Tight results for Next Fit and Worst Fit with resource augmentation

Joan Boyar [*] Leah Epstein[†] Asaf Levin[‡]

Abstract

It is well known that the two simple algorithms for the classic bin packing problem, NF and WF both have an approximation ratio of 2. However, WF seems to be a more reasonable algorithm, since it never opens a new bin if an existing bin can still be used.

Using resource augmented analysis, where the output of an approximation algorithm, which can use bins of size $b > 1$, is compared to an optimal packing into bins of size 1, we give a *complete analysis* of the asymptotic approximation ratio of WF and of NF, and use it to show that WF is strictly better than NF for any $1 < b < 2$, while they have the same asymptotic performance guarantee for all $b \geq 2$, and for $b = 1$.

1 Introduction

Bin packing has been extensively studied in both the offline and the online environments and has numerous applications [8, 11, 7, 4, 3]. In the basic problem, the goal is to pack a sequence of items of sizes $s_1, s_2, \ldots$, where $s_i \in (0, 1]$, into a minimum number of unit-capacity blocks, called *bins*, such that the total size of the items in each bin does not exceed 1. An item is identified with its index, and for a set of items $X \subseteq \{1, 2, \ldots\}$, we denote $s(X) = \sum_{j \in X} s_j$. Thus X can be packed in a bin if $s(X) \leq 1$. If the problem is online, then the items must be packed irrevocably one by one, while future items are unknown at the time of packing. The goal is to minimize the number of bins containing at least one item, also called *used* bins. The operation of assigning a first item to a new bin is called *opening* a new bin.

For an algorithm $\mathcal{A}$, we denote its cost, i.e., the number of used bins in its packing, on an input I, by $\mathcal{A}(I)$ (or simply $\mathcal{A}$). The cost of an optimal solution $\textsc{Opt}$, for the same input, is denoted by $\textsc{Opt}(I)$ (or $\textsc{Opt}$). The *asymptotic approximation ratio* allows to compare the costs for inputs for which the optimal cost is sufficiently large. The asymptotic approximation ratio of $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}}$ is defined as follows.

$$\mathcal{R}_{\mathcal{A}} = \lim_{N \to \infty} \left(\sup_{I: \textsc{Opt}(I) \geq N} \frac{\textsc{Alg}(I)}{\textsc{Opt}(I)} \right) .$$

[*] Department of Mathematics and Computer Science, University of Southern Denmark, 5230 Odense M, Denmark, joan@imada.sdu.dk

[†] Department of Mathematics, University of Haifa, 31905 Haifa, Israel. lea@math.haifa.ac.il.

[‡] Chaya fellow. Faculty of Industrial Engineering and Management, The Technion, 32000 Haifa, Israel. levinas@ie.technion.ac.il.

In this paper we only consider the asymptotic approximation ratio, which is the common measure for bin packing algorithms. Thus we use the term approximation ratio throughout the paper, with the meaning of asymptotic approximation ratio.

In the early days of the study of bin packing, several natural algorithms were introduced. Two such algorithms are NEXT FIT (NF) and WORST FIT (WF) [7]. The two algorithm were presented as offline heuristics, but are in fact online algorithms which process the items as a list. NF keeps a single active bin at each time. If the next unpacked item cannot be packed into the current active bin, then it is closed and never used again, while a new active bin is opened in order to accommodate the item. WF packs the next item in a previously opened bin with the minimum total packed size of items if such a bin can accommodate this item as well. Only if no such bin exists is a new bin opened in order to accommodate the item. Thus WF is intuitively the better algorithm, though NF is more efficient; it is a bounded space algorithm. In fact, Worst-Fit is provably better than Next-Fit. The following result (see also [2]) actually applies to Next-Fit compared to any Any-Fit algorithm.

Proposition 1. *On any sequence of items, I,* NF *will use at least as many bins as* WF.

Proof. Let $B_{\mathrm{NF}}(i)$ denote the bin number where NF places the item that WF places as the first in bin i. We show by induction on i that $B_{\mathrm{NF}}(i) \geq i$ for all i. Both values are 1 for $i = 1$. Suppose it holds for some value t. Then WF opens a new bin, t, with item j and NF places j in some bin $t' \geq t$. Consider the item, k, where WF opens bin $t + 1$. If NF has not already opened bin $t' + 1$, it has packed all items between items j and k in bin t'. WF cannot have put more items in bin t, so bin t' in NF's packing is at least as full as bin t in WF's packing. Thus, NF must open bin $t' + 1$ if it has not already done so. Since $B_{\mathrm{NF}}(i) \geq i$ for all i, NF uses at least as many bins as WF. □

However, both WF and NF have approximation ratios of 2 [7], so the standard measure does not distinguish between these two algorithms.

We use resource augmentation [9, 5] in order to analyze the two algorithms and compare their behavior. In resource augmentation, an approximation (or online) algorithm is equipped with resources which are larger than those of an optimal algorithm which is it compared to. For bin packing, resource augmentation with a ratio $b > 1$ means that the approximation algorithm may use bins which are b times larger than those of the optimal algorithm [5]. Specifically, we assume that the algorithm uses bins of size 1, while, an optimal algorithm uses a bin of size $\frac{1}{b}$. Clearly, all item sizes are in $(0, \frac{1}{b}]$.

Our results. We show that the approximation ratio of WF, $\mathcal{R}_{\mathrm{WF}}(b)$, is:

$$\mathcal{R}_{\mathrm{WF}}(b) = \begin{cases} \frac{2b}{3b-2}\ , & \text{for} \quad b \in [1, 2] \\ \frac{1}{b-1}\ , & \text{for} \quad b \in [2, \infty) \end{cases}$$

We show that the approximation ratio of NF, $\mathcal{R}_{\mathrm{NF}}(b)$, is:

$$\frac{2t_b^2 b - 2t_b^2 - 4t_b + 2 + 2bt_b}{t_b^2 b + 2bt_b - t_b^2 - 3t_b - 2 + 2b}\ ,$$

where $t_b = \lfloor \frac{1}{b-1} \rfloor$. Thus, for $b > 2$, $t_b = 0$ and the ratio becomes $\frac{1}{b-1}$. If $t_b = \frac{1}{b-1}$, i.e., $b = 1 + \frac{1}{k}$ for some integer $k \geq 1$, the ratio becomes $\frac{2}{b}$. Moreover, in the case $t_b = 1$, i.e., $\frac{3}{2} < b \leq 2$, we get the ratio $\frac{4b-4}{5b-6}$, for

$t_b = 2$, i.e., $\frac{4}{3} < b \leq \frac{3}{2}$, we get the ratio $\frac{6b-7}{5b-6}$, for $t_b = 3$, i.e., $\frac{5}{4} < b \leq \frac{4}{3}$, we get the ratio $\frac{24b-28}{17b-20}$ etc. In the case $b = 1$ we have $t_b = \infty$, and the approximation ratio is 2.

For the analysis we use weight functions, which are related to the weight function originally introduced for the analysis of FIRST FIT (FF) [8, 11]. We use clever generalizations of this type of weight function to achieve tight bounds for all values of b.

Previous work. Resource augmentation for bin packing was studied by Csirik and Woeginger [5]. They have studied bounded space bin packing, where a constant number of bins can be available at any time to receive new items (active bins). If the maximum number of open bins is reached, and a new bin needs to be used, one of the active bins must be closed and never used again. They defined a function $\rho(b)$, and extended the Harmonic algorithms of [10] for the case of $b > 1$. The worst case ratio of this algorithm comes arbitrary close to a certain bound $\rho(b)$. They also proved that no online bounded space algorithm can have an approximation ratio smaller than $\rho(b)$ in the worst case. Unbounded space resource augmented bin-packing was studied in [6], where improved algorithms are designed, and lower bounds for general online bin packing algorithms are proved.

2 Some easy cases for the analysis of NF and WF

In this section, we show some simple bounds on the approximation ratio of the two algorithms. These bounds are in fact tight in a part of the intervals. The more difficult cases are discussed later.

Lemma 2. *The approximation ratios of* NF *and* WF *are at most* $\frac{1}{b-1}$*, and at most* $\frac{2}{b}$ *for* $b \leq 2$*.*

Proof. Since no item has a size of more than $\frac{1}{b}$, every bin except for possibly the last opened bin has a total size of items of at least $1 - \frac{1}{b} = \frac{b-1}{b}$. Since a bin of an optimal packing can contain a total of at most $\frac{1}{b}$, an approximation ratio of at most $\frac{1}{b-1}$ follows.

On the other hand, since for both WF and NF, the sum of items in two consecutive bins is more than 1, if q bins are opened, the total size of items is more than $\frac{q-1}{2}$, so these items require more than $\frac{b(q-1)}{2}$ bins in a packing into bins of size $\frac{1}{b}$, that is, at least $\frac{bq}{2}$ for $b \leq 2$, and an approximation ratio of at most $\frac{2}{b}$ follows. □

We next show that in the case $b \geq 2$, both WF and NF have an approximation ratio of exactly $\frac{1}{b-1}$, i.e., the approximation ratio for $b = 2$ is 1, it is monotonically decreasing, and tends to zero as b grows.

Theorem 3. *Let* $b \geq 2$*. The approximation ratio of both* NF *and* WF *is* $\frac{1}{b-1}$*.*

Proof. By Lemma 2, we only need to prove a lower bound. The following lower bound construction is valid for both algorithms for $b \geq 2$. Let N be a large integer. Let $p = \lfloor (b-2)N \rfloor + 1$. Then $(b-2)N < p \leq (b-2)N + 1$ or $2 + \frac{p}{N} > b$ and $\frac{p}{N} \leq b - 2 + \frac{1}{N}$.

The sequence consists of N batches of $p + 1$ items, each of which contains an item of size $\frac{1}{b}$, followed by p items of size $\frac{1}{Nb}$. The total size of the items of one batch is $\frac{1}{b} + \frac{p}{Nb}$. A new item of size $\frac{1}{b}$ cannot be added to a bin which contains all items of one batch since the total size would be at least $\frac{2}{b} + \frac{p}{Nb} > 1$.

Both WF and NF need to open a new bin for every large item, and then all small items are packed together with the larger item.

In an optimal packing into a bins of size $\frac{1}{b}$, N bins are completely filled with items of size $\frac{1}{b}$. Each bin can receive N smaller items, thus p additional bins are used. The approximation ratio is at least $\frac{N}{N+p} \geq \frac{1}{1+b-2+\frac{1}{N}} = \frac{1}{b-1+\frac{1}{N}}$, which tends to $\frac{1}{b-1}$ for large N. $\square$

We next consider the approximation ratio of NF for cases where $b = 1 + \frac{1}{t}$, for some integer $t \geq 2$.

Theorem 4. *Let $b = 1 + \frac{1}{t}$ for an integer $t \geq 2$. The approximation ratio of* NF *is exactly $\frac{2}{b}$.*

Proof. Let $N \geq t$ be a large integer and consider an input with Nt batches of four jobs, of the sizes $\frac{t}{t+1}$, $\frac{t}{(t+1)N}$, $\frac{1}{t+1}$, $\frac{t}{(t+1)N}$. We claim that NF uses two new bins for each batch, and these bins have a total packed size of $\frac{tN+t}{(t+1)N}$ and $\frac{N+t}{(t+1)N}$, respectively. For $N \geq t$, both of these total packed sizes are less than 1. Indeed, the third item cannot be packed into a bin of the first type since $\frac{tN+t}{(t+1)N} + \frac{N}{N(t+1)} > 1$, and the first item cannot be packed into a bin of the second type since $\frac{N+t}{(t+1)N} + \frac{tN}{(t+1)N} > 1$ as well. Thus $2Nt$ bins are used.

An optimal packing into bins of size $\frac{1}{b} = \frac{t}{t+1}$ uses Nt bins for the items of the first type, N bins for the items of the third type, and $2t$ bins for the other items.

Thus, the approximation ratio is at least $\frac{2Nt}{Nt+2t+N}$, which tends to $\frac{2t}{t+1} = \frac{2}{b}$ for large N. $\square$

3 A complete analysis of NEXT FIT

In this section, we analyze NF for values of b which satisfy $\frac{t+2}{t+1} < b < \frac{t+1}{t}$, for some integer $t \geq 1$. An alternative definition of t is $t = \lfloor \frac{1}{b-1} \rfloor$. These are the missing cases for NF.

We define the following weight function of the items. In both the analysis of NF and the analysis of WF for the additional cases, we use piecewise linear functions defined on $(0, \frac{1}{b}]$. Thus the weight of an item is only based on its size. For a set $X \subseteq \{1, 2, \ldots\}$, and any function $g : (0, \frac{1}{b}] \to \mathbb{R}$, we let $g(X) = \sum_{i \in X} g(s_i)$.

Let I_i, for $0 \leq i \leq t$ be defined as

$$I_i = \left(i(1 - \frac{1}{b}), \frac{t-i+1}{b} - (t-i) \right] ,$$

and let J_i, for $1 \leq i \leq t$ be defined as

$$J_i = \left(\frac{t-i+2}{b} - (t-i+1), i(1 - \frac{1}{b}) \right] .$$

Note that for any i, $i(1 - \frac{1}{b}) < \frac{t-i+1}{b} - (t-i)$ holds since $b < \frac{t+1}{t}$, and $\frac{t-i+2}{b} - (t-i+1) < i(1 - \frac{1}{b})$ holds since $b > \frac{t+2}{t+1}$. For $i \geq 1$, $J_i \cup I_i = (\frac{t-i+2}{b} - (t-i+1), \frac{t-i+1}{b} - (t-i)]$, and $I_0 = (0, \frac{t+1}{b} - t]$, therefore $\bigcup_{1 \leq i \leq t} J_i \cup \bigcup_{0 \leq i \leq t} I_i = (0, \frac{1}{b}]$.

We define the weight function w as follows.

$$w(x) = \begin{cases} x + i((t+1) - \frac{t+2}{b}) \ , & \text{for} \quad x \in I_i, \quad 0 \leq i \leq t \\ 2x - i(\frac{t+1}{b} - t) \ , & \text{for} \quad x \in J_i, \quad 1 \leq i \leq t \end{cases}$$

In the proofs of Claim 5 and Lemma 7, the breakpoints between the I's and J's are considered. We let p_{2i} denote the point $i(1 - \frac{1}{b})$ for $0 \leq i \leq t$ and p_{2i+1} is $\frac{t-i+1}{b} - (t-i)$, for $0 \leq i \leq t$. These breakpoints are p_j for $1 \leq j \leq 2t$, while p_0 and p_{2t+1} are the boundaries of the domain of the function w. Note that by definition, $1 - p_{2i} = p_{2(t-i+1)+1}$, for $1 \leq i \leq t$, and $1 - p_{2i+1} = p_{2(t-i+1)}$, for $1 \leq i \leq t$. We analogously define $p_{2t+2} = (t+1)(1 - \frac{1}{b})$ and $p_{2t+3} = \frac{t-(t+1)+1}{b} - (t - (t+1)) = 1$. We have $p_{2t+3} + p_0 = 1$ and $p_{2t+2} + p_1 = 1$, and thus $p_j + p_{2t+3-j} = 1$ for any $0 \leq j \leq 2t+3$. We also let $w(0) = 0$.

Claim 5. *The function w is continuous and monotonically increasing in $(0, \frac{1}{b}]$.*

Proof. Since w is piecewise linear, with positive slopes, it is sufficient to prove that it is continuous at breakpoints.

The value of the function for $p_{2i} = i(1 - \frac{1}{b})$ for $1 \leq i \leq t$ is

$$2i(1 - \frac{1}{b}) - i(\frac{t+1}{b} - t) = i(t + 2 - \frac{t+3}{b}),$$

while the value for $p_{2i} + \varepsilon$, for a sufficiently small value of ε, is

$$i(1 - \frac{1}{b}) + \varepsilon + i((t+1) - \frac{t+2}{b}) = i(t + 2 - \frac{t+3}{b}) + \varepsilon.$$

In the second case, the value of the function for p_{2i+1} for $0 \leq i \leq t-1$ is

$$\frac{t-i+1}{b} - (t-i) + i((t+1) - \frac{t+2}{b}) = i(t + 2 - \frac{t+3}{b}) + \frac{t+1}{b} - t$$

while the value for $p_{2i+1} + \varepsilon$, for a sufficiently small value of ε, is

$$2(\frac{t-i+1}{b} - (t-i) + \varepsilon) - (i+1)(\frac{t+1}{b} - t) = i(t + 2 - \frac{t+3}{b}) + \frac{t+1}{b} - t + 2\varepsilon.$$

Thus the function is continuous and therefore, monotonically increasing. $\square$

Lemma 6. *Let X be a set of items such that $s(X) \leq \frac{1}{b}$, then $w(X) \leq \frac{t^2b+tb-t^2-2t+1}{b}$.*

Proof. Consider a set X. We first show that we can assume without loss of generality that all items come from the intervals I_0, I_1 and J_1.

Consider an item j of I_i for $i > 1$. Replace this item with i items of size $\frac{s_j}{i}$. The resulting items have a size in

$$\left(1 - \frac{1}{b}, \frac{t-i+1}{bi} - \frac{t-i}{i}\right] \subseteq I_1,$$

since $\frac{t-i+1}{bi} - \frac{t-i}{i} \leq \frac{t}{b} - (t-1)$ is equivalent to $t-i+1-bt \leq ti-tbi$ or $bt(i-1) \leq ti-t+i-1 = (t+1)(i-1)$. Using $i > 1$, we get that this is equivalent to $b \leq \frac{t+1}{t}$ which clearly holds. The total weight of the new items is therefore $i \cdot (\frac{s_j}{i} + t + 1 - \frac{t+2}{b})$, which is equal to the weight of the original item.

Consider an item j of J_i for $i > 1$. Replace this item with i items of size $\frac{s_j}{i}$. The resulting items have a size in $(\frac{t-i+2}{bi} - \frac{t-i+1}{i}, 1 - \frac{1}{b}] \subseteq J_1$, since $\frac{t-i+2}{bi} - \frac{t-i+1}{i} \geq \frac{t+1}{b} - t$ is equivalent to $t - i + 2 - bt + ib - b \geq ti + i - tbi$ or

$$b(t+1)(i-1) \geq ti - t + 2i - 2 = (t+2)(i-1).$$

Using $i > 1$, we get that this is equivalent to $b \geq \frac{t+2}{t+1}$ which clearly holds. The total weight of the new items is therefore $i \cdot (2\frac{s_j}{i} - (\frac{t+1}{b} - t))$, which is equal to the weight of the original item.

Let K_J and K_I denote the subsets of items in X, of sizes in J_1 and in I_1, respectively, and let $k_J = |K_J|$ and $k_I = |K_I|$.

$$w(X) = \sum_{j \in X} s_j + \sum_{j \in K_J} s_j - k_J(\frac{t+1}{b} - t) + k_I((t+1) - \frac{t+2}{b}).$$

Clearly, $\sum_{j \in K_J} s_j \leq \frac{1}{b} - \sum_{j \in K_I} s_j \leq \frac{1}{b} - k_I(1 - \frac{1}{b})$ and $\sum_{j \in K_J} s_j \leq k_J(1 - \frac{1}{b})$.

We consider two cases. If $\frac{1}{b} - k_I(1 - \frac{1}{b}) \leq k_J(1 - \frac{1}{b})$, we have $k_I + k_J \geq \frac{1}{b-1}$, and since k_I and k_J are integers and $\frac{1}{b-1}$ is not, we get $k_I + k_J \geq \lceil \frac{1}{b-1} \rceil = t + 1$. Thus using $\sum_{j \in X} s_j \leq \frac{1}{b}$ we get,

$$\begin{aligned} w(X) &\leq \frac{1}{b} + \frac{1}{b} - k_I(1 - \frac{1}{b}) - k_J(\frac{t+1}{b} - t) + k_I((t+1) - \frac{t+2}{b}) \\ &= \frac{2}{b} - (k_I + k_J)(\frac{t+1}{b} - t) \leq \frac{2}{b} - (t+1)(\frac{t+1}{b} - t) = \frac{t^2 b + tb - t^2 - 2t + 1}{b}. \end{aligned}$$

If $\frac{1}{b} - k_I(1 - \frac{1}{b}) \geq k_J(1 - \frac{1}{b})$, we have $k_I + k_J \leq \frac{1}{b-1}$, and since k_I and k_J are integers, we get $k_I + k_J \leq \lfloor \frac{1}{b-1} \rfloor = t$. Thus

$$\begin{aligned} w(X) &\leq \frac{1}{b} + k_J(1 - \frac{1}{b}) - k_J(\frac{t+1}{b} - t) + k_I((t+1) - \frac{t+2}{b}) = \frac{1}{b} + (k_I + k_J)((t+1) - \frac{t+2}{b}) \\ &\leq \frac{1}{b} + t((t+1) - \frac{t+2}{b}) = \frac{t^2 b + tb - t^2 - 2t + 1}{b}. \end{aligned}$$

□

We next analyze the weight in bins of NF. For that, we define a modified weight function w' by $w'(x) = w(x) - \tilde{w}(x)$, where $\tilde{w}(x) = x$. Denote the bins used by NF by $B_1, B_2, \ldots, B_{k'}$, where $k' = \mathrm{NF}$, that is, B_i is the set of items packed into the i-th bin. For a bin B_i let $\gamma_i = s(B_i)$ denote the total size of items in B_i, and let τ_i denote the size of the first item ever packed into B_i. For a bin B_i ($i < k'$) we define a new weight

$$f(B_i) = \tilde{w}(B_i) + w'(\tau_{i+1}) = s(B_i) + w'(\tau_{i+1}).$$

If k' is odd then let $k = k' - 1$ and otherwise $k = k' - 2$. Thus k is even and $\mathrm{NF} \leq k + 2$. Let n denote the number of items in the input. Clearly,

$$\sum_{i=1}^{k} f(B_i) < \sum_{j=1}^{n} \tilde{w}(s_j) + \sum_{j=1}^{n} w'(s_j) = \sum_{j=1}^{n} w(s_j).$$

Lemma 7. *Let $i < k$. Then $f(B_i) + f(B_{i+1}) \geq t^2 + 2t + 2 - \frac{(t+1)(t+2)}{b}$.*

Proof. Recall the breakpoints p_i of the weight function w. Let Y, Z be such that $s(B_i) \in Y = (p_y, p_{y+1}]$ and $s(B_{i+1}) \in Z = (p_z, p_{z+1}]$, where $y, z \leq 2t + 2$. Note that $i + 1 < k'$, thus the bin B_{i+1} is not the last bin, and an item was packed into bin B_{i+2}, so τ_{i+2} is well-defined. We have $\tau_{i+1} > 1 - s(B_i)$ and $\tau_{i+2} > 1 - s(B_{i+1})$.

By definition, $1 - s(B_i) \in [p_{2t+2-y}, p_{2t+3-y})$ and $1 - s(B_{i+1}) \in [p_{2t+2-z}, p_{2t+3-z})$. We next show $y + z \geq 2t + 2$. Using $s(B_i) + s(B_{i+1}) > 1$, we get $p_{y+1} + p_{z+1} > 1$. Since $p_{y+1} = 1 - p_{2t+2-y}$, we get $p_{z+1} > p_{2t+2-y}$. Therefore, $z+1 > 2t+2-y$ or $z+y > 2t+1$. Since z, y are integers, then $z+y \geq 2t+2$. If one of z and y is odd and the other one is even, then $z+y \geq 2t+3$.

We next calculate

$$f(B_i) + f(B_{i+1}) = s(B_i) + s(B_{i+1}) + w'(\tau_{i+1}) + w'(\tau_{i+2}).$$

Consider a bin B_ℓ, where $\ell \in \{i, i+1\}$, and $s(B_\ell) \in (p_v, p_{v+1}]$ (hence $v \in \{y, z\}$). Note that since w' is a continuous piecewise linear function whose slopes are non-negative, we conclude that w' is monotonically non-decreasing function, and hence $f(B_\ell) = s(B_\ell) + w'(\tau_{\ell+1}) \geq s(B_\ell) + w'(1 - s(B_\ell))$. We next obtain a lower bound on $f(B_\ell)$, this bound depends on the parity of v.

- If v is even, then $w'(1 - s(B_\ell)) = \frac{2t+2-v}{2}(t+1-\frac{t+2}{b})$, and

  $$s(B_\ell)+w'(1-s(B_\ell)) \geq \frac{v}{2}(1-\frac{1}{b})+\frac{2t+2-v}{2}(t+1-\frac{t+2}{b}) = (t+1)(t+1-\frac{t+2}{b})+\frac{v}{2}(\frac{t+1}{b}-t).$$

- If v is odd, then

  $$s(B_\ell) + w'(1 - s(B_\ell)) \geq s(B_\ell) + 1 - s(B_\ell) - \frac{2t+3-v}{2}(\frac{t+1}{b} - t) = 1 - \frac{2t+3-v}{2}(\frac{t+1}{b} - t).$$

We consider three cases depending on the parity of y and z, and in each of these cases, we show that $f(B_i) + f(B_{i+1}) \geq t^2 + 2t + 2 - \frac{(t+2)(t+1)}{b}$.

- **Both y and z are odd.** In this case, using $\frac{t+1}{b} - t > 0$ and $y + z \geq 2t + 2$,

  $$\begin{aligned} f(B_i) + f(B_{i+1}) &\geq 1 - \frac{2t+3-y}{2}(\frac{t+1}{b} - t) + 1 - \frac{2t+3-z}{2}(\frac{t+1}{b} - t) \\ &= 2 + \frac{y+z-4t-6}{2}(\frac{t+1}{b} - t) \\ &\geq 2 - (t+2)(\frac{t+1}{b} - t) = t^2 + 2t + 2 - \frac{(t+2)(t+1)}{b} . \end{aligned}$$

- **The sum of y and z is odd.** Consider the case where y is odd and z is even, the other case is symmetric. In this case we have $y + z \geq 2t + 3$. Since $\frac{t+1}{b} - t > 0$, we conclude the following:

  $$\begin{aligned} & f(B_i) + f(B_{i+1}) \geq 1 - \frac{2t+3-y}{2}(\frac{t+1}{b} - t) + (t+1)(t+1-\frac{t+2}{b}) + \frac{z}{2}(\frac{t+1}{b} - t) \\ = \;& t^2 + 2t + 2 - \frac{(t+2)(t+1)}{b} + (\frac{y+z-2t-3}{2})(\frac{t+1}{b} - t) \\ \geq \;& t^2 + 2t + 2 - \frac{(t+2)(t+1)}{b} . \end{aligned}$$

- **Both y and z are even.** If $y + z \geq 2t + 4$, then

  $$\begin{aligned} & f(B_i) + f(B_{i+1}) \\ \geq \;& (t+1)(t+1-\frac{t+2}{b}) + \frac{y}{2}(\frac{t+1}{b} - t) + (t+1)(t+1-\frac{t+2}{b}) + \frac{z}{2}(\frac{t+1}{b} - t) \\ \geq \;& 2t^2 + 4t + 2 - \frac{2(t+1)(t+2)}{b} + (t+2)(\frac{t+1}{b} - t) = t^2 + 2t + 2 - \frac{(t+1)(t+2)}{b} . \end{aligned}$$

Otherwise, since the sum of z and y is even, and $2t+2 \leq y+z \leq 2t+3$, then $y+z = 2t+2$.

$$\begin{aligned}
& f(B_i) + f(B_{i+1}) \\
\geq \;& s(B_i) + s(B_{i+1}) + \frac{2t+2-y}{2}(t+1-\frac{t+2}{b}) + \frac{2t+2-z}{2}(t+1-\frac{t+2}{b}) \\
\geq \;& 1 + \frac{4t+4-y-z}{2}(t+1-\frac{t+2}{b}) = 1+(t+1)(t+1-\frac{t+2}{b}) \\
= \;& t^2+2t+2-\frac{(t+1)(t+2)}{b}\,.
\end{aligned}$$

$\square$

Theorem 8. *The approximation ratio of* NF *for* $\frac{t+2}{t+1} < b < \frac{t+1}{t}$ *is exactly*

$$\frac{2t^2b-2t^2-4t+2+2bt}{t^2b+2bt-t^2-3t-2+2b}\,.$$

Proof. Let D denote the set of items. By Lemma 6, $w(D) \leq \frac{t^2b+tb-t^2-2t+1}{b}$OPT. By Lemma 7 and the definition of k,

$$w(D) \geq \frac{k}{2}\cdot\frac{t^2b+2bt-t^2-3t-2+2b}{b} \geq (\text{NF}-2)\frac{t^2b+2bt-t^2-3t-2+2b}{2b}\,.$$

Thus

$$\text{NF} \leq \frac{2t^2b-2t^2-4t+2+2bt}{t^2b+2bt-t^2-3t-2+2b}\text{OPT}+2.$$

For the lower bound, let N be a large integer, divisible by t. Let $\varepsilon = \frac{t+1-\frac{t+2}{b}}{4N}$. The input first contains N batches. Each of these batches consists of four items of the following sizes: $\frac{1}{b}$, ε, $1-\frac{1}{b}$, ε. Every bin will contain an item of size $\frac{1}{b}$ or $1-\frac{1}{b}$, followed by an item of size ε.

Next, the following sequence of additional items is repeated $\lfloor\frac{t+1-tb}{t(tb+2b-t-3)}N\rfloor$ times (note that $t+1-tb > 0$ and $tb+2b-t-3>0$, by the definition of t and using $b>1$).

These are one item of size $\frac{1}{b}$, $4N+1$ items of size ε, one item of size $\frac{t+1}{b}-t$ and an additional $4N+1$ items of size ε.

Note that $\frac{t+1}{b}-t>0$ and that $(4N+1)\varepsilon = 4N\frac{t+1-\frac{t+2}{b}}{4N}+\varepsilon = t+1-\frac{t+2}{b}+\varepsilon$. Since $\frac{1}{b}+\frac{t+1}{b}-t+t+1-\frac{t+2}{b}+\varepsilon = 1+\varepsilon$, each item of size $\frac{t+1}{b}-t$ or $\frac{1}{b}$, including the first such item of this part of the input, starts a new bin.

The number of bins used by NF is at least $2N+2\frac{t+1-tb}{t(tb+2b-t-3)}N-2$.

Note that $\frac{t+1-tb}{t(tb+2b-t-3)} < \frac{1}{t}$, since this is equivalent to $tb+2b-t-3+tb-t-1>0$ or to $b(t+1)>t+2$, which holds by the definition of t.

Note that there are at most $N+\frac{t+1-tb}{t(tb+2b-t-3)}N$ items of size $\frac{1}{b}$, N items of size $1-\frac{1}{b}$, at most $\frac{t+1-tb}{t(tb+2b-t-3)}N$ items of size $\frac{t+1}{b}-t$, and at most $2N+2(4N+1)(\frac{t+1-tb}{t(tb+2b-t-3)}N)$ items of size ε.

We next consider a packing into bins of size $\frac{1}{b}$. There are at most $N+\frac{t+1-tb}{t(tb+2b-t-3)}N$ bins with one item of size $\frac{1}{b}$. Since $(1-\frac{1}{b})t \leq \frac{1}{b}$, t items of size $1-\frac{1}{b}$ are packed into one bin, resulting in $\frac{N}{t}$ bins. Each such bin can either receive an item of size $\frac{1}{b} - t(1 - \frac{1}{b}) = \frac{t+1}{b} - t$, or at least

$$\left\lfloor \frac{\frac{t+1}{b} - t}{\varepsilon} \right\rfloor = \left\lfloor \frac{\frac{t+1}{b} - t}{\frac{t+1-\frac{t+2}{b}}{4N}} \right\rfloor \geq 4N\frac{t+1-bt}{bt+b-t-2} - 1$$

items of size ε.

Therefore, the total number of items of size ε which are combined into existing bins is at least

$$\begin{aligned}
& \left(\frac{N}{t} - \frac{t+1-tb}{t(tb+2b-t-3)}N\right) \cdot (4N\frac{t+1-bt}{bt+b-t-2} - 1) \\
= \quad & \frac{tb+2b-t-3-t-1+tb}{t(tb+2b-t-3)}N \cdot (4N\frac{t+1-bt}{bt+b-t-2} - 1) \\
= \quad & \frac{2(tb+b-t-2)}{t(tb+2b-t-3)}N \cdot (4N\frac{t+1-bt}{bt+b-t-2} - 1) = \frac{8N^2(t+1-bt)}{t(bt+2b-t-3)} - \frac{2(tb+b-t-2)}{t(tb+2b-t-3)}N\ .
\end{aligned}$$

Therefore, the number of remaining items of size ε is at most $2N + 2\frac{t+1-tb}{t(tb+2b-t-3)}N + \frac{2(tb+b-t-2)}{t(tb+2b-t-3)}N$.

We have $t+1-tb < tb+2b-t-3$ and $tb+b-t-2 < tb+2b-t-3$, so the number of remaining items is at most $2N + \frac{4N}{t} < 6N$, for any $t \geq 1$. Since $t+1-\frac{t+2}{b} < t+1-\frac{t+2}{(t+1)/t} = \frac{t^2+2t+1-t^2-2t}{t+1} = \frac{1}{t+1}$. Therefore, since $\frac{1}{b} \geq \frac{1}{t+1}$, at least $4N$ items of size ε can share a bin, so two new bins are sufficient for the remaining items of size ε.

We get a ratio of at least $\frac{2N+2(\frac{t+1-tb}{t(tb+2b-t-3)}N)-2}{N+\frac{t+1-tb}{t(tb+2b-t-3)}N+\frac{N}{t}+2}$, which tends to $\frac{2t^2b-2t^2-4t+2+2bt}{t^2b+2bt-t^2-3t-2+2b}$ for large enough values of N. $\square$

4 A complete analysis of WORST FIT

In order to complete the analysis of WF, we need to consider the case $1 < b < 2$. In this case, we will show a tight bound of $\frac{2b}{3b-2}$ on the approximation ratio. Thus it is monotonically decreasing in this case as well, and the approximation ratio as a function of b is continuous at $b = 2$.

Theorem 9. *For any $1 < b < 2$, the approximation ratio of* WF *is $\frac{2b}{3b-2}$.*

Proof. We start with the lower bound. Let N be an even large integer. Let $\varepsilon = \frac{2-b}{Nb}$. The input consists of N batches. Each batch starts with an item of size $\frac{1}{b} - \frac{1}{2}$, which is followed by $\lceil 2N\frac{b-1}{2-b}\rceil + 1$ items of size ε. The total size of the items in a single batch is at least $\frac{1}{b} - \frac{1}{2} + (2N\frac{b-1}{2-b} + 1) \cdot \frac{2-b}{Nb} = \frac{1}{b} - \frac{1}{2} + 2 - \frac{2}{b} + \frac{2-b}{Nb} > \frac{3}{2} - \frac{1}{b}$. For large enough N, this total size is also less than 1. Thus each batch of $\lceil 2N\frac{b-1}{2-b}\rceil + 2$ items is packed into a separate bin (once a new bin is opened, the worst fit of the next items of the batch is this new bin, and the total size of a batch together with the large item of the next batch exceeds 1).

After these N batches, there are $\lceil \frac{3b-2}{2-b}N\rceil$ additional pairs of items, each of which consists of items of sizes $\frac{1}{2}$ and ε. Once again, WF packs each pair of items into a dedicated bin.

The number of bins used by WF is at least $N + \lceil \frac{3b-2}{2-b}N\rceil \geq N + \frac{3b-2}{2-b}N = N\frac{2b}{2-b}$.

Note that there are N items of size $\frac{1}{b} - \frac{1}{2}$, $\lceil \frac{3b-2}{2-b}N\rceil$ items of size $\frac{1}{2}$, and at most $N(2N\frac{b-1}{2-b} + 2) + \frac{3b-2}{2-b}N + 1 = 2N^2\frac{b-1}{2-b} + N\frac{b+2}{2-b} + 1$ items of size ε.

We next consider a packing into bins of size $\frac{1}{b}$. There are N bins with one item of size $\frac{1}{b} - \frac{1}{2}$ and one item of size $\frac{1}{2}$. The other items of size $\frac{1}{2}$ are packed into additional bins. A bin which already contains (only) an item of size $\frac{1}{2}$ can receive additional $\frac{N}{2}$ items of size ε, since $\frac{1}{2} + \frac{2-b}{Nb}\frac{N}{2} = \frac{1}{b}$.

The number of items of size ε which can be packed with the remaining items of size $\frac{1}{2}$ is at least $(N\frac{3b-2}{2-b} - N)\frac{N}{2} \geq (N\frac{4(b-1)}{2-b})\frac{N}{2} = \frac{2(b-1)N^2}{2-b}$. Hence, only at most $N\frac{b+2}{2-b} + 1$ unpacked small items remain.

New bins are used for the remaining small items. One bin can hold at least $\lfloor \frac{N}{2-b} \rfloor$ items, since $\lfloor \frac{N}{2-b} \rfloor \cdot \frac{2-b}{Nb} \leq \frac{1}{b}$. The remaining items require at most $\lceil \frac{\frac{b+2}{2-b}N+1}{\lfloor \frac{N}{2-b} \rfloor} \rceil \leq \lceil \frac{\frac{b+2}{2-b}N+1}{\frac{N}{2-b}-1} \rceil = \lceil \frac{(b+2)N+2-b}{N-2+b} \rceil \leq \frac{(b+2)N+2-b}{N-2+b} + 1 = \frac{(b+3)N}{N-2+b} \leq 2(b+3) \leq 10$ additional bins (since $1 < b < 2$).

We get a ratio of at least $\frac{N\frac{2b}{2-b}}{\frac{3b-2}{2-b}N+10}$, which tends to $\frac{2b}{3b-2}$ for large enough values of N.

To prove the upper bound, we use a weight function. In order to define this function, we first define a threshold rule for WF. Consider a set A, which contains items of a total size $1 - \alpha$ (for some $\alpha \geq 0$). The threshold rule for WF is that the largest item in A, has a size of at least α.

The motivation for this threshold rule is that a bin is opened by WF for an item of size α, only if all previously opened bins have a total packed size larger than $1 - \alpha$. Note that in the results of [8] for FF, a threshold rule is used as well, only in the case of FF, a similar situation implies that *all* items will have a size of at least α, while for WF this is not necessarily the case.

We will consider a weight function for which the following three properties hold. The first property is that if the total size of items in a set A is at least $1 - \alpha$, and A satisfies the threshold rule, that is, the size of the largest item in A is at least α, then $w(A) \geq 1$. The second property is that if the total size of items is only $1 - \alpha - \beta$ (for some $\beta > 0$), but the threshold rule is satisfied for α (that is, the size of the largest item in A is at least α, rather than $\alpha + \beta$), then $w(A) \geq 1 - \ell\beta$, where

$$\ell = \sup_{0 < x \leq \frac{1}{b}} w(x)/x.$$

We only consider functions w where ℓ is finite.

The last property which is required is that for any set B, which contains items of a total size of at most $\frac{1}{b}$, it holds that $w(B) \leq \mathcal{R}$, where $\mathcal{R}$ is the approximation ratio $\frac{2b}{3b-2}$.

Given a weight function w which satisfies the three properties, we consider only bins of weight strictly smaller than 1. That is, we remove all bins with weight at least 1 and consider the remaining bins. We define the coarseness of bin i, c_i, (see the analysis of FF in [1]) as the maximum value such that there exists a bin $j < i$ which has a total size of items of $1 - c_i$, that is, the maximum empty space in any preceding bin. We let $c_1 = 0$. Since all bins we consider have a total weight of items smaller than 1, for bin i, the total size of items is some value $1 - \alpha_i$, where the largest item packed in bin i has a size of $\alpha_i - \beta_i$, for some $\beta_i > 0$. We always have $(\alpha_i - \beta_i) > c_i$, as otherwise WF would pack this item in the bin j for which the maximum in the definition of c_i is achieved.

We have $c_{i+1} \geq \alpha_i > c_i + \beta_i$. Let W_i be the total weight of bin i, then we have $W_i \geq 1 - \ell\beta_i$.

If WF uses n bins, the total weight is $\sum\limits_{i=1}^{n} W_i \geq n - \ell(\sum\limits_{i=1}^{n} \beta_i)$.

We calculate $\sum\limits_{i=1}^{n} \beta_i$. $\sum\limits_{i=1}^{n} \beta_i \leq \beta_n + \sum\limits_{i=1}^{n-1} (c_{i+1} - c_i) \leq \beta_n + c_n \leq \alpha_n < 1$. The total weight of all bins is therefore at least WF $-\ \ell$ (note that this inequality holds even when we also consider the bins of weight at least 1, which were removed earlier).

Assume that the third property is satisfied. Each of OPT's bins is filled to at most $\frac{1}{b}$. Hence, the total weight in each of OPT's bins is at most $\frac{2b}{3b-2}$, so $\frac{2b}{3b-2}$OPT is an upper bound on the total weight, and WF $-\ \ell \leq \frac{2b}{3b-2}$OPT. Thus in order to prove the theorem, it suffices to show a weight function w for which the three properties hold, and the value of ℓ is bounded by a fixed constant.

Define the following weight function:

$$w(x) = \begin{cases} \frac{2b}{3b-2}x & \text{, for } \quad x \in (0, \frac{1}{b} - \frac{1}{2}] \\ \frac{4b}{3b-2}x - \frac{2-b}{3b-2} & \text{, for } \quad x \in (\frac{1}{b} - \frac{1}{2}, \frac{1}{2}] \\ 1 & \text{, for } \quad x \in (\frac{1}{2}, \frac{1}{b}] \end{cases}$$

For $x \leq \frac{1}{2}$, $w(x) \geq \frac{2b}{3b-2}x$, since $2bx \geq 2-b$ for $x \geq \frac{1}{b} - \frac{1}{2}$. The function is continuous and monotonically non-decreasing. We next show that this function w satisfies the three required properties.

Lemma 10. *Let A be a set of items of total size $1-\alpha$ ($\alpha \geq 0$), where the largest item in A, i, has a size of at least α. Then $w(A) \geq 1$.*

Proof. If $s_i > \frac{1}{2}$, we are done.

If $s_i \leq \frac{1}{b} - \frac{1}{2}$, then $1-\alpha \geq 1 - s_i \geq \frac{3}{2} - \frac{1}{b} = \frac{3b-2}{2b}$. All items belong to the first case of the weight function, so we get a total weight of at least $\frac{2b}{3b-2} \cdot \frac{3b-2}{2b} = 1$.

We are left with the case $\frac{1}{b} - \frac{1}{2} < s_i \leq \frac{1}{2}$. In this case, the total weight is at least $\frac{2b}{3b-2}(1-\alpha-s_i) + \frac{4b}{3b-2}s_i - \frac{2-b}{3b-2} \geq \frac{2b}{3b-2}(1-2s_i) + \frac{4b}{3b-2}s_i - \frac{2-b}{3b-2} = 1$. □

Lemma 11. *Let A be a set of items, of a total size $1-\alpha-\beta$ (for some $\beta > 0$), which satisfies the threshold rule for α. Then $w(A) \geq 1 - \ell\beta$.*

Proof. Add a dummy item of size β. The threshold rule for WF is still kept with α, and the new total size of items is $1-\alpha$. Let W denote the total weight of original items, and W', the total weight after the modification. By Lemma 10, $W' \geq 1$. We have $W = W' - w(\beta)$. By the definition of ℓ, $w(\beta) \leq \ell\beta$, and the claim follows. □

Lemma 12. *Let B a set of items which can be packed into a bin of size $\frac{1}{b}$ (i.e., $s(B) \leq \frac{1}{b}$). Then $w(B) \leq \frac{2b}{3b-2}$.*

Proof. If the bin contains an item of size $y > \frac{1}{2}$, let $t \leq \frac{1}{b} - y \leq \frac{1}{b} - \frac{1}{2} = \frac{2-b}{2b}$ be the total size of other items, each of which has weight $\frac{2b}{3b-2}$ times its size. The total weight is at most $1 + (\frac{2b}{3b-2})(\frac{2-b}{2b}) = \frac{2b}{3b-2}$.

Otherwise, if the bin contains exactly one item of size $\frac{1}{b} - \frac{1}{2} < y \leq \frac{1}{2}$, then the total weight is at most $\frac{4b}{3b-2}y - \frac{2-b}{3b-2} + (\frac{1}{b} - y)\frac{2b}{3b-2} \leq \frac{2b}{3b-2} - \frac{2-b}{3b-2} + (\frac{1}{b} - \frac{1}{2})\frac{2b}{3b-2} = \frac{2b}{3b-2}$, where the inequality holds since the maximum of the left side is obtained for $y = \frac{1}{2}$.

Finally, if the bin contains at least two items in the interval $(\frac{1}{b} - \frac{1}{2}, \frac{1}{2}]$, such that their total size is y, then the total weight of B is at most $\frac{4b}{3b-2} \cdot y - 2 \cdot \frac{2-b}{3b-2} + \frac{2b}{3b-2} \cdot (\frac{1}{b} - y) = \frac{2b}{3b-2}y - \frac{4}{3b-2} + \frac{2b}{3b-2} + \frac{2}{3b-2} \leq \frac{2}{3b-2} - \frac{4}{3b-2} + \frac{2b}{3b-2} + \frac{2}{3b-2} = \frac{2b}{3b-2}$. □

The value of ℓ is 2, since $\frac{2b}{3b-2} \le 2$, $\frac{4b}{3b-2}x - \frac{2-b}{3b-2} \le 2x$ for any $x \le \frac{1}{2}$, and clearly $1 < 2x$ for $x > \frac{1}{2}$. $\square$

References

[1] A. Borodin and R. El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[2] J. Boyar, G. Dósa, and L. Epstein. On the absolute approximation ratio of well-known bin packing algorithms. Manuscript submitted for publication, 2009.

[3] E. G. Coffman Jr. and J. Csirik. Performance guarantees for one-dimensional bin packing. In T. F. Gonzalez, editor, *Handbook of Approximation Algorithms and Metaheuristics*, chapter 32. Chapman & Hall/Crc, 2007. 18 pages.

[4] J. Csirik and G. J. Woeginger. On-line packing and covering problems. In *A. Fiat and G. J. Woeginger, editors,* Online Algorithms: The State of the Art, pages 147–177, 1998.

[5] J. Csirik and G. J. Woeginger. Resource augmentation for online bounded space bin packing. *Journal of Algorithms*, 44(2):308–320, 2002.

[6] L. Epstein and R. van Stee. Online bin packing with resource augmentation. *Discrete Optimization*, 4(3-4):322–333, 2007.

[7] D. S. Johnson. Fast algorithms for bin packing. *Journal of Computer and System Sciences*, 8:272–314, 1974.

[8] D. S. Johnson, A. Demers, J. D. Ullman, M. R. Garey, and R. L. Graham. Worst-case performance bounds for simple one-dimensional packing algorithms. *SIAM Journal on Computing*, 3:256–278, 1974.

[9] B. Kalyanasundaram and K. Pruhs. Speed is as powerful as clairvoyance. *Journal of the ACM*, 47(4):617–643, 2000.

[10] C. C. Lee and D. T. Lee. A simple online bin packing algorithm. *Journal of the ACM*, 32(3):562–572, 1985.

[11] J. D. Ullman. The performance of a memory allocation algorithm. Technical Report 100, Princeton University, Princeton, NJ, 1971.