ON THE USELESSNESS OF QUANTUM QUERIES

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ABSTRACT

Given a prior probability distribution over a set of possible oracle functions, we define a number of queries to be useless for determining some property of the function if the probability that the function has the property is unchanged after the oracle responds to the queries. A familiar example is the parity of a uniformly random Boolean-valued function over $\{1, 2, ..., N\}$, for which N - 1 classical queries are useless. We prove that if 2k classical queries are useless for some oracle problem, then k quantum queries are also useless. For such problems, which include classical threshold secret sharing schemes, our result also gives a new way to obtain a lower bound on the quantum query complexity, even in cases where neither the function nor the property to be determined is Boolean.

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1. Introduction

Many computational problems involve queries to an oracle (calls to a subroutine) that evaluates some function f at the argument x passed to it and returns the result f(x). Typically, the task is to use the oracle to determine some property of the unknown function. An important example for quantum computation is PERIOD FINDING [1,2] (the ABELIAN HIDDEN SUBGROUP PROBLEM [3,4]), where the function is invariant under addition of some constant to its argument and the task is to find that constant. Another example is CONCEPT LEARNING, where there is some set (the *concept class*) of Boolean-valued functions and the task is to identify exactly which one (the *concept*) the oracle is evaluating [5]. Grover's UNSTRUCTURED SEARCH problem [6] is an instance of concept learning, where the possible functions each take the value 1 for exactly one argument and the value 0 for all other arguments.

A natural goal is to minimize the number of queries to the oracle needed to solve the problem; this minimum is the query complexity of the problem. An alternative goal is to maximize the probability of determining the desired property of f using no more than some fixed number of queries, k. Although this probability is clearly non-decreasing in k, when it does not increase with additional queries, we might say that these queries provide no information, or describe them as useless.

For example, consider Deutsch's problem, in which $f : \{1,2\} \to \mathbb{Z}_2$ and the property to be determined is f(1) + f(2) [7]. If f is chosen uniformly at random, then the prior probabilities for the value of this sum are each 1/2. In this case, a single classical query is useless: the posterior probabilities for the value of the sum are unchanged after the oracle responds to either query. A single quantum query, on the other hand, is not useless: used properly, it identifies the value of the sum with probability 1 [8].

But this raises a natural question: Can quantum queries be useless? In this paper, we formalize the notion of uselessness and study problems for which the answer to this question is "yes". Our main result is a relation between the uselessness of classical and quantum queries: if 2k classical queries provide no relevant information about f, then k quantum queries provide no relevant information about f.

The maximum number of queries that is useless will always be a lower bound for the query complexity; thus our analysis provides a new method for finding a lower bound for the quantum query complexity of any problem for which some number of classical queries is useless.

A familiar problem to which our results apply is PARITY, a generalization of Deutsch's problem in which $N \in \mathbb{N}$ is fixed, an arbitrary function $f : \{1, \ldots, N\} \to \mathbb{Z}_2$ is chosen uniformly at random, and the property to be determined is the modulo 2 sum of the values of f. This problem is an example of a black box oracle problem, in which the values $f(1), \ldots, f(N)$ form an unknown N-bit string. Since N - 1 classical queries reveal no information about the parity of this string, our result says that $\lfloor (N-1)/2 \rfloor$ quantum

queries are also useless. This implies that the quantum query complexity must be at least $\lceil N/2 \rceil$. Beals, *et al.* obtain this same lower bound using the polynomial method, and note that this bound is realized by a quantum algorithm that applies the solution to Deutsch's problem to the function values in pairs [9].*

PARITY is a simple example of a classical threshold secret sharing scheme. In the oracle problem framework, a classical (k, N) threshold secret sharing scheme [11] can be described as a set of functions $f : \{1, \ldots, N\} \to Y$ together with some property of f that can be determined by any k distinct classical queries, but about which no k - 1 classical queries provide any information. So as a corollary of our main theorem, we find that any classical (k, N) threshold secret sharing scheme defines an oracle problem for which $\lfloor (k - 1)/2 \rfloor$ quantum queries are useless and which therefore has quantum query complexity at least $\lfloor k/2 \rfloor$.

Thus our results also give new quantum lower bounds, *e.g.*, for POLYNOMIAL INTERPOLA-TION, a threshold secret sharing scheme introduced by Shamir [11]. Here the function f is a polynomial function of degree k over \mathbb{Z}_p , with k+1 < p, and the problem is to determine f(0). The theory of polynomials easily implies that k + 1 classical queries suffice, but kqueries yield no information. Applying our general results, this implies that $\lfloor k/2 \rfloor$ quantum queries yield no information and thus at least $\lfloor k/2 \rfloor + 1$ quantum queries are necessary.

As this problem exemplifies, our formulation includes oracles that return more than a single bit in response to a query; it is thus more general than the one in which query complexities of Boolean functions are studied. Moreover, as indicated in the discussion of Deutsch's problem above, our formulation also includes a prior probability distribution over possible oracles. As such it includes the more commonly studied cases of total and partial functions as special cases: the former has a constant probability distribution over all functions, while the latter has a two-valued probability distribution that vanishes on disallowed functions.

Furthermore, the methods we use to prove the main theorem are new. In the Appendix, we show how an existing method, the polynomial lower bound method [9], together with an observation of Buhrman, *et al.* [12], can be used to prove a special case of our theorem, namely the case in which we wish to compute a Boolean function, or partial Boolean function, of an *N*-bit string. But these existing methods do not appear to suffice to prove our theorem in complete generality, *i.e.*, in their current form they do not apply to the case in which the set Y has more than 2 elements, nor to the case in which we wish to compute more than just a Boolean classification of the allowed functions.

2. The definition of uselessness

Let X and Y be finite sets, and let $\mathcal{C} \subseteq Y^X$ be a subset of the set of all functions from X to Y. Boolean-valued functions, *i.e.*, $Y = \mathbb{Z}_2$, are commonly studied—in computational

^{*} Farhi, *et al.* obtained the same results using a different method [10].

learning theory, for example, where C is called a *concept class* [5].

Suppose that the class C is partitioned into disjoint subclasses C_j , $j \in J$. In the learning problems $(C, \{C_j \mid j \in J\}, \mu)$ we are considering, an element f is chosen from C according to an arbitrary, but known, prior probability distribution μ , and the task is to determine to which subclass C_j the function f belongs. Information about f is available only *via* an oracle that, given a query $x \in X$, returns the value of f(x). To formalize the action of this oracle we begin by recalling some standard notation:

Let $\mathcal{H} = \mathbb{C}^X \otimes \mathbb{C}^Y \otimes \mathbb{C}^Z$, where Z is a finite set. The three tensor factors represent query, response, and auxiliary registers, respectively. We assume that Y is an abelian group, and that the quantum oracle \mathcal{O}_f acts on \mathcal{H} by addition of f(x) into the response register. (Everything in the following, however, can be carried out more generally in the permutation model introduced in [13].) Thus the action of the oracle $\mathcal{O}_f : \mathcal{H} \to \mathcal{H}$ is specified by the following permutation of the computational basis

$$\mathcal{O}_f : |x, y, z\rangle \mapsto |x, y + f(x), z\rangle.$$

A general k-query quantum learning algorithm can now be described as follows: An initial state is prepared with density matrix $\rho_0 \in \mathcal{H} \otimes \mathcal{H}^{\dagger}$. The algorithm passes this state to the oracle, which acts by \mathcal{O}_f ; then the algorithm acts by some unitary operator U_1 , independent of f; and the state is again passed to the oracle; *etc.* After the k^{th} call to the oracle, the algorithm applies a last unitary operator U_k to arrive in the final state

$$\rho_f = U_k \mathcal{O}_f U_{k-1} \dots U_1 \mathcal{O}_f \rho_0 \mathcal{O}_f^{\dagger} U_1^{\dagger} \dots U_{k-1}^{\dagger} \mathcal{O}_f^{\dagger} U_k^{\dagger}.$$
(1)

The last step is a POVM $\{\Pi_s\}$ indexed by an arbitrary set S. Some map $S \to J$, which is part of the algorithm (and independent of f), specifies the subset C_j to which we conclude fbelongs. (Notice that the unitary operator U_k is unnecessary, since it could be incorporated into the measurement. It is notationally convenient, however, to include it.)

Our main result concerns situations in which no information about the part C_j to which the function f belongs can be derived from some number of classical or quantum queries. We now make this notion precise.

DEFINITION (classical version). Let $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ be a learning problem as described above. Then we say that k classical queries yield no information, or are useless, if for any $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in Y$,

$$\mu(f \in \mathcal{C}_j \mid f(x_i) = y_i, i = 1, \dots, k) = \mu(f \in \mathcal{C}_j), \text{ for all } j \in J.$$

That is, the probability of f being in any of the sets C_j is independent of the knowledge of any k function values.

DEFINITION (quantum version). Let $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ be a learning problem as described above. Then we say that k quantum queries yield no information, or are useless, if for any k-query quantum algorithm with initial state ρ_0 , unitary operations U_1, \ldots, U_k , and measurement $\{\Pi_s\}$,

$$\mu(f \in \mathcal{C}_j \mid s) = \mu(f \in \mathcal{C}_j), \text{ for all } s \in S, j \in J.$$

That is, the probability of f being in C_j is independent of any measurement taken after k calls to the oracle.

3. From classical to quantum uselessness

Having made these definitions precise, we can state our main result:

THEOREM 1. Let $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ be a learning problem. Suppose that 2k classical queries are useless. Then k quantum queries are useless.

EXAMPLE 1 (PARITY). As we described in the introduction, Theorem 1 applies to PARITY: Let $N \in \mathbb{N}$, and let \mathcal{C} be the set of all functions from $\{1, \ldots, N\}$ to \mathbb{Z}_2 with a uniform prior distribution. Partition \mathcal{C} into \mathcal{C}_0 and \mathcal{C}_1 according to the sum of the values of f. Then it is easy to see that N - 1 classical queries are useless. Thus, by Theorem 1, $\lfloor (N-1)/2 \rfloor$ quantum queries are also useless. Since PARITY can be solved with $\lceil N/2 \rceil$ quantum queries (using repeated XORs, *i.e.*, solutions to Deutsch's problem), the quantum query complexity of PARITY for exact solution is exactly $\lceil N/2 \rceil$, reproving a result of Farhi, *et al.* [10] and Beals, *et al.* [9]. Theorem 1 tells us a little more, namely that using 1 fewer query than this there is no quantum algorithm that succeeds with probability greater than 1/2, a result that we show in the Appendix also follows from the analysis of *unbounded error quantum query complexity* of Boolean functions by Montanaro, *et al.* using more complicated machinery [14].

EXAMPLE 2. Generalizing Deutsch's problem in a different direction than does PARITY, let C be the set of all functions from $\{1, 2, 3\}$ to \mathbb{Z}_3 with a uniform prior distribution. Let $C = C_{\text{even}} \sqcup C_{\text{odd}}$, where a function f is defined to be even or odd depending on whether the size of the image of f is even or odd. Notice that the prior probability $\Pr(f \in C_{\text{even}}) = 2/3$, not 1/2. It is straightforward to check that two classical queries yields no information. Thus, by Theorem 1, a single quantum query is useless. (It turns out that two quantum queries suffice to solve this problem with probability 1. This result and generalizations will be the subject of a subsequent publication [15].)

EXAMPLE 3 (POLYNOMIAL INTERPOLATION). Shamir's example of a threshold secret sharing scheme [11] provides a distinct family of examples. Let p be prime; let $p - 1 > k \in \mathbb{N}$; and let

$$\mathcal{C} = \left\{ f : \{1, \dots, p-1\} \to \mathbb{Z}_p \mid f(x) = \sum_{i=0}^k a_i x^i \text{ for } a_i \in \mathbb{Z}_p \right\}.$$

Let μ be the uniform distribution on C; this is equivalent to choosing each a_i independently and uniformly at random in \mathbb{Z}_p . For $j \in \mathbb{Z}_p$, let $\mathcal{C}_j = \{f \in \mathcal{C} \mid f(0) = j\}$. Since the unknown polynomial f has degree k, interpolation of the k values obtained by k classical queries, together with any value for f(0), identifies f. Since the value for f(0) is chosen uniformly at random, this means that any k classical queries alone give no information about C_j . So Theorem 1 tells us that $\lfloor k/2 \rfloor$ quantum queries are useless. As with PARITY, this implies a lower bound for the quantum query complexity of POLYNOMIAL INTERPOLATION:

THEOREM 2. For POLYNOMIAL INTERPOLATION, $\lfloor k/2 \rfloor$ quantum queries are useless, and hence the quantum query complexity of POLYNOMIAL INTERPOLATION is at least $\lfloor k/2 \rfloor + 1$.

4. Proof of the main theorem

The proof of Theorem 1 rests upon the following lemma:

LEMMA. Let $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ be a learning problem. If 2k classical queries are useless, then for any j,

$$\sum_{f \in \mathcal{C}_j} \mu(f) \rho_f = \mu(\mathcal{C}_j) \sum_{f \in \mathcal{C}} \mu(f) \rho_f,$$

where ρ_f is defined by equation (1).

Proof. First note that any matrix $B \in \mathcal{H} \otimes \mathcal{H}^{\dagger}$ has rows and columns indexed by $X \times Y \times Z$. Since \mathcal{O}_f is a permutation matrix, it is easy to express the entries of the matrix $\mathcal{O}_f B \mathcal{O}_f^{\dagger}$ in terms of the matrix B. If L = (x, y, z) and M = (u, v, w), then

$$(\mathcal{O}_f B \mathcal{O}_f^{\dagger})_{L,M} = B_{fL,fM},\tag{2}$$

where for the triple L = (x, y, z), we define fL = (x, y + f(x), z).

Let ρ_i denote the state after the *i*th query and after applying U_i , as in equation (1). Then

$$\rho_i = U_i \mathcal{O}_f \rho_{i-1} \mathcal{O}_f^{\dagger} U_i^{\dagger},$$

and from equation (2) and matrix multiplication, it follows that

$$(\rho_i)_{L,M} = \sum_{L',M'} (U_i)_{L,L'} (\rho_{i-1})_{fL',fM'} (U_i^{\dagger})_{M',M}, \tag{3}$$

with the sum taken over all $L', M' \in X \times Y \times Z$. Now apply equation (3) iteratively:

First,

$$(\rho_1)_{L,M} = \sum_{L_1,M_1} (U_1)_{L,L_1} (\rho_0)_{fL_1,fM_1} (U_1^{\dagger})_{M_1,M}.$$

Note that the quantity being summed depends only on the indices L, M, L_1 and M_1 , and the two function values $f(x_1)$ and $f(u_1)$, where x_1 and u_1 are the first coordinates of L_1 and M_1 , respectively. (It also depends on ρ_0 and the unitary matrix U_0 , but these are fixed.) On the uselessness of quantum queries

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Second,

$$(\rho_2)_{L,M} = \sum_{L_1, M_1, L_2, M_2} (U_2)_{L, L_2} (U_1)_{fL_2, L_1} (\rho_0)_{fL_1, fM_1} (U_1^{\dagger})_{M_1, fM_2} (U_2^{\dagger})_{M_2, M}$$

Here the quantity being summed depends on the indices L, M, L_1 , L_2 , M_1 and M_2 , and the four function values $f(x_1)$, $f(x_2)$, $f(u_1)$ and $f(u_2)$.

Continuing in this manner, the final density matrix after k queries, $\rho_k = \rho_f$, is given by

$$\rho_f = \sum_I Q_I \big(f(x_1), \dots, f(x_k), f(u_1), \dots, f(u_k) \big),$$

where the sum is taken over all tuples $I = (L_1, \ldots, L_k, M_1, \ldots, M_k) \in (X \times Y \times Z)^{2k}$, and $Q_I(f(x_1), \ldots, f(x_k), f(u_1), \ldots, f(u_k)) \in \mathcal{H} \otimes \mathcal{H}^{\dagger}$ is a matrix that depends only on the index I and the 2k function values shown.

Thus, for any $j \in J$,

$$\sum_{f \in \mathcal{C}_j} \mu(f) \rho_f = \sum_I \sum_{f \in \mathcal{C}_j} \mu(f) Q_I \big(f(x_1), \dots, f(x_k), f(u_1), \dots, f(u_k) \big).$$
(4)

Regrouping, the right hand side of equation (4) becomes

$$\sum_{I} \sum_{\{y_i\}, \{v_i\}} \mu(f \in \mathcal{C}_j \text{ and } f(x_i) = y_i, f(u_i) = v_i, i \in \{1, \dots, k\}) Q_I(y_1, \dots, y_k, v_1, \dots, v_k),$$

with the inner sum taken over all $y_1, \ldots, y_k, v_1, \ldots, v_k \in Y$. But by the hypothesis that 2k classical queries yield no information,

$$\mu (f \in \mathcal{C}_j \text{ and } f(x_i) = y_i, f(u_i) = v_i, i \in \{1, \dots, k\})$$

= $\mu(\mathcal{C}_j) \mu (f(x_i) = y_i, f(u_i) = v_i, i \in \{1, \dots, k\})$

Thus equation (4) becomes

$$\sum_{f \in \mathcal{C}_j} \mu(f) \rho_f$$

= $\mu(\mathcal{C}_j) \sum_{I, \{y_i\}, \{v_i\}} \mu(f(x_i) = y_i, f(u_i) = v_i, i \in \{1, \dots, k\}) Q_I(y_1, \dots, y_k, v_1, \dots, v_k).$
(5)

Summing equation (5) over all j gives

$$\sum_{f \in \mathcal{C}} \mu(f) \rho_f = \sum_{I, \{y_i\}, \{v_i\}} \mu(f(x_i) = y_i, f(u_i) = v_i, i \in \{1, \dots, k\}) Q_I(y_1, \dots, y_k, v_1, \dots, v_k),$$

whence the lemma follows.

Proof of Theorem 1. The statement of the theorem is that the probability of f being in C_j does not change if s is observed after k queries, *i.e.*, for any $j \in J$ and $s \in S$, we need to show that

$$\sum_{f \in \mathcal{C}_j} \mu(f \mid s) = \mu(\mathcal{C}_j).$$

To prove this, calculate the probability of f having been the chosen function conditioned on having observed s, using Bayes' Theorem:

$$\mu(f \mid s) = \frac{\operatorname{Tr}(\rho_f \Pi_s)\mu(f)}{\sum_{g \in \mathcal{C}} \operatorname{Tr}(\rho_g \Pi_s)\mu(g)}$$

Thus,

$$\sum_{f \in \mathcal{C}_j} \mu(f \mid s) = \frac{\operatorname{Tr}\left(\left(\sum_{f \in \mathcal{C}_j} \mu(f)\rho_f\right)\Pi_s\right)}{\operatorname{Tr}\left(\left(\sum_{g \in \mathcal{C}} \mu(g)\rho_g\right)\Pi_s\right)}.$$
(6)

Applying the Lemma, the quotient on the right hand side of equation (6) reduces to $\mu(\mathcal{C}_j)$, establishing the theorem.

5. Conclusion

As we noted in the introduction, Theorem 1 implies a lower bound on the quantum query complexity of certain learning problems:

THEOREM 3. Let $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ be a learning problem. Suppose that 2k classical queries are useless. Then the quantum query complexity of the problem is at least k + 1.

The uselessness of some number of quantum queries in learning problems with two subclasses also has a consequence for amplified impatient learning [16]: If in addition to the membership oracle (the oracle that returns function values), we have access to an equivalence oracle (an oracle that answers the questions of the form "Is $f \in C_j$?"), a commonly studied situation in computational learning theory [17], we can implement amplitude amplification [6,18,19,20,21] after any number of quantum queries. If k quantum queries to the membership oracle are useless, however, amplitude amplification works exactly as well if it is implemented immediately, *i.e.*, after no quantum queries, as when it is implemented after k or fewer quantum queries.

These results encourage further investigation of the quantum query complexity of, and quantum algorithms for, learning problems in which some number of classical queries are useless. These include problems in the families exemplified by Examples 2 and 3. We will address some of these questions in a forthcoming paper [15].

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References

- D. R. Simon, "On the power of quantum computation", in S. Goldwasser, ed., Proceedings of the 35th Annual Symposium on Foundations of Computer Science, Santa Fe, NM, 20–22 November 1994 (Los Alamitos, CA: IEEE 1994) 116–123;
 D. R. Simon, "On the power of quantum computation", SIAM J. Comput. 26 (1997) 1474–1483.
- [2] P. W. Shor, "Algorithms for quantum computation: discrete logarithms and factoring", in S. Goldwasser, ed., Proceedings of the 35th Symposium on Foundations of Computer Science, Santa Fe, NM, 20–22 November 1994 (Los Alamitos, CA: IEEE Computer Society Press 1994) 124–134;

P. W. Shor, "Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer", SIAM J. Comput. 26 (1997) 1484–1509.

- [3] D. Boneh and R. J. Lipton, "Quantum cryptanalysis of hidden linear forms", in D. Coppersmith, ed., Proceedings of Crypto '95, Lecture Notes in Computer Science 963 (Berlin: Springer-Verlag 1995) 424–437.
- [4] R. Jozsa, "Quantum algorithms and the Fourier transform", Proc. Roy. Soc. Lond. A 454 (1998) 323–337.
- [5] See, e.g., D. Angluin, "Computational learning theory: Survey and selected bibliography", in Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing (New York: ACM 1992) 351–369.
- [6] L. K. Grover, "A fast quantum mechanical algorithm for database search", in Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing, Philadelphia, PA, 22–24 May 1996 (New York: ACM 1996) 212–219;
 L. K. Grover, "Quantum mechanics helps in searching for a needle in a haystack", Phys. Rev. Lett. **79** (1997) 325–328.
- [7] D. Deutsch, "Quantum theory, the Church-Turing principle and the universal quantum computer", Proc. Roy. Soc. Lond. A 400 (1985) 97–117.
- [8] R. Cleve, A. Ekert, C. Macchiavello and M. Mosca, "Quantum algorithms revisited", Proc. Roy. Soc. Lond. A 454 (1998) 339–354.
- [9] R. Beals, H. Buhrman, R. Cleve, M. Mosca and R. de Wolf, "Quantum lower bounds by polynomials", J. ACM 48 (2001) 778–797.
- [10] E. Farhi, J. Goldstone, S. Gutmann and M. Sipser, "Limit on the speed of quantum computation in determining parity", Phys. Rev. Lett. 81 (1998) 5442–5444.
- [11] A. Shamir, "How to share a secret", Commun. ACM 22 (1979) 612–613.
- [12] H. Buhrman, N. Vereshchagin and R. de Wolf, "On computation and communication with small bias", Proceedings of the 22nd Annual Conference on Computational Complexity, San Diego, CA, 13-16 June 2007 (Los Alamitos, CA: IEEE 2007) 24–32.
- [13] D. A. Meyer and J. Pommersheim, "Single query learning from abelian and non-abeli-

an Hamming distance oracles", arXiv:0912.0583v1 [quant-ph].

- [14] A. Montanaro, H. Nishimura and R. Raymond, "Unbounded error quantum query complexity", in S.-H. Hong, N. Nagamochi and T. Fukunaga, eds., Algorithms and Computation, proceedings of the 19th International Symposium, ISAAC 2008, Gold Coast, Australia, 15-17 December 2008, Lecture Notes in Computer Science 5369 (Berlin: Springer-Verlag 2008) 919–930.
- [15] D. A. Meyer and J. Pommersheim, "Multi-query quantum algorithms for summation", in preparation.
- [16] M. Hunziker, D. A. Meyer, J. Park, J. Pommersheim and M. Rothstein, "The geometry of quantum learning", *Quantum Inform. Processing* DOI: 10.1007/s11128-009-0129-6 (2009) 1–21.
- [17] D. Angluin, "Queries and concept learning", Machine Learning 2 (1988) 319–342.
- [18] G. Brassard and P. Høyer, "An exact quantum polynomial-time algorithm for Simon's problem", Proceedings of 5th Israeli Symposium on Theory of Computing and Systems, Ramat-Gan, Israel 17–19 June 1997 (Los Alamitos, CA: IEEE 1997) 12–23.
- [19] L. K. Grover, "A framework for fast quantum mechanical algorithms", in Proceedings of the 30th Annual ACM Symposium on Theory of Computing, Dallas, TX, 23–26 May 1998 (New York: ACM 1998) 53–62.
- [20] G. Brassard, P. Høyer and A. Tapp, "Quantum counting", Proceedings of the 25th International Colloquium on Automata, Languages, and Programming, Ålborg, Denmark, 13–17 July 1998, Lecture Notes in Computer Science 1443 (Berlin: Springer-Verlag 1998) 820–831.
- [21] G. Brassard, P. Høyer, M. Mosca and A. Tapp, "Quantum amplitude amplification and estimation", in S. J. Lomonaco, Jr. and H. E. Brandt, eds., Quantum Computation and Information, Contemporary Mathematics **305** (Providence, RI: AMS 2002) 53– 74.

Appendix

In this appendix, we focus on the special case in which we are trying to compute a Boolean function, or partial Boolean function, of N-bit strings. That is, we assume that (a) the concept class C consists of Boolean-valued functions, *i.e.*, the codomain is $Y = \{0, 1\}$; and (b) the partition of C has exactly two parts $C = C_0 \sqcup C_1$. We show that the polynomial method of [9], together with an observation of [12], can be used to prove Theorem 1 for this special case. Similar ideas appear in Section 3 of [14]. As noted in the Introduction, it seems that these ideas cannot be used to prove Theorem 1, which is not limited by restrictions (a) or (b).

We base our proof of this special case of Theorem 1 on the following result, which gives a general relation between k-query quantum algorithms and 2k-query classical algorithms.

THEOREM 4. Suppose $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ is a learning problem, as described above, such that $Y = \{0, 1\}$ and $J = \{0, 1\}$. Given a k-query quantum algorithm, for each N-bit string f of function values computed by the oracle, denote by p(f) the probability that the quantum algorithm outputs 0. Then there exists a positive real number T and a 2k-query

(randomized) classical algorithm whose output probability for f is given by

$$p_{\text{classical}}(f) = \frac{1}{T} \left(p(f) - \frac{1}{2} \right) + \frac{1}{2}$$

That is, for each f, the bias of the classical algorithm away from $\frac{1}{2}$ is T^{-1} times the bias of the quantum algorithm away from $\frac{1}{2}$.

Note that this theorem does not require the existence of a prior distribution on C.

Proof. For convenience, we will assume that $X = \{1, \ldots, N\}$, and we will identify Y^X with *N*-bit strings $f = (f(1), \ldots, f(N))$. \mathcal{C} is then a subset of *N*-bit strings. Suppose we are given any *k*-query quantum algorithm. Then the arguments of [9] show that there exists a squarefree polynomial p(f) of degree at most 2k with real coefficients such that for any $f \in \mathcal{C}$ evaluated by the oracle, p(f) is the probability that the quantum algorithm outputs 0.

We now change variables, so as to identify functions from $\{0, 1\}^N \to \{0, 1\}$ with functions $\{-1, 1\}^N \to \{-1, 1\}$. Specifically, we introduce the polynomial

$$q(w_1, \dots, w_N) = 2p\left(\frac{w_1+1}{2}, \dots, \frac{w_N+1}{2}\right) - 1.$$

Then q(w) is a squarefree polynomial of degree at most 2k with real coefficients, and has the property that for any $w \in \{-1, 1\}^N$, the probability that the corresponding $f \in \{0, 1\}^N$ leads to an output of 0 is equal to p(x) = (1 + q(w))/2. Then we have

$$q(w) = \sum_{S} \hat{q}(S) w_S,$$

where the sum is over all subsets S of $\{1, \ldots, N\}$ of size less than or equal to 2k, and w_S denotes the product of w_i with $i \in S$. Let $T = \sum_S |\hat{q}(S)|$.

We now introduce a classical algorithm, following the observation of Buhrman, *et al.* [12]. First note that the absolute value of \hat{q}/T defines a probability distribution on the subsets S of $\{1, \ldots, N\}$ of size less than or equal to 2k. Begin by picking a random subset S according to this distribution. By invoking the classical oracle at most 2k times, compute w_S . Then according to whether sign $(\hat{q}(S))w_S$ is 1 or -1, output 0 or 1, respectively.

We claim that for any $f \in \mathcal{C} \subseteq \{0,1\}^N$, the probability that this classical algorithm outputs 0 equals $p_{\text{classical}}(f) = (p(f) - \frac{1}{2})/T + \frac{1}{2}$. To see this, note that the probability of outputting 0 is

$$\sum_{S} \frac{|\hat{q}(S)|}{T} \delta_S,$$

where $\delta_S = 1$ if sign $(\hat{q}(S))w_S = 1$, and 0 otherwise. This simplifies to

$$\sum_{S} \frac{|\hat{q}(S)|}{T} \left(\frac{\operatorname{sign}(\hat{q}(S))w_{S} + 1}{2} \right) = \frac{1}{2T} \left(T + \sum_{S} \hat{q}(S)w_{S} \right) = \frac{T + q(w)}{2T} = \frac{1}{T} \left(p(f) - \frac{1}{2} \right) + \frac{1}{2},$$

as desired.

As a consequence we have the following special case of Theorem 1:

COROLLARY 5. For any learning problem $(\mathcal{C}, \{\mathcal{C}_j \mid j \in J\}, \mu)$ as described above, with $Y = \{0, 1\}$ and $J = \{0, 1\}$, if 2k classical queries are useless, then k quantum queries are useless.

Proof. Suppose that 2k classical queries are useless. Given any k-query quantum algorithm, consider the corresponding 2k-query classical algorithm given by Theorem 4. Since this algorithm is useless, we have

$$\frac{\sum_{f \in \mathcal{C}_0} \mu(f) \left(p(f) - \frac{1}{2} \right) / T}{\sum_{f \in \mathcal{C}} \mu(f) \left(p(f) - \frac{1}{2} \right) / T} = \sum_{f \in \mathcal{C}_0} \mu(x).$$

It follows that

$$\frac{\sum_{f \in \mathcal{C}_0} \mu(f) p(f)}{\sum_{f \in \mathcal{C}} \mu(f) p(f)} = \sum_{f \in \mathcal{C}_0} \mu(f).$$

In other words, the quantum algorithm is also useless.