Hierarchy and equivalence of multi-letter quantum finite automata^{*}

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Abstract

Multi-letter quantum finite automata (QFAs) were a new one-way QFA model proposed recently by Belovs, Rosmanis, and Smotrovs (LNCS, Vol. 4588, Springer, Berlin, 2007, pp. 60-71), and they showed that multi-letter QFAs can accept with no error some regular languages $((a+b)^*b)$ that are unacceptable by the one-way QFAs. In this paper, we continue to study multi-letter QFAs. We mainly focus on two issues: (1) we show that (k + 1)-letter QFAs are computationally more powerful than k-letter QFAs, that is, (k + 1)-letter QFAs can accept some regular languages that are unacceptable by any k-letter QFA. A comparison with the one-way QFAs is made by some examples; (2) we prove that a k_1 -letter QFA \mathcal{A}_1 and another k_2 -letter QFA \mathcal{A}_2 are equivalent if and only if they are $(n_1 + n_2)^4 + k - 1$ -equivalent, and the time complexity of determining the equivalence of two multi-letter QFAs using this method is $O(n^{12} + k^2n^4 + kn^8)$, where n_1 and n_2 are the numbers of states of \mathcal{A}_1 and \mathcal{A}_2 , respectively, and $k = \max(k_1, k_2)$. Some other issues are addressed for further consideration.

Keywords: Quantum computing; Multi-letter finite automata; Quantum finite automata; Equivalence; Hierarchy

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1. Introduction

Quantum computing is an intriguing and promising research field, which touches on computer science, quantum physics, and mathematics [17, 18, 11, 10]. To a certain extent, quantum computing was motivated by the exponential speed-up of Shor's quantum algorithm for factoring integers in polynomial time [33] and Grover's algorithm of searching in database of size n with only $O(\sqrt{n})$ accesses [16].

Quantum computers—the physical devices complying with the rules of quantum mechanics were first considered by Benioff [8], and then suggested by Feynman [14]. By elaborating and formalizing Benioff and Feynman's idea, in 1985, Deutsch [12] re-examined the Church-Turing Principle and defined *quantum Turing machines* (QTMs). Subsequently, Deutsch [13] considered quantum network models. In 1993, Yao [36] demonstrated the equivalence between QTMs and quantum circuits. Quantum computation from the viewpoint of complexity theory was first studied systematically by Bernstein and Vazirani [7].

Another kind of simpler models of quantum computation is quantum finite automata (QFAs), which can be thought of as theoretical models of quantum computers with finite memory. This kind of computing machines was first studied by Moore and Crutchfield [27]. as well as by Kondacs and Watrous [24] independently. Then it was dealt with in depth by Ambainis and Freivalds [1], Brodsky and Pippenger [5], and the other authors (for example, see the references in [17, 31]). The study of QFAs is mainly divided into two ways: one is one-way quantum finite automata (1QFAs) whose tape heads only move one cell to right at each computation step (1QFAs have been extensively studied [4]), and the other is two-way quantum finite automata (2QFAs), in which the tape heads are allowed to move towards right or left, or to be stationary [24]. (Notably, Amano and Iwama [2] dealt with a decidability problem concerning an intermediate form called 1.5QFAs, whose tape heads are allowed to move right or to be stationary; Hirvensalo [19] investigated a decidability problem related to one-way QFAs.) Furthermore, by considering the number of times of the measurement in a computation, 1QFAs have two different forms: measure-once 1QFAs (MO-1QFAs) proposed by Moore and Crutchfield [27], and, measure-many 1QFAs (MM-1QFAs) studied first by Kondacs and Watrous [24].

MM-1QFAs are strictly more powerful than MO-1QFAs [1, 4] (Indeed, a^*b^* can be accepted by MM-1QFAs with bounded error but not by any MO-1QFA with bounded error). Due to the unitarity of quantum physics and finite memory of finite automata, both MO-1QFAs and MM-1QFAs can only accept proper subclasses of regular languages with bounded error (e.g., [24, 1, 5, 4]). Indeed, it was shown that the regular language $(a + b)^*b$ cannot be accepted by any MM-1QFA with bounded error [24].

Recently, Belovs, Rosmanis, and Smotrovs [6] proposed a new one-way QFA model,

namely, multi-letter QFAs, that can be thought of as a quantum counterpart of more restricted classical one-way multi-head finite automata (see, for example, [20]). Roughly speaking, a k-letter QFA is not limited to seeing only one, the just-incoming input letter, but can see several earlier received letters as well. That is, the quantum state transition which the automaton performs at each step depends on the last k letters received. For the other computing principle, it is similar to the usual MO-1QFAs as described above. Indeed, when k = 1, it reduces to an MO-1QFA. Any given k-letter QFA can be simulated by some k + 1-letter QFA. However, we will prove that the contrary does not hold. Belovs et al. [6] have already showed that $(a + b)^*b$ can be accepted by a 2-letter QFA but, as proved in [24], it cannot be accepted by any MM-1QFA with bounded error. By $\mathcal{L}(QFA_k)$ we denote the class of languages accepted with bounded error by k-letter QFAs. In this paper, we will prove that $\mathcal{L}(QFA_k) \subset \mathcal{L}(QFA_{k+1})$ for k = 1, 2, ..., where the inclusion \subset is proper. Therefore, (k + 1)-letter QFAs are computationally more powerful than k-letter QFAs.

As we know, determining the equivalence for computing models is a very important issue in the theory of classical computation (see, e.g., [28, 34, 32, 9, 22, 21]). Concerning the problem of determining the equivalence for QFAs, there exists some work [5] that deals with the simplest case—MO-1QFAs. For quantum sequential machines (QSMs), Qiu [29] gave a negative outcome for determining the equivalence of QSMs, and then Li and Qiu [25] further gave a method for determining whether or not any two given QSMs are equivalent. This method applies to determining the equivalence between any two MO-1QFAs and also is different from the previous ones. For the equivalence problem of MM-1QFAs, inspired by the work of [35] and [4], Li and Qiu [26] presented a polynomial-time algorithm for determining whether or not any two given MM-1QFAs are equivalent.

In this paper, we will give a polynomial-time algorithm for determining whether or not any two given k_1 -letter QFA \mathcal{A}_1 and k_2 -letter QFA \mathcal{A}_2 for accepting unary languages are equivalent. More specifically, we prove that two multi-letter QFAs \mathcal{A}_1 and \mathcal{A}_2 , are equivalent if and only if they are $(n_1+n_2)^4 + k - 1$ -equivalent, where n_1 and n_2 are the numbers of states of \mathcal{A}_1 and \mathcal{A}_2 , respectively, $k = \max(k_1, k_2)$, and two multi-letter QFAs over the same input alphabet Σ are *n*-equivalent if and only if the accepting probabilities of \mathcal{A}_1 and \mathcal{A}_2 are equal for the input strings of length not more than *n*. This method, generalized appropriately, may apply to dealing with more general cases.

The remainder of the paper is organized as follows. In Section 2, we recall the definition of multi-letter QFAs and other related definitions, and some related results are reviewed. In Section 3, we prove that $\mathcal{L}(QFA_k) \subset \mathcal{L}(QFA_{k+1})$ for k = 1, 2, ..., where the inclusion \subset is proper. More precisely, we show that, for $k \geq 2$, regular language $(a_1 + a_2 + ... + a_k)^* a_1 a_2 \cdots a_{k-1}$ cannot be accepted with bounded error by (k-1)-letter QFAs but can be exactly accepted by some k-letter QFAs. In addition, we present a number of examples to show the relation between multi-letter QFAs and the usual one-way QFAs. In Section 4, we concentrate on the equivalence issue. After proving some useful lemmas, we prove that a k_1 -letter QFA \mathcal{A}_1 and another k_2 -letter QFA \mathcal{A}_2 for accepting unary languages are equivalent if and only if they are $(n_1 + n_2)^4 + k - 1$ -equivalent, and the time complexity of determining the equivalence of two multi-letter DFAs using this method is $O(n^{12}+k^2n^4+kn^8)$, where $n = n_1 + n_2$, n_1 and n_2 are the numbers of states of \mathcal{A}_1 and \mathcal{A}_2 , respectively, and $k = \max(k_1, k_2)$. Finally, in Section 5 we address some related issues for further consideration.

In general, symbols will be explained when they first appear.

2. Preliminaries

In this section, we briefly review some definitions and related properties that will be used in the sequel. For the details, we refer to [6].

First we recall k-letter deterministic finite automata (k-letter DFAs).

Definition 1 ([6]). A k-letter deterministic finite automaton (k-letter DFA) is defined by a quintuple $(Q, Q_{acc}, q_0, \Sigma, \gamma)$, where Q is a finite set of states, $Q_{acc} \subseteq Q$ is the set of accepting states, $q_0 \in Q$ is the initial state, Σ is a finite input alphabet, and γ is a transition function that maps $Q \times T^k$ to Q, where $T = \{\Lambda\} \bigcup \Sigma$ and letter $\Lambda \notin \Sigma$ denotes the blank symbol (like a blank symbol in Turing machines [34]), and $T^k \subset T^*$ consists of all strings of length k.

We describe the computing process of a k-letter DFA on an input string x in Σ^* , where $x = \sigma_1 \sigma_2 \cdots \sigma_n$, and Σ^* denotes the set of all strings over Σ . The k-letter DFA has a tape which contains the letter Λ in its first k - 1 position followed by the input string x. The automaton starts in the initial state q_0 and has k reading heads which initially are on the first k positions of the tape (clearly, the kth head reads σ_1 and the other heads read Λ). Then the automaton transfers to a new state as current state and all heads move right a position in parallel. Now the (k-1)th and kth heads point to σ_1 and σ_2 , respectively, and the others, if any, to Λ . Subsequently, the automaton transfers to a new state and all heads move to the right. This process does not stop until the kth head has read the last letter σ_n . The input string x is accepted if and only if the automaton enters an accepting state after its kth head reading the last letter σ_n .

Clearly, k-letter DFAs are not more powerful than DFAs. The family of languages accepted by k-letter DFAs, for $k \ge 1$, is exactly the family of regular languages.

For the sake of readability, we briefly recall the definitions of MO-1QFAs and MM-1QFAs in the following.

An MO-1QFA is defined as a quintuple $A = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \{U(\sigma)\}_{\sigma \in \Sigma})$, where Q is a set of finite states, $Q_{acc} \subseteq Q$ is the set of accepting states, $|\psi_0\rangle$ is the initial state that is a superposition of the states in Q, Σ is a finite input alphabet, and $U(\sigma)$ is a unitary matrix for each $\sigma \in \Sigma$.

As usual, we identify Q with an orthonormal base of a complex Euclidean space and every state $q \in Q$ is identified with a basis vector, denoted by Dirac symbol $|q\rangle$ (a column vector), and $\langle q|$ is the conjugate transpose of $|q\rangle$. We describe the computing process for any given input string $x = \sigma_1 \sigma_2 \cdots \sigma_m \in \Sigma^*$. At the beginning the machine A is in the initial state $|\psi_0\rangle$, and upon reading σ_1 . The transformation $U(\sigma_1)$ acts on $|\psi_0\rangle$. After that, $U(\sigma_1)|\psi_0\rangle$ becomes the current state and the machine reads σ_2 . The process continues until the machine has read σ_m ending in the state $|\psi_x\rangle = U(\sigma_m)U(\sigma_{m-1})\cdots U(\sigma_1)|\psi_0\rangle$. Finally, a measurement is performed on $|\psi_x\rangle$ and the accepting probability $p_a(x)$ is equal to

$$p_a(x) = \langle \psi_x | P_a | \psi_x \rangle = \| P_a | \psi_x \rangle \|^2$$

where $P_a = \sum_{q \in Q_{acc}} |q\rangle \langle q|$ is the projection onto the subspace spanned by $\{|q\rangle : q_i \in Q_{acc}\}$.

An MM-1QFA is defined as a 6-tuple $A = (Q, Q_{acc}, Q_{rej}, |\psi_0\rangle, \Sigma, \{U(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}})$, where $Q, Q_{acc} \subseteq Q, |\psi_0\rangle, \Sigma, \{U(\sigma)\}_{\sigma \in \Sigma \cup \{\$\}}$ are the same as those in an MO-1QFA defined above, $Q_{rej} \subseteq Q$ represents the set of rejecting states, and $\$ \notin \Sigma$ is a tape symbol denoting the right end-mark. For any input string $x = \sigma_1 \sigma_2 \cdots \sigma_m \in \Sigma^*$, the computing process is similar to that of MO-1QFAs except that after every transition, A measures its state with respect to the three subspaces that are spanned by the three subsets Q_{acc}, Q_{rej} , and Q_{non} , respectively, where $Q_{non} = Q \setminus (Q_{acc} \cup Q_{rej})$. In other words, the projection measurement consists of $\{P_a, P_r, P_n\}$ where $P_a = \sum_{q \in Q_{acc}} |q\rangle\langle q|, P_r = \sum_{q \in Q_{rej}} |q\rangle\langle q|, P_n = \sum_{q \in Q \setminus (Q_{acc} \cup Q_{rej})} |q\rangle\langle q|$. The machine stops after the right end-mark \$ has been read. Of course, the machine may also stop before reading \$ if the current state of the machine reading some σ_i ($1 \le i \le m$) does not contain the states of Q_{non} . Since the measurement is performed after each transition with the states of Q_{non} being preserved, the accepting probability $p_a(x)$ and the rejecting probability $p_r(x)$ are given as follows (for convenience, we denote $\$ = \sigma_{m+1}$):

$$p_a(x) = \sum_{k=1}^{m+1} \|P_a U(\sigma_k) \prod_{i=1}^{k-1} (P_n U(\sigma_i)) |\psi_0\rangle\|^2,$$
$$p_r(x) = \sum_{k=1}^{m+1} \|P_r U(\sigma_k) \prod_{i=1}^{k-1} (P_n U(\sigma_i)) |\psi_0\rangle\|^2.$$

We further recall the definitions of a group finite automaton (GFA) [5] and a one-way reversible finite automaton (1RFA) [1]. A GFA is a DFA whose state transition function, say δ , satisfies that for any input symbol σ , $\delta(\cdot, \sigma)$ is a one-to-one map on the state set, i.e., a permutation. A 1RFA is defined as an MO-1QFA but restricting the values of its state transition function onto $\{0, 1\}$. More specifically, a 1RFA is a DFA whose set of states, input alphabet, and state transition function are Q, Σ, δ , respectively, where δ satisfies that, for any $q \in Q$ and any $\sigma \in \Sigma$, there is at most one $p \in Q$ such that $\delta(p, \sigma) = q$. Qiu [30] proved that GFAs and 1RFAs are equivalent, i.e., any GFA can be simulated by a 1RFA and vice-versa.

Definition 2 ([6]). A k-letter DFA $(Q, Q_{acc}, q_0, \Sigma, \gamma)$ is called a k-letter group finite automaton (k-letter GFA) if and only if for any string $x \in T^k$ the function $\gamma_x(q) = \gamma(q, x)$ is a bijection from Q to Q.

Remark 1. When k = 1, a 1-letter DFA is exactly a DFA [32, 34, 37], and a 1-letter GFA is also the usual GFA [5]. By $\mathcal{L}(GFA_k)$ and $\mathcal{L}(DFA_k)$ we denote the classes of all languages accepted by k-letter GFAs and by k-letter DFAs, respectively. In addition, we denote $\mathcal{L}(GFA_*) = \bigcup_{k=1}^{\infty} \mathcal{L}(GFA_k)$ and $\mathcal{L}(DFA_*) = \bigcup_{k=1}^{\infty} \mathcal{L}(DFA_k)$. In [6] it was shown that

$$\mathcal{L}(GFA) \subset \mathcal{L}(GFA_*) \subset \mathcal{L}(DFA) = \mathcal{L}(DFA_*), \tag{1}$$

where \subset is a proper inclusion.

Now we further recall the definition of multi-letter QFAs [6].

Definition 3 ([6]). A k-letter QFA \mathcal{A} is defined as a quintuple $\mathcal{A} = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$ where Q is a set of states, $Q_{acc} \subseteq Q$ is the set of accepting states, $|\psi_0\rangle$ is the initial unit state that is a superposition of the states in Q, Σ is a finite input alphabet, and μ is a function that assigns a unitary transition matrix U_w on $\mathbb{C}^{|Q|}$ for each string $w \in (\{\Lambda\} \cup \Sigma)^k$, where |Q| is the cardinality of Q.

The computation of a k-letter QFA \mathcal{A} works in the same way as the computation of an MO-1QFA, except that it applies unitary transformations corresponding not only to the last letter but the last k letters received (like a k-letter DFA). When k = 1, it is exactly an MO-1QFA as pointed out before. According to [6], all languages accepted by k-letter QFAs with bounded error are regular languages for any k.

Now we give the probability $P_{\mathcal{A}}(x)$ for k-letter QFA $\mathcal{A} = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$ accepting any input string $x = \sigma_1 \sigma_2 \cdots \sigma_m$. From the definition we know that, for any $w \in (\{\Lambda\} \cup \Sigma)^k$, $\mu(w)$ is a unitary matrix. In terms of the definition of μ , we can define the unitary transition for each string $x = \sigma_1 \sigma_2 \cdots \sigma_m \in \Sigma^*$. By $\overline{\mu}$ we mean a map from Σ^* to the set of all |Q|-order unitary matrices. Indeed, $\overline{\mu}$ is induced by μ in the following way. For $x = \sigma_1 \sigma_2 \cdots \sigma_m \in \Sigma^*$,

$$\overline{\mu}(x) = \begin{cases} \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\Lambda^{k-m}x), & \text{if } m < k, \\ \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\sigma_{m-k+1}\sigma_{m-k+2}\cdots\sigma_m), & \text{if } m \ge k, \end{cases}$$
(2)

which implies the computing process of \mathcal{A} for input string x.

As before, we identify the states in Q with an orthonormal basis of the complex Euclidean space $\mathbb{C}^{|Q|}$, and let P_{acc} denote the projector on the subspace spanned by Q_{acc} . Then we define that

$$P_{\mathcal{A}}(x) = \|\langle \psi_0 | \overline{\mu}(x) P_{acc} \|^2.$$
(3)

Definition 4 ([6]). For $k \geq 1$, a DFA contains a C_k -construction if and only if there are states q_1, q_2, q_3, q_4, q_5 and a string $w = \sigma_1 \sigma_2 \cdots \sigma_k$ of length k such that $q_2 \neq q_5$, and transformation function γ satisfies $\gamma(q_2, \sigma_k) = \gamma(q_5, \sigma_k) = q_3, \gamma^*(q_1, \sigma_1 \cdots \sigma_{k-1}) = q_2$ and $\gamma^*(q_4, \sigma_1 \cdots \sigma_{k-1}) = q_5$.

In the above C_k -construction, if there exists an m > 0 such that $\gamma^*(q_3, w^{m-1}) = q_4$, then we call it a D_k -construction.

Proposition 1 ([6]). If there exists a C_k -construction in a DFA, then there also exists a D_k -construction in this DFA.

Theorem 2 ([6]). The following statements are equivalent:

- A language L is in $\mathcal{L}(QFA_k)$, i.e., L is accepted by a k-letter QFA with bounded error.
- The minimal DFA of L contains no C_k -construction.
- L is accepted by a k-letter GFA.

From Theorem 2 we know that a language is accepted by a k-letter GFA if and only if it is accepted by a k-letter QFA with bounded error. For k = 1, it was proved by Brodsky and Pippenger [5].

3. Hierarchy of multi-letter QFAs and some relations

In this section, we deal with two issues. In Subsection 3.1, we consider the hierarchy of multi-letter QFAs and prove that *j*-letter QFA are strictly more powerful than *i*-letter QFAs for $1 \leq i < j$. In Subsection 3.2, we attempt to clarify the relations between the families of languages accepted by multi-letter QFAs and MO-1QFAs and also between those by multi-letter QFAs and MM-QFAs.

3.1. Hierarchy of multi-letter QFAs

Are k-letter QFAs more powerful than (k-1)-letter QFAs for k = 1, 2, ...? The answer is positive for k = 2 as proved in [6]. In this subsection, we demonstrate that k-letter QFAs are more powerful than (k-1)-letter QFAs for any $k \ge 3$.

Theorem 3. For any $k \ge 3$, there exists a language that can be accepted by a k-letter GFA but cannot be accepted by any (k-1)-letter GFA.

Proof. We consider the regular language $(a_1 + a_2 + \ldots + a_k)^* a_1 a_2 \cdots a_{k-1}$ denoted by L_k over alphabet $\Sigma = \{a_1, a_2, \ldots, a_k\}$, and we will prove that L_k satisfies the theorem. First we construct a minimal DFA for L_k as $A_k = (Q, \Sigma, q_0, \delta, F)$ where:

- $Q = \{q_0, q_1, \dots, q_{k-1}\};$
- $\Sigma = \{a_1, a_2, \ldots, a_k\};$
- $F = \{q_{k-1}\};$
- δ is defined as follows:

$$\begin{aligned} &-\delta(q_0, a_1) = q_1; \ \delta(q_0, a_i) = q_0 \text{ for } i = 2, 3, \dots, k; \\ &-\delta(q_1, a_1) = q_1; \ \delta(q_1, a_2) = q_2; \ \delta(q_1, a_i) = q_0 \text{ for } i = 3, 4, \dots, k; \\ &-\delta(q_l, a_{l+1}) = q_{l+1} \text{ and } \delta(q_l, a_t) = q_0 \text{ for } l = 2, 3, \dots, k-1 \text{ and } t \in \{2, \dots, l, l+2, l+3, \dots, k\}, \text{ where we denote } q_k = q_0. \\ &-\delta(q_i, a_1) = q_1 \text{ for } i = 2, 3, \dots, k-1. \end{aligned}$$

Figure 1 depicts the DFA A_k above described. We prove that A_k is a minimal DFA. It suffices to prove that, for all states q_0, q_1, \dots, q_{k-1} , any two different states are distinguishable [23]. In other words, for any $0 \le i, j \le k-1$ with $i \ne j$, there exists $w \in \Sigma^*$ such that exactly one of $\delta^*(q_i, w)$ and $\delta^*(q_j, w)$ is the accepting state q_{k-1} . Indeed, we can divide it into three cases.

- 1. q_i and q_{k-1} for $0 \le i \le k-2$. Take $w = \epsilon$, empty string. Then $\delta^*(q_i, \epsilon) = q_i$ and $\delta^*(q_{k-1}, \epsilon) = q_{k-1}$.
- 2. q_0 and q_l for $1 \le l \le k-2$. Take $w = a_{l+1}a_{l+2}\cdots a_{k-1}$. Then $\delta^*(q_0, w) = q_0$ and $\delta^*(q_l, w) = q_{k-1}$.
- 3. q_i and q_j for $1 \le i < j \le k-2$. Take $w = a_{j+1}a_{j+2}\cdots a_{k-1}$. Then $\delta^*(q_j, w) = q_{k-1}$ and $\delta^*(q_i, w) = q_0$.

Therefore, we have proved that any two different states of q_0, q_1, \dots, q_{k-1} are distinguishable. Consequently, A_k is minimal.

In fact, we can see that the number k of states is minimal from the number of equivalence classes over Σ^* [23]. This equivalence relation \equiv is defined as: for any $w_1, w_2 \in \Sigma^*$, $w_1 \equiv w_2$ iff for any $z \in \Sigma^*$, either both $w_1 z$ and $w_2 z$ in L_k , or neither $w_1 z$ nor $w_2 z$ in L_k . Then we can divide Σ^* into the following k equivalence classes: $[\epsilon], [a_1], [a_1 a_2], \cdots, [a_1 a_2 \cdots a_{k-1}]$. As a result, k is the number of states of the minimal DFA accepting L_k .

In the state transition figure of A_k , we find a C_{k-1} -construction. In fact, set $w = a_2a_3...a_k$. Since $\delta(q_0, a_i) = q_0$ for i = 2, 3, ..., k, we get $\delta^*(q_0, a_2a_3\cdots a_{k-1}) = q_0$ and $\delta(q_0, a_k) = q_0$. Moreover, $\delta(q_i, a_{i+1}) = q_{i+1}$ for i = 1, 2, ..., k, where we denote $q_k = q_0$. This C_{k-1} -construction is better described by Figure 2. By Theorem 2, we conclude that A_k cannot be accepted by any (k-1)-letter QFA with bounded error.



Figure 1. A state transition diagram of DFA A_k .

However, we will verify that, in the minima DFA A_k , there is no C_k -construction. Therefore, according to Theorem 2, L_k can be accepted by a k-letter QFA with bounded error.



Figure 2. A C_{k-1} -construction in DFA A_k .

Now we check that there is no C_k -construction in A_k . We prove it by contradiction. Indeed, suppose that there is a C_k -construction depicted by Figure 3.



Figure 3. A supposed C_k -construction.

We divide the proof into the following three cases.

1. $q = q_0$.

- $\sigma_k = a_1$: It is impossible since q_0 cannot be accessed by inputting a_1 .
- $\sigma_k = a_2$: In this case, $q_{i_k}, q_{j_k} \in \{q_0, q_2, q_3, \dots, q_{k-1}\}$.

If one of q_{i_k}, q_{j_k} , say q_{i_k} is q_0 , and the other one q_{j_k} belongs to $\{q_2, q_3, \ldots, q_{k-1}\}$, then $\sigma_{k-1} = a_{j_k}$ where $j_k \ge 2$. Thus, $q_{j_{k-1}} = q_{j_{k-1}}$ and $q_{i_{k-1}} = q_0$. In succession, we find that $q_{i_t} = q_0, q_{j_t} = q_1$ for some $2 \le t \le k$. However, there is no $\sigma \in \Sigma$ leading to q_0 and q_1 simultaneously. Therefore, it is impossible.

If $q_{i_k}, q_{j_k} \in \{q_2, q_3, \dots, q_{k-1}\}$, then the above case shows that this is impossible either.

Consequently, this case does not exist.

• $\sigma_k = a_s$ for $3 \le s \le k$:

These cases can be similarly verified as above, and we leave the details out here.

2. $q = q_1$

- $\sigma_k = a_1$: In this case, $q_{i_k}, q_{j_k} \in \{q_0, q_1, q_2, q_3, \dots, q_{k-1}\}$. Similar to the above proof.
- $\sigma_k = a_s$ for $2 \le s \le k$: It is clearly impossible.
- 3. $q = q_s$ for $2 \le s \le k$:

For any $2 \leq s \leq k$, there is no $\sigma \in \Sigma$ and two different states $p_1 \neq p_2$ such that $\delta(p_1, \sigma) = \delta(p_2, \sigma) = q_s$. Consequently, there does not exist such a C_k -construction.

Hence, there does not exist a C_k -construction in A_k , and therefore, by Theorem 2, L_k can be accepted by a k-letter GFA.

From Theorem 2 and Theorem 3 we have the following corollary.

Corollary 4. For $k \geq 2$, $\mathcal{L}(QFA_{k-1}) \subset \mathcal{L}(QFA_k)$, where the inclusion is proper.

3.2. Comparison of multi-letter QFAs with others

In this subsection, we try to compare the relations between the families of languages accepted by multi-letter QFAs and MO-1QFAs and also between those by multi-letter QFAs and MM-QFAs. First we recall the definition of *forbidden construction* in a DFA [1].

In a DFA, a forbidden construction means that there exist string x and states p_1 and p_2 , $p_1 \neq p_2$, such that $\delta^*(p_1, x) = p_2$ and $\delta^*(p_2, x) = p_2$, where p_2 is neither "all-accepting" state, nor "all-rejecting". A state p is neither "all-accepting" state, nor "all-rejecting" whenever there exist $w_1, w_2 \in \Sigma^*$ such that exactly one of $\delta^*(p, w_1)$ and $\delta^*(p, w_2)$ is an accepting state.

Remark 2. Ambainis and Freivalds [1] presented a forbidden construction and showed that, if the minimal DFA for accepting a regular language does not contain a forbidden construction, then this language can be accepted by a one-way reversible finite automaton. In [30], Qiu proved that one-way reversible finite automata are also GFAs and vice versa. Also, Ambainis and Freivalds [1] proved that a regular language is accepted by an MM-1QFA with bounded error and with probability over $\frac{7}{9}$ if and only if this language is accepted by a 1RFA and thus by a GFA as well.

Next we verify that a forbidden construction implies a C_1 -construction.

Proposition 5. In a DFA, if there exists a forbidden construction, then there also exists a C_1 -construction.

Proof. Let $A = (Q, Q_{acc}, q_0, \Sigma, \delta)$ be a DFA. Suppose that there is a forbidden construction, that is, there are states p_1, p_2 and $x \in \Sigma^*$ satisfying $\delta(p_1, x) = p_2$ and $\delta(p_2, x) = p_2$. Suppose that $x = \sigma_1 \sigma_2 \dots \sigma_k$. Then there are states q_1, q_2, \dots, q_k and r_1, r_2, \dots, r_k with $q_k = p_2 = r_k$ such that $\delta(p_1, \sigma_1) = q_1$, $\delta(q_k, \sigma_1) = r_1$, $\delta(q_i, \sigma_{i+1}) = q_{i+1}$, where $i = 1, 2, \dots, k - 1$. This relation can be described by Figure 4.

$$(p_1) \xrightarrow{\sigma_1} (q_1) \xrightarrow{\sigma_2} (q_2) \xrightarrow{\sigma_k} (q_k) \xrightarrow{\sigma_1} (r_1) \xrightarrow{\sigma_k} (r_k)$$

Figure 4. A relation diagram where $q_k = p_2 = r_k$.

Since $p_1 \neq p_2 = q_k$ but $q_k = r_k$, there exists $q_i = r_i$ but $q_{i-1} \neq r_{i-1}$. Therefore, we have $\delta(q_{i-1}, \sigma_i) = q_i = r_i$ and $\delta(r_{i-1}, \sigma_i) = r_i = r_i$, which is a C_1 -construction.

By Remark 2 and Theorem 2 we obtain the following corollary.

Proposition 6. The minimal DFA accepting a regular language L does not contain C_1 construction if and only if L can be accepted by an MM-1QFA with bounded error and with
probability over $\frac{7}{9}$.

Proof. If the minimal DFA accepting a regular language L does not contain C_1 -construction, then, by Theorem 2 we obtain that L can be accepted by a GFA. Therefore, by Remark 2, L can be accepted by an MM-1QFA with bounded error and with probability over $\frac{7}{9}$.

On the other hand, if L is accepted by an MM-1QFA with bounded error and with probability over $\frac{7}{9}$, then, with Remark 2 we know that L can be accepted by a GFA. By Theorem 2, the minimal DFA accepting L does not contain C_1 -construction.

Next, we present a few examples to show that $\mathcal{L}(QFA_*)$ is still a proper subset of all regular languages. Let us first show an example of regular language that can be accepted by an MM-1QFA but not by any multi-letter QFA.

Example 1. The language a^*b^* can be accepted by an MM-1QFA [1] but it cannot be accepted by any k-letter QFA. Indeed, we can describe the minimal DFA M for accepting a^*b^* by Figure 5. In addition, from this figure we can find that there exists a C_k -construction for any $k \geq 2$, which is visualized by Figure 6.



Figure 5. A state transition diagram of DFA M accepting a^*b^* .



Figure 6. A C_k -construction in Figure 6.

Next we provide another example which demonstrates that there exist regular languages acceptable neither by MM-1QFAs nor by multi-letter QFAs with bounded error. However, we need a result from [6]. **Definition 5** ([6]). A DFA with state transition function δ is said to contain an F-construction if and only if there are non-empty words $t, z \in \Sigma^+$ and two distinct states $q_1, q_2 \in Q$ such that $\delta^*(q_1, z) = \delta^*(q_2, z) = q_2, \ \delta^*(q_1, t) = q_1, \ \delta^*(q_2, t) = q_2.$

Proposition 7 ([6]). A language L can be accepted by a multi-letter QFA with bounded error if and only if the minimal DFA of L does not contain any F-construction.

Example 2. We use an example from [3]. Let L be the language consisting of all words that start with any number of letters a and after first letter b (if there is one) there is an odd number of letters a. The minimal DFA G accepting L is depicted by Figure 7. As proved by Ambainis *et al* [3], L cannot be accepted by MM-1QFAs with bounded error. Indeed, L cannot be accepted by any multi-letter QFA, either. Because there exists an F-construction (Figure 8) in the minimal DFA G (Figure 7), we get the result.



Figure 7. Automaton G.



Figure 8. An F-construction in the minimal DFA G.

In conclusion, we can describe the relations between the families of languages accepted by MO-1QFAs, MM-1QFAs, and multi-letter QFAs, denoted by $\mathcal{L}(MO)$, $\mathcal{L}(MM)$, and $\mathcal{L}(QFA_*)$, respectively. We recall that the language $(a + b)^*b$ is accepted with no error by a 2-letter QFA but cannot be accepted by any MM-1QFA with bounded error, while a^*b^* is accepted by an MM-1QFA but cannot be accepted by any multi-letter QFA. Therefore, both $\mathcal{L}(MM) \setminus \mathcal{L}(QFA_*) \neq \emptyset$ and $\mathcal{L}(QFA_*) \setminus \mathcal{L}(MM) \neq \emptyset$ hold. Furthermore, we have that $\mathcal{L}(MO) \subseteq \mathcal{L}(MM) \cap \mathcal{L}(QFA_*)$, where \subseteq may be proper. However, by Example 2, we have known that $\mathcal{L}(MM) \cup \mathcal{L}(QFA_*)$ still is a proper subset of all regular languages.

4. Determining the equivalence between multi-letter quantum finite automata

Determining whether or not two one-way (probabilistic, quantum) finite automata and sequential machines are equivalent is of importance and has been well studied [28, 35, 27, 25, 26]. Concerning multi-letter QFAs, this issue is much more complicated and a new technique is needed. Here, we consider only the case of unary languages, i.e., the input alphabet having one element.

Our goal is to deal with the decidability of equivalence of unary multi-letter QFAs. More specifically, for any given k_1 -letter QFA \mathcal{A}_1 and k_2 -letter QFA \mathcal{A}_2 over the same input alphabet $\Sigma = \{\sigma\}$, our purpose is to determine whether or not they are equivalent.

For a k-letter QFA $\mathcal{A} = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu)$, we recall the probability $P_{\mathcal{A}}(x)$ for \mathcal{A} accepting input string $x = \sigma_1 \sigma_2 \cdots \sigma_m$ and the definition of $\overline{\mu}(x)$ as follows:

$$\overline{\mu}(x) = \begin{cases} \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\Lambda^{k-m}x), & \text{if } m < k, \\ \mu(\Lambda^{k-1}\sigma_1)\mu(\Lambda^{k-2}\sigma_1\sigma_2)\cdots\mu(\sigma_{m-k+1}\sigma_{m-k+2}\cdots\sigma_m), & \text{if } m \ge k, \end{cases}$$
(4)

and then

$$P_{\mathcal{A}}(x) = \|\langle \psi_0 | \overline{\mu}(x) P_{acc} \|^2.$$
(5)

We give the definition of equivalence between two multi-letter QFAs.

Definition 6. A k_1 -letter QFA \mathcal{A}_1 and another k_2 -letter QFA \mathcal{A}_2 over the same input alphabet Σ are said to be equivalent (resp. *t*-equivalent) if $P_{\mathcal{A}_1}(w) = P_{\mathcal{A}_2}(w)$ for any $w \in \Sigma^*$ (resp. for any input string w with $|w| \leq t$).

Before we present a method for determining the equivalence between multi-letter QFAs over the same unary alphabet, we prove a useful lemma that is helpful to the main result. We recall the definition of tensor product of matrices [17]. For $m \times n$ matrix A =

matrix defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

A basic property of tensor product is that, for any $m \times n$ matrix $A, p \times q$ matrix $B, n \times o$ matrix C, and $q \times r$ matrix D,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Now we present the crucial lemma.

Lemma 8. Let $\{U_1, U_2, \ldots, U_k\}$ be a finite set of $n \times n$ unitary matrices, and let \mathbb{M}_{n^2} denote the linear space consisting of all $n^2 \times n^2$ complex square matrices. Denote

$$H^{(i)} = span\{(U_1U_2\cdots U_k) \otimes (U_1^*U_2^*\cdots U_k^*), \dots, (U_1U_2\cdots U_k^i) \otimes (U_1^*U_2^*\cdots (U_k^i)^*)\}$$

for $i = 1, 2, \dots$, where, for any subset A of \mathbb{M}_{n^2} , spanA denotes the minimal subspace spanned by A, and * denotes the conjugate operation. Then, there exists an $i_0 \leq n^4$ such that

$$H^{(i_0)} = H^{(i_0+t)} \tag{6}$$

for any $t \geq 0$.

Proof. Let $\dim(S)$ denote the dimension of subspace S. Due to

$$H^{(i)} \subseteq H^{(i+1)} \subseteq \mathbb{M}_{n^2}$$

for any $i \ge 1$, we have $1 \le \dim(H^{(1)}) \le \dim(H^{(2)}) \le \cdots \le \dim(H^{(n^4+1)}) \le n^4$. Therefore, we obtain that there exists an $i_0 \leq n^4$ such that $H^{(i_0)} = H^{(i_0+1)}$. Next we prove by induction that Eq. (6) holds for $t \ge 0$. First, we have known that it holds for t = 0, 1. Suppose that it holds for $t = i \ge 1$, i.e., $H^{(i_0)} = H^{(i_0+i)}$. This implies that $H^{(i_0)} = H^{(i_0+1)} = \cdots = H^{(i_0+i)}$. Our purpose is to show that it holds for t = i + 1, i.e., $H^{(i_0)} = H^{(i_0+i+1)}$. Indeed, we have

$$(U_{1}U_{2}\cdots U_{k}^{i_{0}+i+1}) \otimes (U_{1}^{*}U_{2}^{*}\cdots (U_{k}^{i_{0}+i+1})^{*})$$

$$= \left[(U_{1}U_{2}\cdots U_{k}^{i_{0}+i}) \otimes (U_{1}^{*}U_{2}^{*}\cdots (U_{k}^{i_{0}+i})^{*}) \right] (U_{k} \otimes U_{k}^{*})$$

$$= \sum_{j=1}^{i_{0}} c_{j} [(U_{1}U_{2}\cdots U_{k}^{j}) \otimes (U_{1}^{*}U_{2}^{*}\cdots (U_{k}^{j})^{*})] (U_{k} \otimes U_{k}^{*})$$
(7)

$$= \sum_{j=1}^{i_0} c_j (U_1 U_2 \cdots U_k^{j+1}) \otimes (U_1^* U_2^* \cdots (U_k^{j+1})^*)$$
(8)

where (7) is due to the assumption $H^{(i_0)} = H^{(i_0+i)}$. Therefore,

$$(U_1U_2\cdots U_k^{i_0+i+1})\otimes (U_1^*U_2^*\cdots (U_k^{i_0+i+1})^*)\in H^{(i_0+1)}=H^{(i_0+i)}.$$

Consequently, $H^{(i_0+i+1)} = H^{(i_0+i)}$. Again, by the assumption of induction $H^{(i_0)} = H^{(i_0+i)}$, we obtain that $H^{(i_0+i+1)} = H^{(i_0)}$. Therefore, (6) holds for any $t \ge 0$.

 \Box

Now we are ready to present the main theorem regarding the equivalence of multi-letter QFAs.

Theorem 9. For $\Sigma = \{\sigma\}$, a k_1 -letter QFA $\mathcal{A}_1 = (Q_1, Q_{acc,1}, |\psi_0^{(1)}\rangle, \Sigma, \mu_1)$ and another k_2 letter QFA $\mathcal{A}_2 = (Q_2, Q_{acc,2}, |\psi_0^{(2)}\rangle, \Sigma, \mu_2)$ are equivalent if and only if they are $(n_1 + n_2)^4 + k - 1$ -equivalent, where n_i is the number of states of Q_i , $i = 1, 2, k = \max(k_1, k_2)$, with $k_1, k_2 \geq 1$.

Proof. Let $P_{acc,1}$ and $P_{acc,2}$ denote the projections on the subspaces spanned by $Q_{acc,1}$ and $Q_{acc,2}$, respectively. For any string $x \in \Sigma^*$, we set $\overline{\mu}(x) = \overline{\mu}_1(x) \oplus \overline{\mu}_2(x)$ and $P_{acc} = P_{acc,1} \oplus P_{acc,2}$, $Q_{acc} = Q_{acc,1} \oplus Q_{acc,2}$, where \oplus denotes the direct sum operation of any two matrices. More precisely, for any $m_1 \times n_1$ matrix A and $m_2 \times n_2$ matrix B, $A \oplus B$ is defined as $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, an $(m_1 + m_2) \times (n_1 + n_2)$ matrix.

In addition, we denote $|\eta_1\rangle = |\psi_0^{(1)}\rangle \oplus \mathbf{0}_2$ and $|\eta_2\rangle = \mathbf{0}_2 \oplus |\psi_0^{(2)}\rangle$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ represent column zero vectors of n_1 and n_2 dimensions, respectively. Then, for any string $x \in \Sigma^*$,

$$P_{\eta_1}(x) = \|\langle \eta_1 | \overline{\mu}(x) P_{acc} \|^2 \tag{9}$$

and

$$P_{\eta_2}(x) = \|\langle \eta_2 | \overline{\mu}(x) P_{acc} \|^2.$$
(10)

Indeed, we further have that

$$P_{\eta_{1}}(x) = \|\langle \eta_{1} | \overline{\mu}(x) P_{acc} \|^{2}$$

$$= \langle \eta_{1} | \overline{\mu}(x) P_{acc} P_{acc}^{\dagger} \overline{\mu}(x)^{\dagger} | \eta_{1} \rangle$$

$$= \langle \eta_{1} | \overline{\mu}(x) P_{acc} \overline{\mu}(x)^{\dagger} | \eta_{1} \rangle$$

$$= \langle \psi_{0}^{(1)} | \overline{\mu}_{1}(x) P_{acc,1} \overline{\mu}_{1}(x)^{\dagger} | \psi_{0}^{(1)} \rangle$$

$$= P_{\mathcal{A}_{1}}(x) \qquad (11)$$

and

$$P_{\eta_{2}}(x) = \|\langle \eta_{2} | \overline{\mu}(x) P_{acc} \|^{2}$$

$$= \langle \eta_{2} | \overline{\mu}(x) P_{acc} P_{acc}^{\dagger} \overline{\mu}(x)^{\dagger} | \eta_{2} \rangle$$

$$= \langle \eta_{2} | \overline{\mu}(x) P_{acc} \overline{\mu}(x)^{\dagger} | \eta_{2} \rangle$$

$$= \langle \psi_{0}^{(2)} | \overline{\mu}_{2}(x) P_{acc,2} \overline{\mu}_{2}(x)^{\dagger} | \psi_{0}^{(2)} \rangle$$

$$= P_{\mathcal{A}_{2}}(x). \qquad (12)$$

Therefore, $P_{\mathcal{A}_1}(x) = P_{\mathcal{A}_2}(x)$ holds if and only if

$$P_{\eta_1}(x) = P_{\eta_2}(x)$$
 (13)

for any string $x \in \Sigma^*$.

On the other hand, we have that

$$P_{\eta_{1}}(x) = \|\langle \eta_{1} | \overline{\mu}(x) P_{acc} \|^{2}$$

$$= \sum_{p_{j} \in Q_{acc}} |\langle \eta_{1} | \overline{\mu}(x) | p_{j} \rangle|^{2}$$

$$= \sum_{p_{j} \in Q_{acc}} \langle \eta_{1} | \overline{\mu}(x) | p_{j} \rangle (\langle \eta_{1} | \overline{\mu}(x) | p_{j} \rangle)^{*}$$

$$= \sum_{p_{j} \in Q_{acc}} \langle \eta_{1} | (\langle \eta_{1} |)^{*} \overline{\mu}(x) \otimes (\overline{\mu}(x))^{*} | p_{j} \rangle (| p_{j} \rangle)^{*}$$

$$= \langle \eta_{1} | (\langle \eta_{1} |)^{*} \overline{\mu}(x) \otimes (\overline{\mu}(x))^{*} \sum_{p_{j} \in Q_{acc}} | p_{j} \rangle (| p_{j} \rangle)^{*}$$
(14)

and

$$P_{\eta_{2}}(x) = \|\langle \eta_{2} | \overline{\mu}(x) P_{acc} \|^{2}$$

$$= \sum_{p_{j} \in Q_{acc}} |\langle \eta_{2} | \overline{\mu}(x) | p_{j} \rangle|^{2}$$

$$= \sum_{p_{j} \in Q_{acc}} \langle \eta_{2} | \overline{\mu}(x) | p_{j} \rangle (\langle \eta_{2} | \overline{\mu}(x) | p_{j} \rangle)^{*}$$

$$= \sum_{p_{j} \in Q_{acc}} \langle \eta_{2} | (\langle \eta_{2} |)^{*} \overline{\mu}(x) \otimes (\overline{\mu}(x))^{*} | p_{j} \rangle (| p_{j} \rangle)^{*}$$

$$= \langle \eta_{2} | (\langle \eta_{2} |)^{*} \overline{\mu}(x) \otimes (\overline{\mu}(x))^{*} \sum_{p_{j} \in Q_{acc}} | p_{j} \rangle (| p_{j} \rangle)^{*}.$$
(15)

Therefore, Eq. (13) holds if and only if

$$\langle \eta_1 | (\langle \eta_1 |)^* \overline{\mu}(x) \otimes (\overline{\mu}(x))^* \sum_{p_j \in Q_{acc}} |p_j\rangle (|p_j\rangle)^*$$

= $\langle \eta_2 | (\langle \eta_2 |)^* \overline{\mu}(x) \otimes (\overline{\mu}(x))^* \sum_{p_j \in Q_{acc}} |p_j\rangle (|p_j\rangle)^*$ (16)

for any string $x \in \Sigma^*$.

Denote

$$D(x) = \overline{\mu}(x) \otimes \overline{\mu}(x)^* \tag{17}$$

where D(x) is an $(n_1 + n_2)^2 \times (n_1 + n_2)^2$ complex square matrix. Then the equivalence between \mathcal{A}_1 and \mathcal{A}_2 depends on whether or not the following equation holds for all string $x \in \Sigma^*$:

$$\langle \eta_1 | (\langle \eta_1 |)^* D(x) \sum_{p_j \in Q_{acc}} | p_j \rangle (| p_j \rangle)^*$$

$$= \langle \eta_2 | (\langle \eta_2 |)^* D(x) \sum_{p_j \in Q_{acc}} | p_j \rangle (| p_j \rangle)^*$$
(18)

Consider the linear space \mathbb{M}_{n^2} consisting of all $(n_1+n_2)^2 \times (n_1+n_2)^2$ complex square matrices. It is clear that the dimension of \mathbb{M}_{n^2} equals $(n_1+n_2)^4 = n^4$.

By $\mathcal{D}^{(i)}$ we denote the subspace of \mathbb{M}_{n^2} spanned by $\{D(x) : x \in \Sigma^*, |x| \leq i\}$, where |x| denotes the length of x. Clearly, we have

$$\mathcal{D}^{(0)} \subseteq \mathcal{D}^{(1)} \subseteq \dots \subseteq \mathcal{D}^{(i)} \subseteq \mathcal{D}^{(i+1)} \subseteq \dots$$
(19)

Since the dimension of $\mathcal{D}^{(i)}$ is not more than $(n_1 + n_2)^4$ for any $i \ge 1$, there exists i_0 such that for any $N \ge i_0$, $\mathcal{D}^{(i_0)} = \mathcal{D}^{(N)}$. In the rest of the proof, our purpose is to fix i_0 .

For the sake of convenience, we denote $A_i = \mu_1(\Lambda^{k_1-i}\sigma^i)$ for $i = 1, 2, ..., k_1$, and $B_j = \mu_2(\Lambda^{k_2-j}\sigma^j)$ for $j = 1, 2, ..., k_2$. Set $k = \max(k_1, k_2)$. If $k_1 \le k_2$, then we denote $A_i = A_{k_1}$ for $i = k_1 + 1, k_1 + 2, ..., k$; if $k_2 \le k_1$, then we denote $B_j = B_{k_2}$ for $j = k_2 + 1, k_2 + 2, ..., k$.

In addition, we denote $C_i = A_i \oplus B_i$ for i = 1, 2, ..., k. Then C_i is an $n = n_1 + n_2$ order unitary matrix for i = 1, 2, ..., k. According to the definition of $\overline{\mu}(x) = \overline{\mu}_1(x) \oplus \overline{\mu}_2(x)$, we know that $C_1 C_2 \cdots C_i = \overline{\mu}(x)$ for $x \in \Sigma^*$ and $|x| = i \leq k$. On the other hand, if $i \geq k$, then $C_1 C_2 \cdots C_k^{i-k+1} = \overline{\mu}(x)$.

Thus, $D(x) = (C_1 C_2 \cdots C_i) \otimes (C_1^* C_2^* \cdots C_i^*)$ for $x \in \Sigma^*$ and $|x| = i \le k$; and if $i \ge k$, then $D(x) = (C_1 C_2 \cdots C_k^{i-k+1}) \otimes (C_1^* C_2^* \cdots (C_k^{i-k+1})^*).$

We set $E^{(i)} = span\{D(x) : x \in \Sigma^*, k \leq |x| \leq k+i\}, i = 0, 1, 2, \dots$ Then, by means of Lemma 8 it follows that, there exists $i_0 \leq n^4 - 1$, such that

$$E^{(i_0)} = E^{(i_0+s)} \tag{20}$$

for any $s \ge 0$.

Eq. (20) implies that, for any $x \in \Sigma^*$ with $|x| \ge k + i_0$, D(x) can be linearly represented by some matrices in $\{D(y) : k \le |y| \le k + i_0\}$. Therefore, if Eq. (18) holds for $|x| \le n^4 + k - 1$, then so does it for any $x \in \Sigma^*$. We have proved this theorem. **Remark 3.** We analyze the complexity of computation in Theorem 11. As in [35], we assume that all the inputs consist of complex numbers whose real and imaginary parts are rational numbers and that each arithmetic operation on rational numbers can be done in constant time. Still we denote $n = n_1 + n_2$. Note that in time $O(in^4)$ we check whether or not Eq. (18) holds for $x \in \Sigma^*$ with |x| = i. Because the length of x to be checked in Eq. (18) is at most $n^4 + k$, the time complexity for checking whether the two multi-letter QFAs are equivalent is $O(n^3(1 + 2 + ... + (n^4 + k)))$, that is at most $O(n^{12} + k^2n^4 + kn^8)$.

5. Concluding remarks

In this paper, we have considered several issues concerning multi-letter QFAs. Our technical contributions mainly contain the following two aspects: (1) we have shown that (k + 1)letter QFAs are strictly more powerful than k-letter QFAs, that is, (k + 1)-letter QFAs can accept some regular languages unacceptable by any k-letter QFA, and some examples of regular languages unacceptable by multi-letter QFAs have been provided. We have known that multi-letter QFAs are strictly more powerful than MO-1QFAs [27], but they are not comparable to MM-1QFAs [24, 1] since the language a^*b^* can be accepted with bounded error by MM-1QFAs but cannot be accepted by multi-letter QFAs, and the language $(a + b)^*b$ shows the opposite direction. Moreover, $a^*b(a^2)^*a$ cannot be accepted by MM-1QFAs and by multi-letter QFAs with bounded error. (2) We have proved that a k_1 -letter QFA \mathcal{A}_1 and another k_2 -letter QFA \mathcal{A}_2 for accepting unary languages are equivalent if and only if they are $(n_1 + n_2)^4 + k - 1$ -equivalent, and the time complexity of this computing method is $O(n^{12} + k^2n^4 + kn^8)$, where $n = n_1 + n_2$, n_1 and n_2 are the numbers of states of \mathcal{A}_1 and \mathcal{A}_2 , respectively, and $k = \max(k_1, k_2)$.

The method presented in the paper may be generalized to deal with more general cases. Another issue worthy of consideration is concerning the state complexity of multi-letter QFAs compared with the usual 1QFAs for accepting some languages (for example, unary regular languages [37, 32]). Also, the power of measure-many multi-letter QFAs, as the relation between MM-1QFAs and MO-1QFAs, is worth being clarified. Whether or not measure-many multi-letter QFAs can recognize non-regular languages may also be considered in the future.

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