where $n \ge 4$. If $\{s, t\} \ne \{u, v\}$, then there exists a Hamiltonian s-t path or an s-t path of length $(n-1)! - 2$ in S_n that contains (u, v).*

Proof. We prove this lemma by induction on n. This lemma holds for S_4, which can be verified by exhaustive search (see [34]). So, we assume that this lemma holds for S_k, and then consider S_{k+1} below, where $k \ge 4$.

Without loss of generality, suppose $u, v \in V(\langle *^k q\rangle_k)$ for some $1 \le q \le k+1$. We also assume $(u, v) \in E^{(l)}(S_{k+1})$, where $2 \le l \le k$. With the following three cases, we show a longest s–t path of length $(k+1)! - 1$ or $(k+1)! - 2$ in S_{k+1} that contains (u, v).

Case 1. $s \in V(\langle *^k q\rangle_k)$ and $t \notin V(\langle *^k q\rangle_k)$ or $s \notin V(\langle *^k q\rangle_k)$ and $t \in V(\langle *^k q\rangle_k)$. We consider the situation of $s \in V(\langle *^k q\rangle_k)$ and $t \notin V(\langle *^k q\rangle_k)$. The discussion for the situation of $s \notin V(\langle *^k q\rangle_k)$ and $t \in V(\langle *^k q\rangle_k)$ is very similar. A desired s–t path can be obtained using the construction method of Fig. 2. We only need to change $n - 1$ to k and $|A|$ to $k + 1$, and set $a_1 = q$. Besides, the node x_1 is selected with $\{s, x_1\} \ne \{u, v\}$. The induction hypothesis assures a Hamiltonian s–x_1 path in $\langle *^k a_1\rangle_k$ that contains (u, v). The desired s–t path has length $(k+1) \times k! - 1 = (k+1)! - 1$ or $(k+1) \times k! - 2 = (k+1)! - 2$.

Case 2. $s, t \in V(\langle *^k q\rangle_k)$. A desired s–t path can be obtained as shown in Fig. 3, where $a_1 = q$ is assumed. The induction hypothesis assures a longest s–t path in $\langle *^k a_1\rangle_k$ that contains (u, v). A link $(w, w') \ne (u, v)$ can be selected from the path. Let $z = e^{(k+1)}(w)$ and $z' = e^{(k+1)}(w')$. A Hamiltonian z–z' path in $S_{k+1} - \langle *^k a_1\rangle_k$ can be obtained using the construction method of Fig. 2 (changing s to z, t to z', $n-1$ to k, $|A|$ to k, and $\langle *^{n-1}a_r\rangle_{n-1}$ to $\langle *^k a_{r+1}\rangle_k$ for all $1 \le r \le |A|$). The Hamiltonian z–z' path, combining with $(w, z), (w', z')$, and the s–w and w'–t paths in $\langle *^k a_1\rangle_k$, forms a desired s–t path in S_{k+1}. The desired s–t path has length $(k \times k! - 1) + 2 + ((k! - 1) - 1) = (k+1)! - 1$ or $(k \times k! - 1) + 2 + ((k! - 2) - 1) = (k+1)! - 2$.

Case 3. $s, t \notin V(\langle *^k q\rangle_k)$. Suppose $s \in V(\langle *^k g\rangle_k)$ and $t \in V(\langle *^k h\rangle_k)$, where $g, h \in \{1, 2, \ldots, k+1\} - \{q\}$. First we assume $g \ne h$. A desired s–t path can be obtained using the construction method of Fig. 2 (changing $n - 1$ to k and $|A|$ to $k + 1$, and letting $a_1 = g$, $a_2 = q$, and $a_{k+1} = h$). The node x_2 is selected with $\{x_2, y_2\} \ne \{u, v\}$. The induction hypothesis assures a Hamiltonian y_2–x_2 path in $\langle *^k a_2\rangle_k$ that contains (u, v). The desired s–t path has length $(k+1) \times k! - 1 = (k+1)! - 1$ or $(k+1) \times k! - 2 = (k+1)! - 2$.

Next we assume $g = h$. A desired s–t path can be obtained by slightly modifying the construction method of Fig. 3. We only need to set $a_1 = g = h$ and $a_2 = q$. The node x_2 is selected with $\{z, x_2\} \ne \{u, v\}$. The induction hypothesis assures a Hamiltonian z–x_2 path in $\langle *^k a_2\rangle_k$ that contains (u, v). A Hamiltonian y_3–z' path in $S_{k+1} - \langle *^k a_1\rangle_k - \langle *^k a_2\rangle_k$ can be obtained, similarly, using the construction method of Fig. 2. The desired s–t path, which consists of the s–w and w'–t paths in $\langle *^k a_1\rangle_k$, the Hamiltonian z–x_2 path, the Hamiltonian y_3–z' path, and three links $(w, z), (w', z'), (x_2, y_3)$, has length $(k+1)! - 1$ or $(k+1)! - 2$. □

Lemma 7. *For any two distinct links (s, t), (u, v) in S_n, there exists a Hamiltonian cycle in S_n that contains both of them, where $n \ge 3$.*

Proof. Since S_3 is a cycle of length 6, this lemma holds for S_3. When $n \ge 4$, since s and t belong to different partite sets of S_n, this lemma holds for S_n, as a consequence of Lemma 6. □

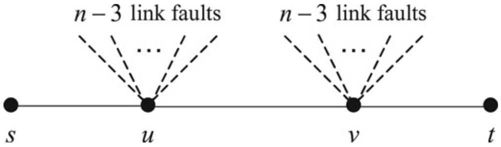

Fig. 4. A distribution of $2n-6$ link faults over S_n.

The main result of this paper is presented in the following theorem whose proof is shown in the next section.

Theorem 1. *With the assumption of two or more fault-free links incident to each node, an n-dimensional star network can tolerate up to* $2n-7$ *link faults, and be strongly (fault-free) Hamiltonian laceable, where* $n \geq 4$.

Theorem 1 is optimal with respect to the number of link faults tolerated. Fig. 4 shows a distribution of $2n-6$ link faults over S_n, where $\langle s, u, v, t\rangle$ is an s–t path and (s, u), (u, v) ((u, v), (v, t)) are the only two fault-free links incident to u (v). It is easy to see that no fault-free Hamiltonian s–t path exists in the faulty S_n.

4. Proof of Theorem 1

With $|F| \leq 2n-7$ and $\delta(S_n - F) \geq 2$, we show by induction that there exists a longest s–t path of length $n!-1$ or $n!-2$ between every two distinct nodes s, t of $S_n - F$. By Lemma 1 ($n-3 = 2n-7$ as $n = 4$), the theorem holds for S_4. So, we assume that this theorem holds for S_k, and then consider S_{k+1} in the rest of this section, where $k \geq 4$.

By Lemma 2, we can partition S_{k+1} along some dimension d such that $|E^{(d)}(S_{k+1}) \cap F| \geq 1$ and $\delta(\langle *^{d-1}q*^{k+1-d}\rangle_k - F) \geq 2$ for all $1 \leq q \leq k+1$, where $1 < d \leq k+1$. Without loss of generality, we assume $d = k+1$. Now that $|E^{(k+1)}(S_{k+1}) \cap F| \geq 1$, we have $|E(\langle *^k q\rangle_k) \cap F| \leq 2k-6$ for all $1 \leq q \leq k+1$. A desired s–t path in $S_{k+1} - F$ is constructed in Section 4.1 if $|E(\langle *^k q\rangle_k) \cap F| \leq 2k-7$ for all $1 \leq q \leq k+1$, and constructed in Section 4.2 else.

4.1. $|E(\langle *^k q\rangle_k) \cap F| \leq 2k-7$ *for all* $1 \leq q \leq k+1$

Suppose that $|E(\langle *^k q\rangle_k) \cap F| \leq 2k-7$ for all $1 \leq q \leq k+1$. Since $|F| \leq 2k-5$, we have $\sum_{i,j\in\{1,2,\ldots,k+1\} \text{ and } i\neq j} |\tilde{E}^{(k+1)}_{i,j}(S_{k+1}) \cap F| \leq 2k-5$, where $k \geq 4$. Notice that $2k-5 = ((k+1)-2)!/2$ when $k = 4$, and $2k-5 < ((k+1)-2)!/2$ when $k > 4$. Two cases are discussed below.

Case 1. $k > 4$. We have $|\tilde{E}^{(k+1)}_{i,j}(S_{k+1}) \cap F| < ((k+1)-2)!/2$ for all $i, j \in \{1, 2, \ldots, k+1\}$ and $i \neq j$. The induction hypothesis assures that $\langle *^k q\rangle_k - F$ is strongly Hamiltonian laceable for all $1 \leq q \leq k+1$. If $s \in V(\langle *^k g\rangle_k)$ and $t \in V(\langle *^k h\rangle_k)$ for some $g, h \in \{1, 2, \ldots, k+1\}$ and $g \neq h$, then by Lemma 5, $S_{k+1} - F$ is strongly Hamiltonian laceable. If $s, t \in V(\langle *^k g\rangle_k)$ for some $1 \leq g \leq k+1$, then a desired s–t path in $S_{k+1} - F$ can be obtained by slightly modifying the construction method of Fig. 3. We only need to set $a_1 = g$. The induction hypothesis assures a longest s–t path in $\langle *^k a_1\rangle_k - F$. By Lemma 4, a link (w, w') with $(w, z) \in \tilde{E}^{(k+1)}_{a_1,a_2} - F$ and $(w', z') \in \tilde{E}^{(k+1)}_{a_1,a_{k+1}} - F$ can be selected from the path. By Lemma 5, a Hamiltonian z–z' path in $S_{k+1} - \langle *^k a_1\rangle_k - F$ can be obtained. The resulting longest s–t path in $S_{k+1} - F$ has length $(k+1)!-1$ or $(k+1)!-2$.

Case 2. $k = 4$. If $|\tilde{E}^{(5)}_{i,j}(S_5) \cap F| < ((4+1)-2)!/2 = 3$ for all $i, j \in \{1, 2, \ldots, 5\}$ and $i \neq j$, then the discussion is the same as Case 1. So, we consider $|\tilde{E}^{(5)}_{i',j'}(S_5) \cap F| = 3$ for $i', j' \in \{1, 2, \ldots, 5\}$ and $i' \neq j'$. Since $|F| \leq 3$ as $k = 4$, all link faults are in $\tilde{E}^{(5)}_{i',j'}(S_5)$. Assume that $s \in V(\langle *^4 g\rangle_4)$ and $t \in V(\langle *^4 h\rangle_4)$, where $g, h \in \{1, 2, \ldots, 5\}$. When $g \neq h$, a desired s–t path in $S_5 - F$ can be obtained using the construction method of Fig. 2. We only need to change $n-1$ to 4 and $|A|$ to 5, set $a_1 = g$ and $a_5 = h$, and set a_2, a_3, a_4 with $\{i', j'\} \neq \{a_r, a_{r+1}\}$ for all $1 \leq r \leq 4$. The induction hypothesis assures a Hamiltonian s–x_1 path in $\langle *^4 a_1\rangle_4 - F$, a Hamiltonian y_r–x_r path in $\langle *^4 a_r\rangle_4 - F$ for all $2 \leq r \leq 4$, and a longest y_5–t path in $\langle *^4 a_5\rangle_4 - F$. The desired s–t path has length $5!-1$ or $5!-2$.

When $g = h$, a desired s–t path in $S_5 - F$ can be obtained using the construction method of Fig. 3. We only need to change $k+1$ to 5, set $a_1 = g = h$, and set a_2, a_3, a_4, a_5 with $\{i', j'\} \neq \{a_r, a_{(r \bmod 5)+1}\}$ for all $1 \leq r \leq 5$. The induction hypothesis assures a longest s–t path in $\langle *^4 a_1\rangle_4 - F$. By Lemma 4, a link (w, w') with $(w, z) \in \tilde{E}^{(5)}_{a_1,a_2} - F$ and $(w', z') \in \tilde{E}^{(5)}_{a_1,a_5} - F$ can be selected from the path. The induction hypothesis assures a Hamiltonian y_r–x_r path in $\langle *^4 a_r\rangle_4 - F$ for all $2 \leq r \leq 5$, where $y_2 = z$ and $x_5 = z'$. The desired s–t path has length $5!-1$ or $5!-2$.

4.2. $|E(\langle *^k \alpha\rangle_k) \cap F| = 2k-6$ *for some* $1 \leq \alpha \leq k+1$

Suppose that $|E(\langle *^k \alpha\rangle_k) \cap F| = 2k-6$, where $1 \leq \alpha \leq k+1$. Now that $|F| \leq 2k-5$ and $|E^{(k+1)}(S_{k+1}) \cap F| \geq 1$, we have $|E^{(k+1)}(S_{k+1}) \cap F| = 1$. Besides, we have $|E(\langle *^k q'\rangle_k) \cap F| = 0$ for all $q' \in \{1, 2, \ldots, k+1\} - \{\alpha\}$, and $|\tilde{E}^{(k+1)}_{i,j}(S_{k+1}) \cap F| \leq 1 (< ((k+1)-2)!/2)$ for any $i, j \in \{1, 2, \ldots, k+1\}$ and $i \neq j$, where $k \geq 4$.

If $s, t \in V(\langle *^k \alpha\rangle_k)$, then a link fault, say (h, h'), is chosen from $E(\langle *^k \alpha\rangle_k) \cap F$ such that $\{s, t\} \neq \{h, h'\}$ and $e^{(k+1)}(h)$, $e^{(k+1)}(h') \notin F$. Imagine that (h, h') is fault-free. Then, by the induction hypothesis, there is a longest s–t path, denoted by P,

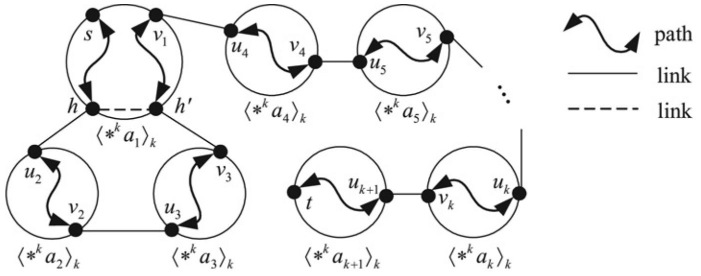

Fig. 5. A longest s–t path in $S_{k+1} - F$ when $h, h' \notin N(V(\langle *^k a_{k+1}\rangle_k))$.

of length $k! - 1$ or $k! - 2$ in $\langle *^k\alpha\rangle_k - (F - \{(h, h')\})$. A desired s–t path in $S_{k+1} - F$ can be obtained using the construction method of Fig. 3. We only need to set $a_1 = \alpha$, and $(w, w') = (h, h')$ if P contains (h, h'). The desired s–t path has length $(k + 1)! - 1$ or $(k + 1)! - 2$.

If $s \notin V(\langle *^k\alpha\rangle_k)$ or $t \notin V(\langle *^k\alpha\rangle_k)$, then a desired s–t path in $S_{k+1} - F$ is constructed in Sections 4.2.1 and 4.2.2, where we use $a_1, a_2, \ldots, a_{k+1}$ to denote the $k + 1$ distinct integers from 1 to $k + 1$ (i.e., $\{a_1, a_2, \ldots, a_{k+1}\} = \{1, 2, \ldots, k + 1\}$).

4.2.1. $s \in V(\langle *^k\alpha\rangle_k)$ and $t \notin V(\langle *^k\alpha\rangle_k)$ or $s \notin V(\langle *^k\alpha\rangle_k)$ and $t \in V(\langle *^k\alpha\rangle_k)$

Suppose that $s \in V(\langle *^k\alpha\rangle_k)$ and $t \notin V(\langle *^k\alpha\rangle_k)$. The discussion for $s \notin V(\langle *^k\alpha\rangle_k)$ and $t \in V(\langle *^k\alpha\rangle_k)$ is similar. We assume $t \in V(\langle *^k\beta\rangle_k)$, where $\beta \neq \alpha$. A link fault (h, h') is chosen from $E(\langle *^k\alpha\rangle_k) \cap F$ such that $s \notin \{h, h'\}$ and $e^{(k+1)}(h)$, $e^{(k+1)}(h') \notin F$. Set $a_1 = \alpha$ and $a_{k+1} = \beta$.

We first consider the situation of $h, h' \notin N(V(\langle *^k a_{k+1}\rangle_k))$. Imagine that (h, h') is fault-free. A vertex $v_1 \in V(\langle *^k a_1\rangle_k)$ is determined such that v_1 and s belong to different partite sets of S_{k+1}, $e^{(k+1)}(v_1) \notin F$, and $v_1 \notin N(V(\langle *^k a_{k+1}\rangle_k)$. The induction hypothesis assures a longest s–v_1 path of length $k! - 1$ in $\langle *^k a_1\rangle_k - (F - \{(h, h')\})$. If the longest s–v_1 path does not contain (h, h'), then a desired s–t path of length $(k + 1)! - 1$ or $(k + 1)! - 2$ in $S_{k+1} - F$ can be obtained using the construction method of Fig. 2. We only need to set $x_1 = v_1$, and change $n - 1$ and $|A|$ to k and $k + 1$, respectively.

If the longest s–v_1 path contains (h, h'), then a desired s–t path in $S_{k+1} - F$ can be obtained as shown in Fig. 5. We set a_2 and a_3 such that $h \in N(V(\langle *^k a_2\rangle_k))$ and $h' \in N(V(\langle *^k a_3\rangle_k))$. There is an additional restriction to $v_1 : v_1 \notin N(V(\langle *^k a_2\rangle_k))$ and $v_1 \notin N(V(\langle *^k a_3\rangle_k))$. Also we set a_4 such that $v_1 \in N(V(\langle *^k a_4\rangle_k))$. Then, three vertices $u_2 \in V(\langle *^k a_2\rangle_k)$, $v_3 \in V(\langle *^k a_3\rangle_k)$, and $u_4 \in V(\langle *^k a_4\rangle_k)$ are determined such that $(u_2, h) = e^{(k+1)}(h)$, $(v_3, h') = e^{(k+1)}(h')$, and $(u_4, v_1) = e^{(k+1)}(v_1)$. By Lemma 5, there is a longest u_4–t path of length $(k - 2) \times k! - 1$ or $(k - 2) \times k! - 2$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k + 1\} - \{a_1, a_2, a_3\}$. Again, by Lemma 5, there is a longest u_2–v_3 path of length $2 \times k! - 1$ in $S_{k+1}[V_A] - F$, where $A = \{a_2, a_3\}$. The desired s–t path has length $((k! - 1) - 1) + (2 \times k! - 1) + ((k - 2) \times k! - 1) + 3 = (k + 1)! - 1$ or $((k! - 1) - 1) + (2 \times k! - 1) + ((k - 2) \times k! - 2) + 3 = (k + 1)! - 2$.

Then we consider the situation of $h \in N(V(\langle *^k a_{k+1}\rangle_k))$, without loss of generality. Notice that at most one of h and h' belongs to $N(V(\langle *^k a_{k+1}\rangle_k))$. Imagine that (h, h') is fault-free. A vertex $u_1 \in V(\langle *^k a_1\rangle_k)$ is determined such that u_1 and s belong to different partite sets of S_{k+1}, $e^{(k+1)}(u_1) \notin F$, and $u_1 \notin N(V(\langle *^k a_{k+1}\rangle_k)$. The induction hypothesis assures a longest s–u_1 path of length $k! - 1$ in $\langle *^k a_1\rangle_k - (F - \{(h, h')\})$. If the longest s–u_1 path does not contain (h, h'), a desired s–t path in $S_{k+1} - F$ can be obtained using the construction method of Fig. 2. We only need to set $x_1 = u_1$, and change $n - 1$ and $|A|$ to k and $k + 1$, respectively. If the longest s–u_1 path contains (h, h'), two cases: $h \in N(t)$ or $h \notin N(t)$, are discussed below.

Case 1. $h \in N(t)$. A desired s–t path in $S_{k+1} - F$ can be obtained as shown in Fig. 6(a). We set a_2 with $h' \in N(V(\langle *^k a_2\rangle_k))$. There is an additional restriction to $u_1 : u_1 \notin N(V(\langle *^k a_2\rangle_k))$ and $u_1 \notin N(V(\langle *^k a_{k+1}\rangle_k))$. Also we set a_k with $u_1 \in N(V(\langle *^k a_k\rangle_k))$. Then, six vertices $u_2, v_2 \in V(\langle *^k a_2\rangle_k)$, $u_3 \in V(\langle *^k a_3\rangle_k)$, $v_k \in V(\langle *^k a_k\rangle_k)$, and $u_{k+1}, v_{k+1} \in V(\langle *^k a_{k+1}\rangle_k)$ are determined such that $(u_2, h') = e^{(k+1)}(h')$, v_2 and u_2 belong to different partite sets of S_{k+1}, $e^{(k+1)}(v_2) \notin F$, $(u_{k+1}, v_2) = e^{(k+1)}(v_2)$, v_{k+1} and u_{k+1} belong to the same partite set of S_{k+1}, $e^{(k+1)}(v_{k+1}) \notin F$, $(u_3, v_{k+1}) = e^{(k+1)}(v_{k+1})$, and $(v_k, u_1) = e^{(k+1)}(u_1)$. By Lemma 1, there are a longest u_2–v_2 path of length $k! - 1$ in $\langle *^k a_2\rangle_k$ and a longest u_{k+1}–v_{k+1} path of length $k! - 2$ in $\langle *^k a_{k+1}\rangle_k - \{t\}$. By Lemma 5, there is a longest u_3–v_k path of length $(k-2) \times k! - 1$ or $(k-2) \times k! - 2$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_{k+1}\}$. The desired s–t path has length $((k!-1)-1)+(k!-1)+(k!-2)+((k-2)\times k!-1)+4 = (k + 1)! - 1$ or $((k! - 1) - 1) + (k! - 1) + (k! - 2) + ((k - 2) \times k! - 2) + 4 = (k + 1)! - 2$.

Case 2. $h \notin N(t)$. A desired s–t path in $S_{k+1} - F$ is shown in Fig. 6(b) when $w' \neq t$ and $(v_2, w') \notin F$, and shown in Fig. 6(c) when $w' = t$ or $(v_2, w') \in F$. We set a_2 such that $h' \in N(V(\langle *^k a_2\rangle_k))$. There is an additional restriction to $u_1 : u_1 \notin N(V(\langle *^k a_2\rangle_k))$ and $u_1 \notin N(V(\langle *^k a_{k+1}\rangle_k))$. Also we set a_k with $u_1 \in N(V(\langle *^k a_k\rangle_k))$. Then, three vertices $u_2 \in V(\langle *^k a_2\rangle_k)$, $v_k \in V(\langle *^k a_k\rangle_k)$, and $w \in V(\langle *^k a_{k+1}\rangle_k)$ are first determined such that $(u_2, h') = e^{(k+1)}(h')$, $(w, h) = e^{(k+1)}(h)$, and $(v_k, u_1) = e^{(k+1)}(u_1)$. By Lemma 3, there exist $v_2 \in V(\langle *^k a_2\rangle_k)$ and $w' \in N(w) \cap V(\langle *^k a_{k+1}\rangle_k)$ such that $(v_2, w') = e^{(k+1)}(w')$ (see Fig. 6(b)).

If $w' \neq t$ and $(v_2, w') \notin F$, then, again by Lemma 3, there exist $u_3 \in V(\langle *^k a_3\rangle_k)$ and $v_{k+1} \in N(t) \cap V(\langle *^k a_{k+1}\rangle_k)$ such that $(u_3, v_{k+1}) = e^{(k+1)}(v_{k+1})$. Besides, $(u_3, v_{k+1}) \notin F$ can be satisfied, because $|E^{(k+1)}(S_{k+1}) \cap F| = 1$. By Lemma 7, there exists a Hamiltonian cycle in $\langle *^k a_{k+1}\rangle_k$ that contains (t, v_{k+1}) and (w, w'). By Lemma 1, there is a longest u_2–v_2 path of length $k! - 1$ in $\langle *^k a_2\rangle_k$. By Lemma 5, there is a longest u_3–v_k path of length $(k - 2) \times k! - 1$ or $(k - 2) \times k! - 2$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_{k+1}\}$. The desired s–t path has length $((k!-1)-1)+(k!-1)+(k!-2)+((k-2)\times k!-1)+4 = (k + 1)! - 1$ or $((k! - 1) - 1) + (k! - 1) + (k! - 2) + ((k - 2) \times k! - 2) + 4 = (k + 1)! - 2$.

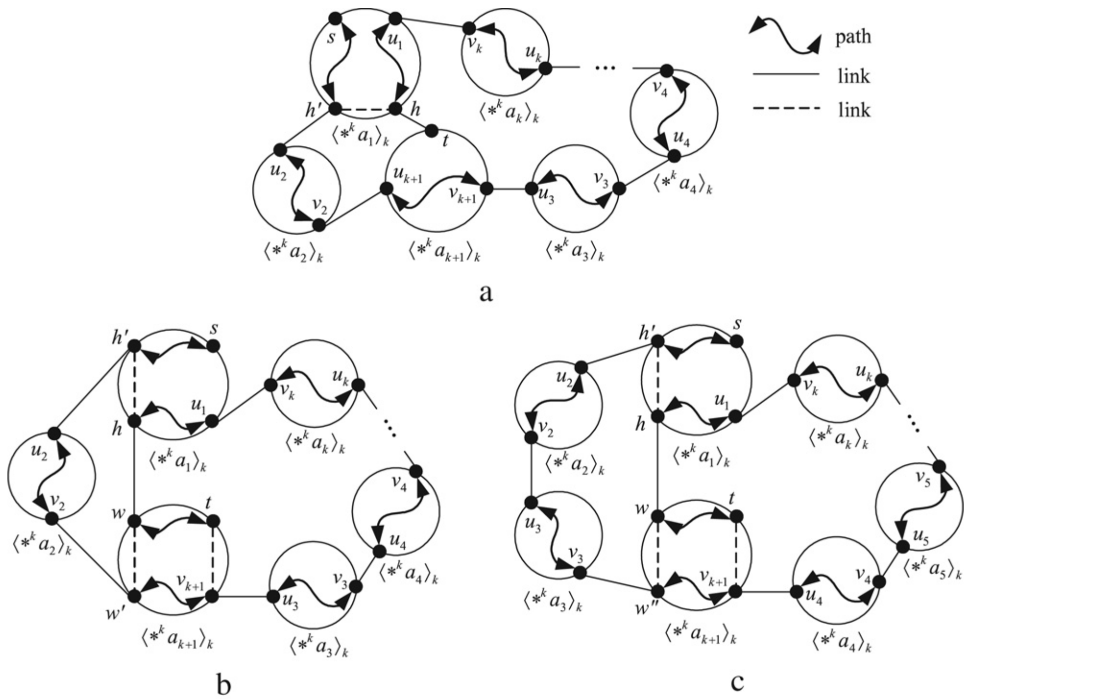

Fig. 6. A longest s–t path in $S_{k+1} - F$ when $h \in N(V(\langle *^k a_{k+1}\rangle_k))$. (a) $h \in N(t)$. (b) $h \notin N(t)$ and ($w' \neq t$ and $(v_2, w') \notin F$). (c) $h \notin N(t)$ and ($w' = t$ or $(v_2, w') \in F$).

If $w' = t$ or $(v_2, w') \in F$, then by Lemma 3, there exist $v_3 \in V(\langle *^k a_3\rangle_k)$ and $w'' \in N(w) \cap V(\langle *^k a_{k+1}\rangle_k)$ such that $(v_3, w'') = e^{(k+1)}(w'')$. Again by Lemma 3, there exist $u_4 \in V(\langle *^k a_4\rangle_k)$ and $v_{k+1} \in N(t) \cap V(\langle *^k a_{k+1}\rangle_k)$ with $(u_4, v_{k+1}) = e^{(k+1)}(v_{k+1})$. Besides, (v_3, w''), $(u_4, v_{k+1}) \notin F$ and $w'' \neq t$ can be satisfied. By Lemma 7, there exists a Hamiltonian cycle in $\langle *^k a_{k+1}\rangle_k$ that contains (t, v_{k+1}) and (w, w''). By Lemma 5, there is a longest u_2–v_3 path of length $2 \times k! - 1$ in $S_{k+1}[V_A] - F$, where $A = \{a_2, a_3\}$. If $k = 4$, Lemma 1 assures a longest u_4–v_4 path of length $4! - 1$ or $4! - 2$ in $\langle *^4 a_4\rangle_4$. If $k > 4$, Lemma 5 assures a longest u_4–v_k path of length $(k-3) \times k! - 1$ or $(k-3) \times k! - 2$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_3, a_{k+1}\}$. The desired s–t path has length $(k+1)! - 1$ or $(k+1)! - 2$.

4.2.2. $s, t \notin V(\langle *^k \alpha\rangle_k)$

Suppose that $s \in V(\langle *^k \beta\rangle_k)$ and $t \in V(\langle *^k \gamma\rangle_k)$, where $\beta, \gamma \in \{1, 2, \ldots, k+1\} - \{\alpha\}$. We first consider the situation of $\beta = \gamma$. A link fault (h, h') is chosen from $E(\langle *^k \alpha\rangle_k) \cap F$ such that $e^{(k+1)}(h), e^{(k+1)}(h') \notin F$. Set $a_1 = \beta = \gamma$, and set $a_3 = \alpha$ if $h, h' \notin N(V(\langle *^k a_1\rangle_k))$, and $a_2 = \alpha$ else. A desired s–t path in $S_{k+1} - F$ can be constructed in a way similar to Fig. 3. We only explain below the construction for $\alpha = a_3$. The construction for $\alpha = a_2$ is similar.

Refer to Fig. 3 again. For the purpose of our construction, we let $A = \{1, 2, \ldots, k+1\}$, $(x_3, y_3) = (h, h')(\in E(\langle *^k a_3\rangle_k))$, and select $(w, w')(\in E(\langle *^k a_1\rangle_k))$ such that $\{w, w'\} \neq \{s, t\}$, $w, w' \notin N(V(\langle *^k a_3\rangle_k))$, w and y_3 belong to different partite sets of S_{k+1}, and $e^{(k+1)}(w), e^{(k+1)}(w') \notin F$. Four vertices $z, x_2 \in V(\langle *^k a_2\rangle_k)$, $y_4 \in V(\langle *^k a_4\rangle_k)$, and $z' \in V(\langle *^k a_{k+1}\rangle_k)$ are determined such that $(z, w) = e^{(k+1)}(w)$, $(x_2, y_3) = e^{(k+1)}(y_3)$, $(y_4, x_3) = e^{(k+1)}(x_3)$, and $(z', w') = e^{(k+1)}(w')$.

There is a longest z–x_2 path of length $k! - 1$ in $\langle *^k a_2\rangle_k$. By Lemma 6, there exists a longest s–t path of length $k! - 1$ or $k! - 2$ in $\langle *^k a_1\rangle_k$ that contains (w, w'). Imagine that (x_3, y_3) is fault-free. The induction hypothesis assures a longest y_3–x_3 path of length $k! - 1$ in $\langle *^k a_3\rangle_k - (F - \{(x_3, y_3)\})$. By Lemma 5, there is a longest y_4–z' path of length $(k-2) \times k! - 1$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_3\}$. The desired s–t path has length $(k! - 1) + ((k! - 1) - 1) + (k! - 1) + ((k-2) \times k! - 1) + 4 = (k+1)! - 1$ or $(k! - 1) + ((k! - 1) - 2) + (k! - 1) + ((k-2) \times k! - 1) + 4 = (k+1)! - 2$.

Then we consider the situation of $\beta \neq \gamma$. Set $a_1 = \beta$ and $a_{k+1} = \gamma$. A link fault (h, h') is chosen from $E(\langle *^k \alpha\rangle_k) \cap F$ such that $e^{(k+1)}(h), e^{(k+1)}(h') \notin F$. Three cases are further discussed below.

Case 1. Neither of h and h' belongs to $N(V(\langle *^k a_1\rangle_k)) \cup N(V(\langle *^k a_{k+1}\rangle_k))$. Set $a_3 = \alpha$. A desired s–t path in $S_{k+1} - F$ can be constructed in a way similar to Fig. 2. Refer to Fig. 2 again. For the purpose of our construction, we let $A = \{1, 2, \ldots, k+1\}$ and $(x_3, y_3) = (h, h')(\in E(\langle *^k a_3\rangle_k))$. We assume that s and y_3 belong to the same partite set of S_{k+1}. In case s and y_3 belong to different partite sets of S_{k+1}, we have s and x_3 in the same partite set of S_{k+1} and exchange x_3 and y_3 in Fig. 2.

Four vertices $x_1 \in V(\langle *^k a_1\rangle_k) - \{s\}$, $y_2, x_2 \in V(\langle *^k a_2\rangle_k)$, and $y_4 \in V(\langle *^k a_4\rangle_k)$ are determined such that $(x_2, y_3) = e^{(k+1)}(y_3)$, y_2 and x_2 belong to different partite sets of S_{k+1}, $e^{(k+1)}(y_2) \notin F$, $(x_1, y_2) = e^{(k+1)}(y_2)$, and $(y_4, x_3) = e^{(k+1)}(x_3)$. There are a longest s–x_1 path in $\langle *^k a_1\rangle_k$ and a longest y_2–x_2 path in $\langle *^k a_2\rangle_k$, each of length $k! - 1$. Imagine that (x_3, y_3) is fault-free. The induction hypothesis assures a longest y_3–x_3 path of length $k! - 1$ in $\langle *^k a_3\rangle_k - (F - \{(x_3, y_3)\})$. By Lemma 5, there is

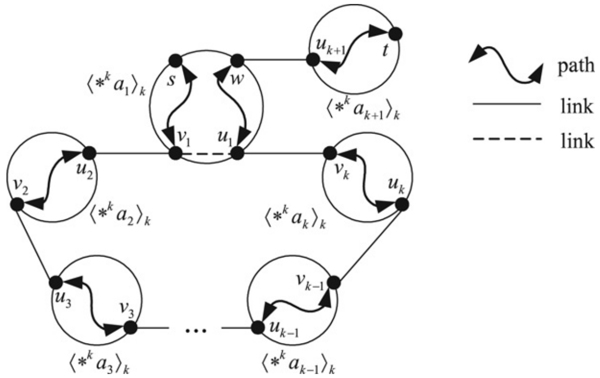

Fig. 7. A longest s–t path in $S_{k+1} - F$ when $\beta \neq \gamma$, $h \in N(V(\langle *^k a_1 \rangle_k))$, and $h' \notin N(V(\langle *^k a_1 \rangle_k)) \cup N(V(\langle *^k a_{k+1} \rangle_k))$.

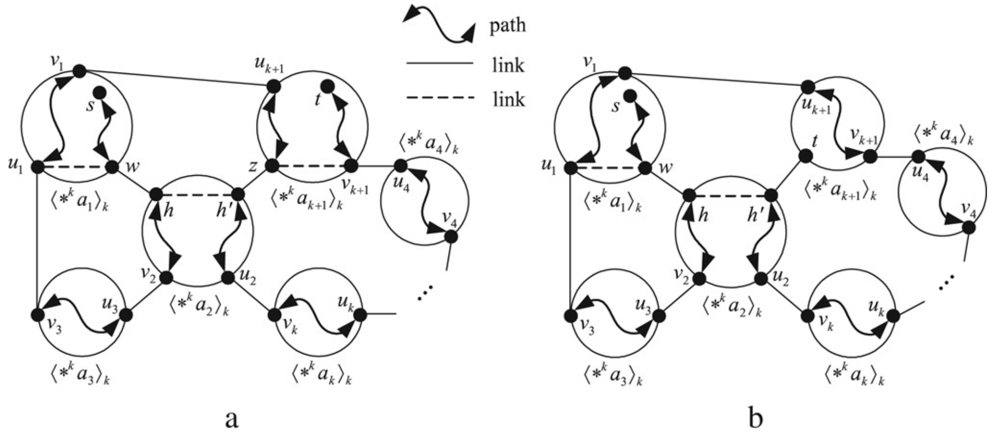

Fig. 8. A longest s–t path in $S_{k+1} - F$ when $\beta \neq \gamma$, $h \in N(V(\langle *^k a_1 \rangle_k))$, and $h' \in N(V(\langle *^k a_{k+1} \rangle_k))$. (a) $z \neq t$. (b) $z = t$.

a longest $y_4 - t$ path of length $(k-2) \times k! - 1$ or $(k-2) \times k! - 2$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_3\}$. The desired s–t path has length $3 \times (k!-1) + ((k-2) \times k! - 1) + 3 = (k+1)! - 1$ or $3 \times (k!-1) + ((k-2) \times k! - 2) + 3 = (k+1)! - 2$.

Case 2. One of h and h' belongs to $N(V(\langle *^k a_1 \rangle_k)) \cup N(V(\langle *^k a_{k+1} \rangle_k))$. Without loss of generality, we assume that $h \in N(V(\langle *^k a_1 \rangle_k))$ and $h' \notin N(V(\langle *^k a_1 \rangle_k)) \cup N(V(\langle *^k a_{k+1} \rangle_k))$. A desired s–t path in $S_{k+1} - F$ can be constructed as shown in Fig. 7. Set $a_2 = \alpha$ and $(u_2, v_2) = (h, h') \in E(\langle *^k a_2 \rangle_k)$. Six vertices $v_1 \in V(\langle *^k a_1 \rangle_k)$, $u_1 \in N(v_1) \cap V(\langle *^k a_1 \rangle_k) - \{s\}$, $w \in V(\langle *^k a_1 \rangle_k)$, $u_3 \in V(\langle *^k a_3 \rangle_k)$, $v_k \in V(\langle *^k a_k \rangle_k)$, and $u_{k+1} \in V(\langle *^k a_{k+1} \rangle_k) - \{t\}$ are determined such that $(v_1, u_2) = e^{(k+1)}(u_2)$, $(u_3, v_2) = e^{(k+1)}(v_2)$, $e^{(k+1)}(u_1) \notin F$, $(v_k, u_1) = e^{(k+1)}(u_1)$, w and s belong to different partite sets of S_{k+1}, $e^{(k+1)}(w) \notin F$, and $(u_{k+1}, w) = e^{(k+1)}(w)$.

Imagine that (u_2, v_2) is fault-free. The induction hypothesis assures a longest u_2–v_2 path of length $k! - 1$ in $\langle *^k a_2 \rangle_k - (F - \{(u_2, v_2)\})$. By Lemma 6, there exists a longest s–w path of length $k! - 1$ in $\langle *^k a_1 \rangle_k$ that contains (u_1, v_1). By Lemma 1, there is a longest u_{k+1}–t path of length $k! - 1$ or $k! - 2$ in $\langle *^k a_{k+1} \rangle_k$. By Lemma 5, there is a longest u_3–v_k path of length $(k-2) \times k! - 1$ in $S_{k+1}[V_A] - F$, where $A = \{1, 2, \ldots, k+1\} - \{a_1, a_2, a_{k+1}\}$. The desired s–t path has length $(k!-1) + ((k!-1)-1) + (k!-1) + ((k-2) \times k! - 1) + 4 = (k+1)! - 1$ or $(k!-1) + ((k!-1)-1) + (k!-2) + ((k-2) \times k! - 1) + 4 = (k+1)! - 2$.

Case 3. Both of h' and h belong to $N(V(\langle *^k a_1 \rangle_k)) \cup N(V(\langle *^k a_{k+1} \rangle_k))$. If $h \in N(V(\langle *^k a_1 \rangle_k)) (\in N(V(\langle *^k a_{k+1} \rangle_k)))$, then $h' \in N(V(\langle *^k a_{k+1} \rangle_k)) (\in N(V(\langle *^k a_1 \rangle_k)))$. Without loss of generality, we assume that $h \in N(V(\langle *^k a_1 \rangle_k))$ and $h' \in N(V(\langle *^k a_{k+1} \rangle_k))$. Set $a_2 = \alpha$. Two vertices $w \in V(\langle *^k a_1 \rangle_k)$ and $z \in V(\langle *^k a_{k+1} \rangle_k)$ are determined such that $(w, h) = e^{(k+1)}(h)$ and $(z, h') = e^{(k+1)}(h')$. A desired s–t path in $S_{k+1} - F$ can be constructed as shown in Fig. 8(a) if $z \neq t$, and as shown in Fig. 8(b) if $z = t$.

If $z \neq t$, then ten vertices $u_1 \in N(w) \cap V(\langle *^k a_1 \rangle_k) - \{s\}$, $v_1 \in V(\langle *^k a_1 \rangle_k) - \{s\}$, $u_2, v_2 \in V(\langle *^k a_2 \rangle_k)$, $u_3, v_3 \in V(\langle *^k a_3 \rangle_k)$, $u_4 \in V(\langle *^k a_4 \rangle_k)$, $v_k \in V(\langle *^k a_k \rangle_k)$, $u_{k+1} \in V(\langle *^k a_{k+1} \rangle_k)$, and $v_{k+1} \in N(z) \cap V(\langle *^k a_{k+1} \rangle_k) - \{t\}$ are further determined such that $e^{(k+1)}(u_1) \notin F$, $(v_3, u_1) = e^{(k+1)}(u_1)$, u_3 and v_3 belong to different partite sets of S_{k+1}, $e^{(k+1)}(u_3) \notin F$, $(v_2, u_3) = e^{(k+1)}(u_3)$, u_2 and v_2 belong to different partite sets of S_{k+1}, $e^{(k+1)}(u_2) \notin F$, $(v_k, u_2) = e^{(k+1)}(u_2)$, v_1 and t belong to the same partite set of S_{k+1}, $e^{(k+1)}(v_1) \notin F$, $(u_{k+1}, v_1) = e^{(k+1)}(v_1)$, $e^{(k+1)}(v_{k+1}) \notin F$, and $(u_4, v_{k+1}) = e^{(k+1)}(v_{k+1})$.

There is a longest u_3–v_3 path of length $k! - 1$ in $\langle *^k a_3 \rangle_k$. Imagine that (h, h') is fault-free. The induction hypothesis assures a longest u_2–v_2 path of length $k! - 1$ in $\langle *^k a_2 \rangle_k - (F - \{(h, h')\})$. By Lemma 6, there exist a longest u_{k+1}–t path of length $k! - 1$ in $\langle *^k a_{k+1} \rangle_k$ that contains (z, v_{k+1}) and a longest s–v_1 path of length $k! - 1$ or $k! - 2$ in $\langle *^k a_1 \rangle_k$ that contains (u_1, w). Since u_4 and v_k

belong to different partite sets of S_{k+1}, Lemma 1 assures a longest u_4–v_4 path of length $4!-1$ in $\langle *^4a_4\rangle_4$ if $k=4$, and Lemma 5 assures a longest u_4–v_k path of length $(k-3)\times k!-1$ in $S_{k+1}[V_A]-F$ if $k>4$, where $A=\{1,2,\ldots,k+1\}-\{a_1,a_2,a_3,a_{k+1}\}$. The desired s–t path has length $(k!-1)+2\times((k!-1)-1)+((k!-1)-1)+((k-3)\times k!-1)+7=(k+1)!-1$ or $(k!-1)+2\times((k!-1)-1)+((k!-2)-1)+((k-3)\times k!-1)+7=(k+1)!-2$.

If $z=t$, then vertices $u_1, v_1, u_2, v_2, u_3, v_3, v_k$, and u_{k+1} are determined all the same as above. Differently, $v_{k+1}\in V(\langle *^ka_{k+1}\rangle_k)$ and $u_4\in V(\langle *^ka_4\rangle_k)$ are determined such that v_{k+1} and u_{k+1} belong to the same partite set of S_{k+1}, $e^{(k+1)}(v_{k+1})\notin F$, and $(u_4,v_{k+1})=e^{(k+1)}(v_{k+1})$. With the same arguments as the situation of $z\neq t$, there are a longest u_3–v_3 path of length $k!-1$ in $\langle *^ka_3\rangle_k$, a longest u_2–v_2 path of length $k!-1$ in $\langle *^ka_2\rangle_k-(F-\{(h,h')\})$, a longest s–v_1 path of length $k!-1$ or $k!-2$ in $\langle *^ka_1\rangle_k$ that contains (u_1,w), and a longest u_4–v_k path of length $(k-3)\times k!-1$ in $S_{k+1}[V_A]-F$, where $A=\{1,2,\ldots,k+1\}-\{a_1,a_2,a_3,a_{k+1}\}$. By Lemma 1, there is a longest u_{k+1}–v_{k+1} path of length $k!-2$ in $\langle *^ka_{k+1}\rangle_k-\{t\}$ (u_{k+1} and t belong to different partite sets of S_{k+1}). The desired s–t path has length $(k!-1)+((k!-1)-1)+((k!-1)-1)+((k-3)\times k!-1)+(k!-2)+7=(k+1)!-1$ or $(k!-1)+((k!-1)-1)+((k!-2)-1)+((k-3)\times k!-1)+(k!-2)+7=(k+1)!-2$.

5. Concluding remarks

Since processor faults and/or link faults may occur to multiprocessor systems, it is both practically significant and theoretically interesting to study the fault tolerance of multiprocessor systems. Most of previous works used the random fault model, which assumed that the faults might occur everywhere without any restriction. There was another fault model, i.e., the conditional fault model, which assumed that the fault distribution must be subject to some constraint.

In this paper, adopting the conditional fault model and assuming that there were two or more fault-free links incident to each node, we constructed a longest fault-free path between two arbitrary nodes of an n-dimensional star network with up to $2n-7$ link faults. The result is optimal with respect to the number of link faults tolerated. The longest path has length $n!-1$ ($n!-2$) if the two end nodes belong to different partite sets (the same partite set) of the star network. Two fundamental construction methods were demonstrated in Figs. 2 and 3. When additional restrictions were imposed, the two construction methods must be adapted. The modified construction methods were demonstrated in Figs. 5–8.

Usually, the problem of embedding fault-free paths in faulty networks under the conditional fault model is more difficult than the same problem under the random fault model. Although the problem of embedding fault-free paths in faulty star networks were solved in [24] under the random fault model, the properties derived in [24] were not sufficient for our purpose. Instead, we need to develop more properties, i.e., Lemmas 3–7, when the conditional fault model is used. These lemmas are useful to those people who are interested in star networks and/or fault-tolerant embedding under the conditional fault model.

Apparently, as a consequence of this paper, there is a fault-free Hamiltonian cycle in an n-dimensional star network with up to $2n-7$ link faults under the conditional fault model and our assumption. For a faulty star network, finding a fault-free Hamiltonian cycle is much easier than finding a longest fault-free path between every two distinct nodes. It is very likely that embedding a fault-free Hamiltonian cycle in a faulty star network may tolerate more than $2n-7$ link faults under the conditional fault model and the same assumption.

Finally, it should be mentioned that the assumption we made in this paper about the conditional fault model is meaningful in practice. Let p_n be the probability that each node of an n-dimensional star network with $2n-7$ link faults is incident with two or more fault-free links. The value of p_n is very close to 1, even if n is small. A formula for estimating a lower bound on p_n was derived in [14]. Based on this formula, we have, for example, $p_5>0.999736296$, $p_{10}>1-3\times10^{-43}$, $p_{15}>1-4\times10^{-140}$, and $p_{20}>1-9\times10^{-305}$.

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References

[1] S.B. Akers, D. Horel, B. Krishnamurthy, The star graph: An attractive alternative to the n-cube, in: Proc. Int. Conference on Parallel Processing, 1987, pp. 393–400.
[2] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (4) (1989) 555–566.
[3] N. Ascheuer, Hamiltonian path problems in the on-line optimization of flexible manufacturing systems, Ph.D. Thesis, University of Technology, Berlin, Germany, 1995. Also available at: ftp://ftp.zib.de/pub/zib-publications/reports/TR-96-03.ps.
[4] Y.A. Ashir, I.A. Stewart, Fault-tolerant embedding of Hamiltonian circuits in k-ary n-cube, SIAM Journal on Discrete Mathematics 15 (3) (2002) 317–328.
[5] N. Bagherzadeh, N. Nassif, S. Latifi, A routing and broadcasting scheme on faulty star graphs, IEEE Transactions on Computers 42 (11) (1993) 1398–1403.
[6] M.Y. Chan, S.J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM Journal on Discrete Mathematics 4 (4) (1991) 511–527.
[7] K. Day, The conditional node connectivity of the k-ary n-cube, Journal of Interconnection Networks 5 (1) (2004) 13–26.
[8] K. Day, A.E. Al-Ayyoub, Fault diameter of k-ary n-cube networks, IEEE Transactions on Parallel and Distributed Systems 8 (9) (1997) 903–907.
[9] K. Day, A.R. Tripathi, A comparative study of topological properties of hypercubes and star graphs, IEEE Transactions on Parallel and Distributed Systems 5 (1) (1994) 31–38.
[10] K. Diks, A. Pele, Efficient gossiping by packets in networks with random faults, SIAM Journal on Discrete Mathematics 9 (1) (1996) 7–18.

[11] A.H. Esfahanian, Generalized measures of fault tolerance with application to N-cube networks, IEEE Transactions on Computers 38 (11) (1989) 1586–1591.
[12] A.H. Esfahanian, S.L. Hakimi, Fault-tolerant routing in de Bruijn communication networks, IEEE Transactions on Computers C-34 (9) (1985) 777–788.
[13] P. Fragopoulou, S.G. Akl, Optimal communication algorithms on star graphs using spanning tree constructions, Journal of Parallel and Distributed Computing 24 (1) (1995) 55–71.
[14] J.S. Fu, Conditional fault-tolerant hamiltonicity of star graphs, Parallel Computing 33 (7–8) (2007) 488–496.
[15] S.Y. Hsieh, G.H. Chen, C.W. Ho, Hamiltonian-laceability of star graphs, Networks 36 (4) (2000) 225–232.
[16] S.Y. Hsieh, G.H. Chen, C.W. Ho, Longest fault-free paths in star graphs with vertex faults, Theoretical Computer Science 262 (1–2) (2001) 215–227.
[17] H.C. Hsu, T.K. Li, J.M. Tan, L.H. Hsu, Fault hamiltonicity and fault Hamiltonian connectivity of the arrangement graphs, IEEE Transactions on Computers 53 (1) (2004) 39–53.
[18] J.S. Jwo, S. Lakshmivarahan, S.K. Dhall, Embedding of cycles and grids in star graphs, Journal of Circuits, Systems, and Computers 1 (1) (1991) 43–74.
[19] S. Latifi, Combinatorial analysis of the fault-diameter of the n-cube, IEEE Transactions on Computers 42 (1) (1993) 27–33.
[20] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE Transactions on Computers 43 (2) (1994) 218–222.
[21] F.T. Leighton, Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes, Morgan Kaufmann, CA, 1991.
[22] M. Lewinter, W. Widulski, Hyper-hamilton laceable and caterpillar-spannable product graphs, Computers & Mathematics with Applications 34 (11) (1997) 99–104.
[23] L. Qiao, Z. Yi, Restricted connectivity and restricted fault diameter of some interconnection networks, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 21 (1995) 267–273.
[24] T.K. Li, J.M. Tan, L.H. Hsu, Hyper Hamiltonian laceability on the edge fault star graph, Information Sciences 165 (2004) 59–71.
[25] A.C. Liang, S. Bhattacharya, W.T. Tsai, Fault-tolerant multicasting on hypercubes, Journal of Parallel and Distributed Computing 23 (1994) 418–428.
[26] V.E. Mendia, D. Sarkar, Optimal broadcasting on the star graph, IEEE Transactions on Parallel and Distributed Systems 3 (4) (1992) 389–396.
[27] Z. Miller, D. Pritikin, I.H. Sudborough, Near embeddings of hypercubes into Cayley graphs on the symmetrical group, IEEE Transactions on Computers 43 (1) (1994) 13–22.
[28] K. Qiu, S.G. Akl, H. Meijer, On some properties and algorithms for the star and pancake interconnection networks, Journal of Parallel and Distributed Computing 22 (1) (1994) 16–25.
[29] Y. Rouskov, S. Latifi, P.K. Srimani, Conditional fault diameter of star graph networks, Journal of Parallel and Distributed Computing 33 (1) (1996) 91–97.
[30] C.H. Tsai, Linear array and ring embeddings in conditional faulty hypercubes, Theoretical Computer Science 314 (2004) 431–443.
[31] S.A. Wong, Hamiltonian cycles and paths in butterfly graphs, Networks 26 (1995) 145–150.
[32] J. Wu, Safety levels — an efficient mechanism for achieving reliable broadcasting in hypercubes, IEEE Transactions on Computers 44 (5) (1995) 702–706.
[33] J. Wu, Fault tolerance measures for m-ary n-dimensional hypercubes based on forbidden faulty sets, IEEE Transactions on Computers 47 (8) (1998) 888–893.
[34] http://inrg.csie.ntu.edu.tw/~bytsai/.