

The Predecessor-Existence Problem for k -Reversible Processes

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Abstract

For $k \geq 1$, we consider the graph dynamical system known as a k -reversible process. In such process, each vertex in the graph has one of two possible states at each discrete time. Each vertex changes its state between the present time and the next if and only if it currently has at least k neighbors in a state different than its own. Given a k -reversible process and a configuration of states assigned to the vertices, the PREDECESSOR EXISTENCE problem consists of determining whether this configuration can be generated by the process from another configuration within exactly one time step. We can also extend the problem by asking for the number of configurations from which a given configuration is reachable within one time step. PREDECESSOR EXISTENCE can be solved in polynomial time for $k = 1$, but for $k > 1$ we show that it is NP-complete. When the graph in question is a tree we show how to solve it in $O(n)$ time and how to count the number of predecessor configurations in $O(n^2)$ time. We also solve PREDECESSOR EXISTENCE efficiently for the specific case of 2-reversible processes when the maximum degree of a vertex in the graph is no greater than 3. For this case we present an algorithm that runs in $O(n)$ time.

Keywords: k -reversible processes, Garden-of-Eden configurations, Predecessor-existence problem, Graph dynamical systems.

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1 Introduction

Let G be a simple, undirected, finite graph with n vertices and m edges. The set of vertices of G is denoted by $V(G) = \{v_1, v_2, \dots, v_n\}$ and its set of edges is denoted by $E(G)$. A k -reversible process on G is an iterative process in which, at each discrete time t , each vertex in G has one of two possible states. A state of a vertex is represented by an integer belonging to the set $Q = \{-1, +1\}$ and each vertex has its state changed from one time to the next if and only if it currently has at least k neighbors in a state different than its own, where k is a positive integer.

Let $Y_t(v_i)$ be the state of vertex v_i at time t . A *configuration* of states at time t for the vertices in $V(G)$ is denoted by $Y_t = (Y_t(v_1), Y_t(v_2), \dots, Y_t(v_n))$. The one-step dynamics in a k -reversible process for graph G can be described by a function $F_G^k : Q^n \rightarrow Q^n$ such that $Y_t = F_G^k(Y_{t-1})$ through a local state update rule for each vertex v_i given by

$$Y_t(v_i) = \begin{cases} Y_{t-1}(v_i), & \text{if } v_i \text{ has fewer than } k \text{ neighbors in state } -Y_{t-1}(v_i) \\ & \text{at time } t-1; \\ -Y_{t-1}(v_i), & \text{otherwise.} \end{cases} \quad (1)$$

The motivation to study k -reversible processes is related to the analysis of opinion dissemination in social networks. For example, suppose that a network is modeled by a graph, each vertex representing a person and each edge between two vertices indicating that the corresponding persons are friends. Suppose further that state -1 represents disagreement on some issue and that the state $+1$ means agreement on the same issue. A k -reversible process is an approach to model opinion dissemination when people are strongly influenced by the opinions of their friends and the society they are part of. Notice that in this model we are assuming that all people act in the same manner. A more complex approach could assume, for example, distinct thresholds for each person or thresholds based on one's number of friends.

Note that k -reversible processes are examples of *graph dynamical systems*; more precisely, of *synchronous dynamical systems*, which extend the notion of a *cellular automaton* to arbitrary graph topologies. The study of graph dynamical systems and cellular automata is multidisciplinary and related to several areas, like optics [9], neural networks [12], statistical mechanics [1], as well as opinion [5] and disease dissemination [14]. Also in distributed computing there are several studies regarding models of graph dynamical systems. An example is the model of majority processes in which each vertex changes its state if and only if at least half of its neighbors have a different state [13]. An application of this model is used for maintaining data consistency [16].

Most of the studies regarding k -reversible processes are related to the MINIMUM CONVERSION SET problem in such processes. This problem consists of determining the cardinality of the minimum set of vertices that, if in state $+1$, lead all vertices in the graph also to state $+1$ after a finite number of time steps. It has been proved that this problem is NP-hard for $k > 1$ [6]. There are also

several interesting results about this problem in the work by Dreyer [7], who also presents some important results regarding the periodic behavior of k -reversible processes, as well as upper bounds on the transient length that precedes periodicity. Most of the results presented by Dreyer are based on reductions from the so-called *threshold processes*, which are broadly studied by Goles and Olivos [11, 10]. Another approach to study the transient and periodic behavior of k -reversible processes is the use of a specific energy function that leads to a much more intuitive proof of the maximum period length and also better bounds on the transient length [15].

A problem that arises in synchronous dynamical systems on graphs is the so-called PREDECESSOR EXISTENCE problem, defined as follows. Given one such dynamical system and a configuration of states, the question is whether this configuration can be generated from another configuration in a single time step using the system's update rule. In the affirmative case, such configuration is called a *predecessor* of the one that was given initially. This problem was studied by Sutner [17] within the context of cellular automata, where configurations lacking a predecessor configuration are known as garden-of-Eden configurations, and was proved to be NP-complete for finite cellular automata. NP-completeness results for related dynamical systems as well as polynomial-time algorithms for some graph classes can also be found in the literature [3]. An extension of the PREDECESSOR EXISTENCE problem is to count the number of predecessor configurations. This is also a hard problem and has been proved to be #P-complete [18].

For k -reversible processes, we address these two problems in this paper. We are interested in determining whether a configuration Y_{t-1} exists for which $Y_t = F_G^k(Y_{t-1})$. Because only time steps $t-1$ and t matter, for simplicity we denote Y_{t-1} and Y_t by Y' and Y , respectively. We henceforth denote this special case of PREDECESSOR EXISTENCE by $\text{PRE}(k)$. We also consider the associated counting problem, $\#\text{PRE}(k)$, which asks for the number of predecessor configurations. Our results include an NP-completeness proof for the general case of k -reversible processes and polynomial-time algorithms for some particular cases.

The remainder of the paper is organized as follows. In Section 2 we show that $\text{PRE}(1)$ is polynomial-time solvable. In Section 3 we provide an NP-completeness proof of $\text{PRE}(k)$ for $k > 1$. In Section 4 we describe two efficient algorithms for trees, one for solving $\text{PRE}(k)$ and the other for solving $\#\text{PRE}(k)$. In Section 5 we show an efficient algorithm to solve $\text{PRE}(2)$ for graphs with maximum degree no greater than 3. Section 6 contains our conclusions.

2 Polynomial-time solvability of $\text{PRE}(1)$

For $k = 1$, if Y' exists then any pair of neighbors u and v for which $Y(u) = Y(v)$ also has $Y'(u) = Y'(v)$. Based on this observation, we start by partitioning G into connected subgraphs that are maximal with respect to the property that each of them contains only vertices whose states in Y are the same. Clearly, all

vertices in the same subgraph must have equal states also in a predecessor of Y . Let us call each of such maximal connected subgraphs an MCS.

Let H be an MCS in which a vertex v exists whose neighbors in G all have the same state as its own. In other words, all of v 's neighbors are also in H . For this vertex, clearly there is no possibility other than $Y'(v) = Y(v)$. Because there is only one choice of state for v in a predecessor configuration, we call both the vertex and its containing MCS *locked*. We refer to all other vertices and MCSs as being *unlocked* (so there may exist unlocked vertices in a locked MCS). We also say that any two MCSs are neighbors whenever they contain vertices that are themselves neighbors.

Theorem 1. *PRE(1) is solved affirmatively if and only if the following two conditions hold:*

- *No two locked MCSs are neighbors;*
- *Every vertex in an unlocked MCS has at least one neighbor in another unlocked MCS.*

In this case, Y' is obtained from Y by changing the state of all vertices in unlocked MCSs.

Proof. The case of a single (necessarily locked) MCS is trivial. If, on the other hand, more than one MCS exists in G , then clearly the two conditions suffice for Y' to exist and be as stated: in one time step from Y' , the state of every vertex in a locked MCS remains unchanged and that of every vertex in an unlocked MCS changes, thus yielding Y .

It remains for necessity to be shown. We do this by noting that, should the first condition fail and at least two locked MCSs be neighbors, any prospective Y' would have to differ from Y in all vertices of each of these MCSs, but the presence of locked vertices in them would make it impossible for Y to be obtained in one time step. Should the second condition be the one to fail and at least one vertex in an unlocked MCS have neighbors outside its MCS only in locked MCSs, any prospective Y' would have to differ from Y in all vertices of such an unlocked MCS. Once again it would be impossible to obtain Y in one time step due to the locked vertices. It follows that both conditions are necessary for Y' to exist. \square

By Theorem 1, we can easily solve PRE(1) in $O(n + m)$ time.

3 NP-completeness of PRE(k) for $k > 1$

We first note that the NP-completeness proof of PREDECESSOR EXISTENCE for finite cellular automata [17] cannot be directly extended to k -reversible processes in graphs for $k > 1$. Sutner's proof only shows that there exist finite cellular automata for which PREDECESSOR EXISTENCE is NP-complete. In other words, it depends on the vertex update rule being used in the cellular automaton. A different approach is then needed.

We present a reduction from a satisfiability problem known as 3SAT EXACTLY-TWO. This problem is the variation of the 3SAT problem in which each clause must be satisfied by exactly two positive literals. We start by proving that 3SAT EXACTLY-TWO is NP-complete.

Lemma 2. 3SAT EXACTLY-TWO is NP-complete.

Proof. The problem is trivially in NP. We proceed with the reduction from another variation of the 3SAT problem, known as 3SAT EXACTLY-ONE [8], which is NP-complete and asks whether there exists an assignment of variables satisfying each clause by exactly one positive literal.

The reduction is simple and consists of inverting all literals in all clauses of an instance S of 3SAT EXACTLY-ONE, resulting in an instance S' of 3SAT EXACTLY-TWO. It is easy to check that a solution for S directly gives a solution for S' , and conversely, a solution for S' directly gives a solution for S . \square

Theorem 3. $\text{PRE}(k)$ is NP-complete for $k > 1$.

Proof. Given two configurations Y and Y' , verifying whether Y' is a predecessor configuration of Y is straightforward and can be done by simulating one step of the k -reversible process starting with configuration Y' . Simulating one step of the process takes $O(n+m)$ time and the final comparison between the resulting configuration and Y takes $O(n)$ time; thus, $\text{PRE}(k)$ is in NP. The remainder of the proof is a reduction from 3SAT EXACTLY-TWO, which by Lemma 2 is NP-complete.

Let S be an arbitrary instance of 3SAT EXACTLY-TWO with the M clauses c_1, c_2, \dots, c_M and the N variables x_1, x_2, \dots, x_N . We construct an instance (G, Y) of $\text{PRE}(k)$ from S as follows.

Vertex set $V(G)$ is the union of:

- $\{x_i, \neg x_i\}$, for each variable x_i in S ;
- $\{z_i, z'_i\}$, for each variable x_i in S ;
- $\{u_{i,1}, \dots, u_{i,2k-3}\}$, for each variable x_i in S ;
- $\{p_{i,1}, \dots, p_{i,2k-3}\}$, for each variable x_i in S ;
- $\{w_{i,1}, \dots, w_{i,k-2}\}$, for each variable x_i in S , provided $k > 2$;
- $\{w'_{i,1}, \dots, w'_{i,k-2}\}$, for each variable x_i in S , provided $k > 2$;
- $\{c_i, c'_i\}$, for each clause c_i in S ;
- $\{b_{i,1}, \dots, b_{i,k-2}\}$, for each clause c_i in S , provided $k > 2$;
- $\{b'_{i,1}, \dots, b'_{i,k-1}\}$, for each clause c_i in S .

Vertices x_i and $\neg x_i$ are called *literal vertices* and vertices c_i and c'_i are called *clause vertices*. If x is a neighbor of u and x is a literal vertex, then we say that x is a literal neighbor of u . Similarly, if x is a neighbor of u and x is a clause vertex, then x is a clause neighbor of u .

Edge set $E(G)$ is the union of:

- $\{(x_i, z_i), (x_i, z'_i), (\neg x_i, z_i), (\neg x_i, z'_i)\}$, for each variable x_i in S ;
- $\{(x_i, u_{i,1}), \dots, (x_i, u_{i,2k-3})\}$, for each variable x_i in S ;
- $\{(\neg x_i, p_{i,1}), \dots, (\neg x_i, p_{i,2k-3})\}$, for each variable x_i in S ;
- $\{(z_i, w_{i,1}), \dots, (z_i, w_{i,k-2})\}$, for each variable x_i in S , provided $k > 2$;
- $\{(z'_i, w'_{i,1}), \dots, (z'_i, w'_{i,k-2})\}$, for each variable x_i in S , provided $k > 2$;
- $\{(c'_j, b'_{j,1}), \dots, (c'_j, b'_{j,k-1})\}$, for each clause c_j in S ;
- $\{(c_j, b_{j,1}), \dots, (c_j, b_{j,k-2})\}$, for each clause c_j in S , provided $k > 2$;
- $\{(c_j, x_i), (c'_j, x_i)\}$, for each literal x_i occurring in clause c_j ;
- $\{(c_j, \neg x_i), (c'_j, \neg x_i)\}$, for each literal $\neg x_i$ occurring in clause c_j .

We finish the construction by defining the target configuration Y :

- $Y(x_i) = Y(\neg x_i) = +1$, for $1 \leq i \leq N$;
- $Y(z_i) = +1$, $Y(z'_i) = -1$, for $1 \leq i \leq N$;
- $Y(u_{i,j}) = Y(p_{i,j}) = +1$, for $1 \leq i \leq N$ and $1 \leq j \leq k-1$;
- $Y(u_{i,j}) = Y(p_{i,j}) = -1$, for $1 \leq i \leq N$ and $k \leq j \leq 2k-3$, provided $k > 2$;
- $Y(w_{i,j}) = -1$, $Y(w'_{i,j}) = +1$, for $1 \leq i \leq N$ and $1 \leq j \leq k-2$, provided $k > 2$;
- $Y(c_i) = +1$, $Y(c'_i) = -1$, for $1 \leq i \leq M$;
- $Y(b_{i,j}) = -1$, for $1 \leq i \leq M$ and $1 \leq j \leq k-2$, provided $k > 2$;
- $Y(b'_{i,j}) = -1$, for $1 \leq i \leq M$ and $1 \leq j \leq k-1$, provided $k > 2$.

Figure 1 illustrates the case of $k = 3$, $M = 1$, and $N = 3$, the single clause being $c_1 = x_1 \vee \neg x_2 \vee \neg x_3$.

For each variable in S , at most $6k-6$ vertices are created, and for each clause at most $2k-1$ vertices, resulting in at most $n = N(6k-6) + M(2k-1)$ vertices. Likewise, the total number of edges is at most $m = N(6k-6) + M(2k+3)$. Since k is a constant, we have a polynomial-time reduction.

We proceed by showing that if S is satisfiable then Y has at least one predecessor configuration. In fact, given any satisfying truth assignment for S , we can construct a predecessor configuration Y' of Y in the following manner:

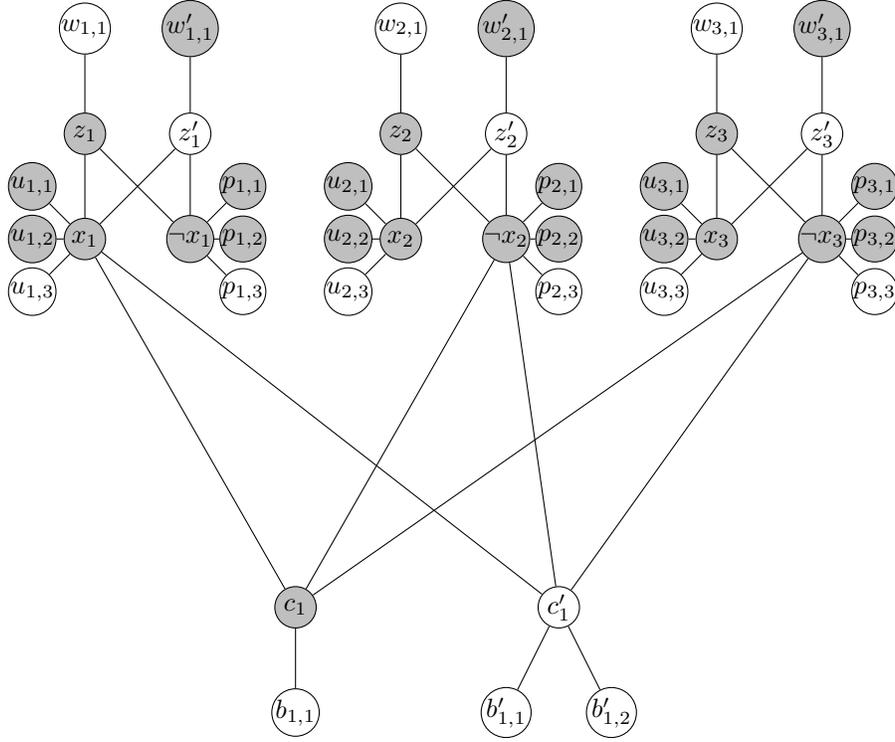


Figure 1: Graph G in the instance of PRE(3) having $M = 1$ and $N = 3$ for which $c_1 = x_1 \vee \neg x_2 \vee \neg x_3$. Shaded circles indicate state +1 in configuration Y ; empty circles indicate state -1.

- $Y'(x_i) = +1$, if variable x_i is true in the given assignment;
- $Y'(x_i) = -1$, if variable x_i is false in the given assignment;
- $Y'(\neg x_i) = -Y'(x_i)$;
- $Y'(z_i) = +1$, $Y'(z'_i) = -1$, for $1 \leq i \leq N$;
- $Y'(u_{i,j}) = Y'(p_{i,j}) = +1$, for $1 \leq i \leq N$ and $1 \leq j \leq k - 1$;
- $Y'(u_{i,j}) = Y'(p_{i,j}) = -1$, for $1 \leq i \leq N$ and $k \leq j \leq 2k - 3$;
- $Y'(w_{i,j}) = -1$, $Y'(w'_{i,j}) = +1$, for $1 \leq i \leq N$ and $1 \leq j \leq k - 2$, provided $k > 2$;
- $Y'(c_i) = Y'(c'_i) = +1$, for $1 \leq i \leq M$;
- $Y'(b_{i,j}) = -1$, for $1 \leq i \leq M$ and $1 \leq j \leq k - 2$, provided $k > 2$;
- $Y'(b'_{i,j}) = -1$, for $1 \leq i \leq M$ and $1 \leq j \leq k - 1$.

Figure 2 shows predecessor configuration Y' for the Y given in Figure 1. As an example, we have let all of x_1 , $\neg x_2$, and x_3 be true in the satisfying assignment.

By construction, and considering configuration Y' as described above, each vertex x_i or $\neg x_i$ has at least k neighbors in state $+1$ in Y' and exactly $k - 1$ neighbors in state -1 . This condition is sufficient to guarantee that every literal vertex reaches state $+1$ in one time step, regardless of its state in configuration Y' .

Each of vertices $u_{i,j}$, $p_{i,j}$, $w_{i,j}$, $w'_{i,j}$, $b_{i,j}$, and $b'_{i,j}$ has only one neighbor, and will obviously keep its state in the next configuration. Each of vertices z_i and z'_i has at most $k - 1$ neighbors in the opposite state, since vertices x_i and $\neg x_i$ have mutually opposite states. So z_i and z'_i remain unchanged as well.

In order for configuration Y to be obtained after one time step, in configuration Y' no vertex c_i can have more than one literal neighbor in state -1 , which would lead to state -1 for c_i in the next configuration. Since S is satisfiable, there are exactly two positive literals for each clause in S , and by construction of Y' , each vertex c_i has exactly one literal neighbor in state -1 . Similarly, every vertex c'_i needs at least one literal neighbor in state -1 , and as the construction guarantees, c'_i has exactly one literal neighbor in state -1 .

Hence, given the configuration Y' as constructed, we see that the next configuration is precisely configuration Y . We then conclude that whenever S is satisfiable, Y has at least one predecessor configuration.

Conversely, we now show that if Y has at least one predecessor configuration then S is satisfiable.

In any predecessor configuration of Y , vertices $u_{i,j}$, $p_{i,j}$, $w_{i,j}$, $w'_{i,j}$, $b_{i,j}$, and $b'_{i,j}$ should all be in the same states as in configuration Y , since they all have only one neighbor each.

Vertices z_i and z'_i should also have the same states as in configuration Y in any predecessor configuration. For suppose that, in a predecessor configuration, z_i is in state -1 ; then necessarily x_i and $\neg x_i$ should be in state $+1$, and consequently vertex z'_i would reach state $+1$ in the next configuration, which is different from its state in configuration Y . An analogous argument holds for vertex z'_i . This condition also forces x_i to have a state different than $\neg x_i$ in any predecessor configuration of Y , since if both have the same state then in the next configuration z_i and z'_i will also have the same state.

Each vertex c_i must have state $+1$ in a predecessor configuration of Y , otherwise every literal neighbor of c_i would need to have state -1 in the predecessor configuration and consequently vertex c_i would not change to state $+1$, which is its state in Y . Also, it must be the case that at least two of the literal neighbors of c_i have state $+1$, otherwise c_i would have state -1 in the next configuration, thus not matching configuration Y .

Each vertex c'_i has the same literal neighbors as vertex c_i . As we know that some of these neighbors have state $+1$ in a predecessor configuration, c'_i must have state $+1$, otherwise all of these neighbors would need to have state -1 in the predecessor configuration, contradicting the fact that some of them have state $+1$. Along with the restriction imposed by c_i , we have that exactly two

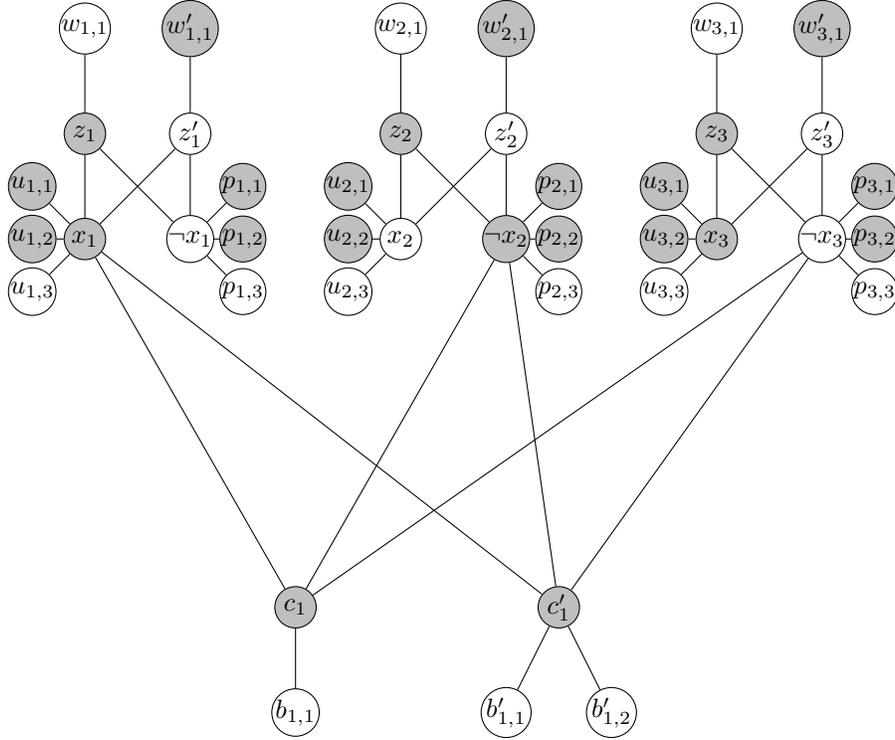


Figure 2: Predecessor configuration Y' for the G and Y of Figure 1 when all of x_1 , $\neg x_2$ and x_3 are true. Shaded circles indicate state +1 in Y' ; empty circles indicate state -1.

of the three literal vertices associated with clause c_i must have state +1 in a predecessor configuration.

Hence, given any predecessor configuration of Y , we can associate with any literal the value true if its vertex has state +1, and value false otherwise. The construction guarantees that this will satisfy every clause with exactly two positive literals and that no opposite literals will have the same assignment.

We conclude that if Y has a predecessor configuration then S is satisfiable. \square

Corollary 4. $\text{PRE}(k)$ on bipartite graphs is NP-complete for $k > 1$.

Proof. The graph constructed in the proof of Theorem 3 is bipartite for the node sets $\bigcup_{i,j} \{x_i, \neg x_i, b_{i,j}, b'_{i,j}, w_{i,j}, w'_{i,j}\}$ and $\bigcup_{i,j} \{c_i, c'_i, u_{i,j}, p_{i,j}, z_i, z'_i\}$. \square

4 Polynomial-time algorithms for trees

In this section, we consider a tree T rooted at an arbitrary vertex called *root*. For each vertex v in T , $parent_v$ denotes the parent of v , and $children_v$ denotes the set of children of v . A subtree of T will be denoted by T_u , where u is the root of the subtree.

It will be helpful to adopt a notation for a configuration of a subtree of T . Denote by $Y_{v,t} = (Y_t(w_1), Y_t(w_2), \dots, Y_t(w_{|T_v|}))$ the configuration of states of the vertices in subtree T_v at time t , where each w_i is one of the $|T_v|$ vertices of T_v . Notice that $Y_{v,t}$ is a subsequence of Y_t and its purpose is to refer to states of vertices in subtree T_v only; thus, if $v = root$, $Y_{v,t} = Y_t$.

The one-step dynamics in a k -reversible process in subtree T_v can be described using the function $F_{G,v}^k : Q^{|T_v|} \rightarrow Q^{|T_v|}$ such that

$$Y_{v,t} = F_{G,v}^k(Y_{v,t-1}) = (Y_t(w_1), Y_t(w_2), \dots, Y_t(w_{|T_v|})). \quad (2)$$

As in Section 1, we omit, for simplicity, the subscript referring to time, using notation Y' to refer to the configuration at time $t-1$ and Y to refer to the configuration at time t . Hence, $Y_v = Y_{v,t}$ and $Y'_v = Y_{v,t-1}$.

So long as we take into account the influence of $parent_v$ on the dynamics of T_v , then it is easy to see that the following holds. If a configuration Y' exists for which $Y = F_G^k(Y')$, then we have $Y_v = F_{G,v}^k(Y'_v)$ as well. That is, the subsequence of Y' that corresponds to T_v is a predecessor configuration of the subsequence of Y that corresponds to T_v .

Section 4.1 presents an algorithm that solves $PRE(k)$ in polynomial time for any k when the graph is a tree. Section 4.2 presents an algorithm that solves the associated counting problem $\#PRE(k)$, also in polynomial time, for any k when the graph is a tree.

4.1 Polynomial-time algorithm for $PRE(k)$

We start by defining the function $vstate(target, current, p, k)$ that determines whether a vertex in state *current* can reach state *target* in one time step assuming that it has p neighbors in the state different than *current* in a k -reversible process. This is a simple Boolean function that only checks whether a given state transition is possible. It can be calculated in $O(1)$ time, since

$$vstate(target, current, p, k) = \begin{cases} false, & \text{if } current = target \text{ and } p \geq k, \\ & \text{or } current \neq target \text{ and } p < k; \\ true, & \text{otherwise.} \end{cases} \quad (3)$$

The algorithm tries to build a predecessor configuration Y' of Y by determining possible configurations for its subtrees and testing states on vertices. Suppose we traverse the tree in a top-down fashion attaching states to each vertex we visit. Let v be the vertex of T that the algorithm is visiting. Suppose that for $parent_v$ the algorithm attached state c . We define the function $fstate(v, c)$ that returns the state that v must have in a configuration Y'_v such that $F(Y'_v) = Y_v$

or returns ∞ if there is no such configuration Y'_v when $Y'(parent_v) = c$. If both states are possible for v , the function simply returns $Y(parent_v)$ or $+1$ when v is the root.

We assume that function $fstate$ will be called with parameters v and 0 when v is the root of the tree. Hence, $fstate(root, 0)$ is simply the state that $root$ must have in a predecessor configuration of Y . If $fstate(root, 0)$ is different than ∞ , configuration Y has a predecessor configuration; otherwise, it does not.

Given the function $fstate(v, c)$ one can easily test whether the state c attached during the algorithm is possible or not in a predecessor configuration of Y . When visiting vertex $parent_v$, the algorithm calls the function $fstate(w, c)$ for each child w of $parent_v$, and then by using function $vstate$ it decides whether a transition is possible from state c to state $Y(parent_v)$.

Notice that when it is possible for both states to be returned by $fstate(v, c)$, we define the function to return state $Y(parent_v)$. This choice is always correct in this case, since it is not important for the state to be actually assigned to $parent_v$ in the predecessor configuration. What it does is to increase the number of neighbors that can help $parent_v$ to reach state $Y(parent_v)$ in the next time step. We maximize the number of neighbors in state $Y(parent_v)$, and if this maximum number is not enough to make $parent_v$ reach state $Y(parent_v)$, then a lower number of neighbors would certainly not be either.

Now, the problem is to calculate $fstate(root, 0)$ in a correct and efficient way. Assume that for each child f of v the values of $fstate(f, +1)$ and $fstate(f, -1)$ are correctly calculated. In other words, we know the states of every child of v when v has state $+1$ or state -1 in a predecessor configuration. Let l be the number of children of v with state other than the $Y'(v)$ calculated by the $fstate$ function. As we set the state of $parent_v$ we can update l if $parent_v$ also has a state different than $Y'(v)$, meaning that l represents the number of vertices in a state different than v in the configuration we are trying to construct. Then we can verify $vstate(Y(v), st, l, k)$ to check whether st is a valid state for v in a predecessor configuration Y' such that $F_{G,v}^k(Y'_v) = Y_v$.

An important observation to have in mind is that, for the case in which there is at least one child f of v that does not have a possible state (which means that $fstate(f, st) = \infty$), then the state st is not valid for v .

Given that we already know all the valid states for v when $Y'(parent_v) = c$, function $fstate$ can be calculated like this:

$$fstate(v, c) = \begin{cases} Y(parent_v), & \text{if both states are valid for } v; \\ +1, & \text{if only state } +1 \text{ is valid for } v; \\ -1, & \text{if only state } -1 \text{ is valid for } v; \\ \infty, & \text{otherwise.} \end{cases} \quad (4)$$

Since we already know which the valid states are, $fstate(v, c)$ is easily determined in $O(1)$ time. To check whether a given state is valid or not we simply need to count the number of neighbors with state opposite to the one being checked using function $fstate$, and then call function $vstate$ to verify whether the state is valid. This can be done in $O(d(v))$ time.

It is possible to calculate $fstate$ for all vertices in T using a recursive algorithm similar to depth-first search. The algorithm, when visiting vertex v with $parent_v$ in state c , recursively calculates $fstate(f, +1)$ and $fstate(f, -1)$ for each child f of v , and once the algorithm returns from the recursion, all values needed to calculate $fstate(v, c)$ are available. Hence, we simply need to calculate $fstate(root, 0)$ recursively and check whether the returned value is ∞ or not.

This is what Algorithm 1 does. The algorithm maintains a table $fstate$ that contains all the function values. This table is initialized with the null value for all vertices and states.

For each vertex v , the algorithm tries to assign state $Y(parent_v)$, and in case this is a valid state, this value is stored in the table and returned. Otherwise, the algorithm tries to assign the opposite state to $Y(parent_v)$ and does assign it in case it is a valid state. If none of the states is valid then the algorithm stores ∞ in the table and returns. Notice that when the algorithm accesses a position in the table it increments the c value by 1, so it does not try to access a negative position when c has value -1 . Table handling is very important to make the algorithm run in polynomial time. Without verification in line 2, it would be a simple backtracking algorithm with time complexity $O(2^n)$. We can verify this by analyzing the case in which T is the graph P_n containing a single path with n vertices and $k = 2$. Let $H(n)$ be the worst-case time complexity of the algorithm in this case, without using the table. Suppose we choose the root to be one of the vertices with degree 1. $H(n)$ is easily verified to be

$$H(n) = 2H(n - 1) + O(1), \tag{5}$$

considering that the algorithm tries both states for each vertex, which happens when the given input configuration does not have a predecessor configuration. Additionally, the structure of P_n allows us to calculate the number of steps as a function of the number of steps to solve the problem on the subtree P_{n-1} . By solving the above recurrence relation from $H(1) = O(1)$ we obtain $H(n) = O(2^n)$. Using the table amounts to employing memoization [4] to avoid the exponential running time.

Suppose that the last function call is $calcstate(v, c)$ and that this is the first time the function is called with these parameters. In the worst case, the algorithm calls $calcstate(f, +1)$ and $calcstate(f, -1)$ for each child f of v and calculates $fstate[v, c + 1]$. For each child f of v , the algorithm makes two visits, and any other call to $calcstate(v, c)$ will not result in any other visit to v 's children since the algorithm returns $fstate[v, c + 1]$ in line 3. When the algorithm calls $calcstate(v, -c)$, again in the worst case it will call $calcstate(f, +1)$ and $calcstate(f, -1)$ for each child f of v , incrementing to four the number of visits to each child of v . After this call to $calcstate(v, -c)$, any other visit to vertex v will not produce any other visit to any child of v since both $fstate(v, c)$ and $fstate(v, -c)$ will be stored in the table and the algorithm will return in line 3. We conclude that each vertex will be visited at most four times. We also conclude that each edge will be traversed at most four times, twice for each state being tested. Thus, the time complexity of Algorithm 1 is $O(n + m)$. Using $m = n - 1$ yields a time complexity of $O(n)$.

Algorithm 1: $\text{calcstate}(v, c)$

```
1 begin
2   if  $fstate[v, c + 1] \neq NIL$  then
3      $\lfloor$  return  $fstate[v, c + 1]$ ;
4    $count \leftarrow 0$ ;
5    $state \leftarrow Y(\text{parent}_v)$ ;
6   while  $count \neq 2$  do
7      $count \leftarrow count + 1$ ;
8      $l \leftarrow 0$ ;
9      $ret \leftarrow true$ ;
10    if  $c = -state$  then
11       $\lfloor$   $l \leftarrow l + 1$ ;
12    foreach  $f \in \text{children}_v$  do
13      if  $\text{calcstate}(f, state) = -state$  then
14         $\lfloor$   $l \leftarrow l + 1$ ;
15      if  $\text{calcstate}(f, state) = \infty$  then
16         $\lfloor$   $ret \leftarrow false$ ;
17    if  $vstate(Y(v), state, l, k) = true$  and  $ret \neq false$  then
18       $\lfloor$   $fstate[v, c + 1] \leftarrow state$ ;
19       $\lfloor$  return  $state$ ;
20     $state \leftarrow -state$ ;
21   $fstate[v, c + 1] \leftarrow \infty$ ;
22  return  $\infty$ ;
```

Algorithm 2: *buildstate*(v, c)

```
1 begin
2    $y[v] = fstate[v, c + 1];$ 
3   foreach  $f \in children_v$  do
4      $buildstate(f, y[v]);$ 
```

Algorithm 2 implements the recursive idea to recover a predecessor configuration Y' once we know that one exists. It traverses the tree accessing values in table *fstate* according to the vertex being visited and to the state assigned to its parent. Once a state is assigned to a vertex, the algorithm looks at the table to check the only option of recursive call to make. Algorithm 2 traverses the tree exactly once, assigning states; thus, its time complexity is $O(n)$.

4.2 Polynomial-time algorithm for $\#PRE(k)$

Suppose a class of trees constructed in the following manner:

- A vertex v_1 connected to $p > 1$ vertices v_2, v_3, \dots, v_{p+1} ;
- Each vertex v_i , with $2 \leq i \leq p + 1$, connected to two other vertices, denoted by d_{i-1} and e_{i-1} .

Each tree in this class has $3p + 1$ vertices. Assume a 2-reversible process and a configuration Y in which all states are $+1$. In this case, any configuration in which vertex v_1 has state -1 and at least two vertices out of v_2, v_3, \dots, v_{p+1} have state $+1$ is a predecessor configuration of Y . Notice that all vertices d_i and e_i must have state $+1$ in a predecessor configuration of Y , since they have degree 1. A lower bound on the number of predecessor configurations of Y for this class of trees is given by

$$\sum_{i=2}^p \binom{p}{i} = 2^p - p - 1. \quad (6)$$

Figure 3 illustrates the construction of a tree in this class for $p = 3$ and the respective configuration Y . We also illustrate some of the possible predecessor configurations for a 2-reversible process.

Given this lower bound on the number of predecessor configurations, it turns out that modifying the previous algorithm in order to store all possible predecessor configurations and then reconstruct them is an exponential-time task. However, solving the associated counting problem can be done in time $O(n^2)$ for every k .

In order to do this, we use a function similar to *fstate*. However, a more robust function must be used, one that will contain not only the state that a vertex must have in a predecessor configuration, but also the number of predecessor configurations of the subtree for the cases in which the vertex has states

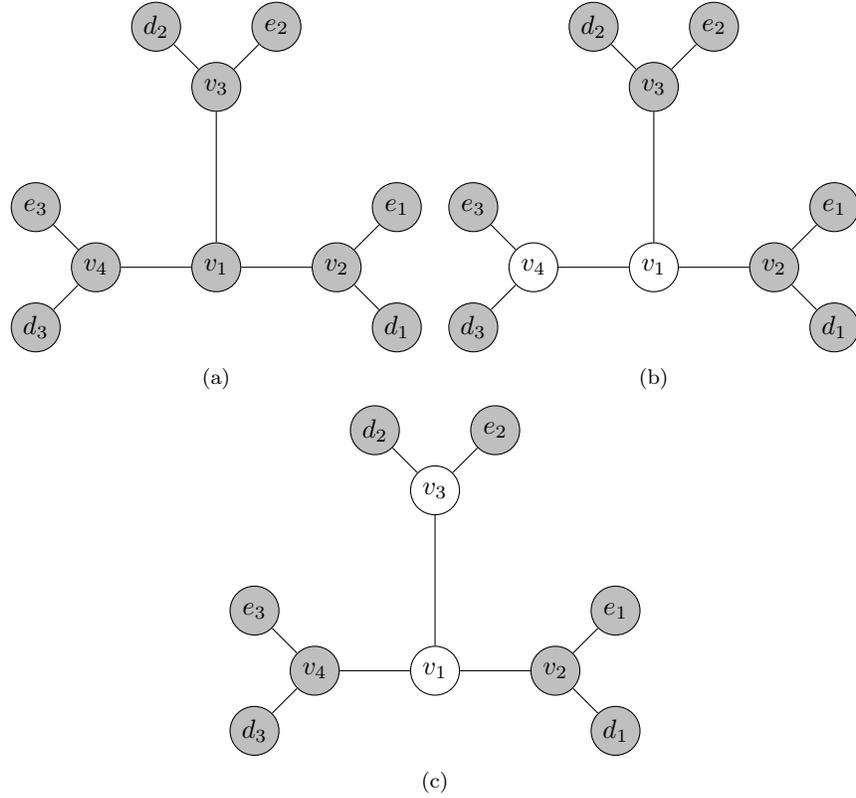


Figure 3: Configuration Y with a possibly exponential number of predecessor configurations. Shaded circles indicate state $+1$; empty circles indicate state -1 . (a) Configuration Y ; (b) Predecessor configuration Y' ; (c) Predecessor configuration Y'' .

$+1$ and -1 . We define the function $cfstate(v, c)$ as an ordered pair whose first element is the number of predecessor configurations of subtree T_v when $parent_v$ has state c and v has state $+1$, the second element being the number of predecessor configurations of subtree T_v when $parent_v$ has state c and v has state -1 .

Similarly to function $fstate$, when v is the root of the tree, function $cfstate$ is called with parameters v and 0 . Thus, the total number of predecessor configurations of subtree T_v , when $parent_v$ has state c , is the sum of the two elements in the ordered pair $cfstate(v, c)$, and in case v is the root, the sum of the two elements in the ordered pair $cfstate(v, 0)$.

The natural way to calculate $cfstate(v, c)$ is quite simple. Without loss of generality, suppose that we are calculating the first element of pair $cfstate(v, c)$ and that, in this case, at least l children of v must have an arbitrary state st in a predecessor configuration. For simplicity, the first element in the pair

$cfstate(v, c)$ will be denoted by $cfstate(v, c)_{+1}$, whereas the second element will be denoted by $cfstate(v, c)_{-1}$. Then

$$cfstate(v, c)_{+1} = \sum_{X \in C_v^l} calc(X, st), \quad (7)$$

where C_v^l is the set of all subsets of children of v with at least l elements and

$$calc(X, st) = \begin{cases} 1, & \text{if } X = \emptyset; \\ \prod_{f \in X} cfstate(f, +1)_{st} \prod_{f \in V(G) \setminus X} cfstate(f, +1)_{-st}, & \text{otherwise.} \end{cases} \quad (8)$$

In other words, we simply test all possibilities of state assignment to all children of v such that at least l of them have state $+1$. For each one of these possibilities we calculate the total number of predecessor configurations, multiplying the number of predecessor configurations for each subtree. The value $cfstate(v, c)_{+1}$ is the sum obtained in each possibility.

Notice that it is possible to calculate $cfstate(v, c)_{-1}$ likewise, again assuming that in each predecessor configuration at least l children of v have state st . We just need to redefine $calc(X, st)$ to be

$$calc(X, st) = \begin{cases} 1, & \text{if } X = \emptyset; \\ \prod_{f \in X} cfstate(f, -1)_{st} \prod_{f \in V(G) \setminus X} cfstate(f, -1)_{-st}, & \text{otherwise.} \end{cases} \quad (9)$$

A point to note in this approach is that the total number of configurations to iterate through is $O(2^{d(v)-1})$. For example, for $l = 1$,

$$|C_v^1| = \sum_{i=1}^{d(v)-1} \binom{d(v)-1}{i} = 2^{d(v)-1} - 1. \quad (10)$$

We can, however, use dynamic programming [4] to calculate $cfstate(v, c)$ in polynomial time without needing to iterate through all possible configurations.

Assume that the children of v are ordered as $child_{v,0}, \dots, child_{v,d(v)-2},$ where v is an internal vertex of the tree. If v is the root, the order is: $child_{v,0}, \dots, child_{v,d(v)-1}$. For simplicity, denote the number of children of v by $d'(v)$.

Define the function $g_v^{rt}(i, j)$ as the total number of predecessor configurations for subtree T_v in which v has state rt and, moreover, exactly j of the vertices $child_{v,0}, child_{v,1}, \dots, child_{v,i-1}$ have state $+1$. Similarly, define $h_v^{rt}(i, j)$ as the total number of predecessor configurations for subtree T_v in which v has state rt and, moreover, exactly j of the vertices $child_{v,0}, child_{v,1}, \dots, child_{v,i-1}$ have state -1 . Then we can calculate $cfstate(v, c)_{rt}$ in the following way.

If at least l children of v are required to have state $+1$ in a predecessor configuration of T_v , then

$$cfstate(v, c)_{rt} = \sum_{i=l}^{d'(v)} g_v^{rt}(d'(v), i), \quad (11)$$

where $g_v^{rt}(i, j)$ is defined recursively, as in

$$g_v^{rt}(i, j) = \begin{cases} 0, & \text{if } i = 0 \text{ and } j > 0; \\ 1, & \text{if } i = 0 \text{ and } j = 0; \\ g_v^{rt}(i-1, j)a_i + g_v^{rt}(i-1, j-1)b_i, & \text{if } i > 0 \text{ and } j > 0; \\ g_v^{rt}(i-1, j)a_i, & \text{if } i > 0 \text{ and } j = 0, \end{cases} \quad (12)$$

with $a_i = cfstate(child_{v,i}, rt)_{-1}$ and $b_i = cfstate(child_{v,i}, rt)_{+1}$.

If, instead, at least l children of v are required to have state -1 in a predecessor configuration of T_v , then

$$cfstate(v, c)_{rt} = \sum_{i=l}^{d'(v)} h_v^{rt}(d'(v), i), \quad (13)$$

where $h_v^{rt}(i, j)$ is such that

$$h_v^{rt}(i, j) = \begin{cases} 0, & \text{if } i = 0 \text{ and } j > 0; \\ 1, & \text{if } i = 0 \text{ and } j = 0; \\ h_v^{rt}(i-1, j)b_i + h_v^{rt}(i-1, j-1)a_i, & \text{if } i > 0 \text{ and } j > 0; \\ h_v^{rt}(i-1, j)b_i, & \text{if } i > 0 \text{ and } j = 0. \end{cases} \quad (14)$$

As given above, the calculation of $g_v^{rt}(i, j)$ involves four cases that depend on both state possibilities for vertex $child_{v,i-1}$. They are the following (the cases for $h_v^{rt}(i, j)$ are analogous):

- $i = 0$ and $j > 0$: In this case there is no vertex to be considered and there should exist $j > 0$ vertices in state $+1$. Hence, no predecessor configuration exists.
- $i = 0$ and $j = 0$: This is the only case in which a predecessor configuration exists with $i = 0$, since there is no need to have a vertex in state $+1$. Hence, there exists exactly one predecessor configuration.
- $i > 0$ and $j > 0$: In this case we add the number of predecessor configurations for the first i subtrees, considering that vertex $child_{v,i-1}$ has state $+1$, to the total number of predecessor configurations when this vertex has state -1 . Notice that if we assume that $child_{v,i-1}$ has state $+1$, then exactly $j-1$ of vertices $child_{v,0}, \dots, child_{v,i-2}$ must have state $+1$ as well. Otherwise, if we assume $child_{v,i-1}$ to have state -1 , then j of the first $i-1$ children of v must have state $+1$.
- $i > 0$ and $j = 0$: In this case none of vertices $child_{v,0}, \dots, child_{v,i-1}$ is required to have state $+1$. We calculate the total number of predecessor configurations of the subtrees rooted at vertices $child_{v,0}, \dots, child_{v,i-2}$ with all of them having state -1 , which is given by $g_v^{rt}(i-1, j)$, and multiply it by the total number of predecessor configurations of subtree $T_{child_{v,i-1}}$ with vertex $child_{v,i-1}$ having state -1 , which is given by $cfstate(child_{v,i-1}, rt)_{-1}$.

Algorithm 3: *countfstate*(v, c)

```
1 begin
2   if  $cfstate[v, c + 1] \neq NIL$  then
3      $\lfloor$  return  $cfstate[v, c + 1]$ ;
4    $d'(v) \leftarrow d(v) - 2$ ;
5   if  $v = root$  then
6      $\lfloor$   $d'(v) \leftarrow d(v) - 1$ ;
7    $count \leftarrow 0$ ;  $state \leftarrow +1$ ;
8   while  $count \neq 2$  do
9      $i \leftarrow 0$ ;  $count \leftarrow count + 1$ ;
10    foreach  $f \in parent_v$  do
11       $\lfloor$   $par[i] \leftarrow countfstate(f, state)$ ;  $i \leftarrow i + 1$ ;
12     $l \leftarrow threshold_v(state, c, k)$ ;
13    if  $v = root$  then
14       $\lfloor$   $l \leftarrow threshold_v(state, \infty, k)$ ;
15     $tab[0, 0] \leftarrow 1$ ;
16    for  $i \leftarrow 1$  to  $d'(v)$  do
17       $\lfloor$   $tab[0, j] \leftarrow 0$ ;
18    for  $i \leftarrow 1$  to  $d'(v)$  do
19      for  $j \leftarrow 0$  to  $d'(v)$  do
20        if  $Y(v) = +1$  then
21           $\lfloor$   $tab[i, j] \leftarrow tab[i - 1, j]par[i]_-$ ;
22          if  $j > 0$  then
23             $\lfloor$   $tab[i, j] \leftarrow tab[i, j] + tab[i - 1, j - 1]par[i]_+$ ;
24        else
25           $\lfloor$   $tab[i, j] \leftarrow tab[i - 1, j]par[i]_+$ ;
26          if  $j > 0$  then
27             $\lfloor$   $tab[i, j] \leftarrow tab[i, j] + tab[i - 1, j - 1]par[i]_-$ ;
28     $cfstate[v, c + 1]_{state} \leftarrow \sum_{i=l}^{d'(v)} tab[d'(v), i]$ ;
29     $state \leftarrow -state$ ;
30   $\lfloor$  return  $cfstate[v, c + 1]$ ;
```

Applying the recursion directly results in an algorithm whose number of operations grows very fast. Thus, once again we resort to dynamic programming to calculate g_v^{rt} and h_v^{rt} .

Algorithm 3 implements the recursive scheme given above. Similarly to Algorithm 1, we keep *cfstate* values in a table to avoid exponential times. The algorithm does not use the functions g_v^{rt} and h_v^{rt} explicitly, but instead a table *tab* for checking the value of $Y(v)$ to decide which multiplication to perform in lines 23 and 27. We use g_v^{rt} if $Y(v) = +1$ and h_v^{rt} if $Y(v) = -1$. Suppose that $Y(v) = +1$. If in a predecessor configuration of Y vertex v has state -1 , then, depending on the state of $parent_v$, vertex v will need k or $k - 1$ children with state $+1$ in that configuration. If v has state $+1$ in the predecessor configuration, then, depending on the state of $parent_v$, vertex v will need $d(v) - k + 1$ or $d(v) - k$ children with state $+1$. The analysis for the case of $Y(v) = -1$ is analogous.

Besides choosing which function to use, we also need to calculate the value of l . Given a vertex v and that $parent_v$ has state c in the predecessor configuration, we define the function $threshold_v(current, c, k)$ as the least number of children of v in state $Y(v)$ in the predecessor configuration such that in the next step v has state $Y(v)$, assuming that v has state $current$ in the predecessor configuration. Thus,

$$threshold_v(current, c, k) = \begin{cases} \min\{d(v) - k + 1, 0\}, & \text{if } Y(v) = current \\ & \text{and } c \neq Y(v); \\ \min\{d(v) - k, 0\}, & \text{if } Y(v) = current \\ & \text{and } c = Y(v); \\ k, & \text{if } Y(v) \neq current \\ & \text{and } c \neq Y(v); \\ k - 1, & \text{if } Y(v) \neq current \\ & \text{and } c = Y(v). \end{cases} \quad (15)$$

The time spent for each vertex v in the double loop of line 18 is $O(d'(v)^2)$. Summing up over all vertices we get an $O(n^2)$ time complexity.

5 Polynomial-time algorithm for PRE(2) on graphs with maximum degree no greater than 3

In this section, we show that PRE(2) is in P when $\Delta(G) \leq 3$, where $\Delta(G)$ is the maximum degree in G . Hence, this result covers the case of cubic graphs.

We show how to reduce the problem to 2SAT, solvable in $O(N + M)$ time; as before, N is the number of variables and M is the number of clauses [2]. That is, given a configuration Y we want to create a 2SAT instance S such that S is satisfiable if and only if Y has a predecessor configuration.

We start creating S by adding literals x_v and $\neg x_v$ for each vertex v in the graph. We will construct the clauses of S in such a way that whenever Y has a predecessor configuration Y' , S is satisfied by letting each x_v with $Y'(v) = +1$ be

true and each x_v with $Y'(v) = -1$ be false. Similarly, from any satisfying truth assignment for S we construct a predecessor configuration of Y by assigning state $+1$ to v whenever x_v is true and assigning state -1 whenever x_v is false. Because $\Delta(G) \leq 3$, we can construct a set of clauses in the following way.

For each vertex v such that $Y(v) = +1$:

- If $d(v) = 1$: In the predecessor configuration Y' , v must have the same state as in configuration Y , since the process is 2-reversible. Thus, we add the clause:

- x_v .

Clearly, this clause is satisfied whenever Y has a predecessor configuration.

- If $d(v) = 2$ with neighbors u and w : If in the predecessor configuration Y' vertex v has state $+1$, then in Y' at least one of its neighbors must also have state $+1$. If in the predecessor configuration Y' vertex v has state -1 , then both its neighbors must have state $+1$. We can encode these conditions by adding the following clauses:

- $\neg x_v \rightarrow x_u \equiv x_v \vee x_u$;
- $\neg x_v \rightarrow x_w \equiv x_v \vee x_w$;
- $x_v \rightarrow x_u \vee x_w \equiv \neg x_v \vee x_u \vee x_w$.

Analyzing these three clauses reveals that when x_v is true then x_u or x_w is also true in order to make all three clauses satisfiable. In case x_v is false, we force x_u and x_w to have value true. We simplify the clauses as follows:

- $x_v \vee x_u$;
- $x_v \vee x_w$;
- $x_u \vee x_w$.

- If $d(v) = 3$ with neighbors u , w , and z : If in the predecessor configuration Y' vertex v has state $+1$, then at least two of its neighbors must have state $+1$ in Y' . If in Y' vertex v has state -1 , then again at least two of its neighbors must have state $+1$ in Y' . Therefore, we add the following clauses:

- $\neg x_v \rightarrow x_u \vee x_w \equiv x_v \vee x_u \vee x_w$;
- $\neg x_v \rightarrow x_u \vee x_z \equiv x_v \vee x_u \vee x_z$;
- $\neg x_v \rightarrow x_w \vee x_z \equiv x_v \vee x_w \vee x_z$;
- $x_v \rightarrow x_u \vee x_w \equiv \neg x_v \vee x_u \vee x_w$;
- $x_v \rightarrow x_u \vee x_z \equiv \neg x_v \vee x_u \vee x_z$;
- $x_v \rightarrow x_w \vee x_z \equiv \neg x_v \vee x_w \vee x_z$.

As the value assigned to x_v is not important in this subset of clauses, we can easily simplify them:

- $x_u \vee x_w$;
- $x_u \vee x_z$;
- $x_w \vee x_z$.

For each vertex v such that $Y(v) = -1$, the cases are analogous:

- If $d(v) = 1$, create the clause:
 - $\neg x_v$.
- If $d(v) = 2$ with neighbors u and w , create the clauses:
 - $\neg x_v \vee \neg x_u$;
 - $\neg x_v \vee \neg x_w$;
 - $\neg x_u \vee \neg x_w$.
- If $d(v) = 3$ with neighbors u , w , and z , create the clauses:
 - $\neg x_u \vee \neg x_w$;
 - $\neg x_u \vee \neg x_z$;
 - $\neg x_w \vee \neg x_z$.

We have constructed the set of clauses for S in such a way that it is directly satisfiable if Y has at least one predecessor configuration. It now remains for us to argue that the configuration Y' obtained from a satisfying truth assignment as explained above is indeed a predecessor of Y .

Suppose, to the contrary, that such a Y' is not a predecessor of Y . In other words, at least one vertex v exists that does not reach state $Y(v)$ within one time step. Analyzing the case of $Y(v) = +1$ we have the following possibilities:

- $Y'(v) = +1$: In order to force v to change its state v must have at least two neighbors in state -1 . Hence, v is necessarily a vertex with degree at least 2.

If $d(v) = 2$, then both neighbors of v have state -1 and we have clause $x_u \vee x_w$ not satisfied, which is a contradiction.

If $d(v) = 3$, then in a similar way there is an unsatisfied clause.

- $Y'(v) = -1$: In order to force v to remain in state -1 in the next time step, there must be at most one neighbor in state $+1$. Notice that we must have $d(v) \neq 1$, since $Y(v) = +1$, and then the satisfied clause x_v forces $Y'(v) = +1$.

If $d(v) = 2$ with neighbors u and w , then at least one of these two vertices must have state -1 ; hence v does not change its state to $+1$, but this implies that one of the clauses $x_v \vee x_w$ or $x_v \vee x_u$ is not satisfied.

If $d(v) = 3$, then since we have clauses with two literals involving all the three neighbors of v , at least one of them is not satisfied.

The case of $Y(v) = -1$ is analogous. We conclude that if S is satisfiable then configuration Y' is a predecessor configuration of Y .

To summarize, given a graph with n vertices, we create n variables and in the worst case $3n$ clauses. Each clause has at most two literals. Thus, S is indeed a 2SAT instance and we can solve PRE(2) in $O(n)$ time when $\Delta(G) \leq 3$.

6 Conclusions

In this paper we have dealt with PRE(k) and #PRE(k), our denominations for the PREDECESSOR EXISTENCE problem and its counting variation for k -reversible processes. We have shown that PRE(1) is solvable in polynomial time, that PRE(k) is NP-complete for $k > 1$ even for bipartite graphs, and that it can be solved in polynomial time for trees. For trees we have also shown that #PRE(k) is polynomial-time solvable. We have also demonstrated the polynomial-time solvability of PRE(2) if $\Delta(G) \leq 3$.

We identify two problems worth investigating:

- Identify other cases in which PRE(k) can be solved in polynomial time.
- Study the complexity properties of #PRE(2) for $\Delta(G) \leq 3$.

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